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## **Nonexistence of Petrov type III space-times on which Weyl's neutrino equation or Maxwell's equations satisfy Huygens' principle**

by

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**ABSTRACT.** – Extending previous results we show that there are no Petrov type III space-times on which either the Weyl neutrino equation or Maxwell's equations satisfy Huygens' principle. We prove the result by using Maple's `NPspinor` package to convert the five-index necessary condition obtained by Alvarez and Wunsch to dyad form. The integrability conditions of the problem lead to a system of polynomial equations. We then apply Maple's `grobner` package to show that this system has no admissible solutions.

**RÉSUMÉ.** – En prolongeant des résultats précédents, on démontre qu'il n'existe aucun espace-temps de type III de Petrov sur lequel l'équation de neutrino de Weyl ou les équations de Maxwell satisfait au principe d'Huygens. Nous prouvons le résultat par utilisant le logiciel `NPspinor` de Maple pour transformer la nouvelle condition à cinq indices obtenue par Alvarez et Wunsch en composants de repère spinoriel. A partir des conditions d'intégrabilité du problème, on obtient un système d'équations polynômes. Nous employons donc le logiciel `grobner` de Maple pour démontrer qu'il n'existe aucune solution admissible de ce système.

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## 1. INTRODUCTION

This paper is the sixth in a series devoted to the solution of Hadamard's problem for the conformally invariant scalar wave equation, Weyl's neutrino equation and source-free Maxwell's equations. These equations can be written respectively as

$$\square u + \frac{1}{6}Ru = 0, \quad (1)$$

$$\nabla^{AA'}\varphi_A = 0, \quad (2)$$

$$\nabla^{AA'}\varphi_{AB} = 0, \quad (3)$$

The conventions and formalism used in this paper are those of Carminati and McLenaghan [7] (CM in the sequel). All considerations in this paper are entirely local. Part of the calculations presented here were performed using the `NPspinor` package available in Maple [8], [9].

Huygens' principle is valid for (1), (2) and (3) if and only if for every Cauchy initial value problem and every  $x_0 \in V_4$ , the solution depends only on the Cauchy data in an arbitrarily small neighbourhood of  $S \cap C^-(x_0)$ , where  $S$  denotes the initial surface and  $C^-(x_0)$  the past null conoid from  $x_0$  [14], [22], [12]. Hadamard's problem for equations (1), (2) or (3), originally posed only for scalar equations, is that of determining *all* space-times for which Huygens' principle is valid for a particular equation. As a consequence of the conformal invariance of the validity of Huygens' principle, the determination may only be effected up to an arbitrary conformal transformation on the metric in  $V_4$ .

$$\tilde{g}_{ab} = e^{2\varphi}g_{ab}, \quad (4)$$

where  $\varphi$  is an arbitrary real function.

Huygens' principle is valid for (1), (2) and (3) on any space-time conformally related to the exact plane-wave [12], [15], [23], with metric

$$ds^2 = 2dv[du + [D(v)z^2 + \overline{D}(v)\bar{z}^2 + e(v)z\bar{z}]dv] - 2dzd\bar{z}, \quad (5)$$

in a special coordinate system where  $D$  and  $e$  are arbitrary functions. These are the only *known* space-times on which Huygens' principle is valid for these equations. Furthermore, it has been shown [13], [16], [23] that these are the only conformally empty space-times on which Huygens' principle is valid.

In the non-conformally empty case several results are known. In particular for Petrov type N, the following result has been proved [4], [5], [2]: *Every Petrov type N space-time on which the equations (1), (2) or (3) satisfy Huygens' principle is conformally related to an exact plane wave space-time (5).* For the case of a Petrov type D the following result has been established [6], [24], [19]: *There exist no Petrov type D space-times on which equations (1), (2) or (1) satisfy Huygens' principle.*

In this paper we complete the analysis for Petrov type III spacetimes given by CM, in the case of equations (1) and (2). The main result is contained in the following theorem:

**THEOREM 1.** – *There exist no Petrov type III space-times on which Weyl's equation (2) or Maxwell's equations (3) satisfy Huygens' principle.*

### 2. PREVIOUS RESULTS

Let  $\Psi_{ABCD}$  denote the Weyl spinor. Petrov type III space-times are characterized by the existence of a spinor field  $o^A$  satisfying

$$\Psi_{ABCD}o^C o^D = 0, \quad \Psi_{ABCD}o^D \neq 0. \tag{6}$$

Such spinor field is called a *repeated principal spinor* of the Weyl spinor and is determined by the latter up to an arbitrary variable complex factor. Let  $\iota^A$  be any spinor field satisfying

$$o_A \iota^A = 1. \tag{7}$$

The ordered set  $o_A, \iota_A$ , called a dyad, defines a basis for the 1-spinor fields on  $V_4$ .

The main results obtained by CM can be stated as follows:

**THEOREM 2.** – *The validity of Huygens' principle for the conformally invariant scalar wave equation (1), or Maxwell's equations (2), or Weyl's neutrino equation (3) on any Petrov type III space-time implies that the space-time is conformally related to one in which every repeated principal spinor field  $o_A$  of the Weyl spinor is recurrent, that is*

$$o_{A;B\dot{B}} = o_A I_{B\dot{B}}, \tag{8}$$

where  $I_{B\dot{B}}$  is a 2-spinor, and

$$\Psi_{ABCD;E\dot{E}} \iota^A \iota^B \iota^C o^D o^E \bar{o}^{\dot{E}} = 0, \tag{9}$$

$$R = 0, \quad \Phi_{AB\dot{A}\dot{B}} o^A o^B = 0. \tag{10}$$

THEOREM 3. – *If any one of the following three conditions*

$$\Psi_{ABCD;E\dot{E}} \iota^A \iota^B \iota^D \iota^E \bar{o}^{\dot{E}} = 0, \tag{11}$$

$$\Psi_{ABCD;E\dot{E}} \iota^A \iota^B o^D o^E \bar{\iota}^{\dot{E}} = 0, \tag{12}$$

$$\Psi_{ABCD;E\dot{E}} \iota^A \iota^B \iota^D o^E \bar{o}^{\dot{E}} = 0, \tag{13}$$

*is satisfied, then there exist no Petrov type III space-times on which the conformally invariant scalar wave equation (1) or Maxwell's equations (2), or Weyl's neutrino equation (3) satisfies Huygens' principle.*

It will be proved here that conditions (11) to (13) are superfluous in the cases of equations (2) and (3), i.e., they are consequences of the necessary conditions for the validity of Huygens' principle, in particular of the new five index necessary conditions derived by Alvarez and Wunsch.

