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# An almost-Robertson-Walker universe model and the equivalence classes of perturbations: Nonbarotropic perfect fluids

by

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**ABSTRACT.** – A new covariant and gauge-invariant treatment of perturbations, which is applicable to the case of an almost-Robertson-Walker universe dominated by a general perfect fluid with two essential thermodynamic variables, is presented. Beginning from the geometrical foundation, this paper proposes to define gauge-invariant perturbations as the equivalence classes of tangents to one-parameter families of exact solutions of the nonlinear field equations: two tangents  $\delta\mathcal{G}_0$  and  $\delta\mathcal{G}'_0$  are said to be equivalent if there is a transformation of the Lie type which carries  $\delta\mathcal{G}_0$  into  $\delta\mathcal{G}'_0$  and vice-versa. Denoting by  $[\delta\mathcal{G}_0]$  the equivalence class of  $\delta\mathcal{G}_0$ , it is demonstrated explicitly in the context of an almost-Robertson-Walker universe model that, for nonbarotropic perfect fluids, the precise definition of  $[\delta\mathcal{G}_0]$  is equivalent to defining and solving an appropriate system of linear propagation equations for the basic set of variables. This set consists of seventeen linearly independent, not identically vanishing gauge-invariant and covariantly defined quantities. A simple example illustrating the above result is given and elementary comparisons with other covariant and gauge-invariant approaches are also made. Finally, the paper discusses several new features associated with the so-called scalar perturbation theory.

RÉSUMÉ. – Nous présentons un nouveau traitement covariant et invariant de jauge de la théorie des perturbations, susceptible de s'appliquer au cas d'un univers presque Robertson-Walker dominé par un fluide parfait à deux variables thermodynamiques essentielles. A partir d'un formalisme géométrique, cette étude définit des perturbations invariantes de jauge comme des classes d'équivalence des espaces tangents à une famille de solutions exactes d'équations de champs non linéaires : de tels espaces tangents sont appelés équivalents s'il existe une transformation de Lie transformant l'un dans l'autre et réciproquement. En notant  $[\delta\mathcal{G}_0]$  la classe d'équivalence de  $\delta\mathcal{G}_0$  nous démontrons explicitement dans le contexte d'un modèle univers presque de Robertson-Walker, pour un fluide parfait non barotropique, que la définition précise de  $[\delta\mathcal{G}_0]$  équivaut à définir et à résoudre un système approprié d'équations linéaires de propagation pour l'ensemble des variables de base. Cet ensemble est constitué de dix-sept variables linéairement indépendantes, non identiquement nulles, définies de façon covariante et invariante de jauge. Nous donnons un exemple simple illustrant le résultat précédents et nous le comparons à d'autres approches covariantes et invariantes de jauge. Enfin, cet article discute plusieurs faits nouveaux associés avec la théorie de perturbation dite scalaire.

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## 1. INTRODUCTION

All the current generally accepted concepts of physical cosmology [1] are based on the Friedmann-Robertson-Walker universe models. Structure in the universe (galaxies, clusters of galaxies) is thought to have formed as a result of the growth, through gravitational collapse, of small spatial inhomogeneities in an otherwise homogeneous and isotropic cosmological model. The evolution of these small irregularities is governed by the equations of linear perturbation theory [2-4]. However, it has been known for a long time that the gauge problem plagues the study of perturbations in cosmology. Because of this, much attention has been paid to the introduction of a set of covariantly defined gauge-invariant quantities with a simple geometrical and physical meaning, that code the information we need to discuss inhomogeneities in an almost-Robertson-Walker universe model. Considerations in [5-7] are a selection of the more important comprehensive treatments.

The present paper develops, from new points of view, a totally gauge-invariant and covariant formulation of perturbation theory applicable to the case of a general perfect fluid with two essential thermodynamic variables. Beginning from the geometrical foundation, we propose to define perturbations as tangents to one-parameter families of exact solutions of the nonlinear field equations [8]. We then divide these perturbations into physically natural equivalence classes: two infinitesimal perturbations  $\delta\mathcal{G}_0$  and  $\delta\mathcal{G}'_0$  are said to be equivalent if there is a vector field  $v$  on the space-time manifold such that  $\delta\mathcal{G}'_0$  differs from  $\delta\mathcal{G}_0$  by the action of the Lie derivative  $\mathcal{L}_v$  on the background solution  $\mathcal{G}_0$ . The point of this discussion is that instead of concentrating on just one perturbation  $\delta\mathcal{G}_0$ , with the ambiguities that implies, we can deal directly with a whole equivalence class of all perturbations  $\delta\mathcal{G}'_0$  which are equivalent to  $\delta\mathcal{G}_0$ . This equivalence class is denoted  $[\delta\mathcal{G}_0]$  and is called the *gauge-invariant perturbation* associated with  $\delta\mathcal{G}_0$ .

Clearly, these definitions do not tell us how to use  $[\delta\mathcal{G}_0]$  in practical calculations, or whether such calculations are possible at all. However, to every gauge-invariant perturbation  $[\delta\mathcal{G}_0]$  there is an associated set  $\mathbb{D}$  of basic gauge-invariant quantities. More precisely, this set consists of seventeen linearly independent, not identically vanishing gauge-invariant and covariantly defined variables. One can think of  $\mathbb{D}$  as having three aspects. First,  $\mathbb{D}$  provides a mathematically simplest representation of the gauge-invariant perturbation  $[\delta\mathcal{G}_0]$ . In fact,  $[\delta\mathcal{G}_0]$  is uniquely determined from  $\mathbb{D}$  and vice-versa. Second, any gauge-invariant quantity is obtainable directly from the basic variables  $\mathbb{D}$  through purely algebraic and differential operations. Third, a complete set of propagation equations can be derived that involves only  $\mathbb{D}$ . These equations are physically more transparent than the usual ones, because spurious “gauge mode” solutions are automatically excluded.

As noted already, the approach developed here is both fully covariant and gauge invariant; thus it sidesteps the usual problems. Moreover, by formulating the theory in such a fashion one can give a clear discussion of the geometric and physical meaning of the variables introduced. We also mention the following: our basic definitions and equations are independent of any harmonic analysis, but they can be harmonically decomposed if desired. One good reason for considering such decompositions is simply to present an explanation of how the gauge-invariant potential technique of Bardeen [5] and Mukhanov *et al.* [7] relates to our method.

A covariant and gauge-invariant formalism in many respects alternative to this one has been developed by Ellis and Bruni [6], although closer

inspection shows that there are also some important similarities. Contact with their results is made in section 4.2, in order to understand the theory of cosmological perturbations from still another viewpoint.

The program of this paper is as follows. Sections 2 and 3 define the cosmological model and then recall the more relevant aspects of the “naive” version of linear perturbation theory. This will serve to establish notation for the subsequent sections and to indicate the type of results that one would hope to recover from a gauge-dependent description. The aim of section 4 is to present the theory of cosmological perturbations in a covariant and gauge-invariant form. In treating cosmological inhomogeneities, the analysis is commonly restricted to scalar perturbations, as these are the only ones relevant to the formation of galaxies. Thus section 5 proposes a new covariant and gauge-invariant treatment of scalar perturbations. Section 6 then shows how the resulting propagation equations can be solved in the simplest possible case when  $p_0 = 0$ ,  $\Lambda = 0$ , and  $k = 0$ <sup>1</sup>. Section 7 is for discussion and conclusion. The appendices address some important problems motivated by the body of the paper. Thus, appendices A and B prove that the definition of gauge-invariant perturbations  $[\delta\mathcal{G}_0]$  is equivalent to defining and solving the linear propagation equations for the basic set  $\mathbb{D}$  of variables. A representative example illustrating the above result is discussed in appendix C. The analysis of appendix A applies to the general case ( $k = 0, \pm 1$ ). The same remark concerns the discussion in appendix B, although the rigorous proof of *theorem 3* is given when the spatial three-geometry is flat ( $k = 0$ ). However, this theorem is expected to hold also for the  $k = \pm 1$  case (*see remark 2* in appendix B).

One final word regarding this paper. We consider nonbarotropic perfect fluids when there are two essential thermodynamic variables, so that our dynamical equations can be used to model the evolution of density and temperature irregularities after the entire radiation era. However, it is also possible to extend this framework to situations more complex than those discussed here. For example, our method can be refined to apply to an almost-Robertson-Walker universe model containing both radiation and relativistic or nonrelativistic matter.

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1. We denote by  $p_0$  the background pressure. The symbols  $\Lambda$  and  $k$  represent the cosmological constant and the background spatial curvature, respectively. By an appropriate choice of units, the value of  $k$  can be made to be 0 or  $\pm 1$ .

## 2. AN ALMOST-ROBERTSON-WALKER UNIVERSE MODEL

### 2.1. Preliminaries

The fundamental equations of general relativity are Einstein's equations given by

$$R_{ab} - \frac{1}{2} R_c^c g_{ab} + \Lambda g_{ab} = T_{ab}, \quad (2.1)$$

where  $R_{ab}$  is the Ricci tensor,  $g_{ab}$  is the metric of space-time,  $\Lambda$  is the cosmological constant, and  $T_{ab}$  is the stress-energy tensor of the source. We choose units so that the Einstein gravitational constant equals one ( $8\pi G = c = 1$ ); the space-time metric  $g_{ab}$  has signature  $(-, +, +, +)$ . For a perfect fluid, the stress-energy tensor  $T_{ab}$  takes the form

$$T_{ab} = (e + p) u_a u_b + p g_{ab}. \quad (2.2)$$

As usual,  $e$  is the energy density,  $p$  is the pressure, and  $u_a$  is the normalized fluid four-velocity ( $u^a u_a = -1$ ).

In addition to  $T_{ab}$ , we introduce a number flux density  $I^a$  which is required to satisfy the conservation equation

$$I^a_{;a} = 0, \quad (2.3)$$

where a semicolon denotes the covariant derivative of  $I^a$  with respect to  $g_{ab}$ . Clearly,  $I^a$  can be decomposed as

$$I^a = n u^a. \quad (2.4)$$

In the rest frame of the fluid,  $n$  is the number density. Another useful quantity is the temperature of the fluid. We denote this temperature by  $T$ . In the case of a general perfect fluid with two essential thermodynamic variables, it is conventional to express  $e$  and  $p$  in terms of  $n$  and  $T$  by using the equilibrium equations of state:

$$e = e(n, T), \quad p = p(n, T). \quad (2.5)$$

Because of the first and second laws of thermodynamics, these equations of state are not independent and one may obtain  $e = e(n, T)$  and  $p = p(n, T)$  from the specific free energy. However, the present general form of Equations (2.5) suffices for our purposes here.

Denoting by  $g^{ab}$  the contravariant image of  $g_{ab}$ , the fluid is described by specifying

$$\mathcal{G} := (g^{ab}, u^a, n, T), \quad (2.6)$$

and there are eleven dynamical Equations (2.1) and (2.3). This is in fact the correct number of equations to determine  $\mathcal{G}$ , since  $u^a u_a = -1$  and four of

the ten components of the metric can be given arbitrary values by use of the four degrees of freedom to make diffeomorphism [9].

Finally, we would like to mention that Einstein's equations (2.1) are compatible with the following equation

$$T_{;b}^{ab} = 0, \tag{2.7}$$

which is the equation for the conservation of mass-energy and momentum. Here, of course, the symbol  $T^{ab}$  represents the contravariant image of  $T_{ab}$ .

### 2.2. One-parameter families of exact solutions

Consider an open interval  $(-d, +d)$  of  $\mathbb{R}$ ,  $d > 0$ . Adapting the universal ideas of Ehlers [8] and Wald [10] (*see* also the discussion by D'Eath [11] and Banach and Piekarski [12, 13]), we assume that for each  $\varepsilon \in (-d, +d)$  there exists a classical solution  $\mathcal{G}_\varepsilon(x^c)$  of Equations (2.1) and (2.3):

$$\mathcal{G}_\varepsilon(x^c) := \{g^{ab}(\varepsilon, x^c), u^a(\varepsilon, x^c), n(\varepsilon, x^c), T(\varepsilon, x^c)\}. \tag{2.8}$$

Here  $(x^c)$  is an arbitrary set of coordinates for the description of space-time points and the parameter  $\varepsilon$  measures the size of perturbation in the sense that  $\mathcal{G}_\varepsilon(x^c)$  depends continuously on  $\varepsilon \in (-d, +d)$  for each  $(x^c)$  and

$$\mathcal{G}_0(x^c) := \lim_{\varepsilon \rightarrow 0} \mathcal{G}_\varepsilon(x^c) \tag{2.9}$$

is a background solution of Equations (2.1) and (2.3). Note that all fields in Equation (2.8) are defined on the same space-time manifold  $X$ . Thus, just as in the usual theory of partial differential equations, we regard  $g^{ab}$  or  $g_{ab}$  as the dependent variable on the same footing as  $u^a$ ,  $n$ , and  $T$ . This is the so-called passive approach to the equations of general relativity [8]. Another approach, in a sense equivalent to this one, is presented in [6].

