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A representation independent propagator II: Lie groups with square integrable representations

by

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ABSTRACT. – Recently a representation independent propagator has been introduced for the case of a real, compact Lie group. In this paper we prove that such a propagator can be introduced for a real, connected and simply connected Lie group with square integrable representations. Even though the configuration space is generally a curved manifold the lattice regularization for this propagator, nonetheless, corresponds to a propagator on a flat manifold.

RÉSUMÉ. – Nous avons récemment introduit un propagateur dans le cas d'un groupe de Lie compact réel, indépendant de la représentation. Dans cet article, nous prouvons qu'un tel propagateur peut être défini pour un groupe de Lie connexe et simplement connexe admettant des représentations de carré intégrable. Bien que l'espace des configurations soit, en général, une variété courbe, la régularisation sur réseau de ce propagateur correspond néanmoins à un propagateur sur une variété plate.

1. INTRODUCTION

In [23] a representation independent propagator has been constructed for real, compact Lie groups. In this paper the construction of such a propagator is extended to the case of a real, connected and simply connected Lie group with square integrable representations.

Prior to constructing the representation independent propagator for any real, connected and simply connected Lie group G with square integrable representations, its construction is first outlined for the Heisenberg Weyl group; for an alternative construction in this case *see* [16]. Let P, Q, and I be an irreducible, self-adjoint representation of the Heisenberg-algebra, $[Q,P] = iI$, $\hbar = 1$, $[Q,I] = [P,I] = 0$ on some Hilbert space $\mathbf{H}$. Then, any normalized vector $\eta \in \mathbf{H}$ gives rise to a set of states of the form:

$$\eta(p,q) \equiv \frac{1}{\sqrt{2\pi}} V(p,q)\eta, \quad (p,q) \in \mathbb{R}^2$$

where $V(p,q) \equiv \exp(-iqP)\exp(ipQ)$. In fact these states are the familiar *canonical coherent states* which form a strongly continuous, overcomplete family of states for a fixed, normalized fiducial vector $\eta \in \mathbf{H}$. The map $\mathcal{C}_\eta : \mathbf{H} \to L^2(\mathbb{R}^2, dpdq)$, defined for any $\psi \in \mathbf{H}$ by:

$$[\mathcal{C}_\eta\psi](p,q) = \psi_\eta(p,q) \equiv \langle \eta(p,q), \psi \rangle = \left\langle \frac{1}{\sqrt{2\pi}}\eta, V^*(p,q)\psi \right\rangle,$$

yields a representation of the Hilbert space $\mathbf{H}$ by bounded, continuous, square integrable functions on the reproducing kernel Hilbert space $L^2_\eta(\mathbb{R}^2)$ which is a proper subspace of $L^2(\mathbb{R}^2)$. Let $\widetilde{\mathbf{D}}$ be the common dense invariant domain of P and Q, which is also invariant under $V(p,q)$; consequently, one can easily show that the following relations hold on $\widetilde{\mathbf{D}}$:

$$-i\partial_q V^*(p,q) = V^*(p,q)P, \tag{1}$$

$$(q + i\partial_p)V^*(p,q) = V^*(p,q)Q. \tag{2}$$

Notice, that the operator $V^*(p,q)$ intertwines the representation of the Heisenberg-algebra on the Hilbert space $\mathbf{H}$, with the representation of the Heisenberg-algebra by right invariant differential operators on any one of the reproducing kernel Hilbert spaces $L^2_\eta(\mathbb{R}^2)$. An appropriate core for these operators is given by the continuous representation of $\widetilde{\mathbf{D}}$, $\widetilde{\mathbf{D}}_\eta = \mathcal{C}_\eta(\widetilde{\mathbf{D}})$. Let $\mathcal{H}(P,Q)$ be the essentially self-adjoint Hamilton operator of a quantum system on $\mathbf{H}$; then using the intertwining relations (1) and (2) one finds for the time evolution of an arbitrary element $\psi_\eta(p,q,t)$ in any one $\widetilde{\mathbf{D}}_\eta \subset L^2_\eta(\mathbb{R}^2)$ the following

$$\begin{aligned}\psi_\eta(p,q,t) &= \langle \eta(p,q), \exp[-i(t-t')\mathcal{H}(P,Q)]\psi(t')\rangle \\ &= \left\langle \frac{1}{\sqrt{2\pi}}\eta, V^*(p,q)\exp[-i(t-t')\mathcal{H}(P,Q)]\psi(t') \right\rangle \\ &= \exp[-i(t-t')\mathcal{H}(-i\partial_q, q + i\partial_p)]\psi_\eta(p,q,t'),\end{aligned}$$

where the closure of the Hamilton operator has been denoted by the same symbol. This equation can be written in the following alternative form

$$\psi_\eta(p,q,t) = \int K(p,q,t;p',q',t')\psi_\eta(p',q',t')dp'dq', \tag{3}$$

with $K(p,q,t;p',q',t')$ given by:

$$K(p,q,t;p',q',t') \equiv \exp(-it\mathcal{H}(-i\partial_q, q+i\partial_p))\delta(p-p')\delta(q-q'). \tag{4}$$

This propagator is clearly independent of the chosen fiducial vector. A sufficiently large set of test functions for this propagator is given by $C(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$, where $C(\mathbb{R}^2)$ is the set of all continuous functions on $\mathbb{R}^2$. Hence, every element of $L^2_\eta(\mathbb{R}^2)$ is an allowed test function for this propagator. Taking in (4) the limit $t \to t'$ one finds the following initial condition:

$$\lim_{t\to t'} K(p,q,t;p',q',t') = \delta(p-p')\,\delta(q-q').$$

Taking into account that the Feynman propagator is given in the Schrödinger representation and that the limit propagator $K(.,.,t;.,.,t)$ is a product of two delta distributions, the propagator K may be interpreted as a two-dimensional Feynman propagator on $L^2(\mathbb{R}^2, dpdq)$, p and q being the *position variables*. In fact the operators given by equation (1) and (2) are elements of the right invariant enveloping algebra of a two dimensional Schrödinger representation. Based on this interpretation following standard procedures (*cf.* [14]) one can give the representation independent propagator the following regularized *standard* phase space lattice prescription:

$$\begin{aligned} &K(p,q,t;p',q',t') \\ &= \lim_{N\to\infty} \int\cdots\int \exp\left\{ i\sum_{j=0}^{N} [x_{j+\frac{1}{2}}(p_{j+1}-p_j) + k_{j+\frac{1}{2}}(q_{j+1}-q_j) \right. \\ &\left. - \epsilon\mathcal{H}(k_{j+\frac{1}{2}}, (q_{j+1}+q_j)/2 - x_{j+\frac{1}{2}})] \right\} \prod_{j=1}^{N} dp_j dq_j \prod_{j=0}^{N} \frac{dk_{j+\frac{1}{2}}dx_{j+\frac{1}{2}}}{2\pi^2}, \end{aligned}$$

where $(p_{N+1}, q_{N+1}) = (p,q)$, $(p_0, q_0) = (p',q')$, and $\epsilon \equiv (t-t')/(N+1)$. Observe that the Hamiltonian has been used in the special form dictated by the differential operators in equations (1) and (2) and that Weyl ordering has been adopted. Taking an improper limit by interchanging the limit with respect to N with the integrals one finds the following formal *standard* phase space path integral

$$\begin{aligned} K(p,q,t,p',q',t') = \mathcal{M}\int \exp\left\{ i\int [x\dot{p} + k\dot{q} - \mathcal{H}(k, q-x)]dt \right\} \\ \times \mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x, \end{aligned}$$

here "k" and "x" denote "momenta" conjugate to the "coordinates" "q" and "p", respectively. When a similar construction is extended to more general groups it is shown in section 4 that the construction of a regularized lattice phase-space path integral is possible and moreover that the resulting phase-space path integral has the form of a lattice phase-space path integral on a multidimensional *flat* euclidian space.

Despite the fact that the representation independent propagator has been constructed as a propagator appropriate to two (canonical) degrees of freedom, it is nonetheless true that its classical limit refers to a single (canonical) degree of freedom (*cf.* [16]).

2. COHERENT STATES AND COHERENT STATE PATH INTEGRALS

This section serves as an introduction to group coherent states, the continuous representation using group coherent states and the construction of coherent state path integrals based on group coherent states. Readers already familiar with this material can skip ahead to section 3 without serious loss of continuity.

2.1. Coherent States: Minimum Requirements

Let us denote by $\mathbf{H}$ a complex separable Hilbert space, and by $\mathcal{L}$ a topological space, whose finite dimensional subspaces are locally euclidian. For a family of vectors $\{|l\rangle\}_{l\in\mathcal{L}}$ on $\mathbf{H}$ to be a set of coherent states it must fulfill the following two conditions. The first condition is:

Continuity: *The vector $|l\rangle$ is a strongly continuous function of the label l.* That is for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$\||l'\rangle - |l\rangle\| < \epsilon \text{ for all } l' \in \mathcal{L} \text{ with } |l' - l| < \delta.$$

Or stated differently, the family of vectors $\{|l\rangle\}_{l\in\mathcal{L}}$ on $\mathbf{H}$ form a continuous (usually connected) submanifold of $\mathbf{H}$. We assume that $\langle l|l\rangle > 0$ for all $l \in \mathcal{L}$. In the applications we are considering the continuity property is always fulfilled.

The second condition a set of coherent states has to fulfill is:

Completeness (Resolution of the Identity): *There exists a σ-finite positive measure $d\mu(l)$ on $\mathcal{L}$ such that the identity operator $I_{\mathbf{H}}$ admits* *the following* <u>resolution of identity</u>

$$I_{\mathbf{H}} = \int_{\mathcal{L}} |l\rangle\langle l| \cdot \rangle d\mu(l) \tag{5}$$

2.2. Group Coherent States

To avoid unnecessary mathematical complication at this point we restrict our discussion to compact Lie groups. However, we would like to point out to the reader that the discussion applies to a general Lie group, as defined in section 4. Let us denote by G a connected, compact d-dimensional Lie group. It is well known that for compact groups all representations of the group are bounded and that all irreducible representations are finite dimensional. Moreover, one can always choose a scalar product on the representation space in such a way that every representation of G is unitary, (*cf.* [3, Theorem 7.1.1]). Therefore, without loss in generality we assume that we are dealing with a finite dimensional strongly continuous irreducible unitary representation U^ζ of G realized on a d_ζ-dimensional representation space $\mathbf{H}_\zeta$. Let us denote by $\{X_k\}_{k=1}^d$ the set of finite dimensional self-adjoint generators of the representation U^ζ. The X_k, $k = 1, \ldots, d$, form an irreducible representation of the Lie algebra L associated with G, whose commutation relations are given by

$$[X_i, X_j] = i \sum_{k=1}^{d} c_{ij}{}^k X_k,$$

where $c_{ij}{}^k$ denote the structure constants. The physical operators are defined by $\widehat{X}_k \equiv \hbar X_k$. For definiteness it is assumed that there exists a parameterization for G such that

$$U^\zeta_{g(l)} = \exp(-il^1 X_1) \ldots \exp(-il^k X_k),$$

up to some ordering and where $l \in \mathcal{L}$. Here $\mathcal{L}$ denotes the compact parameter space for G. For all $l \in \mathcal{L}$ and a fixed normalized fiducial vector $\eta \in \mathbf{H}_\zeta$ we define the following set of vectors on $\mathbf{H}_\zeta$

$$\eta(l) = \sqrt{d_\zeta} U^\zeta_{g(l)} \eta. \tag{6}$$

It follows from the strong continuity of $U^\zeta_{g(l)}$ that the set of vectors defined in (6) forms a family of strongly continuous vectors on $\mathbf{H}_\zeta$. Furthermore, let us consider the operator

$$O = \int_{\mathcal{L}} \eta(l')\langle \eta(l'), \cdot \rangle dg(l'), \tag{7}$$

where $dg(l)$ denotes the normalized, invariant measure on G. It is not hard to show, using the invariance of dg, that the operator O commutes with all $U^{\varsigma}_{g(l)}$, $l \in \mathcal{L}$. Since $U^{\varsigma}_{g(l)}$ is a unitary irreducible representation one has by Schur's Lemma that $O = \lambda I_{\mathbf{H}_\varsigma}$. Taking the trace on both sides of (7) we learn that

$$\begin{aligned} \operatorname{tr}(\lambda I_{\mathbf{H}_\varsigma}) = \lambda d_\varsigma &= \int \operatorname{tr}[\eta(l')\langle\eta(l'),\,\cdot\,\rangle] dg(l') \\ \lambda d_\varsigma &= d_\varsigma \|\eta\|^2 \int dg(l') \\ \lambda &= 1. \end{aligned}$$

Hence, the family of vectors defined in (6) gives rise to the following resolution of identity:

$$I_{\mathbf{H}_\varsigma} = \int_{\mathcal{L}} \eta(l)\langle\eta(l),\,\cdot\,\rangle dg(l). \tag{8}$$

Therefore, we find that the family of vectors defined in (6) satisfies the requirements set forth in subsection 2.1 for a set of vectors to be a set of coherent states. So we conclude that the vectors defined in (6) form a set of coherent states for the compact Lie group G, corresponding to the irreducible unitary representation $U^{\varsigma}_{g(l)}$.