### 3. NECESSARY CONDITIONS

In order to prove the Theorem 1 we shall need the following necessary conditions for the validity of Huygens' principle [10], [20], [18], [17], [22]:

$$(III) \quad S_{abk; \quad}^k - \frac{1}{2} C^k_{ab}{}^l L_{kl} = 0, \tag{14}$$

$$(V) \quad TS[k_1 C^k_{ab}{}^l{}^m C_{kcdl;m} + 2k_2 C^k_{ab}{}^l{}_{;c} S_{kld} + 2(8k_1 - k_2) S_{ab}{}^k S_{cdk} - 2k_2 C^k_{ab}{}^l S_{klc;d} - 8k_1 C^k_{ab}{}^l S_{cdk;l} + k_2 C^k_{ab}{}^l C_l{}^m{}_{ck} L_{dm} + 4k_1 C^k_{ab}{}^l C^m{}_{cdl} L_{km}] = 0, \tag{15}$$

where

$$C_{abcd} := R_{abcd} - 2g_{[a[d} L_{b]c]}, \tag{16}$$

$$L_{ab} := -R_{ab} + \frac{1}{6} R g_{ab}, \tag{17}$$

$$S_{abc} := L_{a[b;c]}. \tag{18}$$

In the above  $C_{abcd}$  denotes the Weyl tensor,  $R_{ab}$  the Ricci tensor and  $TS[ \ ]$  the operator which takes the trace free symmetric part of the enclosed tensor. In (15)  $k_1$  takes values 3, 8, 5 and  $k_2$  the values 4, 13, 16, depending

on whether the equation under consideration is the conformally invariant scalar equation, Weyl's equation or Maxwell's equations respectively.

We shall also need a necessary condition involving five free indices, valid for Weyl's equation (2) and Maxwell's equations (3), that was found by Alvarez and Wunsch [1], [2]. It can be written in the form

$$T^{(1)}_{abcde} + \sigma_1 T^{(2)}_{abcde} + \sigma_2 T^{(3)}_{abcde} = 0, \tag{19}$$

where  $\sigma_1$  and  $\sigma_2$  are fixed real numbers, and

$$\begin{aligned} T^{(1)}_{abcde} = & TS[4^* C^k_{ab}{}^l C^{ru}_{del;ukc} - 6^* C^k_{bc}{}^l{}_{;a} C^{ru}_{del;uk} \\ & + 26^* C^{ru}_{ab}{}^k{}_{;u} C^v_{dek;vc} \\ & + ^* C^k_{ab}{}^l{}_{;n} C^{kdel;nc} + 5^* C^k_{ab}{}^l C^n_{del;c} L_{kn} \\ & + 4^* C^k_{ab}{}^l C^n_{cdl;k} L_{en} \\ & + 4^* C^k_{ab}{}^l C^u_{klc;u} L_{de} - 21^* C^k_{ab}{}^l C^u_{cdk;u} L_{el}], \end{aligned} \tag{20}$$

$$T^{(2)}_{abcde} := TS[-12^* C^k_{ab}{}^l C^{kcd}{}^n C^u_{lne;u} - C_k{}^{nh}{}_a C_{lnhb}{}^* C^k{}_{de}{}^l{}_{;c}], \tag{21}$$

$$T^{(3)}_{abcde} := TS[-8^* C^k_{ab}{}^l C^{kcd}{}^n C^u_{lne;u} + ^* C_k{}^{nh}{}_a C_{lnhb} C^k{}_{de}{}^l{}_{;c}], \tag{22}$$

where

$$^* C_{abcd} := \frac{1}{2} \epsilon_{ab}{}^{ef} C_{efcd}. \tag{23}$$

We note that, in our conventions, the Riemann tensor, Ricci tensor and the Ricci scalar have opposite sign to those used by Alvarez and Wunsch [2]. The spinor equivalents of conditions III and V are given, respectively, by [19], [5]

$$\Psi_{ABKL}{}^K{}_A{}^L{}_B + \Psi_{AB}{}^{KL} \Phi_{KL}{}^{\dot{A}\dot{B}} = 0, \tag{24}$$

$$\begin{aligned} & k_1 \Psi_{ABCD}{}^K{}_K{}^{\dot{A}\dot{B}\dot{C}\dot{D}} \bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}{}^{\dot{K}\dot{K}} + k_2 \Psi^K{}_{(ABC;D)}{}^{\dot{A}\dot{B}\dot{C}\dot{D}} \bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}}{}^{\dot{L}}{}_{\dot{L}}{}^{\dot{K}} \\ & + k_2 \bar{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C};\dot{D})}{}^{\dot{A}\dot{B}\dot{C}\dot{D}} \Psi_{BCD}{}^L{}_{;L}{}^{\dot{K}} \\ & - 2(8k_1 - k_2) \Psi_{ABC|K|}{}^K{}_{(\dot{A}\dot{B}\dot{C}\dot{D})\dot{K}}{}^{\dot{K}}{}_{\dot{D}} \\ & - k_2 \Psi^K{}_{(ABC} \bar{\Psi}_{\dot{A}\dot{B}\dot{C}|\dot{L}|}{}^{\dot{L}}{}_{|K|D)\dot{D}} - k_2 \bar{\Psi}^{\dot{K}}{}_{(\dot{A}\dot{B}\dot{C}} \Psi_{(ABC|L|}{}^L{}_{|K|D)\dot{D}} \\ & + 4k_1 \Psi^K{}_{(ABC} \bar{\Psi}_{\dot{A}\dot{B}\dot{C}|\dot{L}|}{}^{\dot{L}}{}_{D)K\dot{D}} + 4k_1 \bar{\Psi}^{\dot{K}}{}_{\dot{A}\dot{B}\dot{C}} \Psi_{ABC|L|}{}^L{}_{D)D)\dot{K}} \\ & + 2(k_2 - 4k_1) \Psi^K{}_{(ABC} \Phi_D)K\dot{K}(\dot{A}\dot{B}\dot{C}\dot{D})^{\dot{K}} \\ & - 2(4k_1 + k_2) \Lambda \Psi_{ABCD} \bar{\Psi}_{\dot{A}\dot{B}\dot{C}\dot{D}} = 0. \end{aligned} \tag{25}$$

#### 4. PROOF OF THEOREM 1

In CM, Theorem 2 was proved by using conditions III and V. The explicit form of these necessary conditions is obtained by first converting the spinorial expressions to the dyad form and then contracting them with appropriate products of  $o^A$  and  $\iota^A$  and their complex conjugates. In particular, it was shown, that there exists a dyad  $o_A, \iota_A$  and a conformal transformation such that

$$\kappa = \sigma = \rho = \tau = \epsilon = 0, \quad (26)$$

$$\Psi_0 = \Psi_1 = \Psi_2 = \Psi_4 = 0, \quad \Psi_3 = -1, \quad (27)$$

$$\Phi_{00} = \Phi_{01} = \Phi_{02} = R = 0, \quad (28)$$

$$D\alpha = D\beta = 0. \quad (29)$$

$$\Phi_{11} = c, \quad (30)$$

where  $c$  is a constant.