For an almost-Robertson-Walker universe model, a one-parameter family of exact solutions, namely  $\{\mathcal{G}_\varepsilon; \varepsilon \in (-d, +d)\}$ , is understood to satisfy the following conditions:

$$\lim_{\varepsilon \rightarrow 0} g^{ab} = q^{ab}, \quad \lim_{\varepsilon \rightarrow 0} u^a = w^a, \tag{2.10a}$$

$$\lim_{\varepsilon \rightarrow 0} n = n_0, \quad \lim_{\varepsilon \rightarrow 0} T = T_0, \tag{2.10b}$$

where  $q^{ab}$  is the contravariant Robertson-Walker metric,  $w^a$  is the geometrically preferred four-velocity,  $n_0$  is the background number density, and  $T_0$  is the background temperature. In defining  $w^a$  or  $w_a := \lim_{\varepsilon \rightarrow 0} u_a$ , we postulate that the Ricci tensor for the Robertson-Walker metric  $q_{ab}$  is isotropic about  $w^a$ . Moreover, we assume that the forms of  $n_0(x^c)$  and  $T_0(x^c)$  are consistent with the background space-time geometry which is that of a Robertson-Walker space-time.

If  $\mathcal{G}_\varepsilon$  depends differentiably on  $\varepsilon \in (-d, +d)$ , it will be possible to define the perturbation

$$\delta\mathcal{G}_0 := \{G^{ab}, U^a, n_0 M, T_0 K\} \quad (2.11)$$

of

$$\mathcal{G}_0 = (q^{ab}, w^a, n_0, T_0) \quad (2.12)$$

as follows:

$$G^{ab} := \lim_{\varepsilon \rightarrow 0} \left( \frac{\partial g^{ab}}{\partial \varepsilon} \right), \quad U^a := \lim_{\varepsilon \rightarrow 0} \left( \frac{\partial u^a}{\partial \varepsilon} \right), \quad (2.13a)$$

$$M := \frac{1}{n_0} \lim_{\varepsilon \rightarrow 0} \left( \frac{\partial n}{\partial \varepsilon} \right), \quad K := \frac{1}{T_0} \lim_{\varepsilon \rightarrow 0} \left( \frac{\partial T}{\partial \varepsilon} \right). \quad (2.13b)$$

We call  $\delta\mathcal{G}_0$  the infinitesimal perturbation of  $\mathcal{G}_0$ . It is important to stress that the perturbation  $\delta\mathcal{G}_0$  *so defined* [8, 10] has the absolute geometrical meaning independent of any particular choice of the coordinate system  $(x^c)$  in  $X$ .

The next stage in the analysis is to introduce a projection tensor into the tangent three-spaces orthogonal to the background flow vector  $w^a$ :

$$\gamma_{ab} := q_{ab} + w_a w_b. \quad (2.14a)$$

Correspondingly, we set

$$\gamma_b^a := q_b^a + w^a w_b, \quad \gamma^{ab} := q^{ab} + w^a w^b, \quad (2.14b)$$

where  $q_b^a := q^{ac} q_{cb} = \delta_b^a$  (in the standard notation). Given Equations (2.14), it is natural to represent the metric and velocity perturbations in terms of rescaled variables. We define these variables by

$$Q := w_c w_d G^{cd}, \quad Q^a := -w_c \gamma_d^a G^{cd}, \quad (2.15a)$$

$$D := \frac{1}{6} \gamma_{cd} G^{cd}, \quad F^{ab} := \frac{1}{2} \gamma_c^a \gamma_d^b G^{cd} - D \gamma^{ab}, \quad (2.15b)$$

$$V := -w_c U^c, \quad V^a := \gamma_c^a U^c. \quad (2.15c)$$

Elementary inspection shows that

$$w_c Q^c = 0, \quad w_c F^{cd} = w_c F^{dc} = 0, \quad (2.16a)$$

$$\gamma_{cd} F^{cd} = 0, \quad w_c V^c = 0. \quad (2.16b)$$

Associated with  $\delta\mathcal{G}_0$  is the object

$$J(\delta\mathcal{G}_0) := (Q, Q^a, D, F^{ab}, V, V^a, M, K) \quad (2.17)$$

which we also call the infinitesimal perturbation of  $\mathcal{G}_0$ . Interpreting these transformations, we propose to use  $J(\delta\mathcal{G}_0)$  in place of  $\delta\mathcal{G}_0$ . Of course, these two descriptions appear on an equal footing, and we can choose either one to suit the problem at hand. Such is indeed the case because  $\delta\mathcal{G}_0$  is uniquely determined from  $J(\delta\mathcal{G}_0)$  and conversely.



### 3. EVOLUTION OF COSMOLOGICAL PERTURBATIONS

#### 3.1. Dynamical equations for the background

To carry on the intended analysis of the dynamical equations for  $\mathcal{G}_0$ , it is necessary to define a few mathematical quantities. First, we introduce the time derivative of any tensor  $A^{ab\dots c}$  along the fundamental fluid flow lines

$$\dot{A}^{ab\dots c} := w^d \partial_d A^{ab\dots c}, \quad (3.1)$$

where  $\partial_d A^{ab\dots c}$  is the covariant derivative of  $A^{ab\dots c}$  with respect to  $q_{ab}$ . With the use of  $w^a$  and  $\partial_a$ , we get an explicit expression for Hubble's parameter  $H$ :

$$H := \frac{1}{3} \partial_c w^c. \quad (3.2)$$

Now, denote by  $\mathcal{R}_{bcd}^a$  the Riemann tensor for the Robertson-Walker metric  $q_{ab}$ . Then another useful object is given by the following:

$$\frac{k}{R^2} := -H^2 - \frac{1}{6} \gamma_b^a \gamma^{cd} \mathcal{R}_{cda}^b. \quad (3.3)$$

The constant quantity  $k$  represents the spatial curvature and  $R$  is the expansion factor related to  $H$  by  $H := \dot{R}/R$ . Without any loss of generality, the constant  $k$  in Equation (3.3) can take the values  $k = -1, 0, +1$ , giving three different kinds of Robertson-Walker metrics. Of course, if one chooses to treat the  $k = 0$  case only, then the complete description of  $\mathcal{G}_0$  does not entail any canonical definition of  $R$  and the whole perturbation theory can be understood without making any explicit or implicit reference to the expansion factor  $R$ . In other words, we are under the necessity of introducing  $R$  and  $H = \dot{R}/R$  if and only if  $k \neq 0$ .

With this preparation behind us, the *covariant equations* governing the evolution of the "background" are given by

$$3H^2 + 3\frac{k}{R^2} = e_0 + \Lambda, \quad (3.4a)$$

$$-6(\dot{H} + H^2) = e_0 + 3p_0 - 2\Lambda, \quad (3.4b)$$

$$\dot{n}_0 + 3n_0 H = 0, \quad (3.4c)$$

where [see Equations (2.5)]

$$e_0 := e(n_0, T_0), \quad p_0 := p(n_0, T_0). \quad (3.5)$$

Here and henceforth, we shall refer to  $e_0$  as the "background" energy density and to  $p_0$  as the "background" pressure.

To sum up, from our calculations it follows that in order to discuss the cosmological equations for  $\mathcal{G}_0$ , there is no need to introduce a particular

slicing into the physical almost-Robertson-Walker universe, based on the standard Friedmann-Robertson-Walker coordinate charts. These charts are very useful and natural, however, because the symmetry is broken by the existence of a background solution for  $u^a$ .

### 3.2. Linear propagation equations

Since  $\mathcal{G}_\varepsilon$  depends differentiably on  $\varepsilon \in (-d, +d)$ , it is always possible to differentiate Einstein's equations (2.1) and the equation of balance of number density (2.3) with respect to  $\varepsilon$  and then set  $\varepsilon$  equal to zero:

$$\lim_{\varepsilon \rightarrow 0} \left[ \frac{\partial}{\partial \varepsilon} \left( R_{ab} - \frac{1}{2} R_c^c g_{ab} + \Lambda g_{ab} - T_{ab} \right) \right] = 0, \quad (3.6a)$$

$$\lim_{\varepsilon \rightarrow 0} \left[ \frac{\partial}{\partial \varepsilon} (I^a_{;a}) \right] = 0. \quad (3.6b)$$

Equations (3.6a) and (3.6b) are linear equations for  $\delta\mathcal{G}_0$ , i.e., they can be expressed in the form [8, 10]

$$\mathbb{L}(\delta\mathcal{G}_0) = 0, \quad (3.7)$$

where  $\mathbb{L}$  is a linear differential space-time operator acting on  $\delta\mathcal{G}_0$ . If we can solve Equation (3.7) for  $\delta\mathcal{G}_0$ , then  $\mathcal{G}_0 + \varepsilon\delta\mathcal{G}_0$  should yield a good approximation to  $\mathcal{G}_\varepsilon$ , and issues of cosmological interest thus can be investigated.

If one chooses to work in terms of  $J(\delta\mathcal{G}_0)$  rather than  $\delta\mathcal{G}_0$ , the starting point for the analysis is of course a system of linear propagation equations governing the evolution of  $J(\delta\mathcal{G}_0)$ . However, before considering this system, we propose to introduce some useful notation. *Let a slash denote the spatially totally projected covariant background derivative operator orthogonal to  $w^a$ .* For example, we write

$$A_{|a} := \gamma_a^c \partial_c A, \quad A^a_{|b} := \gamma_c^a \gamma_b^d \partial_d A^c, \quad (3.8a)$$

$$A^a_{|c} \dots^b := \gamma_d^a \dots \gamma_e^b \gamma_c^f \partial_f A^{d\dots e}, \quad (3.8b)$$

$$A^a_{|cd} \dots^b := (A^a_{|c} \dots^b)_{|d}. \quad (3.8c)$$

Moreover, we define  $(A^{a\dots b})^\cdot$  by

$$(A^{a\dots b})^\cdot := (\dot{A}^{a\dots b})^\cdot \quad (3.9a)$$

and  $\dot{A}^a_{|c} \dots^b$  by

$$\dot{A}^a_{|c} \dots^b := (\dot{A}^{a\dots b})_{|c} \neq (A^a_{|c} \dots^b)^\cdot, \quad (3.9b)$$

where an overdot indicates the “proper time” derivative along the fundamental fluid flow lines [see Equation (3.1)]. Finally, we introduce the infinitesimal perturbations of  $e_0$  and  $p_0$  as follows:

$$E := \lim_{\varepsilon \rightarrow 0} (\partial e / \partial \varepsilon) = n_0 e_M M + T_0 e_T K, \quad (3.10a)$$

$$P := \lim_{\varepsilon \rightarrow 0} (\partial p / \partial \varepsilon) = n_0 p_M M + T_0 p_T K, \quad (3.10b)$$

where

$$e_M := \partial e_0 / \partial n_0, \quad e_T := \partial e_0 / \partial T_0, \quad (3.11a)$$

$$p_M := \partial p_0 / \partial n_0, \quad p_T := \partial p_0 / \partial T_0. \quad (3.11b)$$

Given these preliminaries, a careful analysis of Equations (3.6) shows that the dynamical equations for  $J(\delta \mathcal{G}_0)$  can be written in the form

$$\begin{aligned} \frac{2}{H} \dot{D} - \frac{2k}{R^2 H^2} D + \frac{2}{3H} Q_{|a}^a + Q + \frac{1}{3H^2} (F_{|ab}^{ab} - 2\gamma^{ab} D_{|ab}) \\ = -\frac{1}{3H^2} E, \end{aligned} \quad (3.12a)$$

$$\begin{aligned} (D)^{\cdot\cdot} + 3H \dot{D} - k R^{-2} D + \frac{1}{2} H \dot{Q} + \left( \dot{H} + \frac{3}{2} H^2 \right) Q + \frac{1}{6} \gamma^{ab} Q_{|ab} \\ + \frac{1}{3} \dot{Q}_{|a}^a + \frac{2}{3} H Q_{|a}^a + \frac{1}{6} (F_{|ab}^{ab} - 2\gamma^{ab} D_{|ab}) = \frac{1}{2} P, \end{aligned} \quad (3.12b)$$

$$\begin{aligned} \dot{F}_{|b}^{ab} - 2\gamma^{ab} \dot{D}_{|b} - \frac{1}{2} (\gamma^{ac} Q_{|cb}^b - \gamma^{bc} Q_{|bc}^a) - H \gamma^{ab} Q_{|b} \\ = -2 [\dot{H} (V^a + Q^a) - k R^{-2} V^a], \end{aligned} \quad (3.12c)$$

$$\begin{aligned} (F^{ab})^{\cdot\cdot} + 3H \dot{F}^{ab} - \frac{k}{R^2} F^{ab} + \frac{1}{2} (\gamma^{ac} \dot{Q}_{|c}^b + \gamma^{bc} \dot{Q}_{|c}^a) - \frac{1}{3} \gamma^{ab} \dot{Q}_{|c}^c \\ + H (\gamma^{ac} Q_{|c}^b + \gamma^{bc} Q_{|c}^a) - \frac{2}{3} H \gamma^{ab} Q_{|c}^c + \frac{1}{2} \left( \gamma^{ac} \gamma^{bd} - \frac{1}{3} \gamma^{ab} \gamma^{cd} \right) Q_{|cd} \\ = \gamma^{cd} F_{|cd}^{ab} - \gamma^{ac} F_{|cd}^{db} - \gamma^{bc} F_{|cd}^{ad} + \frac{2}{3} \gamma^{ab} F_{|cd}^{cd} \\ + \left( \gamma^{ac} \gamma^{bd} - \frac{1}{3} \gamma^{ab} \gamma^{cd} \right) D_{|cd}, \end{aligned} \quad (3.12d)$$

$$-3\dot{D} + \dot{M} + V_{|a}^a = 0. \quad (3.12e)$$

Here, of course, Latin indices take values of 0, 1, 2, 3 and repeated Latin indices are to be summed over these values.