2.3. Continuous Representation

Analogously to standard quantum mechanics one can use the set of coherent states defined in (6) to give a functional representation of the space $\mathbf{H}_\varsigma$. Let us define the map

$$\begin{aligned} \mathcal{C}_\eta : \mathbf{H}_\varsigma &\to L^2(G, dg) \\ \psi &\mapsto [\mathcal{C}_\eta \psi](l) = \psi_\eta(l) \equiv \langle \eta(l), \psi\rangle. \end{aligned}$$

This yields a representation of the space $\mathbf{H}_\varsigma$ by bounded, continuous, square integrable functions on some closed subspace $L^2_\eta(G)$ of $L^2(G)$. Let us denote by B any bounded operator on $\mathbf{H}_\varsigma$, then using the map $\mathcal{C}_\eta$ and the resolution of identity given in (8) we find that:

$$\langle \eta(l), B\psi\rangle = \int \langle \eta(l), B\eta(l')\rangle\langle\eta(l'), \psi\rangle dg(l') \tag{9}$$

holds. Choosing $B = I_{\mathbf{H}_\varsigma}$ we find

$$\psi_\eta(l) = \int \mathcal{K}_\eta(l; l')\psi_\eta(l') dg(l'), \tag{10}$$

where

$$\mathcal{K}_\eta(l;l') = \langle \eta(l), \eta(l') \rangle.$$

One calls (10) the reproducing property. Furthermore, the kernel $\mathcal{K}_\eta(l';l)$ is an element of $L^2_\eta(G)$ for fixed $l \in \mathcal{L}$. Therefore, the kernel $\mathcal{K}_\eta(l';l)$ is a reproducing kernel and $L^2_\eta(G)$ is a reproducing kernel Hilbert space. Note that a reproducing kernel Hilbert space can never have more than one reproducing kernel (*cf.* [20, p. 43]). Therefore, since $L^2_\eta(G)$ is a space of continuous functions, $\mathcal{K}_\eta(l';l)$ is unique. Moreover, since the coherent states are strongly continuous, the reproducing kernel $\mathcal{K}_\eta(l';l)$ is a jointly continuous function, nonzero for $l = l'$, and therefore, nonzero in a neighborhood of $l = l'$. This means that (10) is a real restriction on the admissible functions in the continuous representation of $\mathbf{H}_\zeta$. Of course a similar equation holds for the Schrödinger representation, however there one has $\langle q|q'\rangle = \delta(q - q')$ which poses no restriction on the allowed functions. In fact, the reproducing kernel $\mathcal{K}_\eta(l';l)$ is the integral kernel of a projection operator from $L^2(G)$ onto the reproducing kernel Hilbert space $L^2_\eta(G)$ (*cf.* [20, p. 47]). This ends our discussion of the kinematics (framework) and brings us to the subject of dynamics.

2.4. The Coherent State Propagator for Group Coherent States

Let $\psi \in \mathbf{H}_\zeta$ and denote by $\mathcal{H}(\widehat{X}_1, \ldots, \widehat{X}_d)$ the bounded Hamilton operator of the quantum system under discussion, then the Schrödinger equation on $\mathbf{H}_\zeta$ is given by

$$i\hbar\partial_t\psi = \mathcal{H}(\widehat{X}_1, \ldots, \widehat{X}_d)\psi$$

since $\mathcal{H}$ is assumed to be self-adjoint and does not explicitly depend on time, a solution to Schrödinger's equation is given by:

$$\psi(t'') = \exp\left[-\frac{i}{\hbar}(t'' - t')\mathcal{H}\right]\psi(t').$$

Now making use of (9) we find

$$\psi_\eta(l'', t'') = \int K_\eta(l'', t''; l', t')\psi_\eta(l', t')dg(l'),$$

where

$$K_\eta(l'', t''; l', t') = \left\langle \eta(l''), \exp\left[-\frac{i}{\hbar}(t'' - t')\mathcal{H}\right]\eta(l') \right\rangle.$$

Note that the coherent state propagator $K_\eta(l'', t''; l', t')$ satisfies the following initial condition

$$\lim_{t'' \to t'} K_\eta(l'', t''; l', t') = \mathcal{K}_\eta(l''; l').$$

Hence, as $t'' \to t'$ we obtain the reproducing kernel $\mathcal{K}_\eta(l''; l')$ which, as it has been mentionned above, is the integral kernel of a projection operator from $L^2(G)$ onto $L^2_\eta(G)$. Moreover, since $\mathcal{K}_\eta(l''; l')$ is unique, we see that if we change the fiducial vector from η to η', save for a change of phase, the resulting coherent state propagator is no longer a propagator for the elements of the reproducing kernel Hilbert space $L^2_\eta(G)$, but is a propagator for the elements of the reproducing kernel Hilbert space $L^2_{\eta'}(G)$. Hence, we see that the coherent state propagator $K_\eta(l'', t''; l', t')$ depends strongly on the fiducial vector η.