By contracting condition (III) with  $\iota^{AB}\bar{o}^{\dot{A}\dot{B}}$  and  $\iota^A o^B \bar{\iota}^{\dot{A}\dot{B}}$  (where the notation  $o_{A_1 \dots A_p} = o_{A_1} \dots o_{A_p}$ , etc. has been used) we get, respectively,

$$D\pi = 0, \quad (31)$$

$$\delta\beta = -\beta(\bar{\alpha} + \beta). \quad (32)$$

From the Bianchi identities, using the above conditions, we obtain

$$D\Phi_{12} = 2\bar{\pi}\Phi_{11}, \quad (33)$$

$$D\Phi_{22} = -2(\beta + \bar{\beta}) + 2\Phi_{21}\bar{\pi} + 2\Phi_{12}\pi, \quad (34)$$

$$\delta\Phi_{12} = 2\bar{\alpha} + 4\bar{\pi} + 2\bar{\lambda}\Phi_{11} - 2\bar{\alpha}\Phi_{12}, \quad (35)$$

$$\bar{\delta}\Phi_{12} = -2\beta + 2\bar{\mu}\Phi_{11} - 2\bar{\beta}\Phi_{12}. \quad (36)$$

$$\begin{aligned} \delta\Phi_{22} = & \Delta\Phi_{12} + 2\bar{\gamma} + 4\bar{\mu} - 2\bar{\nu}\Phi_{11} + 2\bar{\lambda}\Phi_{21} \\ & + 2\Phi_{12}\mu - 2\Phi_{22}\beta - 2\Phi_{22}\bar{\alpha} + 2\bar{\gamma}\Phi_{12}. \end{aligned} \quad (37)$$

From the Ricci identities we get

$$D\gamma = \bar{\pi}\alpha + \beta\pi + \Phi_{11}, \quad (38)$$

$$\delta\bar{\pi} = D\bar{\lambda} - \bar{\pi}^2 - \bar{\pi}\bar{\alpha} + \bar{\pi}\beta, \tag{39}$$

$$D\bar{\nu} = \Delta\bar{\pi} + \bar{\pi}\bar{\mu} + \bar{\lambda}\pi + \bar{\pi}\bar{\gamma} - \bar{\pi}\gamma - 1 + \Phi_{12}, \tag{40}$$

$$\delta\alpha = \bar{\delta}\beta + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\beta\alpha + \Phi_{11}, \tag{41}$$

$$\delta\pi = D\mu - \bar{\pi}\pi + \pi\bar{\alpha} - \beta\pi. \tag{42}$$

We can obtain useful integrability conditions for the above identities by using NP commutation relations. Using (33), (34), (37), (32), (38), (39), (40) and (42) in the commutator expression  $[\delta, D]\Phi_{22} - [\Delta, D]\Phi_{12}$ , gives

$$\delta\bar{\beta} = -2\Phi_{11} - \bar{\beta}\bar{\alpha} - 4\bar{\beta}\bar{\pi} - 2D\bar{\mu} - \beta\bar{\beta} + 2\bar{\pi}\pi. \tag{43}$$

From now on we shall consider both Maxwell and Weyl cases separately. We begin with the Maxwell equations, i.e.,  $k_1 = 5$  and  $k_2 = 16$ . By contracting condition V with  $\iota^{ABCD}\bar{\iota}^{\dot{A}\dot{B}}\bar{o}^{\dot{C}\dot{D}}$ , we get

$$62\bar{\beta}\pi + 40\bar{\beta}\alpha + 6\pi\alpha + 3\alpha^2 + \bar{\delta}\alpha + 2\bar{\delta}\pi + \bar{\delta}\bar{\beta} + \bar{\beta}^2 = 0. \tag{44}$$

Substituting (32) into this equation we get

$$\delta(2\bar{\pi} + \bar{\alpha}) = -62\bar{\pi}\beta - 39\beta\bar{\alpha} - 6\bar{\pi}\bar{\alpha} - 3\bar{\alpha}^2. \tag{45}$$

From (41), (42) and (43) we obtain

$$\delta(2\pi + \alpha) = 2\pi\bar{\alpha} + \alpha\bar{\alpha} - 6\beta\pi - 3\beta\alpha - \Phi_{11}. \tag{46}$$

Contracting condition V with  $\iota^{ABC}o^D\bar{\iota}^{\dot{A}\dot{B}}\bar{o}^{\dot{C}\dot{D}}$ , using (38), (42) and the complex conjugate of (43), we get

$$148\Phi_{11} + 152\bar{\beta}\bar{\pi} + 76(D\mu + D\bar{\mu}) - 8\bar{\pi}\alpha - 104\beta\bar{\beta} - 8\alpha\bar{\alpha} - 232\bar{\pi}\pi + 152\beta\pi - 8\pi\bar{\alpha} = 0. \tag{47}$$

Using (45), (46), (32), (41), (42), (43) and (47) in  $[\bar{\delta}, \delta](\alpha + 2\pi) = (\alpha - \bar{\beta})\delta(\alpha + 2\pi) + (-\bar{\alpha} + \beta)\bar{\delta}(\alpha + 2\pi)$ , we obtain one expression for  $D\bar{\mu}$ . Substituting it in (47) and solving for  $D\mu$  we obtain

$$D\mu = \frac{1}{152} \left( -1520\alpha\beta\pi + 208\alpha\bar{\alpha}\pi - 152\bar{\beta}\bar{\pi}\pi + 1040\beta\bar{\beta}\alpha + 1968\bar{\beta}\beta\pi + 1216\bar{\beta}\pi\bar{\alpha} - 1228\pi\Phi_{11} + 1688\alpha\pi\bar{\pi} - 739\alpha\Phi_{11} + 80\alpha^2\bar{\alpha} + 380\bar{\beta}\Phi_{11} + 2496\bar{\pi}\pi^2 - 2432\beta\pi^2 + 128\pi^2\bar{\alpha} + 80\bar{\pi}\alpha^2 + 760\bar{\beta}\alpha\bar{\alpha} \right) / (-\bar{\beta} + 5\alpha + 8\pi), \tag{48}$$



where we assumed that  $\bar{\beta} - 5\alpha - 8\pi \neq 0$ . By substituting this equation into (46), we find:

$$D\bar{\mu} = \frac{1}{152} \left( 776\alpha\bar{\beta}\bar{\alpha} - 304\bar{\beta}^2\bar{\pi} + 208\bar{\beta}^2\beta + 84\bar{\beta}\Phi_{11} - 1216\bar{\pi}\pi^2 \right. \\ \left. + 741\alpha\Phi_{11} + 1536\alpha\bar{\pi}\bar{\beta} - 760\alpha\bar{\pi}\pi + 2744\bar{\pi}\pi\bar{\beta} + 1332\pi\bar{\beta}\bar{\alpha} \right. \\ \left. + 741\alpha\Phi_{11} + 1140\pi\Phi_{11} \right) \\ / (\bar{\beta} - 5\alpha - 8\pi). \quad (49)$$

We notice that (48) and (49) have the same denominator. So, in what follows we shall use the Pfaffians  $\delta\alpha$ ,  $\delta\pi$ , and  $\delta\bar{\beta}$ , given by (41), (42) and (43), respectively, and their complex conjugates, in such a way that they have all the same denominator. This procedure simplifies the expressions to be obtained from the integrability conditions.