We have thus obtained the desired system of linear propagation equations for the determination of  $J(\delta\mathcal{G}_0)$ . The above results are *exact* but rather tedious consequences of Equations (3.6). [For lack of space, we will not comment on the technical details leading to Equations (3.12). These details, however, are available on request.] Here it is also important to mention the following: Throughout this paper we use a covariant formalism. Consequently, Equations (3.12) have a clear geometrical meaning independent of any particular coordinate chart chosen.

Now, we can verify that every classical solution of Equations (3.12) obeys

$$\dot{E} - 2\left(\frac{k}{R^2} - \dot{H}\right)\dot{M} + 3H(E + P) = 0. \quad (3.13a)$$

A further study of Equations (3.12) yields the supplementary balance law, interpreted as the equation of balance of  $V^a + Q^a$ :

$$\begin{aligned} \dot{V}^a + \dot{Q}^a + 2H(V^a + Q^a) + \frac{1}{2}\gamma^{ab}Q_{|b} \\ = \left(\frac{k}{R^2} - \dot{H}\right)^{-1} \left\{ [(H)\ddot{\cdot} + 2H\dot{H}](V^a + Q^a) - \frac{1}{2}\gamma^{ab}P_{|b} \right\}. \end{aligned} \quad (3.13b)$$

Equations (3.13) can also be derived by directly differentiating the equation of motion of the matter  $T_{;b}^{ab} = 0$  with respect to  $\varepsilon \in (-d, +d)$  and then evaluating the result for  $\varepsilon = 0$ ; thus these equations represent the balance law which is a local conservation of energy and momentum. Similarly, one can prove that Equation (3.12e) represents the continuity law (2.3) in its linearized form.

Another remark is also in order. Even for a simple gas of material particles, all having the same proper mass  $m$ , the equations of state  $e = e(n, T)$  and  $p = p(n, T)$  depend in a rather complicated way on both  $n$  and  $T$  [14-16]. However, in the limiting case of  $T$  small we have a convenient approximation [14]

$$e = mn + \frac{3}{2}nT + \dots, \quad p = nT + \dots \quad (3.14)$$

The terms not written out explicitly are much smaller than those shown (the Boltzmann constant  $k_B$  equals one in our system of units). Consequently, if the temperature  $T$  vanishes in the background ( $T_0 = 0$ ), we find from

Equations (3.10) and (3.11) that the infinitesimal perturbations of  $e_0 = m n_0$  and  $p_0 = 0$  simplify to

$$E = m n_0 \left( M + \frac{3}{2} \mathbb{K} \right), \quad P = m n_0 \mathbb{K}, \quad (3.15)$$

where the dimensionless quantity  $\mathbb{K}$  is given by [13]

$$\mathbb{K} := \frac{1}{m} \lim_{\varepsilon \rightarrow 0} \left( \frac{\partial T}{\partial \varepsilon} \right). \quad (3.16)$$

With the variable  $\mathbb{K}$  playing the role previously played by  $K$  (cf. Equation (2.13b)), we now see how our general propagation Equations (3.12) and (3.13) can be adapted to situations where the pressure  $p$  vanishes in the background ( $p_0 = 0$ ). In this case, it is also possible to replace the quantity  $(H)^\cdot + 2H \dot{H}$  which appears in Equation (3.13b) by  $H [(k/R^2) - \dot{H}]$  or simply by  $\frac{1}{2} m n_0 H$ .

### 3.3. Equivalence classes of perturbations

There is a gauge freedom in general relativity corresponding to the group of diffeomorphisms of space-time [9]. Because of this, two different objects  $\mathcal{G}_{(1)} := \{g_{(1)}^{ab}, u_{(1)}^a, n_{(1)}, T_{(1)}\}$  and  $\mathcal{G}_{(2)} := \{g_{(2)}^{ab}, u_{(2)}^a, n_{(2)}, T_{(2)}\}$  defined on  $X$  are physically equivalent if there is a diffeomorphism  $\sigma : X \Rightarrow X$  which takes  $\mathcal{G}_{(1)}$  into  $\mathcal{G}_{(2)}$  [ $\sigma^* \mathcal{G}_{(1)} = \mathcal{G}_{(2)}$ ], and clearly  $\mathcal{G}_{(1)}$  satisfies the nonlinear field equations if and only if  $\mathcal{G}_{(2)}$  does. Thus the solutions  $\mathcal{G}$  of Equations (2.1) and (2.3) can be unique only up to a diffeomorphism. Within the framework of a linear approximation, this implies that two perturbations  $\delta\mathcal{G}_0$  and  $\delta\mathcal{G}'_0$  satisfying Equation (3.7) represent the same perturbation of  $\mathcal{G}_0$  if (and only if) they differ by the action of an “infinitesimal diffeomorphism” [10] on the background solution  $\mathcal{G}_0$ . An infinitesimal diffeomorphism and its action on  $\mathcal{G}_0$  are most conveniently described in terms of a vector field  $v$  on the space-time manifold  $X$ . More precisely, using one-parameter groups of diffeomorphisms of  $X$  and one-parameter families of exact solutions of Equations (2.1) and (2.3), one can construct “new” one-parameter families of exact solutions obeying the conditions (2.10) and hence verify that the change in a perturbation induced by  $v$  is the *Lie derivative*  $\mathcal{L}_v \mathcal{G}_0$  of  $\mathcal{G}_0 := (q^{ab}, w^a, n_0, T_0)$  with respect to  $v$  [17]:

$$\mathcal{L}_v \mathcal{G}_0 := \{(\mathcal{L}_v q)^{ab}, (\mathcal{L}_v w)^a, \mathcal{L}_v n_0, \mathcal{L}_v T_0\}. \quad (3.17)$$

Thus  $\delta\mathcal{G}_0$  and  $\delta\mathcal{G}_0 + \mathcal{L}_v \mathcal{G}_0$  represent the same perturbation, and clearly  $\delta\mathcal{G}_0$  satisfies the linearized field equation (3.7) if and only if  $\delta\mathcal{G}_0 + \mathcal{L}_v \mathcal{G}_0$  does.

The set consisting of  $\mathcal{L}_v \mathcal{G}_0$  for all vector fields  $v$  on  $X$  of class  $C^r$  ( $r$  sufficiently large) is written  $\mathcal{P}_0$ ; this set carries a natural structure of a vector space. Clearly,  $\mathcal{P}_0$  is a subspace of the space  $\mathcal{P}$  whose elements are classical

solutions of Equation (3.7); thus by definition  $\mathcal{L}_v \mathcal{G}_0 \in \mathcal{P}_0$  and  $\delta \mathcal{G}_0 \in \mathcal{P}$  satisfy Equation (3.7). Given the object  $J(\delta \mathcal{G}_0)$  as in Equation (2.17), we denote by  $\mathcal{W}$  the collection of all  $J(\delta \mathcal{G}_0)$  where  $\delta \mathcal{G}_0 \in \mathcal{P}$  and by  $\mathbb{W}$ ,  $\mathbb{W}'$ , and similar symbols the elements of  $\mathcal{W}$ . It follows from these definitions that  $\mathbb{W} \in \mathcal{W}$  is a *classical solution* of Equations (3.12). By way of digression, it is frequently unnecessary to distinguish between  $\delta \mathcal{G}_0$  and  $\mathbb{W} = J(\delta \mathcal{G}_0)$ , since they have the same geometrical meaning, but merely apply to different descriptions of an almost-Robertson-Walker universe model. Because of this, we do not hesitate to call both  $\delta \mathcal{G}_0$  and  $\mathbb{W}$  the infinitesimal perturbation of  $\mathcal{G}_0$ . A function from  $\mathcal{P}$  onto  $\mathcal{W}$ , denoted  $J : \mathcal{P} \Rightarrow \mathcal{W}$ , is a linear map which assigns to each  $\delta \mathcal{G}_0 \in \mathcal{P}$  an element  $J(\delta \mathcal{G}_0)$ . Then we introduce the subspace  $\mathcal{W}_0$  of  $\mathcal{W}$ , the subspace which is the image of  $\mathcal{P}_0$  under  $J$ . Clearly,  $\mathbb{W}$  belongs to  $\mathcal{W}_0$  if and only if  $\mathbb{W}$  equals  $J(\mathcal{L}_v \mathcal{G}_0)$  for some  $v$ . In order to simplify our notation, we abbreviate  $J(\mathcal{L}_v \mathcal{G}_0)$  as  $L_v \mathcal{G}_0$ .

We are now in a position to describe explicitly the action of  $L_v$  on  $\mathcal{G}_0$ . First, given the preferred timelike four-vector field  $w^a$ , we split  $v$  into its timelike and spacelike parts relative to  $w^a$ :

$$\vartheta := -w_b v^b, \quad \vartheta^a := \gamma_b^a v^b. \quad (3.18)$$

By using the definition of the Lie derivative  $\mathcal{L}_v$  [17] and exploiting Equations (2.17) and (3.17), we then conclude that  $L_v \mathcal{G}_0$  may be written as

$$L_v \mathcal{G}_0 = (Q_v, Q_v^a, D_v, F_v^{ab}, V_v, V_v^a, M_v, K_v), \quad (3.19)$$

where

$$Q_v := 2(\vartheta)', \quad Q_v^a := (\vartheta^a)' - H \vartheta^a - \gamma^{ab} \vartheta|_b, \quad (3.20a)$$

$$D_v := -H \vartheta - \frac{1}{3} \vartheta|_c, \quad (3.20b)$$

$$F_v^{ab} := -\frac{1}{2} (\gamma^{ac} \vartheta|_c^b + \gamma^{bc} \vartheta|_c^a) + \frac{1}{3} \gamma^{ab} \vartheta|_c, \quad (3.20c)$$

$$V_v := -(\vartheta)', \quad V_v^a := -(\vartheta^a)' + H \vartheta^a, \quad (3.20d)$$

$$M_v := -3 \vartheta H, \quad K_v := \vartheta \frac{1}{T_0} \dot{T}_0. \quad (3.20e)$$

In obtaining Equations (3.20), we have used  $\partial_b w^a = H \gamma_b^a$  and the fact that  $\dot{n}_0 = -3n_0 H$ . If  $T_0 = 0$ ,  $K$  must be replaced by  $\mathbb{K}$  and  $K_v$  by  $\mathbb{K}_v := 0$ .

The situation may therefore be summarized as follows. The object of most physical interest is not just one perturbation  $\mathbb{W} \in \mathcal{W}$  but a whole equivalence class of all perturbations  $\mathbb{W}'$  which are equivalent to  $\mathbb{W}$ :

two infinitesimal perturbations  $W \in \mathcal{W}$  and  $W' \in \mathcal{W}$  will be taken to be equivalent if there is a vector field  $v$  on  $X$  such that  $W' = W + \mathcal{L}_v \mathcal{G}_0$ . The equivalence class of  $W$  is denoted  $[W]$  and is called the *gauge-invariant perturbation* associated with  $W$ . In this way, we verify that the gauge-invariant perturbations are elements of  $\mathcal{W}/\mathcal{W}_0$ , the *quotient space* of  $\mathcal{W}$  by  $\mathcal{W}_0$ . The essential point in the theory of gauge-invariant perturbations is to describe the elements of this quotient space explicitly. These issues will be considered in section 4.

Another route to discussing the gauge problem is to introduce the equivalence class  $[\delta\mathcal{G}_0]$  of  $\mathcal{G}_0$ : two infinitesimal perturbations  $\delta\mathcal{G}_0 \in \mathcal{P}$  and  $\delta\mathcal{G}'_0 \in \mathcal{P}$  are said to be equivalent if  $\delta\mathcal{G}'_0$  equals  $\delta\mathcal{G}_0 + \mathcal{L}_v \mathcal{G}_0$  for some  $v$ . Accordingly with this, we have the quotient space  $\mathcal{P}/\mathcal{P}_0$  which consists of  $[\delta\mathcal{G}_0]$  for all  $\delta\mathcal{G}_0 \in \mathcal{P}$ . However, the theory based on  $[\delta\mathcal{G}_0]$  and  $\mathcal{P}/\mathcal{P}_0$  seems to be somewhat less convenient than that based on  $[W]$  and  $\mathcal{W}/\mathcal{W}_0$ .