Using standard methods (*see* e.g. [14] and [17]) we now derive a coherent state path integral representation for the coherent state propagator. We start from the identity

$$\exp\left[-\frac{i}{\hbar}(t'' - t')\mathcal{H}\right] = \left[\exp\left(-\frac{i}{\hbar}\epsilon\mathcal{H}\right)\right]^{N+1}, \qquad \forall N \geq 0$$

where $\epsilon = (t'' - t')/(N + 1)$, therefore, we find

$$\begin{aligned} K_\eta(l'', t''; l', t') &= \left\langle \eta(l''), \exp\left[-\frac{i}{\hbar}(t'' - t')\mathcal{H}\right]\eta(l')\right\rangle \\ &= \left\langle \eta(l''), \left[\exp\left(-\frac{i}{\hbar}\epsilon\mathcal{H}\right)\right]^{N+1} \eta(l')\right\rangle. \end{aligned}$$

Inserting the resolution of identity (8) N-times this becomes

$$K_\eta(l'', t''; l', t') = \int \cdots \int \prod_{j=0}^{N} \left\langle \eta(l_{j+1}), \exp\left(-\frac{i}{\hbar}\epsilon\mathcal{H}\right)\eta(l_j)\right\rangle \prod_{j=1}^{N} dg(l_j),$$

where $l_{N+1} = l''$ and $l_o = l'$. This expression holds for any N, and therefore, it holds as well in the limit $N \to \infty$ or $\epsilon \to 0$, *i.e.*

$$\begin{aligned} &K_\eta(l'', t''; l', t') \\ &= \lim_{\epsilon \to 0} \int \cdots \int \prod_{j=0}^{N} \left\langle \eta(l_{j+1}), \exp\left(-\frac{i}{\hbar}\epsilon\mathcal{H}\right)\eta(l_j)\right\rangle \prod_{j=1}^{N} dg(l_j). \end{aligned} \quad (11)$$

Hence, one has to evaluate $\langle\eta(l_{j+1}), \exp\left(-\frac{i}{\hbar}\epsilon\mathcal{H}\right)\eta(l_j)\rangle$ for small ϵ. For small ϵ one can use the approximation

$$\begin{aligned}\left\langle\eta(l_{j+1}), \exp\left(-\frac{i}{\hbar}\epsilon\mathcal{H}\right)\eta(l_j)\right\rangle &\approx \left\langle\eta(l_{j+1}), \left(1-\frac{i}{\hbar}\epsilon\mathcal{H}\right)\eta(l_j)\right\rangle \\ &= \langle\eta(l_{j+1}), \eta(l_j)\rangle\left[1-\frac{i}{\hbar}\epsilon\frac{\langle\eta(l_{j+1}), \mathcal{H}\eta(l_j)\rangle}{\langle\eta(l_{j+1}), \eta(l_j)\rangle}\right] \\ &= \mathcal{K}_\eta(l_{j+1}; l_j)\left[1-\frac{i}{\hbar}\epsilon H_\eta(l_{j+1}; l_j)\right] \\ &\approx \mathcal{K}_\eta(l_{j+1}; l_j)\exp\left[-\frac{i}{\hbar}\epsilon H_\eta(l_{j+1}; l_j)\right], \quad (12)\end{aligned}$$

where

$$H_\eta(l_{j+1}; l_j) \equiv \frac{\langle\eta(l_{j+1}), \mathcal{H}\eta(l_j)\rangle}{\langle\eta(l_{j+1}), \eta(l_j)\rangle}.$$

Inserting (12) into (11) yields

$$\begin{aligned}&K_\eta(l'', t''; l', t') \\ &= \lim_{\epsilon\to 0}\int\cdots\int\prod_{j=0}^{N}\mathcal{K}_\eta(l_{j+1}; l_j)\exp\left[-\frac{i}{\hbar}\epsilon H_\eta(l_{j+1}; l_j)\right]\prod_{j=1}^{N}dg(l_j). \quad (13)\end{aligned}$$

This is the form of the coherent state path integral one typically encounters in the literature. It is worth reemphasing that the coherent state path integral representation of the coherent state propagator (13) depends strongly on the fiducial vector.

2.4.1. Formal Coherent State Path Integral

Even though there exists no mathematical justification whatsoever we now take an improper limit of (13) by interchanging the operation of integration with the limit $\epsilon \to 0$. As pointed out in [17, p. 63] one can imagine as $\epsilon \to 0$ that the set of points l_j, $j = 1, \ldots, N$, defines in the limit a (possibly generalized) function $l(t)$, $t' \le t \le t''$. Following [17, pp. 63-64] we now derive an expression for the integrand in (13) valid for *continuous* and *differentiable* paths $l(t)$. Note that the set of coherent states $\eta(l)$ we have defined in (6) is not normalized, but is of constant norm given by $d_\zeta^{1/2}$. We now rewrite the reproducing kernel $\mathcal{K}_\eta(l_{j+1}; l_j) = \langle\eta(l_{j+1}), \eta(l_j)\rangle$ in the following way

$$\begin{aligned}\langle\eta(l_{j+1}), \eta(l_j)\rangle &= \langle\eta(l_{j+1}), \eta(l_{j+1})\rangle - \langle\eta(l_{j+1}), \eta(l_{j+1}) - \eta(l_j)\rangle \\ &= d_\zeta[1 - d_\zeta^{-1}\langle\eta(l_{j+1}), \eta(l_{j+1}) - \eta(l_j)\rangle] \\ &\approx d_\zeta\exp[-d_\zeta^{-1}\langle\eta(l_{j+1}), \eta(l_{j+1}) - \eta(l_j\rangle],\end{aligned}$$

this approximation is valid whenever $\|\eta(l_{j+1}) - \eta(l_j)\| \ll 1$, $j = 0, \ldots, N$. Hence, as $\epsilon \longrightarrow 0$ the approximation becomes increasingly better since the $\eta(l)$ form a continuous family of vectors. Therefore, one finds

$$\mathcal{K}_\eta(l_{j+1}; l_j) \approx d_\zeta \exp[-d_\zeta^{-1}\langle \eta(l_{j+1}), \eta(l_{j+1}) - \eta(l_j)\rangle], \tag{14}$$

for $\|\eta(l_{j+1}) - \eta(l_j)\| \ll 1$, $j = 0, \ldots, N$. Using (14) in (13) and taking the limit $\epsilon \longrightarrow 0$ the integrand in (13) takes for *continuous* and *differentiable* paths the following form:

$$d_\zeta \exp\left[-\frac{1}{d_\zeta} \int_{l'}^{l''} \langle \eta(l), d\eta(l)\rangle - \frac{i}{\hbar d_\zeta} \int_{t'}^{t''} H_\eta(l(t))dt \right],$$