We need to convert (19) to the spinor form in the dyad basis. The resulting expression, obtained using Maple's package `NPSPINOR` [9], has a considerable size, specially due to the term in (20) containing a third order derivative of the Weyl tensor, and will not be presented here. However, among the twenty one independent spinor components, we found a relatively simple one, obtained by contracting the dyad expression with  $\iota^{AB} \iota_{CDE} \bar{\iota}^{\dot{A}\dot{B}\dot{C}\dot{D}\dot{E}}$ . It has the following form:

$$- 14\bar{\alpha} \delta\bar{\pi} - 12\bar{\alpha}^3 - \delta(\delta(\bar{\alpha} + 2\bar{\pi})) \\ - 21\beta\bar{\alpha}^2 + 7\bar{\lambda}\beta^2 + 14\beta^2\bar{\pi} \\ - 7\beta\delta(\bar{\alpha} + 2\bar{\pi}) - 42\beta\bar{\alpha}\bar{\pi} - (10\bar{\alpha} + 6\bar{\pi})\delta\bar{\alpha} - 24\bar{\alpha}^2\bar{\pi} = 0. \quad (50)$$

We observe that the terms (21) and (22) did not contribute to this component. Using (45) to eliminate  $\delta\pi$  from this equation, and solving for  $\delta\bar{\alpha}$  we get

$$\delta\bar{\alpha} = 192\beta\bar{\pi} - 3\bar{\alpha}^2 + 121\beta\bar{\alpha}, \quad (51)$$

and

$$\delta\bar{\pi} = -127\bar{\pi}\beta - 80\beta\bar{\alpha} - 3\bar{\pi}\bar{\alpha}. \quad (52)$$

We have now all the Pfaffians we need to complete the proof. The integrability conditions provided by the NP commutation relations can now be used. Let us consider the NP commutator  $[\bar{\delta}, \delta]\alpha$ . Using the Pfaffians

calculated previously, and solving for  $\Phi_{11}$ , we obtain

$$\begin{aligned} \Phi_{11} := & -8\bar{\beta}(8\pi + 5\alpha)(172736\bar{\pi}\pi^2 - 7776\pi^2\bar{\alpha} \\ & - 13294\alpha\pi\bar{\alpha} + 7866\bar{\beta}\pi\bar{\alpha} \\ & + 211556\alpha\bar{\pi}\pi + 9568\bar{\beta}\beta\pi - 22572\bar{\beta}\bar{\pi}\pi \\ & + 5330\bar{\beta}\beta\alpha + 4845\bar{\beta}\alpha\bar{\alpha} \\ & - 1805\bar{\beta}^2\bar{\alpha} - 2470\beta\bar{\beta}^2 - 13110\bar{\pi}\alpha\bar{\beta} \\ & - 5290\alpha^2\bar{\alpha} + 65010\bar{\pi}\alpha^2)/ \\ & (-772320\pi^2\bar{\beta} - 2048352\alpha\pi^2 - 1158240\pi\alpha^2 \\ & + 4085\alpha\bar{\beta}^2 + 10868\pi\bar{\beta}^2 \\ & - 335985\alpha^2\bar{\beta} - 1020276\alpha\pi\bar{\beta} \\ & + 214700\alpha^3 + 1191680\pi^3). \end{aligned} \tag{53}$$

On the other hand, from the commutator  $[\bar{\delta}, \delta](\alpha + \bar{\beta})$  the following expression for  $\Phi_{11}$  results:

$$\begin{aligned} \Phi_{11} := & -8\bar{\beta}(-20672\bar{\beta}\bar{\pi}\pi - 12920\bar{\pi}\alpha\bar{\beta} \\ & + 8056\bar{\beta}\pi\bar{\alpha} + 172736\bar{\pi}\pi^2 \\ & - 7776\pi^2\bar{\alpha} + 9568\bar{\beta}\beta\pi \\ & + 5035\bar{\beta}\alpha\bar{\alpha} - 5290\alpha^2\bar{\alpha} + 65010\bar{\pi}\alpha^2 \\ & - 13294\alpha\pi\bar{\alpha} + 211556\alpha\bar{\pi}\pi + 5330\bar{\beta}\beta\alpha)/ \\ & (-13047\bar{\beta}\alpha + 42940\alpha^2 + 162944\pi\alpha \\ & + 10412\bar{\beta}^2 - 18564\pi\bar{\beta} + 148960\pi^2). \end{aligned} \tag{54}$$

Using the fact that  $\bar{\delta}(\Phi_{11}) = 0$ , we obtain, from (55), a third expression for  $\Phi_{11}$ :

$$\begin{aligned} \Phi_{11} := & 8\bar{\beta}(-48191081692160\pi^3\bar{\alpha}\alpha^2 + 180931104170496\bar{\pi}\pi^3\alpha^2 \\ & + 7968511840\bar{\beta}^4\pi\bar{\alpha} - 265004094784\bar{\beta}^3\beta\pi^2 \\ & - 5268590200832\bar{\pi}\pi^4\bar{\beta} + 24894675520\bar{\beta}^4\beta\pi \\ & + 221453789400\bar{\pi}\alpha^2\bar{\beta}^3 - 20739582848\bar{\beta}^3\pi^2\bar{\alpha} \\ & + 4980319900\bar{\beta}^4\alpha\bar{\alpha} - 30330200378072\pi^2\bar{\alpha}\alpha^3 \\ & - 1208861900450\alpha^5\bar{\alpha} + 5225292181050\bar{\pi}\alpha^5 \\ & + 724772046800\bar{\beta}^3\bar{\pi}\pi\alpha - 3675701240760\bar{\beta}^2\bar{\pi}\pi\alpha^2 \\ & - 6145239989312\bar{\beta}^2\bar{\pi}\pi^2\alpha - 7649757648780\bar{\beta}\bar{\pi}\pi\alpha^3 \end{aligned}$$