## 4. GAUGE-INVARIANT APPROACH TO COSMOLOGICAL PERTURBATIONS

### 4.1. Construction of the “coordinate system” on $\mathcal{W}/\mathcal{W}_0$

To every gauge-invariant perturbation  $[W] \in \mathcal{W}/\mathcal{W}_0$  there is an associated set

$$\varphi([W]) := \{\chi, \Gamma, \Omega, \Omega^a, \Theta, \Theta^{ab}, S^{abcd}\} \tag{4.1}$$

of basic gauge-invariant variables, defined by

$$\chi := Q + 2V, \quad \Gamma := K + \frac{1}{3HT_0} \dot{T}_0 M, \tag{4.2a}$$

$$\Omega := -\frac{1}{2} Q + \frac{1}{3H^2} (\dot{H} M - H \dot{M}), \tag{4.2b}$$

$$\Omega^a := -3H(V^a + Q^a) + \gamma^{ac} M_{|c}, \tag{4.2c}$$

$$\Theta := -\frac{3}{2} Q - \frac{3}{H} \dot{D} + \frac{1}{H} V_{|c}^c + \frac{1}{H^2} \dot{H} M, \tag{4.2d}$$

$$\Theta^{ab} := \frac{1}{H} \dot{F}^{ab} - \frac{1}{2H} (\gamma^{ac} V_{|c}^b + \gamma^{bc} V_{|c}^a) + \frac{1}{3H} \gamma^{ab} V_{|c}^c, \tag{4.2e}$$

$$S^{abcd} := \frac{k}{R^2} Z^{e[a} (\gamma^{b]c} \gamma_e^d - \gamma^{b]d} \gamma_e^c) + \gamma^{df} \gamma^{e[a} Z_{|ef}^{b]c} - \gamma^{cf} \gamma^{e[a} Z_{|ef}^{b]d}, \tag{4.2f}$$

where

$$Z^{ab} := 2 \left( \frac{1}{3} M - D \right) \gamma^{ab} - 2 F^{ab} \quad (4.3)$$

and where the process of alternation over two upper indices  $a$  and  $b$  in Equation (4.2f) is denoted by square brackets. We emphasize that the set  $\mathbb{D} := \varphi([W])$  consists of covariantly defined gauge-invariant variables. Given  $\mathbb{D}$ , it is evident from Equations (4.2) and (4.3) that in order to define  $\varphi([W])$ , we have used one representative member of  $[W]$ , namely, the infinitesimal perturbation  $W = J(\delta\mathcal{G}_0)$  characterized by Equation (2.17). The “value” of  $\varphi([W])$  is completely independent of this choice and Equation (4.1) defines a function on  $\mathcal{W}/\mathcal{W}_0$ . In fact, recalling the definition of  $L_v \mathcal{G}_0$  [see Equations (3.19) and (3.20)], we get

$$\varphi([L_v \mathcal{G}_0]) = 0. \quad (4.4)$$

The key step in the derivation of this result is the observation that  $S^{abcd}$  vanishes if  $Z^{ab}$  equals  $Z_v^{ab}$ :

$$Z_v^{ab} := 2 \left( \frac{1}{3} M_v - D_v \right) \gamma^{ab} - 2 F_v^{ab} = \gamma^{ac} \vartheta_{|c}^b + \gamma^{bc} \vartheta_{|c}^a. \quad (4.5)$$

We refer to appendix A and the literature quoted there for more details.

*The physical and geometrical meaning of  $\chi$ ,  $\Gamma$ ,  $\Omega$ ,  $\Omega^a$ ,  $\Theta$ ,  $\Theta^{ab}$ , and  $S^{abcd}$  will be explained in section 4.2.*

*Remark.* – If  $T_0 = 0$ , the gauge-invariant quantity  $\Gamma$  is defined by  $\Gamma := \mathbb{K}$ .

We denote by  $\mathcal{D}$  the set consisting of  $\varphi([W])$  for all  $[W] \in \mathcal{W}/\mathcal{W}_0$  and by  $\mathbb{D}$ ,  $\mathbb{D}'$ , and similar symbols the elements of  $\mathcal{D}$ . A function from  $\mathcal{W}/\mathcal{W}_0$  onto  $\mathcal{D}$ , denoted  $\varphi := \mathcal{W}/\mathcal{W}_0 \Rightarrow \mathcal{D}$ , is a linear map which assigns to each  $[W] \in \mathcal{W}/\mathcal{W}_0$  an element  $\varphi([W]) \in \mathcal{D}$ ; thus  $\mathcal{D}$  carries a natural structure of a vector space induced by that of  $\mathcal{W}/\mathcal{W}_0$ . More precisely,  $\mathcal{D}$  is a function space in which the usual operations of addition and scalar multiplication are introduced. The importance of  $\varphi : \mathcal{W}/\mathcal{W}_0 \Rightarrow \mathcal{D}$  can be explained with the help of the following theorem:

**THEOREM 1.** – *For every  $\mathbb{D} \in \mathcal{D}$  there is just one  $[W] \in \mathcal{W}/\mathcal{W}_0$  such that  $\mathbb{D} = \varphi([W])$ , then  $\varphi$  is said to be one-to-one, abbreviated 1-1. In this case we can define the inverse of  $\varphi$ ,  $\varphi^{-1} : \mathcal{D} \Rightarrow \mathcal{W}/\mathcal{W}_0$ , by setting  $(\varphi^{-1} \varphi)([W]) = [W]$ .*



*Proof.* – The proof of this theorem reduces to showing that if  $\varphi([\mathbb{W}])$  is a zero-vector of  $D$ , then  $[\mathbb{W}] = [L_v \mathcal{G}_0]$  where  $v$  is an arbitrary vector field<sup>2</sup> on  $X$  (see appendix A for more details).

COROLLARY. – From our theorem we infer that  $\varphi([\mathbb{W}'])$  equals  $\varphi([\mathbb{W}])$  if (and only if)  $[\mathbb{W}']$  equals  $[\mathbb{W}]$ , i.e., if (and only if)  $\mathbb{W}' = \mathbb{W} + L_v \mathcal{G}_0$  for some  $v$ .

The set  $\mathbb{D} = \varphi([\mathbb{W}])$  is basic and complete for at least two reasons. First,  $[\mathbb{W}] = \varphi^{-1}(\mathbb{D})$  is a gauge-invariant perturbation and the objects appearing on the right-hand side of Equation (4.1) are “coordinates” of  $[\mathbb{W}]$ . Thus  $[\mathbb{W}]$  is uniquely determined from  $\mathbb{D}$  and vice-versa. This fact enables us to interpret  $\varphi : \mathcal{W}/\mathcal{W}_0 \Rightarrow \mathcal{D}$  as a “coordinate system” on the quotient space  $\mathcal{W}/\mathcal{W}_0$ . Second, any gauge-invariant quantity can be construct *directly* from the basic variables  $\mathbb{D}$  through *purely* algebraic and differential operations. We will consider some aspects of this problem in section 4.2.

Now, an equation for  $\chi$  can be obtained from the relation

$$\chi = - \lim_{\varepsilon \Rightarrow 0} \left[ \frac{\partial}{\partial \varepsilon} (u^a u_a) \right], \quad (4.6)$$

with the result  $\chi = Q + 2V = 0$  which is a direct consequence of  $u^a u_a = -1$ . Thus the gauge-invariant quantity  $\chi$  will not be physically significant to us in considering linearization about the Robertson-Walker universe models. This conclusion, however, does not mean that the identity  $\chi = 0$  is not mathematically important; it holds and it will be used in appendix A.

A complete set of symmetry conditions for  $S^{abcd}$  is  $S^{abcd} = S^{[ab][cd]}$  and  $S^a{}^{[bcd]} = 0$ ; thus there are six linearly independent, not identically vanishing components in  $\{S^{abcd}; a, b, c, d = 0, 1, 2, 3\}$ . Such is indeed the case because  $S^{abcd}$  satisfies the constraints  $w_a S^{abcd} = w_a S^{bcad} = 0$ . In fact, we can uniquely recover  $S^{abcd}$  from

$$S := \gamma_{cd} \gamma_{ab} S^{cabd} \quad (4.7a)$$

and

$$S^{ab} := \gamma_{cd} \left( S^{cabd} - \frac{1}{3} \gamma^{ab} \gamma_{ef} S^{cefd} \right) = S^{ba}. \quad (4.7b)$$

2. Precisely speaking,  $v$  must be of class  $C^r$  ( $r$  sufficiently large); otherwise  $L_v \mathcal{G}_0$  cannot be a classical solution of Equations (3.12).

Note that  $\gamma_{ab} S^{ab} = 0$ . Similarly, we have  $\gamma_{ab} \Theta^{ab} = 0$ . It then follows that since  $\chi = 0$ , the total number of independent components in  $\mathbb{D}$  is 17.

One final word concerning this section. Here we discuss the gauge problem in a pure geometrical way, i.e., without explicitly using Equations (3.12). A full analysis of almost-Robertson-Walker universe models must of course examine these equations (*see* section 4.3).

#### 4.2. The physical and geometrical meaning of basic variables

Let  $\{A(\varepsilon, x^c); \varepsilon \in (-d, +d)\}$  be a curve of geometrical objects (matter variables, tensor fields, etc.) obtainable tensor-algebraically from  $\mathcal{G}_\varepsilon(x^c)$  and its covariant derivatives with respect to  $g_{ab}(\varepsilon, x^c)$ , and suppose that  $A(\varepsilon, x^c)$  depends differentiably on  $\varepsilon$ . It is then natural to define the quantity  $\delta A$  which represents the “first variation” on  $A$ :

$$\delta A := \lim_{\varepsilon \rightarrow 0} \left( \frac{\partial A}{\partial \varepsilon} \right). \quad (4.8)$$

As shown already by Ehlers [8], this first variation is invariant under the action of an “infinitesimal diffeomorphism” [10] if and only if

$$\mathcal{L}_v \left( \lim_{\varepsilon \rightarrow 0} A \right) = 0. \quad (4.9)$$

In order to satisfy Equation (4.9), it is necessary to use a scalar  $A$  that is constant in the “unperturbed space-time”  $(X, q_{ab})$ , or any tensor  $A^{ab}_{\dots cd}$  that vanishes in  $(X, q_{ab})$ , or a tensor whose “background value” is a constant linear combination of products of Kroneckers deltas  $\delta_b^a$  [18].

So much for general definitions. Let us now turn to the analysis of some representative examples of Equation (4.9) for nonbarotropic perfect fluids. However, before considering these examples, we first introduce the tensor

$$h_{ab} := g_{ab} + u_a u_b \quad (4.10)$$

which projects the tangent vector-space at each point perpendicularly onto the three-dimensional subspace orthogonal to  $u^a$ .

From the above discussion plus the definition of  $h_{ab}$  we conclude that the simplest physical objects  $A$  satisfying Equation (4.9) can be described as follows:

- (1) The specific entropy:

$$s := \text{entropy per particle}. \quad (4.11)$$

(2) The “normalized curvature scalars:”

$$\mathcal{A} := \frac{1}{n^{2/3}} \left[ h^{ab} R_{ab} - \frac{1}{2} R_a^a - \frac{1}{3} (u_{;a}^a)^2 \right], \quad (4.12a)$$

$$\mathcal{B} := \frac{1}{n^{2/3}} \left[ \Lambda + e - \frac{1}{3} (u_{;a}^a)^2 \right]. \quad (4.12b)$$

(3) The orthogonal spatial gradient of  $n$ :

$$X^a := h^{ab} n_{;b}. \quad (4.13)$$

(4) The vorticity, shear, and acceleration:<sup>3</sup>

$$\omega_{ab} := h_a^c h_b^d u_{[c;d]}, \quad (4.14a)$$

$$\sigma_{ab} := h_a^c h_b^d u_{[c;d]} - \frac{1}{3} (u_{;c}^c) h_{ab}, \quad (4.14b)$$

$$a^b := u^c u_{;c}^b. \quad (4.14c)$$

(5) The electric and magnetic parts  $E_{ab}$ ,  $H_{ab}$  of the Weyl tensor  $C_{abcd}$ :<sup>3</sup>

$$E_{ab} := C_{acbd} u^c u^d, \quad H_{ab} := \frac{1}{2} C_{acde} u^c \eta_{bf}^{de} u^f. \quad (4.15)$$

When Equation (2.3) is combined with the equation of balance of  $s$ , we derive that the perfect fluid is locally adiabatic:  $u^a s_{;a} = 0$ . That is, entropy is constant along the flow lines of the fluid. In this way, we arrive at the following conclusion: the specific entropy  $s$  is a scalar that is constant in the unperturbed space-time  $(X, q_{ab})$ . The same remark concerns  $\mathcal{A}$  and  $\mathcal{B}$ . In fact,  $\mathcal{A}$  is dynamically related to  $\mathcal{B}$  by  $\mathcal{A} = \mathcal{B}$ . If  $g_{ab} = q_{ab}$ , we have  $\mathcal{A} = \mathcal{B} = 3k/n_0^{2/3} R^2$  and the Lie derivatives  $\mathcal{L}_v \mathcal{A}$  and  $\mathcal{L}_v \mathcal{B}$  vanish because of  $\dot{n}_0 = -3n_0 H$  [see Equation (3.4c)]. In the case  $\omega_{ab} = 0$ , the quantity  $2n^{2/3} \mathcal{A}$  acquires a special significance [6]: it is a Ricci scalar of the three-dimensional spaces everywhere orthogonal to the fluid flow vector  $u^a$ . For this reason, we call  $\mathcal{A}$  or  $\mathcal{B}$  the normalized curvature scalar. An important physical object is also the orthogonal density gradient  $X^a$  defined by Equation (4.13). Ellis and Bruni [6] gave the first systematic treatment of the properties of  $X^a$ . Interpreting Equations (4.14), these equations define the standard kinematic quantities which vanish in the background. As regards  $E_{ab}$  and  $H_{ab}$ , the physical and geometrical meaning of these tensor fields is well known [19].