where

$$H_\eta(l(t)) = \langle \eta(l), \mathcal{H}(\widehat{X}_1, \ldots, \widehat{X}_k)\eta(l)\rangle$$

and where we have introduced the coherent state differential

$$d\eta(l) \equiv \eta(l + dl) - \eta(l).$$

Hence, we find the following formal coherent state path integral expression for the coherent state propagator:

$$K_\eta(l'', t''; l', t') = \int \exp\left\{ \frac{i}{\hbar} \int_{t'}^{t''} \left[i\hbar \left\langle \eta(l), \frac{d}{dt}\eta(l) \right\rangle - H_\eta(l) \right] dt \right\} \mathcal{D}g(l), \tag{15}$$

where

$$\eta(l) = U_{g(l)}^\zeta \eta \quad \text{and} \quad \mathcal{D}g(l) \equiv \lim_{N \to \infty} (d_\zeta)^{N+1} \prod_{j=1}^{N} dg(l_j).$$

A discussion of what is right and what is wrong with (15) can be found in [17, 64-66] we only remark here that (15) depends strongly on the choice of the fiducial vector and on the choice of the irreducible unitary representation of G. Hence, one has to reformulate the path integral representation for the coherent state propagator every time one changes the fiducial vector and keeps the irreducible representation the same, or if one changes the irreducible unitary representation of G. Now in many applications it is often convenient to choose the fiducial vector as the ground state of the Hamilton operator $\mathcal{H}$ of the quantum system one considers; *see* for instance Troung [24, 25]. Hence, one has to face the problem of various fiducial vectors. In section 4 we develop a representation independent propagator, which nevertheless, propagates the elements of *any* reproducing kernel Hilbert space $L_\eta^2(G)$ associated with *any* irreducible, square integrable unitary representation of G. Hence, we can overcome the above limitation.

3. NOTATIONS AND PRELIMINARIES

3.1. Notation

In this chapter, G is a real, separable, connected and simply connected, locally compact Lie group with fixed left invariant Haar measure dg. Let $\mathcal{D}(G)$ be the space of regular Bruhat functions with compact support on G (*cf.* [5] and [19, pp. 68-69]). Let T be a closable operator on some Hilbert space **H**, then we denote its closure by $\overline{T}$. Given a basis $x_1, \ldots, x_d$ of the Lie algebra L, we shall denote by $X_i = U(x_i)$, $i = 1, \ldots, d$ a representation of the basis of the Lie algebra L by symmetric operators on some Hilbert space **H** with common dense invariant domain **D**. The commutation relations take the form $[X_i, X_j] = i \sum_{k=1}^{d} c_{ij}{}^k X_k$. We say that the representation U of the Lie algebra L satisfies *Hypothesis (A)* if and only if U is a representation of the Lie algebra L on a dense invariant domain **D** of vectors that are analytic for all symmetric representatives $X_k = U(x_k)$, $k = 1, \ldots, d$, of a basis x_k, $k = 1, \ldots, d$. If Hypothesis (A) is satisfied then by Theorem 3 of Flato *et al.* [10] the representation X_k, $k = 1, \ldots, d$, of the Lie algebra L on **H** is integrable to a unique *unitary* representation of the corresponding connected and simply connected Lie group G on **H**. We will always assume that a representation of L by symmetric operators satisfies Hypothesis (A). Therefore, the representation of L by symmetric operators is integrable to a unique global unitary representation of the associated connected and simply connected Lie group G on **H**. Let there exist a parameterization of G such that the unitary representation U of G can be written in terms of the $\overline{X}_k$ as

$$U_{g(l)} = \prod_{j=1}^{d} \exp(-il^j \overline{X}_j) \equiv \exp(-il^1 \overline{X}_1) \ldots \exp(-il^d \overline{X}_d), \quad (16)$$

$$U^*_{g(l)} = \coprod_{j=1}^{d} \exp(il^j \overline{X}_j) \equiv \exp(il^d \overline{X}_d) \ldots \exp(il^1 \overline{X}_1), \quad (17)$$

for some ordering, where l is an element of a d-dimensional parameter space $\mathcal{G}$. The parameter space $\mathcal{G}$ is all of $\mathbb{R}^d$ if the group is non-compact and a subset of $\mathbb{R}^d$ if the group is compact or has a compact subgroup.

In what follows we shall need a common dense invariant domain for $\overline{X}_1, \ldots, \overline{X}_d$ that is also invariant under the one-parameter groups $\exp(it\overline{X}_k)$, $k = 1, \ldots, d$. Define $\widetilde{\mathbf{D}}$ as the intersection of the domains of all monomials $\overline{X}_{i_1} \ldots \overline{X}_{i_k}$ for all $1 \leq i_1, \ldots, i_k \leq d$. By definition $\widetilde{\mathbf{D}}$ contains **D**, hence is dense in **H**. Then by Lemma 3 of [10] the restriction of $\overline{X}_1, \ldots, \overline{X}_d$ to $\widetilde{\mathbf{D}}$ is a representation of L and by Lemma 4 of [10] $\widetilde{\mathbf{D}}$ is invariant under all one-parameter groups $\exp(it\overline{X}_k)$, $k = 1, \ldots, d$.

Let ${\lambda_m}^k(g(l))$ and ${\rho_m}^k(g(l))$ be functions such that on $\widetilde{\mathbf{D}}$ the following relations hold:

$$\coprod_{a=m+1}^{d} \exp(il^a\overline{X}_a)\,\overline{X}_m \prod_{b=m+1}^{d} \exp(-il^b\overline{X}_b) = \sum_{k=1}^{d} {\lambda_m}^k(g(l))\overline{X}_k, \quad (18)$$

$$\prod_{a=1}^{m-1} \exp(-il^a\overline{X}_a)\,\overline{X}_m \coprod_{b=1}^{m-1} \exp(il^b\overline{X}_b) = \sum_{k=1}^{d} {\rho_m}^k(g(l))\overline{X}_k. \quad (19)$$

Note that, the parameterization of the Lie group G is chosen in such a way that $\det[{\lambda_m}^k(g(l))] \neq 0$ and $\det[{\rho_m}^k(g(l))] \neq 0$, respectively. Therefore, the inverse matrices $[{\lambda^{-1}}_m{}^k(g(l))]$ and $[{\rho^{-1}}_m{}^k(g(l))]$ exist. Furthermore, let $U(l)$ be the $d \times d$ matrix whose mk-element is ${U_m}^k(l)$ such that on $\widetilde{\mathbf{D}}$

$$U^*_{g(l)}\overline{X}_m U_{g(l)} = \sum_{k=1}^{d} {U_m}^k(l)\overline{X}_k, \quad (20)$$

$$U_{g(l)}\overline{X}_m U^*_{g(l)} = \sum_{k=1}^{d} {U^{-1}}_m{}^k(l)\overline{X}_k, \quad (21)$$

holds. One can easily check that $U(l)$ is given by exponentiating the adjoint representation of L,

$$U(l) = \prod_{k=1}^{d} \exp(l^k c_k),$$

here c_k denotes the matrix formed from the structure constants such that $c_k = -c_k({}_i{}^j)$.