$$\begin{aligned}
& - 16323677160464 \bar{\beta} \bar{\pi} \pi^2 \alpha^2 - 15274850502912 \bar{\beta} \bar{\pi} \pi^3 \alpha \\
& - 26601021440 \bar{\beta}^3 \pi \bar{\alpha} \alpha - 1040399202440 \bar{\beta}^2 \pi \bar{\alpha} \alpha^2 \\
& - 1635701635136 \bar{\beta}^2 \pi^2 \bar{\alpha} \alpha + 5632482563850 \bar{\beta} \pi \bar{\alpha} \alpha^3 \\
& + 13162176133400 \bar{\beta} \pi^2 \bar{\alpha} \alpha^2 + 13687783600768 \bar{\beta} \pi^3 \bar{\alpha} \alpha \\
& - 329130147840 \bar{\beta}^3 \beta \pi \alpha + 265058659320 \bar{\beta}^2 \beta \pi \alpha^2 \\
& + 469041329536 \bar{\beta}^2 \beta \pi^2 \alpha + 1952913512680 \bar{\beta} \beta \pi \alpha^3 \\
& + 4375379783424 \bar{\beta} \beta \pi^2 \alpha^2 + 4338917366784 \bar{\beta} \beta \pi^3 \alpha \\
& - 102469061500 \bar{\beta}^3 \beta \alpha^2 + 49609754350 \bar{\beta}^2 \beta \alpha^3 \\
& + 325649974650 \bar{\beta} \beta \alpha^4 + 15559172200 \bar{\beta}^4 \beta \alpha \\
& - 38360907652096 \pi^4 \bar{\alpha} \alpha + 39628423187260 \bar{\pi} \alpha^4 \pi \\
& - 1329967209650 \bar{\pi} \alpha^4 \bar{\beta} - 858758431040 \bar{\beta}^2 \pi^3 \bar{\alpha} \\
& - 20447502080 \bar{\beta}^4 \bar{\pi} \pi - 8524238850 \bar{\beta}^3 \alpha^2 \bar{\alpha} \\
& - 12779688800 \bar{\pi} \alpha \bar{\beta}^4 + 905003032125 \bar{\beta} \alpha^4 \bar{\alpha} \\
& + 136081885849600 \bar{\pi} \pi^4 \alpha + 119915073751888 \bar{\pi} \pi^2 \alpha^3 \\
& - 9564050953390 \alpha^4 \bar{\alpha} \pi + 5344911334400 \pi^4 \bar{\alpha} \bar{\beta} \\
& - 12237656133632 \pi^5 \bar{\alpha} + 1606267826176 \bar{\beta} \beta \pi^4 \\
& + 40801870077952 \bar{\pi} \pi^5 - 730927683900 \bar{\pi} \alpha^3 \bar{\beta}^2 \\
& - 220969445675 \bar{\beta}^2 \alpha^3 \bar{\alpha} + 274949282816 \bar{\beta}^2 \beta \pi^3 \\
& + 592713574016 \bar{\beta}^3 \bar{\pi} \pi^2 - 3416931669632 \bar{\beta}^2 \bar{\pi} \pi^3) / \\
& [(-13047 \bar{\beta} \alpha + 42940 \alpha^2 + 162944 \pi \alpha \\
& + 10412 \bar{\beta}^2 - 18564 \pi \bar{\beta} + 148960 \pi^2) \\
& (102596352 \pi^3 - 7684576 \pi^2 \bar{\beta} + 194465152 \alpha \pi^2 \\
& + 24320 \pi \bar{\beta}^2 - 9460852 \alpha \pi \bar{\beta} + 123050612 \pi \alpha^2 \\
& + 25995895 \alpha^3 - 133000 \alpha \bar{\beta}^2 - 2915745 \alpha^2 \bar{\beta})]. \tag{55}
\end{aligned}$$

We now suppose that the denominators in these three expressions for  $\Phi_{11}$  are different from zero. Later we consider the cases in which each of them is different from zero. We also suppose that spin coefficients  $\alpha$ ,  $\beta$ ,  $\pi$  are different from zero. It was shown in CM that if one of these spin coefficients is equal to zero, the others must be zero too, and Huygens' principle is violated.

The next step consists in proving that (53), (54), and (55) imply that  $\alpha$ ,  $\beta$  and  $\pi$  are proportional to each other. In order to get a system of with

only two complex variables, instead of three, new variables are defined as follows:

$$x_1 x_1 := \frac{\alpha}{\pi} \quad x_2 := \frac{\beta}{\bar{\pi}}. \tag{56}$$

By subtracting (53) from (54), taking the numerator and dividing by  $(8 - x_2 - 5x_1)(5776\bar{x}_2^2\pi\bar{\pi})$ , we get

$$\begin{aligned} N_1 := & 178100 x_1 x_2 \bar{x}_2^2 + 284960 x_2 \bar{x}_2^2 + 208240 \bar{x}_2^2 \bar{x}_1 \\ & + 130150 \bar{x}_1 \bar{x}_2^2 x_1 + 109825 \bar{x}_1 \bar{x}_2 x_1^2 + 252850 x_2 \bar{x}_2 x_1^2 \\ & + 523744 x_2 \bar{x}_2 + 3451480 x_1 \bar{x}_2 + 341900 \bar{x}_2 x_1 \bar{x}_1 \\ & + 265888 \bar{x}_2 \bar{x}_1 + 735800 x_2 \bar{x}_2 x_1 + 2915264 \bar{x}_2 \\ & + 1018400 \bar{x}_2 x_1^2 - 879008 \bar{x}_1 + 18263488 - 408050 x_1^3 \bar{x}_1 \\ & + 4864450 x_1^3 + 35335248 x_1 + 22731900 x_1^2 \\ & - 2101032 x_1 \bar{x}_1 - 1622550 x_1^2 \bar{x}_1 = 0. \end{aligned} \tag{57}$$