3. As usual,  $A_{(ab)}$  represents the symmetric part of  $A_{ab}$ . We denote by  $\eta^{abcd}$  the Levi-Civita alternating tensor. The “background value” of this tensor will also be denoted by  $\eta^{abcd}$ .

Now, it is only a matter of labour to prove that the first variations of  $s$ ,  $\mathcal{A}$ ,  $\mathcal{B}$ ,  $X^a$ ,  $\sigma_{ab}$ , and  $E_{ab}$  are related to  $\Gamma$ ,  $\Omega$ ,  $\Omega^a$ ,  $\Theta^{ab}$ ,  $S$ , and  $S^{ab}$  by

$$\delta s = \frac{e_T}{n_0} \Gamma, \quad (4.16a)$$

$$\delta \mathcal{A} = \frac{1}{n_0^{2/3}} \left( \frac{2}{3} \Omega_{|a}^a - \frac{1}{2} S \right), \quad (4.16b)$$

$$\delta \mathcal{B} = \frac{1}{n_0^{2/3}} (-6H^2 \Omega + T_0 e_T \Gamma), \quad (4.16c)$$

$$\delta X^a = n_0 \Omega^a, \quad (4.16d)$$

$$\delta \sigma_{ab} = -\gamma_{ac} \gamma_{bd} H \Theta^{cd}, \quad (4.16e)$$

$$\begin{aligned} \delta E_{ab} = \gamma_{ac} \gamma_{bd} \left[ -H^2 \Theta^{cd} + \frac{1}{6} (\gamma^{ce} \Omega_{|e}^d + \gamma^{de} \Omega_{|e}^c) \right. \\ \left. - \frac{1}{9} \gamma^{cd} \Omega_{|e}^e - S^{cd} \right]. \end{aligned} \quad (4.16f)$$

Because of Equation (2.3),  $\Theta = 3\Omega$  and we may see the above transformations in another way by noticing that  $\Gamma$ ,  $\Omega$ ,  $\Omega^a$ ,  $\Theta$ ,  $\Theta^{ab}$ , and  $S^{abcd}$  are uniquely determined from  $\delta s$ ,  $\delta \mathcal{A}$ ,  $\delta \mathcal{B}$ ,  $\delta X^a$ ,  $\delta \sigma_{ab}$ , and  $\delta E_{ab}$ . The physical and geometrical meaning of our original gauge-invariant variables  $\mathbb{D}$  is characterized by this fact and the results of section 4.1 (especially *theorem 1*).

*Remark 1.* – In order to obtain Equation (4.16a), we have used the first law of thermodynamics:  $de = (p/n^2) dn + T ds$ .

*Remark 2.* – If  $T_0 = 0$ , we must replace Equation (4.16c) by  $\delta \mathcal{B} = n_0^{-2/3} (-6H^2 \Omega + \frac{3}{2} mn_0 \Gamma)$  where  $\Gamma = \mathbb{K}$ . In this case, Equation (4.16a) does not hold.

We have mentioned in the discussion of section 4.1 that any gauge-invariant quantity can be constructed directly from the basic variables  $\mathbb{D}$  through purely algebraic and differential operations. The question of a definition of gauge-invariant quantities in terms of  $\mathbb{D}$  is a highly complex one. In fact, the precise theorems are very technical and need a great deal of preliminary apparatus from differential geometry. We shall here illustrate only some aspects of these general theorems by showing that, to first order in the deviations from the background solution, the vorticity tensor  $\omega_{ab}$ , the acceleration  $a^b$ , and the magnetic part  $H_{ab}$  of the Weyl tensor  $C_{abcd}$

are uniquely determined from  $\mathbb{D}$ . Indeed, after a fair amount of algebra, the following differential equations are obtained:

$$\delta\omega_{ab} = -\frac{1}{3H} \gamma_{c[a} \Omega_{|b]}, \quad (4.17a)$$

$$\delta a^b = -\gamma^{bc} \Omega_{|c} - \frac{1}{3H} \dot{\Omega}^b + \frac{1}{3} \left( \frac{\dot{H}}{H^2} - 1 \right) \Omega^b, \quad (4.17b)$$

$$\delta H_{ab} = w^e \eta_{e(a}^{cd} \left[ H \gamma_{b)f} \gamma_{cg} \Theta_{|d}^{fg} - \frac{1}{6H} (\Omega_{|b)d}^f \gamma_{fc} - \gamma_{b)f} \Omega_{|cd}^f \right]. \quad (4.17c)$$

The above results should be useful because they show that  $\delta\omega_{ab}$ ,  $\delta a^b$ ,  $\delta H_{ab}$ , and many further gauge-invariant quantities (e.g., the orthogonal spatial gradients of  $T$ ,  $p$ ,  $u_{;a}^a$ , and  $2n^{2/3} \mathcal{A}$ ) are not necessary to define an almost-Robertson-Walker universe model.

The upshot of this discussion may be stated very neatly. There are basically two different approaches by which linear perturbation theory can be elaborated: one of them makes use of the “standard” Einstein’s equations [2, 4, 5] and the other is based on the so-called quasi-Maxwellian description [3, 6, 20]. Within the framework here set up, Equations (4.16d), (4.16e), (4.16f), and (4.17) give an interpretation of how the important variables of this quasi-Maxwellian description relate to our formalism.

### 4.3. Perturbation equations in basic gauge-invariant variables

The object now is to obtain a closed set of equations for the evolution of  $\mathbb{D}$ . This will in fact be easy, because the following theorem holds:

**THEOREM 2.** – *Every dynamical or constraint equation of linear perturbation theory is gauge invariant and can be written in a manifestly gauge-invariant form.*

*Sketch of the proof.* – The general propagation or constraint equation is given by  $\text{Lop}(\mathbb{W}) = 0$ , where  $\text{Lop}$  is a linear operator acting on  $\mathbb{W}$ . Clearly,  $\mathbb{W}$  satisfies  $\text{Lop}(\mathbb{W}) = 0$  if and only if  $\mathbb{W} + L_v \mathcal{G}_0$  does. Setting  $\mathbb{W} = 0$ , we find that  $\text{Lop}(L_v \mathcal{G}_0) = 0$ ; thus  $\text{Lop}(\mathbb{W})$  is gauge invariant. Hence to complete the proof, it suffices to show that there exists a linear operator  $\text{Lop}_{(\text{gi})}$  such that  $\text{Lop}(\mathbb{W}) = \text{Lop}_{(\text{gi})}(\mathbb{D})$ . Since  $\text{Lop}(\mathbb{W} + L_v \mathcal{G}_0) = \text{Lop}(\mathbb{W})$ , the existence of  $\text{Lop}_{(\text{gi})}(\mathbb{D})$  follows from an analysis of section 4.2 (see the text directly after *remark 2*). ■

Given this theorem, it should be clear what Equations (3.12) really are: these are basic gauge-invariant equations which may be re-expressed in

a manifestly gauge-invariant form. Indeed, using the definition (4.2) of  $\varphi$  ([W]), we calculate that

$$\Omega + \frac{1}{9H^2} \Omega_{|a}^a - \frac{1}{12H^2} S = \frac{1}{6H^2} T_0 e_T \Gamma, \quad (4.18a)$$

$$\begin{aligned} \dot{\Omega} + 2 \left( H + \frac{\dot{H}}{H} \right) \Omega + \frac{1}{9H^2} \left( \dot{\Omega}^a - \frac{\dot{H}}{H} \Omega^a \right)_{|a} + \frac{1}{9H} \Omega_{|a}^a \\ + \frac{1}{3H} \gamma^{ab} \Omega_{|ab} = -\frac{1}{2H} T_0 \left( p_T + \frac{1}{3} e_T \right) \Gamma, \end{aligned} \quad (4.18b)$$

$$\Theta_{|b}^{ab} + \frac{1}{6H^2} (\gamma^{ab} \Omega_{|bc}^c - \gamma^{bc} \Omega_{|bc}^a) + 2\gamma^{ab} \Omega_{|b} = \frac{2\dot{H}}{3H^2} \Omega^a, \quad (4.18c)$$

$$\begin{aligned} \dot{\Theta}^{ab} + \left( 3H + \frac{\dot{H}}{H} \right) \Theta^{ab} - \frac{1}{6H^2} (\gamma^{ac} \dot{\Omega}_{|c}^b + \gamma^{bc} \dot{\Omega}_{|c}^a) + \frac{1}{9H^2} \gamma^{ab} \dot{\Omega}_{|c}^c \\ + \frac{\dot{H}}{6H^3} (\gamma^{ac} \Omega_{|c}^b + \gamma^{bc} \Omega_{|c}^a) - \frac{\dot{H}}{9H^3} \gamma^{ab} \Omega_{|c}^c \\ - \frac{1}{3H} (\gamma^{ac} \Omega_{|c}^b + \gamma^{bc} \Omega_{|c}^a) + \frac{2}{9H} \gamma^{ab} \Omega_{|c}^c \\ - \frac{1}{H} \left( \gamma^{ac} \gamma^{bd} - \frac{1}{3} \gamma^{ab} \gamma^{cd} \right) \Omega_{|cd} = -\frac{1}{H} S^{ab}, \end{aligned} \quad (4.18d)$$

$$\Theta = 3\Omega. \quad (4.18e)$$

Moreover, differentiation of Equation (4.2f) with respect to “time” yields the equation of balance of  $S^{abcd}$ :

$$\begin{aligned} \dot{S}^{abcd} + 2H S^{abcd} = -2H \left\{ \frac{k}{R^2} \Theta^{e[a} (\gamma^{b]c} \gamma_e^d - \gamma^{b]d} \gamma_e^c) \right. \\ \left. + \gamma^{df} \gamma_e^{[a} \Theta_{|ef}^{b]c} - \gamma^{cf} \gamma_e^{[a} \Theta_{|ef}^{b]d} \right\}. \end{aligned} \quad (4.18f)$$

In order to obtain this result, we have used the symmetry properties of  $S^{abcd}$  as well as Equations (3.12e), (4.2f), and (4.3). In addition to Equation (4.18f), we can also derive an important constraint equation for

$S^{abcd}$ . More precisely, using the definition (4.2f), we verify that  $S^{abcd}$  has to obey the “linearized Bianchi identities:”

$$\gamma_{ce} \gamma_{df} S_{|g}^{abef} + \gamma_{ge} \gamma_{cf} S_{|d}^{abef} + \gamma_{de} \gamma_{gf} S_{|c}^{abef} = 0. \quad (4.18g)$$

However, since the gauge-invariant tensor field  $S^{abcd}$  depends on  $M$  through  $Z^{ab}$  [see Equations (4.2f) and (4.3)], this tensor field has directly nothing to do with the linearized Riemann tensor of some perturbed three-metric.

Of all the possible differential consequences that one can derive from Equations (3.12), the balance laws (3.13a) and (3.13b) have proved to be particularly useful. In terms of our basic gauge-invariant variables, these balance laws are given by

$$\begin{aligned} \dot{\Gamma} = \frac{3}{e_T} \left[ \frac{1}{T_0} (e_0 + p_0 - n_0 e_M) \left( 1 + T_0 \frac{e_{TT}}{e_T} \right) \right. \\ \left. - (e_T + p_T - n_0 e_{MT}) \right] H \Gamma, \end{aligned} \quad (4.19a)$$

$$\begin{aligned} \dot{\Omega}^a + \left[ 2H - \frac{\dot{H}}{H} - \left( \frac{k}{R^2} - \dot{H} \right)^{-1} ((H)^\cdot + 2H \dot{H}) \right] \Omega^a \\ = \frac{3}{2} H \left( \frac{k}{R^2} - \dot{H} \right)^{-1} T_0 p_T \gamma^{ab} \Gamma_{|b} - 3H \gamma^{ab} \Omega_{|b}, \end{aligned} \quad (4.19b)$$

where

$$e_{MT} := \partial e_M / \partial T_0, \quad e_{TT} := \partial e_T / \partial T_0. \quad (4.20)$$

*Remark.* – If  $T_0 = 0$ , it will be possible to replace Equation (4.19a) by  $\dot{\Gamma} + 2H \Gamma = 0$  with  $\Gamma = \mathbb{K}$ . Moreover, in this case, some terms in Equations (4.18) and (4.19b) will assume the comparatively simple form:

$$T_0 e_T \Gamma \Rightarrow \frac{3}{2} m n_0 \mathbb{K} = \frac{3}{2} m n_0 \Gamma, \quad (4.21a)$$

$$T_0 \left( p_T + \frac{1}{3} e_T \right) \Gamma \Rightarrow \frac{3}{2} m n_0 \mathbb{K} = \frac{3}{2} m n_0 \Gamma, \quad (4.21b)$$

$$\left( \frac{k}{R^2} - \dot{H} \right)^{-1} ((H)^\cdot + 2H \dot{H}) \Rightarrow H. \quad (4.21c)$$

In the description of nonbarotropic perfect fluids, Equations (4.18a)-(4.18g) and (4.19) are of most physical interest because they determine the evolution of basic gauge-invariant variables. In fact, with the help of a “coordinate system” on  $\mathcal{W}/\mathcal{W}_0$ , we easily verify that every  $\mathbb{D} \in \mathcal{D}$  is a classical solution of Equations (4.18) and (4.19).