3.2. Preliminaries

We shall need the following two results in the sequel. Their proofs if not indicated otherwise may be found in [23].

LEMMA 3.2.1. – *The functions* ${\lambda_m}^k(g(l))$ *and* ${\rho_m}^k(g(l))$ *are related as follows:*

$$ {\lambda_m}^k(g(l)) = \sum_{c=1}^{d} {\rho_m}^c(g(l)){U_c}^k(l).$$

LEMMA 3.2.2. – *The functions* $\rho_m{}^k(g(l))$ *and* $\lambda_m{}^k(g(l))$ *satisfy the following equations*

$$(i) \sum_{n=1}^{d} \left\{ \partial_{l^n}[\rho^{-1}{}_j{}^a(g(l))]\rho^{-1}{}_k{}^n(g(l)) - \partial_{l^n}[\rho^{-1}{}_k{}^a(g(l))]\rho^{-1}{}_j{}^n(g(l)) \right\} = \sum_{f=1}^{d} c_{jk}{}^f \rho^{-1}{}_f{}^a(g(l)),$$

$$(ii) \sum_{n=1}^{d} \left\{ \partial_{l^n}[\lambda^{-1}{}_j{}^a(g(l))]\lambda^{-1}{}_k{}^n(g(l)) - \partial_{l^n}[\lambda^{-1}{}_k{}^a(g(l))]\lambda^{-1}{}_j{}^n(g(l)) \right\} = \sum_{f=1}^{d} c_{jk}{}^f \lambda^{-1}{}_f{}^a(g(l)),$$

$$(iii) \sum_{s=1}^{d} \lambda_n{}^s(g(l))\partial_{l^m}[\lambda^{-1}{}_s{}^b(g(l))] = \sum_{s=1}^{d} \rho_m{}^s(g(l))\partial_{l^n}[\rho^{-1}{}_s{}^b(g(l))].$$

where $c_{jk}{}^f$ *are the structure constants for* G.

Proof. – (iii) Using Lemma 3.2.1 (iii) can be rewritten as

$$\sum_{t=1}^{d} \partial_{l^m}[U^{-1}{}_h{}^t(l)\rho^{-1}{}_t{}^b(g(l))]$$
$$= - \sum_{f,s,t=1}^{d} U^{-1}{}_h{}^t(l)\rho^{-1}{}_t{}^f(g(l))\partial_{l^f}[\rho_m{}^s(g(l))]\rho^{-1}{}_s{}^b(g(l))$$

This equation can be simplified as follows

$$\sum_{h,j,t=1}^{d} \rho_n{}^j(g(l))U_j{}^h(l)\partial_{l^m}[U^{-1}{}_h{}^t(l)\rho^{-1}{}_t{}^b(g(l))]$$
$$= - \sum_{s=1}^{d} \partial_{l^n}[\rho_m{}^s(g(l))]\rho^{-1}{}_s{}^b(g(l))$$
$$\sum_{h,j=1}^{s} U^{-1}{}_h{}^f(l)\partial_{l^m}[\rho_n{}^j(g(l))U_j{}^h(l)] = \partial_{l^n}[\rho_m{}^f(g(l))].$$

Differentiating the product and rearranging the terms yields:

$$\partial_{l^m}[\rho_n{}^f(g(l))] - \partial_{l^n}[\rho_m{}^f(g(l))] = - \sum_{j,h=1}^{d} \rho_n{}^j(g(l))\partial_{l^m}[U_j{}^h(l)]U^{-1}{}_h{}^f(l).$$

Next using $\partial_{l^m} U_j{}^h(l) = \rho_m{}^s(g(l)) c_{js}{}^n U_n{}^h(l)$, which is proved along the same lines as Theorem 2.1 (ii) in [23], we find

$$\partial_{l^m}[\rho_n{}^f(g(l))] - \partial_{l^n}[\rho_m{}^f(g(l))] = -\sum_{j,s=1}^{d} \rho_n{}^j(g(l))\rho_m{}^s(g(l))c_{jd}{}^f, \quad (22)$$

Now contracting both sides of (22) with ${\rho^{-1}}_f{}^a(g(l))$ yields

$$\sum_{f=1}^{d}\{\partial_{l^m}[{\rho^{-1}}_f{}^a(g(l))]\rho_n{}^f(g(l)) - \partial_{l^n}[{\rho^{-1}}_f{}^a(g(l))]\rho_m{}^f(g(l))\}$$
$$= \sum_{f=1}^{d}\sum_{s,t=1}^{d} \rho_n{}^s(g(l))\rho_m{}^t(g(l))c_{st}{}^f {\rho^{-1}}_f{}^a(g(l)),$$

where,

$$\sum_{k=1}^{d} \partial_{l^c}[\rho_m{}^k(g(l))]{\rho^{-1}}_k{}^n(g(l)) = -\sum_{k=1}^{d} \partial_{l^c}[{\rho^{-1}}_k{}^n(g(l))]\rho_m{}^k(g(l)),$$

has been used. Finally contract both sides with ${\rho^{-1}}_k{}^m(g(l)){\rho^{-1}}_j{}^n(g(l))$ to obtain the relation,

$$\sum_{n=1}^{d}\{\partial_{l^n}[{\rho^{-1}}_j{}^a(g(l))]{\rho^{-1}}_k{}^n(g(l)) - \partial_{l^n}[{\rho^{-1}}_k{}^a(g(l))]{\rho^{-1}}_j{}^n(g(l))\}$$
$$= \sum_{f=1}^{d} c_{jk}{}^f(g(l)){\rho^{-1}}_f{}^a(g(l)).$$

which is (i), and therefore, establishes (iii). □

4. THE REPRESENTATION INDEPENDENT PROPAGATOR FOR A GENERAL LIE GROUP

4.1. Coherent States for General Lie Groups

In the following we mean by a *general Lie group* G a finite dimensional real, separable, locally compact, connected and simply connected Lie group for which the set $\widehat{G}_d$ of (classes of) square integrable unitary representations is non empty.