By subtracting (54) from (55), taking the numerator and dividing by  $(8 - x_2 - 5x_1)(152\pi\bar{\pi})$ , we get

$$\begin{aligned} N_2 := & -11651821200 x_1^3 - 26531539120 x_1^2 - 10132263424 \\ & - 26800626944 x_1 + 242619584 \bar{x}_2 + 593671488 \bar{x}_2^2 \\ & + 2256829184 \bar{x}_1 - 35155250 \bar{x}_2^2 x_2 x_1^2 + 21550100 \bar{x}_2^3 x_2 x_1 \\ & - 128589500 \bar{x}_2 x_2 x_1^3 + 11036720 \bar{x}_2^3 \bar{x}_1 + 21755825 \bar{x}_2^2 x_1^2 \bar{x}_1 \\ & + 130839250 x_1^3 \bar{x}_2 - 216631750 \bar{x}_2 x_1^3 \bar{x}_1 - 17700400 \bar{x}_2^3 x_1 \\ & + 372958500 x_1^4 \bar{x}_1 + 220600700 \bar{x}_2^2 x_1^2 + 34480160 x_2 \bar{x}_2^3 \\ & - 1915591500 x_1^4 - 28320640 \bar{x}_2^3 + 6897950 \bar{x}_2^3 x_1 \bar{x}_1 \\ & + 724005800 \bar{x}_2^2 x_1 - 112640320 x_1 x_2 \bar{x}_2^2 - 90878112 x_2 \bar{x}_2^2 \\ & + 59839936 \bar{x}_2^2 \bar{x}_1 + 72209280 \bar{x}_1 \bar{x}_2^2 x_1 - 1002142990 \bar{x}_1 \bar{x}_2 x_1^2 \\ & - 585808600 x_2 \bar{x}_2 x_1^2 - 448045312 x_2 \bar{x}_2 + 632690016 x_1 \bar{x}_2 \\ & - 1546809184 \bar{x}_2 x_1 \bar{x}_1 - 796702240 \bar{x}_2 \bar{x}_1 \\ & - 887976960 x_2 \bar{x}_2 x_1 + 509836460 \bar{x}_2 x_1^2 + 2328634300 x_1^3 \bar{x}_1 \\ & + 5728848896 x_1 \bar{x}_1 + 5470002280 x_1^2 \bar{x}_1 = 0. \end{aligned} \tag{58}$$

At this point we shall consider  $x_1, x_2, \bar{x}_1, \bar{x}_2$  as independent variables, and use the package `grobner` in Maple, for the polynomial system formed by the polynomials  $N_1, N_2$ . This package computes a collection of reduced

(lexicographic) Gröbner bases corresponding to a set of polynomials. The result is a list of reduced subsystems whose roots are those of the original system, but whose variables have been successively eliminated and separated as far as possible. In the present case we obtain four subsystems given by

$$\begin{aligned}
 G_1 &:= [42 - 3\bar{x}_1 + 65x_2\bar{x}_2, 8 + 5x_1] \\
 G_2 &:= [205049562510\bar{x}_1x_2 - 2072817918600x_2\bar{x}_2 + 529175067720x_2 \\
 &\quad + 5001500073283\bar{x}_1^2 - 3239213905470\bar{x}_1 + 3029503111800\bar{x}_2 \\
 &\quad - 26163100475032, 23707187714600x_2\bar{x}_2^2 \\
 &\quad - 12070111345240x_2\bar{x}_2 - 15004500219849\bar{x}_1 \\
 &\quad - 34648966659800\bar{x}_2^2 + 11975391986580\bar{x}_2 \\
 &\quad + 41386627076564, -1113092 - 431311\bar{x}_1 \\
 &\quad + 1909780\bar{x}_2 + 954890\bar{x}_2\bar{x}_1, 205x_1 + 368] \\
 G_3 &:= [2175607695654600868570\bar{x}_1x_2 \\
 &\quad - 244429060944194171242925x_2\bar{x}_2x_1 \\
 &\quad - 362016456337543432617920x_2\bar{x}_2 \\
 &\quad - 25492004395136420363950x_2x_1 \\
 &\quad - 33777552239002428460240x_2 \\
 &\quad - 352210319977170626297190x_1\bar{x}_1^2 \\
 &\quad - 527515033185400238012371\bar{x}_1^2 \\
 &\quad - 372609773697867989940085x_1\bar{x}_1 \\
 &\quad - 568758266358009596992694\bar{x}_1 \\
 &\quad + 357242473687668404124275x_1\bar{x}_2 \\
 &\quad + 529100974647178863056960\bar{x}_2 \\
 &\quad + 421165196163010815629650x_1 \\
 &\quad + 613523694569903050334320, 74421671368200x_2\bar{x}_2^2 \\
 &\quad - 372108356841000x_2\bar{x}_2x_1 - 595373370945600x_2\bar{x}_2 \\
 &\quad + 202992871981785x_1\bar{x}_1 + 309188233840256\bar{x}_1 \\
 &\quad - 108770135076600\bar{x}_2^2 + 408707348737250x_1\bar{x}_2 \\
 &\quad + 731690446259960\bar{x}_2 - 363026773505180x_1 \\
 &\quad - 558465136160528, \\
 &\quad 139740\bar{x}_2\bar{x}_1 - 497365x_1\bar{x}_1 - 799324\bar{x}_1 + 279480\bar{x}_2 \\
 &\quad - 1187280x_1 - 1879248, 43975x_1^2 + 137900x_1 + 107824]
 \end{aligned}$$

$$\begin{aligned}
 G_4 := & [15138500 \bar{x}_1 x_2 \bar{x}_2 - 200682625 x_2 x_1^2 \bar{x}_1 - 589775940 x_2 x_1 \bar{x}_1 \\
 & - 425769864 \bar{x}_1 x_2 + 30277000 x_2 \bar{x}_2 - 677199250 x_2 x_1^2 \\
 & - 2061417280 x_2 x_1 - 1553711328 x_2 - 34073270 \bar{x}_2 \bar{x}_1^2 \\
 & - 2202762525 x_1^2 \bar{x}_1^2 - 6761833290 x_1 \bar{x}_1^2 \\
 & - 5193018500 \bar{x}_1^2 - 13933181776 - 6178746050 x_1^2 \\
 & - 18555219840 x_1 - 180544080 \bar{x}_2 - 17865753288 \bar{x}_1 \\
 & - 158418580 \bar{x}_2 \bar{x}_1 - 23445716600 x_1 \bar{x}_1 \\
 & - 7696469075 x_1^2 \bar{x}_1, 38482345375 x_1^3 \bar{x}_1 + 30893730250 x_1^3 \\
 & + 11013812625 x_1^3 \bar{x}_1^2 + 1003413125 x_2 x_1^3 \bar{x}_1 + 3385996250 x_2 x_1^3 \\
 & + 51211999525 x_1^2 \bar{x}_1^2 + 16053417900 x_2 x_1^2 + 4651759450 x_2 x_1^2 \bar{x}_1 \\
 & + 177479507850 x_1^2 \bar{x}_1 + 140245780600 x_1^2 + 212140488080 x_1 \\
 & + 7133355840 x_2 x_1 \bar{x}_1 + 272934958520 x_1 \bar{x}_1 + 25260582880 x_2 x_1 \\
 & + 79415365840 x_1 \bar{x}_1^2 + 106967929856 + 13183919424 x_2 \\
 & + 41078949112 \bar{x}_1^2 + 139995956352 \bar{x}_1 \\
 & + 3612843312 \bar{x}_1 x_2, 427238747000 x_2 \bar{x}_2^2 - 80593740760 x_2 \bar{x}_2 \\
 & - 961615825940 \bar{x}_2^2 \bar{x}_1 + 72041680708 \bar{x}_2 \bar{x}_1 \\
 & + 31587988349750 x_1^2 + 9360849735875 x_1^2 \bar{x}_1 + 90434478667240 x_1 \\
 & + 26404464403460 x_1 \bar{x}_1 + 549197297800 x_1 \bar{x}_2 + 64702426970096 \\
 & + 18636073790528 \bar{x}_1 + 1397883090496 \bar{x}_2 \\
 & - 2547657512880 \bar{x}_2^2, 31185310 x_2 \bar{x}_2 x_1 + 52924196 x_2 \bar{x}_2 \\
 & - 6814654 \bar{x}_2 \bar{x}_1 - 716849880 x_1^2 \\
 & - 440552505 x_1^2 \bar{x}_1 - 2156806048 x_1 \\
 & - 1352366658 x_1 \bar{x}_1 - 45578530 x_1 \bar{x}_2 \\
 & - 1620399064 - 1038603700 \bar{x}_1 \\
 & - 90980056 \bar{x}_2, 505750 x_1^2 + 149875 x_1^2 \bar{x}_1 \\
 & + 1539520 x_1 + 440460 x_1 \bar{x}_1 + 116450 \bar{x}_2 x_1 \bar{x}_1 \\
 & + 232900 x_1 \bar{x}_2 + 1160352 \\
 & + 317976 \bar{x}_1 + 372640 \bar{x}_2 + 186320 \bar{x}_2 \bar{x}_1]. \tag{59}
 \end{aligned}$$