Another welcome features of these equations can be described as follows: (a) they are deterministic, i.e., they lead to a unique solution of the “Cauchy-problem;” (b) their form is independent of any particular coordinate chart chosen; (c) Equations (4.18) and (4.19) are gauge invariant and thus none of the solutions of these equations can be annulled by a gauge transformation; (d) every solution of class  $C^r$  of Equations (4.18) ( $r$  sufficiently large) is an element of  $\mathcal{D}$  and thus satisfies the balance laws (4.19) and any condition  $\text{Lop}_{(\text{gi})}(\mathbb{D}) = 0$  which can be derived from Equations (3.12); (e) the evolution of  $(\Gamma, \Omega, \Omega^a)$  is completely decoupled from that of  $(\Theta, \Theta^{ab}, S^{abcd})$  [see Equations (4.18b), (4.19a), and (4.19b)]; (f) a considerable simplification takes place, since Equations (4.18) and (4.19) involve only the first derivatives of  $\mathbb{D}$  with respect to “time” and Equations (4.18a), (4.18c), (4.18e), and (4.18g) are “algebraic” or “constraint” equations; (g) our basic definitions and equations are independent of any harmonic analysis, but they can be harmonically decomposed if desired; (h) the splitting of Equations (4.18) and (4.19) into “scalar, vector, and tensor parts” [18] can always be given in a coordinate-free manner (see section 5.1).

Turning now to (d), in appendix B we shall prove the following theorem, which is expected to hold also in the case when  $k = \pm 1$ .

**THEOREM 3.** – *For  $k = 0$ , every  $C^2$  solution of Equations (4.18) belongs to  $\mathcal{D}$ , the image space of  $\mathcal{W}/\mathcal{W}_0$  under  $\varphi$ . (For  $k = \pm 1$ , see remark 2 in appendix B).*

Such a proof requires very careful examination if a sound and consistent development is to be achieved for the theory of perturbations at the level of nonbarotropic perfect fluids. It must be stressed that as the theory presently stands, the validity of *theorem 3* is not evident, and we would like to show that the solution of Equations (3.12) is equivalent to solving Equations (4.18). The gist of the point made by *theorems 1* and *3* is that the information content contained in the gauge-invariant perturbation does not contract as the level of description is passed from  $[\mathbb{W}] \in \mathcal{W}/\mathcal{W}_0$  to  $\mathbb{D} \in \mathcal{D}$ , since the passage essentially involves a complete set of variables and every classical solution of Equations (4.18) defines the equivalence class of perturbations.

After solving Equation (4.19a) with respect to  $\Gamma$ , Equations (4.18b) and (4.19b) combine into a single equation for  $\Omega^a$ , equivalent to the central result of Bardeen [5] and Ellis *et al.* [21]; see Equation (4.9) in [5] and Equation (28) in [21]. In fact, of the equations presented, the one of most physical interest is that for  $\Omega^a$ . The gauge-invariant quantity  $H^{-1} \Omega^a$  and its magnitude  $H^{-1} (\gamma_{ab} \Omega^a \Omega^b)^{1/2}$  most closely correspond to the intention of



the usual  $M$  in representing the fractional density increase in a comoving density fluctuation.

## 5. SCALAR PERTURBATION THEORY

### 5.1. Equations for scalar variables

We turn now to the equations for scalar variables<sup>4</sup>, as these are the only ones relevant to the growth of structures in the universe. In this case, we find that

$$\Omega^a = \gamma^{ab} A_{|b}, \quad (5.1a)$$

$$\Theta^{ab} = \left( \gamma^{ac} \gamma^{bd} - \frac{1}{3} \gamma^{ab} \gamma^{cd} \right) B_{|cd}, \quad (5.1b)$$

$$S = \frac{3k}{R^2} C + \gamma^{ab} C_{|ab}, \quad (5.1c)$$

$$S^{ab} = \frac{1}{4} \left( \gamma^{ac} \gamma^{bd} - \frac{1}{3} \gamma^{ab} \gamma^{cd} \right) C_{|cd}, \quad (5.1d)$$

where  $A$  is a potential for  $\Omega^a$ ,  $B$  is a potential for  $\Theta^{ab}$ , and  $C$  is a potential for both  $S$  and  $S^{ab}$ . Because of the “linearized Bianchi identities” (4.18g), in scalar perturbation theory  $S$  and  $S^{ab}$  (and hence  $S^{abcd}$ ) are uniquely determined by specifying  $C$ . The derivation of this technical result is not entirely trivial, however. Note that  $A$ ,  $B$ , and  $C$  are gauge-invariant objects, for essentially obvious reasons. In a similar manifestly covariant fashion, we can introduce the “vector part” of  $\Omega^a$  and the “vector and tensor parts” of  $\Theta^{ab}$  and  $S^{ab}$  [18].

Given these notions as well as Equations (4.18) and (4.19), it is straightforward to verify that  $A$ ,  $B$ ,  $C$ ,  $\Omega$ ,  $\Theta$ , and  $\Gamma$  are constrained to satisfy the following system of equations:

$$\Omega - \frac{k}{4R^2 H^2} C - \frac{1}{12H^2} \gamma^{ab} \left( C - \frac{4}{3} A \right)_{|ab} = \frac{1}{6H^2} T_0 e_T \Gamma, \quad (5.2a)$$

---

4. We assume that this notion is sufficiently well understood in perturbation theory. Nevertheless, we refer the interested reader to Stewart [18] for a detailed definition of scalar, vector, and tensor perturbations.

$$\begin{aligned} \dot{\Omega} + 2 \left( H + \frac{\dot{H}}{H} \right) \Omega + \frac{1}{9H^2} \gamma^{ab} \left( \dot{A} - \frac{\dot{H}}{H} A \right)_{|ab} + \frac{1}{3H} \gamma^{ab} \Omega_{|ab} \\ = -\frac{1}{2H} T_0 \left( p_T + \frac{1}{3} e_T \right) \Gamma, \end{aligned} \quad (5.2b)$$

$$\frac{k}{R^2} B + \frac{1}{3} \gamma^{ab} B_{|ab} + \Omega = \frac{\dot{H}}{3H^2} A, \quad (5.2c)$$

$$-\frac{1}{3} A + H^2 B + \left( HB - \frac{1}{3H} A \right) \dot{\phantom{A}} - \Omega + \frac{1}{4} C = 0, \quad (5.2d)$$

$$\dot{C} - \frac{4}{3} H \gamma^{ab} B_{|ab} = 0, \quad (5.2e)$$

$$\Theta = 3\Omega, \quad (5.2f)$$

$$\left( \frac{e_T}{n_0} \Gamma \right) \dot{\phantom{A}} = 0, \quad (5.2g)$$

$$\begin{aligned} \dot{A} + \left[ H - \frac{\dot{H}}{H} - \left( \frac{k}{R^2} - \dot{H} \right)^{-1} \left( (H) \ddot{\phantom{A}} + 2H \dot{H} \right) \right] A \\ = \frac{3}{2} H \left( \frac{k}{R^2} - \dot{H} \right)^{-1} T_0 p_T \Gamma - 3H \Omega. \end{aligned} \quad (5.2h)$$

Equations (5.2f) and (5.2g) are equivalent to Equations (4.18e) and (4.19a); Equations (5.2a) and (5.2b) are the scalar counterparts of Equations (4.18a) and (4.18b); the ‘‘Bianchi identities’’ (4.18g) are automatically satisfied, because of  $S^{abcd} = -\frac{1}{3} S \gamma^{c[a} \gamma^{b]d} - 2 \gamma^{c[a} S^{b]d} + 2 \gamma^{d[a} S^{b]c}$  and Equations (5.1c) and (5.1d); the remaining equations in the system (5.2) are the simplest ‘‘first and second integrals’’ of Equations (4.18c), (4.18d), (4.18f), and (4.19b).

In summary, we see that there exists a manifestly covariant approach to scalar perturbation theory which does not entail any particular coordinate chart. This approach is very simple because it shows that a complete set of scalar equations can be obtained directly in terms of the gauge-invariant potentials for  $\mathbb{D}$ , i.e., without making any explicit or implicit reference to the gauge-dependent variables  $\delta\mathcal{G}_0$  or  $\mathbb{W}$ . In fact, following these lines, one can easily show that all coordinate systems (and associated tensor bases) are

equally as good as each other to understand the whole Bardeen formalism [5], even though this formalism was originally defined *only* with respect to a geometrically preferred family of space-time coordinates.

## 5.2. A minimal closed set of equations

In the case of scalar perturbations, there exists a gauge-invariant potential  $\Phi$  for  $\delta E_{ab}$  such that

$$\delta E_{ab} = \left( \gamma_a^c \gamma_b^d - \frac{1}{3} \gamma_{ab} \gamma^{cd} \right) \Phi_{|cd}. \quad (5.3)$$

From an analysis of Equation (4.16f) we conclude that one possible choice for  $\Phi$  is as follows:

$$\Phi = \frac{1}{3} A - H^2 B - \frac{1}{4} C. \quad (5.4)$$

However, this choice is not forced on us. By using the definition of  $\Phi$  and exploiting the perturbed field equations (5.2) we then show that

$$\Delta \Phi + 3 \left( \frac{k}{R^2} - H^2 \right) \Phi - 3H \dot{\Phi} = \frac{1}{2} [(e_0 + p_0) \delta + T_0 e_T \Gamma], \quad (5.5a)$$

$$\dot{\Phi} + H \Phi = \frac{1}{2} H (e_0 + p_0) B, \quad (5.5b)$$

$$(\Phi)'' + 4H \dot{\Phi} + (3H^2 + 2\dot{H} - kR^{-2}) \Phi = -\frac{1}{6H} \dot{p}_0 \delta + \frac{1}{2} T_0 p_T \Gamma, \quad (5.5c)$$

where

$$\Delta \Phi := \gamma^{ab} \Phi_{|ab}, \quad \delta := A - 3H^2 B. \quad (5.6)$$

In scalar perturbation theory, the gauge-invariant quantity  $\delta$  seems to encapsulate much of the information we want to discuss density inhomogeneities in an almost-Robertson-Walker universe model.

In a sense, Equations (5.5) were first established by Bardeen [5]. However, as remarked already by Stewart [18] and Mukhanov *et al.* [7], Bardeen makes heavy use of scalar, vector, and tensor harmonic functions. Moreover, his basic gauge-invariant variables are defined with respect to a particular coordinate chart. The purpose of this section was to explain in a covariant manner what Bardeen's major paper is about.

Following Bardeen [5], we stress the fact that Equation (5.5a) has exactly the same form [except for the extra terms due to the expansion of the universe ( $H \neq 0$ ), the spatial curvature ( $k = 0, \pm 1$ ), the temperature effects ( $\Gamma \neq 0$ ), and the time variation of  $\Phi$  ( $\dot{\Phi} \neq 0$ )] as the corresponding Poisson

equation in an expanding background, with  $\frac{1}{2}(e_0 + p_0)\delta$  the analogue of the Newtonian source term and  $\Phi$  the analogue of the Newtonian gravitational potential.

After solving Equation (5.2g) with respect to  $\Gamma$ , a minimal closed set of scalar equations consists of Equations (5.5a) and (5.5c) for  $\Phi$  and  $\delta$ .