For continuous, irreducible, square integrable, unitary representations the following relations hold (*see* [6, 8]):

(i) Let $\phi, \xi \in \mathbf{H}$, $\phi \neq 0$. Then $\langle U_g\xi, \phi\rangle$ is square integrable if and only if $\xi \in \mathbf{D}(K^{-1/2})$. Where K is a unique self-adjoint, positive, semi-invariant operator on $\mathbf{H}$ with weight $\Delta^{-1}(g)$.

(ii) Let $\chi, \chi' \in \mathbf{H}$, and $\xi, \xi' \in \mathbf{D}(K^{-1/2})$. Then one has

$$\int \langle \chi, U_g\xi\rangle\langle U_g\xi', \chi'\rangle dg = \langle \chi, \chi'\rangle\langle K^{-1/2}\xi', K^{-1/2}\xi\rangle. \tag{23}$$

Let $\{X_i\}_{i=1}^d$ be an irreducible representation of the basis of the Lie algebra L corresponding to G, by symmetric operators on $\mathbf{H}$ satisfying Hypothesis (A), then L is integrable to a unique unitary representation of G on $\mathbf{H}$. Let there exist a parameterization of G such that,

$$U_{g(l)} = \prod_{k=1}^{d} \exp(-il^k\overline{X}_k) = \exp(-il^1\overline{X}_1)\ldots\exp(-il^d\overline{X}_d);$$

where $l \in \mathcal{G}$.

Now let $\eta \in \mathbf{D}(K^{1/2})$; then we define the set of coherent states for G, corresponding to the fixed continuous, irreducible, square integrable, unitary representation $U_{g(l)}$ as:

$$\eta(l) \equiv U_{g(l)}K^{1/2}\eta; \quad \eta \in \mathbf{D}(K^{1/2}) \quad \text{and} \quad \|\eta\| = 1. \tag{24}$$

It follows directly from (23) that these states give rise to a resolution of identity of the form

$$I_{\mathbf{H}} = \int_{\mathcal{G}} \eta(l)\langle\eta(l), \cdot\rangle dg(l), \tag{25}$$

where $dg(l)$ is the left invariant Haar measure of G given in the chosen parameterization by

$$dg(l) = \gamma(l)\prod_{k=1}^{d} dl^k, \tag{26}$$

where $\gamma(l) \equiv |\det[\lambda_m{}^k(g(l))]|$.

The map $\mathcal{C}_\eta : \mathbf{H} \to L^2(G)$, defined for any $\psi \in \mathbf{H}$ by:

$$[\mathcal{C}_\eta\psi](l) = \psi_\eta(l) \equiv \langle\eta(l), \psi\rangle = \langle U_{g(l)}K^{1/2}\eta, \psi\rangle, \tag{27}$$

yields a representation of the Hilbert space $\mathbf{H}$ by bounded, continuous, square integrable functions on a proper closed subspace $L^2_\eta(G)$ of $L^2(G)$. Using the resolution of identity one finds

$$\psi_\eta(l) = \int \mathcal{K}_\eta(l;l')\psi_\eta(l')dg(l'), \tag{28}$$

where

$$\mathcal{K}_\eta(l;l') \equiv \langle \eta(l), \eta(l')\rangle = \langle \eta, \overline{K^{1/2}U_{g^{-1}(l)g(l')}K^{1/2}\eta}\rangle$$

One calls (28) the reproducing property. Furthermore, the kernel $\mathcal{K}_\eta(l';l)$ is an element of $L^2_\eta(G)$ for fixed $l \in \mathcal{G}$. Therefore, the kernel $\mathcal{K}_\eta(l';l)$ is a reproducing kernel and $L^2_\eta(G)$ is a reproducing kernel Hilbert space. One easily verifies that the map $\mathcal{C}_\eta$ is an isometric isomorphism from $\mathbf{H}$ to $L^2_\eta(G)$. Note that $(U, \mathbf{H})$ is unitarily equivalent to a subrepresentation of $(\Lambda, L^2(G))$, since $\mathcal{C}_\eta$ intertwines $U_{g(l)}$ on $\mathbf{H}$ with a subrepresentation of $\Lambda_{g(l)}$ on $L^2_\eta(G)$.

LEMMA 4.1.1. – *The unitary representation $U_{g(l)}$ 'intertwines' the operator representation $\{\overline{X}_m\}_{m=1}^d$ of L on $\mathbf{H}$, with the representation of L by right and left invariant differential operators on any one of the reproducing kernel Hilbert spaces $L^2_\eta(G) \subset L^2(G)$. In fact setting $\nabla_l = (\partial_{l^1}, \ldots, \partial_{l^d})$ the following relations hold:*

(i) Let $\tilde{x}_k(-i\nabla_l, l) \equiv \sum_{m=1}^{d} {\rho^{-1}}_k{}^m(g(l))(-i\partial_{l^m}),\ k = 1, \ldots, d,$ *then:*

$$\tilde{x}_k(-i\nabla_l, l)U^*_{g(l)}\psi = U^*_{g(l)}\overline{X}_k\psi, \quad \forall\psi \in \widetilde{\mathbf{D}}.$$

(ii) Let $\tilde{\tilde{x}}_k(i\nabla_l, l) \equiv \sum_{m=1}^{d} {\lambda^{-1}}_k{}^m(g(l))(i\partial_{l^m}),\ k = 1, \ldots, d,$ *then:*

$$\tilde{\tilde{x}}_k(i\nabla_l, l)U_{g(l)}\psi = U_{g(l)}\overline{X}_k\psi, \quad \forall\psi \in \widetilde{\mathbf{D}}.$$

A common dense invariant domain for these differential operators on any one of the $L^2_\eta(G) \subset L^2(G)$ is given by the continuous representation of $\widetilde{\mathbf{D}}$, i.e. $\widetilde{\mathbf{D}}_\eta \equiv \mathcal{C}_\eta(\widetilde{\mathbf{D}})$.