The only sets where the solutions  $x_1 = const.$ ,  $x_2 = const.$  are not obvious are  $G_1$  and  $G_4$ . For  $G_1$ , if we substitute  $x_1 = -8/5$  into  $N_1$  we get  $195x_2\bar{x}_2 + 702/5 = 0$ , which is incompatible with the first equation of this set.

Consider now the fourth and fifth equations in set  $G_4$  given, respectively, by

$$\begin{aligned}
 & 31185310 \overline{x_2} x_2 x_1 + 52924196 \overline{x_2} x_2 - 6814654 \overline{x_2} \overline{x_1} \\
 & \quad - 716849880 x_1^2 - 440552505 x_1^2 \overline{x_1} \\
 & \quad - 1352366658 x_1 \overline{x_1} - 2156806048 x_1 \\
 & \quad - 45578530 x_1 \overline{x_2} - 1038603700 \overline{x_1} \\
 & \quad - 1620399064 - 90980056 \overline{x_2} = 0 \quad (60)
 \end{aligned}$$

and

$$\begin{aligned}
 & 505750 x_1^2 + 149875 x_1^2 \overline{x_1} + 440460 x_1 \overline{x_1} \\
 & \quad + 116450 \overline{x_2} x_1 \overline{x_1} + 1539520 x_1 \\
 & \quad + 232900 x_1 \overline{x_2} 317976 \overline{x_1} + 186320 \overline{x_2} \overline{x_1} \\
 & \quad + 1160352 + 372640 \overline{x_2} = 0. \quad (61)
 \end{aligned}$$

By applying `grobner` to (60), (61) and their complex conjugates (in this case, two new equations), we obtain a system of polynomials for which all solutions have  $x_1$  and  $x_2$  constant.

Let us consider now the special cases in which the denominators in the previous expressions for  $\Phi_{11}$  are zero.

The denominators of (54), (53), and (55) are given, respectively, by

$$\begin{aligned}
 d_1 := & 10868 \overline{x_2}^2 + 772320 \overline{x_2} + 1020276 x_1 \overline{x_2} + 4085 \overline{x_2}^2 x_1 \\
 & + 335985 \overline{x_2} x_1^2 - 1158240 x_1^2 - 2048352 x_1 - 214700 x_1^3 \\
 & - 1191680, \quad (62)
 \end{aligned}$$

$$\begin{aligned}
 d_2 := & 148960 + 42940 x_1^2 + 10412 \overline{x_2}^2 - 18564 \overline{x_2} \\
 & - 13047 x_1 \overline{x_2} + 162944 x_1, \quad (63)
 \end{aligned}$$

$$\begin{aligned}
 d_3 := & (148960 + 42940 x_1^2 + 10412 \overline{x_2}^2 - 18564 \overline{x_2} \\
 & - 13047 x_1 \overline{x_2} + 162944 x_1) \\
 & (133000 \overline{x_2}^2 x_1 - 24320 \overline{x_2}^2 + 7684576 \overline{x_2} + 9460852 x_1 \overline{x_2} \\
 & + 2915745 \overline{x_2} x_1^2 - 194465152 x_1 \\
 & - 123050612 x_1^2 - 102596352 \\
 & - 25995895 x_1^3). \quad (64)
 \end{aligned}$$

Applying  $\bar{\delta}$  to (62) we obtain

$$\begin{aligned}
 &63729588 \bar{x}_2 x_1^2 - 670120 x_1^2 \bar{x}_2^2 + 12371205 x_1^3 \bar{x}_2 \\
 &\quad - 617652 \bar{x}_2^3 + 18435168 x_1^2 + 108263616 x_1 \bar{x}_2 \\
 &\quad + 10424160 x_1^3 + 10725120 x_1 - 1048608 \bar{x}_2^2 \\
 &\quad + 60746496 \bar{x}_2 - 383325 \bar{x}_2^3 x_1 \\
 &\quad - 1749232 \bar{x}_2^2 x_1 + 1932300 x_1^4 = 0. \tag{65}
 \end{aligned}$$

Applying  $\bar{\delta}$  again on (65), gives

$$\begin{aligned}
 &- 71849032 \bar{x}_2^3 x_1 - 2917197592 x_1^2 \bar{x}_2^2 + 2126429184 \bar{x}_2 x_1^2 \\
 &\quad + 3683108352 x_1 \bar{x}_2 + 357185400 x_1^3 \bar{x}_2 - 128701440 x_1^2 \\
 &\quad - 125089920 x_1^4 - 67408896 \bar{x}_2^3 - 18590290 x_1^2 \bar{x}_2^3 + 6696360 \bar{x}_2^4 \\
 &\quad + 4179810 x_1 \bar{x}_2^4 - 221222016 x_1^3 - 2418547200 \bar{x}_2^2 \\
 &\quad + 2059223040 \bar{x}_2 - 22411650 x_1^4 \bar{x}_2 \\
 &\quad - 614629870 x_1^3 \bar{x}_2^2 - 23187600 x_1^5 - 4605932928 \bar{x}_2^2 x_1 = 0. \tag{66}
 \end{aligned}$$

Using grobner package on (62), (65) and (66) gives the empty set solution.