## 6. EXPLICIT COVARIANT AND GAUGE-INVARIANT SOLUTIONS IN THE SIMPLEST POSSIBLE CASE

Suppose that  $\Lambda = 0$ . When  $k = 0$  and  $T_0 = 0$ , we find with the use of  $\dot{\Gamma} + 2H\Gamma = 0$  and (4.21) that the solution of Equations (5.2) can be written as

$$A = -\frac{3}{2}c_1 H^{4/3} - \frac{1}{H^{2/3}}\Delta c_1 - \frac{1}{10H^2}\Delta c_2 - \frac{2}{3H^{1/3}}\Delta c_3, \quad (6.1a)$$

$$B = c_1 H^{-2/3} - \frac{1}{10H^2}c_2 + c_3 H^{-1/3}, \quad (6.1b)$$

$$C = c_2 - \frac{2}{15H^2}\Delta c_2 - \frac{4}{3H^{2/3}}\Delta c_1 - \frac{8}{9H^{1/3}}\Delta c_3, \quad (6.1c)$$

$$\Theta = 3\Omega = \frac{9}{4}c_1 H^{4/3} + \frac{1}{2H^{2/3}}\Delta c_1 + \frac{1}{4H^2}\Delta c_2, \quad (6.1d)$$

$$\Gamma = c_1 H^{4/3}, \quad (6.1e)$$

where  $\Delta$  is the Laplacian in the “three-space.” Included in this solution are three functions of space-time positions (denoted  $c_1$ ,  $c_2$ , and  $c_3$ ), such that these functions are constant on each world line:

$$\dot{c}_n = 0, \quad n = 1, 2, 3. \quad (6.2)$$

As  $c_1$ ,  $c_2$ , and  $c_3$  are “independent of time,” we also have

$$(H^{-4/3}\Delta c_n)' = 0. \quad (6.3)$$

Because of these properties of  $c_n$ , Equation (6.1a) becomes for  $c_1 = 0$  the “standard” equation for zero-pressure or zero-temperature perturbation growth relative to proper time along the flow lines in an expanding universe, giving the “expected” modes  $H^p$  with powers of  $-\frac{2}{3}$  and 1. Since *theorem 3* is valid in this case ( $k = 0$ ) and Equations (6.1) are covariant and gauge invariant, it is quite clear in our analysis that these are geometrically and physically well defined modes of growth and decay. The modes  $H^p$  in

Equation (6.1a) with powers of  $\frac{2}{3}$  and  $\frac{4}{3}$  are also physical ( $c_1 \neq 0$ ,  $c_2 = 0$ ,  $c_3 = 0$ ).

Using Equations (5.1) and (6.1), we obtain  $\mathbb{D}$ . In appendix C we shall explicitly verify that this particular  $\mathbb{D}$  belongs to  $\mathcal{D}$ , the image space of  $\mathcal{W}/\mathcal{W}_0$  under  $\varphi$ .

## 7. DISCUSSION AND CONCLUSION

In this paper, we have exploited the viewpoint that the direct way to formulate linear (or higher-order) perturbation theory for Einstein's field equations and the equation of balance of number density is to use one-parameter families of exact solutions to these equations [8, 10-13]. As we are naturally assuming that each metric  $g(\varepsilon, x^c)$  from  $\{g(\varepsilon', x^c); \varepsilon' \in (-d, +d)\}$  is defined on the same space-time manifold  $X$ , two metrics  $g_{(1)}(\varepsilon, x^c)$  and  $g_{(2)}(\varepsilon, x^c)$  are said to be physically equivalent if there is a diffeomorphism  $\sigma_\varepsilon : X \Rightarrow X$  which takes  $g_{(1)}$  into  $g_{(2)}$  [ $g_{(2)} = \sigma^*(g_{(1)})$ ], and clearly  $g_{(1)}$  satisfies the nonlinear field equations if and only if  $g_{(2)}$  does [9]. Consequently, the infinitesimal perturbation  $\delta\mathcal{G}_0$  of  $\mathcal{G}_0$  depends on the choice of a one-parameter group of diffeomorphisms of space-time. As is by now well known [18, 22], this choice is most conveniently specified by giving a vector field  $v$  on  $X$ . Following these lines, we verify that the solutions of the linearized field equations can be unique only up to an "infinitesimal diffeomorphism" ( $\delta\mathcal{G}_0 \Rightarrow \delta\mathcal{G}_0 + \mathcal{L}_v \mathcal{G}_0$ ). This is the so-called gauge problem of linear perturbation theory.

After briefly characterizing an almost-Robertson-Walker universe model, we have found a complete set of basic gauge-invariant variables for the description of nonbarotropic perfect fluids. It is convenient to think of this set as having four aspects. First,  $\mathbb{D}$  gives covariantly defined gauge-invariant quantities with a simple geometrical and physical meaning, that code the information we need to discuss energy-density inhomogeneities. Second,  $\mathbb{D}$  provides a unique representation of the gauge-invariant perturbations  $[\mathbb{W}] \in \mathcal{W}/\mathcal{W}_0$ . Third, any gauge-invariant quantity is obtainable directly from the basic variables  $\mathbb{D} \in \mathcal{D}$  through purely algebraic and differential operations. Finally, when  $\mathbb{D}$  is a classical solution of Equations (4.18), one will be able to construct  $\mathbb{W}$  which satisfies Equations (3.12), agrees with the definitions (4.1) and (4.2) of  $\varphi([\mathbb{W}])$ , and is such that  $\varphi([\mathbb{W}]) = \mathbb{D}$ . We believe that these four results demonstrate the utility of our method in understanding situations of importance in cosmology.

Among the problems that can be studied systematically with this sort of approach, a precise analysis of the formal structure of perturbation theory beyond leading order presents a most interesting challenge. In our two previous papers [12, 13], some representative higher-order equations have been derived from the Einstein-Boltzmann system. But these equations have a gauge-dependent character, i.e., they are not written in a manifestly gauge-invariant form. As a matter of fact, we have used particular gauges expressed in terms of particular coordinate choices. Just as in [23], the basic philosophy was that if observational quantities are calculated, and one keeps track of the gauge freedom involved, then all will be fine and the results will have a physical meaning locally. Nevertheless, the fact remains that far greater care must be exercised in interpreting approaches based on particular coordinate systems, since what one must finally do is to show that the results do not depend on the coordinate choice made. In nonlinear perturbation theory, this problem is most suggestively illustrated by the following theorem (which holds trivially):

**THEOREM 4.** – *Let  $\{A(\varepsilon, x^c); \varepsilon \in (-d, +d)\}$  be a curve of geometrical objects (matter variables, tensor fields, etc.), and suppose that  $A(\varepsilon, x^c)$  is two times continuously differentiable with respect to  $\varepsilon$ . Consider a situation in which  $A(\varepsilon, x^c)$  obeys Equation (4.9). Then the “second variation” of  $A$  as given by*

$$\delta^2 A := \lim_{\varepsilon \rightarrow 0} \left( \frac{\partial^2 A}{\partial \varepsilon^2} \right) \quad (7.1)$$

*is gauge invariant if and only if  $\mathcal{L}_v(\delta A) = 0$  for any vector field  $v$  on the space-time manifold  $X$ . (The proof is based on lemma 2.2 of Stewart and Walker [22]; this lemma gives an explicit expression for  $A_0 := \lim_{\varepsilon \rightarrow 0} A$ .)*

Because of this theorem, it should be clear that much work remains to be done to develop the nonlinear theory of perturbations into a fully effective mathematical tool<sup>5</sup>. However, we do not wish to give the impression that this theory necessarily dictates use of coordinates chosen from an atlas which is specifically adapted to the particular problem at hand. Just as in the linear case, a covariant and gauge-invariant description of general cosmological perturbations is certainly possible.

**APPENDIX A.** – *Proof of the main theorem for  $k = 0, \pm 1$ .*

In section 4.1, we have defined a linear map  $\varphi : \mathcal{W}/\mathcal{W}_0 \Rightarrow \mathcal{D}$  which assigns to each  $[W] \in \mathcal{W}/\mathcal{W}_0$  an element  $\varphi([W]) \in \mathcal{D}$ ; this element consists of basic gauge-invariant variables. In order to show that  $\varphi$  is a

5. In this context, see our discussion directly after Equation (5.13) in [13].

bijection from  $\mathcal{W}/\mathcal{W}_0$  onto  $\mathcal{D}$  (see *theorem 1*), it is sufficient to prove the following lemma:

LEMMA. – Let  $\mathbb{W} := \{Q, Q^a, D, F^{ab}, V, V^a, M, K\}$  be a classical solution of Equations (3.12), and suppose that  $\mathbb{W}$  so defined satisfies the conditions of the form

$$\chi = 0, \quad \Gamma = 0, \quad \Omega = 0, \quad \Theta = 0, \quad (\text{A.1a})$$

$$\Omega^a = 0, \quad \Theta^{ab} = 0, \quad S^{abcd} = 0, \quad (\text{A.1b})$$

where the objects  $\chi$  through  $S^{abcd}$  are given by Equations (4.2) and (4.3). Then there exists a vector field  $v^a = \vartheta w^a + \vartheta^a$  on the space-time manifold  $X$  such that

$$Q = 2(\vartheta)^\cdot, \quad (\text{A.2a})$$

$$Q^a = (\vartheta^a)^\cdot - H\vartheta^a - \gamma^{ab}\vartheta_{|b}, \quad (\text{A.2b})$$

$$D = -H\vartheta - \frac{1}{3}\vartheta_{|c}^c, \quad (\text{A.2c})$$

$$F^{ab} = -\frac{1}{2}(\gamma^{ac}\vartheta_{|c}^b + \gamma^{bc}\vartheta_{|c}^a) + \frac{1}{3}\gamma^{ab}\vartheta_{|c}^c, \quad (\text{A.2d})$$

$$\dot{V} = -(\vartheta)^\cdot, \quad (\text{A.2e})$$

$$V^a = -(\vartheta^a)^\cdot + H\vartheta^a, \quad (\text{A.2f})$$

$$M = -3\vartheta H, \quad (\text{A.2g})$$

$$K = \vartheta \frac{1}{T_0} \dot{T}_0. \quad (\text{A.2h})$$

*Remark 1.* – Interpreting this lemma, Equations (3.20) and (A.2) imply that  $\mathbb{W}$  is a gauge mode solution of Equations (3.12):  $\mathbb{W} = \mathcal{L}_v \mathcal{G}_0$ . Consequently, if  $\varphi([\mathbb{W}]) = 0$ , then  $[\mathbb{W}]$  is a zero-vector of  $\mathcal{W}/\mathcal{W}_0$ .

*Remark 2.* – In the case  $T_0 = 0$ ,  $K$  must be replaced by  $\mathbb{K}$  and Equation (A.2h) by  $\mathbb{K} = 0$ .

*Proof.* – Write

$$\vartheta := -\frac{1}{3H} M, \quad (\text{A.3})$$

and define the spacelike part  $\vartheta^a$  of  $v^a$  by saying that  $\vartheta^a$  obeys the differential equations of the form<sup>6</sup>

$$(\vartheta^a)^\cdot - H \vartheta^a = -V^a, \quad a = 0, 1, 2, 3. \quad (\text{A.4})$$

This trivially proves Equation (A.2g) for  $M$  and Equation (A.2f) for  $V^a$ . Because of  $\Gamma = 0$  and  $\Omega = 0$ ,  $K$  equals  $T_0^{-1} \vartheta \dot{T}_0$ ,  $Q$  equals  $2 (\vartheta)^\cdot$ , and Equations (A.2h) and (A.2a) hold. Then we may conclude from  $\chi = 0$  that Equation (A.2e) is valid for  $V$  as well. After substituting Equations (A.2f) and (A.2g) into the left-hand side of  $\Omega^a = 0$  [see Equation (4.2c) for the definition of  $\Omega^a$ ], we immediately arrive at Equation (A.2b). Turning now to Equation (4.3) as given by

$$Z^{ab} := 2 \left( \frac{1}{3} M - D \right) \gamma^{ab} - 2 F^{ab} \quad (\text{A.5})$$

and using Equations (4.2d) and (4.2e), we conclude from Equations (A.2a), (A.2f), and (A.2g) that the conditions  $\Theta = 0$  and  $\Theta^{ab} = 0$  can be written as

$$\begin{aligned} \dot{Z}^{ab} &= \gamma^{ac} [(\vartheta^b)^\cdot - H \vartheta^b]_{|c} + \gamma^{bc} [(\vartheta^a)^\cdot - H \vartheta^a]_{|c} \\ &= (\gamma^{ac} \vartheta_{|c}^b + \gamma^{bc} \vartheta_{|c}^a)^\cdot. \end{aligned} \quad (\text{A.6})$$

This result is equivalent to

$$Z^{ab} = \gamma^{ac} \vartheta_{|c}^b + \gamma^{bc} \vartheta_{|c}^a + z^{ab}, \quad (\text{A.7})$$

where  $\{z^{ab}\}$  is the second-rank, symmetric spacelike tensor whose components  $z^{ab}$  are “independent of time:”

$$\dot{z}^{ab} = 0. \quad (\text{A.8})$$

Further, as an explicit application of the condition  $S^{abcd} = 0$  and the observation formulated directly before Equation (4.5), we derive the following differential constraint for  $z^{ab}$ :

$$\frac{k}{R^2} z^{e[a} (\gamma^{b]c} \gamma_e^d - \gamma^{b]d} \gamma_e^c) + \gamma^{df} \gamma^{e[a} z_{|ef}^{b]c} - \gamma^{cf} \gamma^{e[a} z_{|ef}^{b]d} = 0. \quad (\text{A.9})$$

However, this constraints is a *necessary* and *sufficient* condition [see Equation (84.12) on p. 352 in [24)] that, given a symmetric spacelike

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6. Without any loss of generality, it is possible to choose coordinates so that  $V^0 = 0$ ,  $\vartheta^0 = 0$ ,  $(\vartheta^0)^\cdot - H \vartheta^0 = 0$ , and  $(\vartheta^r)^\cdot - H \vartheta^r = \partial \vartheta^r / \partial x^0$  ( $r = 1, 2, 3$ ). Thus Equation (A.4) is equivalent to  $\partial \vartheta^r / \partial x^0 = -V^r$  and may be integrated to give  $\vartheta^a = (0, \vartheta^r)$ . These coordinates are also useful to prove the condition (A.11).



tensor field  $z^{ab}$  on  $X$  satisfying Equation (A.8), there exists a spacelike vector field  $z^a$  on  $X$  such that<sup>6,7</sup>

$$z^{ab} = \gamma^{ac} z^b_{|c} + \gamma^{bc} z^a_{|c} \tag{A.10}$$

and

$$\dot{z}^a - H z^a = 0. \tag{A.11}$$

Of course, since  $(\vartheta^a) \cdot -H \vartheta^a = (\vartheta^a + z^a) \cdot -H (\vartheta^a + z^a)$ , we may always choose to replace  $\vartheta^a$  by  $\vartheta^a + z^a$ . Consequently, if we suppose the vector field  $\vartheta^a$  so adjusted that  $z^a = 0$ , by combining Equations (A.2g), (A.5), (A.7), and (A.10) we obtain

$$D \gamma^{ab} + F^{ab} = -H \vartheta \gamma^{ab} - \frac{1}{2} (\gamma^{ac} \vartheta^b_{|c} + \gamma^{bc} \vartheta^a_{|c}) \tag{A.12}$$

and hence Equations (A.2c) and (A.2d). This completes the proof of our lemma. ■

*Theorem 1* proposed in section 4.1 is a direct consequence of this lemma.

APPENDIX B. – *Knowledge about [W] by means of a basic set of covariant and gauge-invariant propagation equations*

Let us ask now to what extent a knowledge of the solutions of Equations (4.18) determines the equivalence classes of perturbations [W]. To this end, suppose that

$$\mathbb{D} := \{\chi, \Gamma, \Omega, \Omega^a, \Theta, \Theta^{ab}, S^{abcd}\} \tag{B.1}$$

is a classical solution of Equations (4.18). Then by means of Equations (4.2a)-(4.2e), it is always possible, provided  $\mathbb{D}$  is of class  $C^r$  ( $r$  sufficiently large) to choose

$$\mathbb{W} := \{Q, Q^a, D, F^{ab}, V, V^a, M, K\} \tag{B.2}$$

such that Equations (4.2a)-(4.2e) are satisfied: after setting  $Q = 0$  and  $Q^a = 0$ , one simply solves Equations (4.2a)-(4.2e) with respect to  $D, F^{ab}, V, V^a, M,$  and  $K$ . However, the conditions  $Q = 0$  and  $Q^a = 0^a := 0$  do not specify  $\mathbb{W}$  uniquely. In fact, if

$$\mathbb{W} = \{0, 0^a, D, F^{ab}, 0, V^a, M, K\} \tag{B.3}$$

is a solution of Equations (4.2a)-(4.2e), then another solution

$$\overline{\mathbb{W}} = \{\overline{Q}, \overline{Q}^a, \overline{D}, \overline{F}^{ab}, \overline{V}, \overline{V}^a, \overline{M}, \overline{K}\} \tag{B.4}$$

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7. The notation of Truesdell and Toupin [24] differs from that of ours as follows: they denote by  $A^{a_b \dots c}$  the covariant derivative  $A^a_{b \dots c}$ .

can be found, not necessarily satisfying the conditions  $\overline{Q} = \overline{V} = 0$  and  $\overline{Q}^a = 0$ , for which the “time derivative” of

$$\overline{Z}^{cd} := 2 \left( \frac{1}{3} \overline{M} - \overline{D} \right) \gamma^{cd} - 2 \overline{F}^{cd} \quad (\text{B.5})$$

becomes

$$(\overline{Z}^{cd}) \cdot = 2 \left( \frac{1}{3} \dot{M} - \dot{D} \right) \gamma^{cd} - 2 \dot{F}^{cd} + \gamma^{ce} \mathfrak{F}_{|e}^d + \gamma^{de} \mathfrak{F}_{|e}^c. \quad (\text{B.6})$$

Here, of course,  $\mathfrak{F}^a$  is an arbitrary spacelike vector field on  $X$ . This result follows from an analysis of only Equations (4.2a)-(4.2e) and thus is weaker than demanding that  $\overline{W} = W + L_v \mathcal{G}_0$ .

Having chosen some  $W$  as a solution of Equations (4.2a)-(4.2e) and remembering that  $\mathbb{D}$  satisfies by definition Equations 4.18), it can be verified from Equations (4.2a)-(4.2e) and (4.18e)-(4.18g) that  $S^{abcd}$  is of the form (4.2f) with  $Z^{ab}$  characterized by

$$Z^{ab} = 2 \left( \frac{1}{3} M - D \right) \gamma^{ab} - 2 F^{ab} + z^{ab} \quad (\text{B.7})$$

and<sup>8</sup>

$$(z^{ab}) \cdot = \gamma^{ac} \vartheta_{|c}^b + \gamma^{bc} \vartheta_{|c}^a, \quad w_a \vartheta^a = 0. \quad (\text{B.8})$$

*Remark 1.* – Observe that Equation (4.18g) is the only natural condition that the spacelike quantity<sup>8</sup>  $S^{abcd}$  with the properties  $S^{abcd} = S^{[ab][cd]}$  and  $S^{a[bcd]} = 0$  takes the form (4.2f) for some symmetric spacelike tensor field  $Z^{ab}$ . In order to obtain Equations (B.7) and (B.8), we have used this condition as well as the fact [24] that<sup>9</sup>

$$A^{abcd} := \frac{k}{R^2} A^{e[a} (\gamma^{b]c} \gamma_e^d - \gamma^{b]d} \gamma_e^c) + \gamma^{df} \gamma^{e[a} A_{|ef}^{b]c} - \gamma^{ef} \gamma^{e[a} A_{|ef}^{b]d} \quad (\text{B.9})$$

vanishes if and only if  $A^{ab} = \gamma^{ac} A_{|c}^b + \gamma^{bc} A_{|c}^a$  and  $w_a A^a = 0$  (see appendix A). However, when the fulfilment of Equation (4.18g) does not automatically imply the relation (4.2f), the derivation of Equations (B.7)

8. In Equation (B.8),  $\vartheta^a$  is a certain spacelike vector field on  $X$ . By definition,  $S^{abcd}$  belongs to the three-dimensional space, everywhere orthogonal to the observer with four-velocity  $w^a$ . Also  $\Omega^a$  and  $\Theta^{ab}$  are orthogonal to  $w^a$ .

9. We postulate that  $A^{abcd}$  and  $A^{ab} = A^{ba}$  are the spacelike objects.

and (B.8) reduces to finding an appropriate additional condition and to supplementing Equation (4.18g) by this condition.

*Remark 2.* – For  $k = 0$ , Trautman [25] and Pirani [26] were able to deduce from Equation (4.18g) and the symmetry properties of  $S^{abcd}$  the existence of  $Z^{ab}$  satisfying Equation (4.2f). Thus in this case, our argument based on Equation (4.18g) is complete. For  $k = \pm 1$ , all that one need do to see the existence of  $Z^{ab}$  is first separate Equations (4.2f) and (4.18g) into “scalar, vector, and tensor parts” [5, 18] and then exploit the fact [27] that  $\gamma_{bc} S^a{}_{|a} = \frac{1}{6} S_{|c}$  contain all the information in the “three-dimensional Bianchi identities” (4.18g). Because of this, it may prove quite easy to generalize the original result of Trautman [25] and Pirani [26].

Now, using the formulas (B.5) and (B.6) of interest for general solutions of Equations (4.2a)-(4.2e), we can define  $\mathbb{W}$  associated with  $\mathbb{D}$  in such a way that

$$\dot{z}^{ab} = 0. \quad (\text{B.10})$$

But  $D$  and  $F^{ab}$  appear in Equations (4.2a)-(4.2e) only through  $\dot{D}$  and  $\dot{F}^{ab}$ . Consequently, if we suppose  $\mathbb{W}$  so adjusted that  $z^{ab} = 0$ , as is always possible, we arrive at the following two conclusions: (i) for every classical solution  $\mathbb{D}$  of Equations (4.18), there exists  $\mathbb{W}$  which determines  $\mathbb{D}$  from the relations (4.2) and (4.3); (ii) by substituting these relations into Equations (4.18a)-(4.18e), one obtains or recovers the linearized field equations (3.12) for  $\mathbb{W}$ .

The above considerations suffice to prove *theorem 3* as formulated in section 4.3. This theorem and the results of appendix A establish one possible sense in which the classical solutions  $\mathbb{D}$  of Equations (4.18) determine “everything”, namely that one can extract  $[\mathbb{W}]$  from  $\mathbb{D}$  in a unique way.

APPENDIX C. – *Explicit solution for  $\mathbb{W}$  in the simplest possible case* ( $k = 0, \Lambda = 0, T_0 = 0$ ).

Restricting attention to the so-called scalar perturbations [18], we find that  $Q^a$ ,  $V^a$ , and  $F^{ab}$  can be written in the form

$$Q^a = \gamma^{ab} N_{|b}, \quad V^a = \gamma^{ab} W_{|b}, \quad (\text{C.1a})$$

$$F^{ab} = \left( \gamma^{ac} \gamma^{bd} - \frac{1}{3} \gamma^{ab} \gamma^{cd} \right) F_{|cd}, \quad (\text{C.1b})$$

where  $N$  is a potential for  $Q^a$ ,  $W$  is a potential for  $V^a$ , and  $F$  is a potential for  $F^{ab}$ .

For comparison with the results of section 6, we consider the simplest expanding solution, the Einstein-de Sitter universe with  $k = 0$ ,  $\Lambda = 0$ , and  $T_0 = 0$ . Moreover, to obtain an explicit form of  $W$ , we fix a gauge by setting  $Q = 0 = V$  and  $N = 0$ . As usual, it will be convenient to refer to this gauge as a *synchronous gauge*. Using the propagation equations for  $W$  (see section 3.2), we then find that  $\mathbb{K}$  and the gauge-dependent quantities  $M$ ,  $D$ ,  $F$ , and  $W$  are given by

$$\mathbb{K} = c_1 H^{4/3}, \quad (\text{C.2a})$$

$$M = a_1 H + \frac{9}{2} c_1 H^{4/3} - \frac{1}{H^{2/3}} \Delta c_1 - \frac{1}{10 H^2} \Delta c_2, \quad (\text{C.2b})$$

$$D = \frac{1}{3} a_1 H + a_2 - \frac{2}{9H} \Delta a_1 + \frac{3}{2} c_1 H^{4/3} - \frac{1}{H^{2/3}} \Delta c_1 - \frac{1}{30 H^2} \Delta c_2 - \frac{4}{27 H^{7/3}} \Delta (\Delta c_3), \quad (\text{C.2c})$$

$$F = -\frac{2}{3H} a_1 + a_3 H^{-4/3} - \frac{3}{H^{2/3}} c_1 - \frac{1}{10 H^2} c_2 - \frac{2}{3 H^{1/3}} c_3 - \frac{4}{9 H^{7/3}} \Delta c_3, \quad (\text{C.2d})$$

$$W = \frac{1}{3} a_1 + 2 c_1 H^{1/3} + \frac{2}{9 H^{4/3}} \Delta c_3, \quad (\text{C.2e})$$

where the coefficients  $c_1$ ,  $c_2$ , and  $c_3$  have exactly the same meaning as in Equations (6.1) and where the objects  $a_1$ ,  $a_2$ , and  $a_3$  are scalar fields constant on each world line:

$$\dot{a}_1 = 0, \quad \dot{a}_2 = 0, \quad \dot{a}_3 = 0. \quad (\text{C.3})$$

It should be now clear that the solution of Equations (3.12) in a synchronous gauge implies the existence of extra, unphysical modes. These modes can be easily identified if we set  $c_1 = 0$ ,  $c_2 = 0$ , and  $c_3 = 0$  in Equations (C.2). Note that our solution for  $F$  gives two independent modes with the same asymptotic behaviour ( $t \Rightarrow \infty$ ):

$$F = -\frac{2}{3H} a_1, \quad (\text{C.4a})$$

$$F = -\frac{2}{3 H^{1/3}} c_3 - \frac{4}{9H} (H^{-4/3} \Delta c_3). \quad (\text{C.4b})$$

Only one of them, Equation (C.4b), is physical. Ellis and Bruni [6] were certainly right in stressing that the last term on the right-hand side of Equation (6.1a) cannot be annulled by a gauge transformation: this is a gauge-invariant term.

*Note Added in Proof.* – An alternative discussion of the notion of a gauge-invariant variable, based on the theory of vector bundles, is given in [28]. This enables many of the concepts introduced earlier [5-7] to be reformulated in a geometrical way. In [29] we study the gauge problem in a broader context, *i.e.*, for different general-relativistic models such as the Einstein-Liouville system.

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