Applying  $\bar{\delta}$  to (63), gives

$$\begin{aligned}
 &- 128832 \bar{x}_2^2 - 101344 \bar{x}_2^2 x_1 - 2591852 \bar{x}_2 x_1^2 \\
 &\quad - 8248048 x_1 \bar{x}_2 - 257640 x_1^3 - 6550592 \bar{x}_2 - 977664 x_1^2 \\
 &\quad - 20824 \bar{x}_2^3 - 893760 x_1 = 0. \tag{68}
 \end{aligned}$$

Applying  $\bar{\delta}$  again on (68), gives

$$\begin{aligned}
 &62472 \bar{x}_2^4 - 60094064 \bar{x}_2 x_1^2 + 35714228 x_1^2 \bar{x}_2^2 \\
 &\quad + 2832764 x_1^3 \bar{x}_2 - 2838720 \bar{x}_2^3 + 8043840 x_1^2 \\
 &\quad - 210698752 x_1 \bar{x}_2 + 8798976 x_1^3 + 86775744 \bar{x}_2^2 \\
 &\quad - 171601920 \bar{x}_2 - 1690904 \bar{x}_2^3 x_1 \\
 &\quad + 111204048 \bar{x}_2^2 x_1 + 2318760 x_1^4 = 0. \tag{69}
 \end{aligned}$$

Using grobner package on (63), (68) and (69) we obtain the empty set solution. We observe now that one of the factors in  $d_3$  is  $d_2$ . Thus, if



$d_3 = 0$ , we need to consider just the expression

$$\begin{aligned}
 & 133000 \overline{x_2}^2 x_1 - 24320 \overline{x_2}^2 + 7684576 \overline{x_2} \\
 & \quad + 9460852 x_1 \overline{x_2} + 2915745 \overline{x_2} x_1^2 \\
 & \quad - 194465152 x_1 - 123050612 x_1^2 - 102596352 \\
 & \quad - 25995895 x_1^3 = 0.
 \end{aligned} \tag{70}$$

By applying  $\bar{\delta}$  on this equation, two times successively, we obtain

$$\begin{aligned}
 & 1923742456 \overline{x_2} x_1^2 - 54838615 x_1^2 \overline{x_2}^2 + 387128860 x_1^3 \overline{x_2} \\
 & \quad + 28673280 \overline{x_2}^3 + 1750186368 x_1^2 \\
 & \quad + 3181762656 x_1 \overline{x_2} \\
 & \quad + 1107455508 x_1^3 + 923367168 x_1 \\
 & \quad - 143083296 \overline{x_2}^2 + 1751900928 \overline{x_2} \\
 & \quad + 17772600 \overline{x_2}^3 x_1 - 175990444 \overline{x_2}^2 x_1 \\
 & \quad + 233963055 x_1^4 = 0
 \end{aligned} \tag{71}$$

and

$$\begin{aligned}
 & 3071183072 \overline{x_2}^3 x_1 - 65381238064 x_1^2 \overline{x_2}^2 + 363466389312 \overline{x_2} x_1^2 \\
 & \quad + 414477092352 x_1 \overline{x_2} + 141775882688 x_1^3 \overline{x_2} \\
 & \quad - 11080406016 x_1^2 - 13289466096 x_1^4 \\
 & \quad + 2839158528 \overline{x_2}^3 + 811332320 x_1^2 \overline{x_2}^3 \\
 & \quad - 315187200 x_2^4 - 196695600 x_1 \overline{x_2}^4 \\
 & \quad - 21002236416 x_1^3 - 58327724544 \overline{x_2}^2 \\
 & \quad + 177286496256 \overline{x_2} + 20770389380 x_1^4 \overline{x_2} \\
 & \quad - 13320040240 x_1^3 \overline{x_2}^2 - 2807556660 x_1^5 \\
 & \quad - 106950649152 \overline{x_2}^2 x_1 = 0.
 \end{aligned} \tag{72}$$

By applying grobner package to (71) and (72) we obtain again the empty solution set.

We need now to study the case in which  $x_1$  and  $x_2$  are constants. From  $\bar{\delta}x_1 = \bar{\delta}(\alpha/\pi) = 0$ , and  $\delta x_2 = \delta(\beta/\pi) = 0$  we get, respectively,

$$31 x_1 + 10 x_1^2 + 2 = 0, \tag{73}$$

and

$$63x_2 + 40x_2\bar{x}_1 + \bar{x}_1 = 0. \tag{74}$$

We thus have two solutions, given by  $x_1 = -3/2$ ,  $x_2 = 1/2$  and  $x_1 = x_2 = -8/5$ . The first one satisfies  $5x_1 + \bar{x}_2 - 8 = 0$ , which will be considered next. The second case is impossible, since these values don't make  $N_1$  equal to zero.

Let us now consider the case:

$$\beta - 5\bar{\alpha} - 8\bar{\pi} = 0, \tag{75}$$

or, using variables  $x_1$  and  $x_2$ , and dividing by  $\pi$ ,

$$8 - x_2 - 5x_1 = 0. \tag{76}$$

Applying  $\delta$  to this equation, using (32), (51) and (52), we get

$$34x_2\bar{x}_1 - x_2^2 + 56x_2 + 15\bar{x}_1^2 + 24\bar{x}_1 = 0. \tag{77}$$

Thus, the only solution for both equations is given by

$$x_1 = -3/2, \quad x_2 = 1/2. \tag{78}$$

Since the numerator on the right side of (48) must be zero, we obtain, solving for  $\Phi_{11}$ ,

$$\Phi_{11} = -\frac{86}{141}\pi\bar{\pi}. \tag{79}$$

Applying  $\delta$  on this equation, we obtain

$$\delta\pi = -\pi\bar{\pi}. \tag{80}$$

From (79) and (80) we get

$$\bar{\delta}\pi = \pi^2. \tag{81}$$

If we calculate the commutator  $[\bar{\delta}, \delta]\pi$ , using (80) and (81) we get no solution other than  $\pi = 0$ .

## 5. CONCLUSIONS

Theorem 1 was proved for the case of Maxwell equations, i.e., there are no Petrov type III space-times on which Maxwell's equations satisfy Huygens' principle. For the neutrino case,  $k_1 = 8$ ,  $k_2 = 13$ , the proof is similar. The fact that one of the spinor components of Alvarez-Wünsch five-index condition turned out to be simple allowed us to determine all necessary Pfaffians. The use of the package `NPpsinor` in Maple was essential for the conversion of the Alvarez-Wünsch five-index necessary condition from the tensorial to dyad form. The polynomial system obtained from the integrability conditions was simplified by using Maple's package `grobner`. Since a direct application of the algorithm seems impossible, due to the large size of the polynomial system, a "divide and conquer" approach was applied with success to solve this problem, showing that the first six necessary conditions for the validity of Huygens' principle cannot be simultaneously satisfied for Maxwell's equations in Petrov type III space-times.

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