(For the proof *see* [23] Corollary 2.4.)

COROLLARY 4.1.2. – *The differential operators $\{\tilde{x}_k(-i\nabla_l, l)\}_{k=1}^d$ $(\{\tilde{\tilde{x}}_k(i\nabla_l, l)\}_{k=1}^d)$ are essentially self-adjoint on any one of the reproducing kernel Hilbert spaces $L^2_\eta(G)$ and can be identified with the generators*

$\{\Lambda(\overline{X}_k)\}_{k=1}^d$ ($\{\mathrm{P}(\overline{X}_k)\}_{k=1}^d$) *of a subrepresentation of the left (right) regular representation of* G *on* $L^2_\eta(G)$.

Proof. – By Corollary 2.5 in [23] the differential operators $\{\tilde{x}_k(-i\nabla_l, l)\}_{k=1}^d$ are symmetric on any one of the reproducing kernel Hilbert spaces $L^2_\eta(G)$ and can be identified with the generators $\{\Lambda(\overline{X}_k)\}_{k=1}^d$ of a subrepresentation of the left regular representation of G on $L^2_\eta(G)$.

The essential self-adjointness of the operators $\tilde{x}_k(-i\nabla_l, l)$, $k = 1, \ldots, d$, on $L^2_\eta(G)$ is established as follows. Since each of the operators $\overline{X}_k$, $k = 1, \ldots, d$, has a dense set $\mathbf{D} \subset \mathbf{H}$ of analytic vectors, *see* section 3, we have by Lemma 5.1 in [21] that each $\overline{X}_k$, $k = 1, \ldots, d$, is self-adjoint. Hence, the restriction of each $\overline{X}_k$, $k = 1, \ldots, d$, to $\widetilde{\mathbf{D}}$ is essentially self-adjoint. Since, $\mathcal{C}_\eta$ is an isometric isomorphism from $\mathbf{H}$ onto $L^2_\eta(G)$ we have that the closure of each $\tilde{x}_k(-i\nabla_l, l)$, $k = 1, \ldots, d$, contains a dense set of analytic vectors, namely, $\mathcal{C}_\eta(\mathbf{D})$, hence, is by Lemma 5.1 in [21] self-adjoint. In particular, each $\tilde{x}_k(-i\nabla_l, l)$, $k = 1, \ldots, d$, is essentially self-adjoint on $\widetilde{\mathbf{D}}_\eta$.

Similarly one can prove that the operators $\{\tilde{\tilde{x}}_k(i\nabla_l, l)\}_{k=1}^d$ are essentially self-adjoint and that they can be identified with the generators $\{\mathrm{P}(\overline{X}_k)\}_{k=1}^d$ of a subrepresentation of the right regular representation of G on $L^2_\eta(G)$. □

COROLLARY 4.1.3. – *The family of right invariant differential operators* $\{\tilde{x}_k(-i\nabla_l, l)\}_{k=1}^d$ *commutes with the family of left invariant differential operators* $\{\tilde{\tilde{x}}_k(i\nabla_l, l)\}_{k=1}^d$.

Proof. – Let $\tilde{x}_i(-i\nabla_l, l)$ and $\tilde{\tilde{x}}_j(i\nabla_l, l)$ be arbitrary, then

$$\begin{aligned}
&[\tilde{x}_i(-i\nabla_l, l), \tilde{\tilde{x}}_j(i\nabla_l, l)] \\
&= \sum_{m,n=1}^d [\rho^{-1}{}_i{}^m(g(l))(-i\partial_{l^m}), \lambda^{-1}{}_j{}^n(g(l))(i\partial_{l^n})] \\
&= \sum_{n=1}^d \left(\sum_{m=1}^d \{\rho^{-1}{}_i{}^m(g(l))\partial_{l^m}[\lambda^{-1}{}_j{}^n(g(l))] \right. \\
&\qquad \left. - \lambda^{-1}{}_j{}^m(g(l))\partial_{l^m}[\rho^{-1}{}_i{}^n(g(l))]\} \right) \partial_{l^n}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{d} \left(\sum_{m=1}^{d} \rho^{-1}{}_i{}^m(g(l)) \sum_{f,s=1}^{d} \{\lambda^{-1}{}_j{}^f(g(l))\partial_{l^f}[\rho^{-1}{}_s{}^n(g(l))]\rho_m{}^s(g(l))\} \right. \\
&\quad \left. - \sum_{m=1}^{d} \lambda^{-1}{}_j{}^m(g(l))\partial_{l^m}[\rho^{-1}{}_i{}^n(g(l))] \right) \partial_{l^n} \\
&= \sum_{n=1}^{d} \left\{ \sum_{f=1}^{d} \lambda^{-1}{}_j{}^f(g(l))\partial_{l^f}[\rho^{-1}{}_i{}^n(g(l))] \right. \\
&\quad \left. - \sum_{m=1}^{d} \lambda^{-1}{}_j{}^m(g(l))\partial_{l^m}[\rho^{-1}{}_i{}^n(g(l))] \right\} \partial_{l^n} \\
&= 0,
\end{aligned}
$$

where we have used Lemma 3.2.2 (iii) in the fourth line. Therefore,

$$[\tilde{x}_i(-i\nabla_l, l), \tilde{\tilde{x}}_j(i\nabla_l, l)] = 0,$$

and since, $\tilde{x}_i(-i\nabla_l, l)$ and $\tilde{\tilde{x}}_j(i\nabla_l, l)$ were arbitrary, this establishes the Corollary. □

4.2. The Representation Independent Propagator for General Lie Groups

4.2.1. Construction of the Representation Independent Propagator

Let G be a general Lie group and let U be a fixed element in $\widehat{G}_d$. Then it is a direct consequence of Lemma 4.1.1 (i) that for any $\psi \in \widetilde{\mathbf{D}}$

$$\tilde{x}_k(-i\nabla_l, l)[\mathcal{C}_\eta\psi](l) = [\mathcal{C}_\eta \overline{X}_k \psi](l), \quad k = 1, \ldots, d.$$

holds *independently* of η. Therefore, the isometric isomorphism $\mathcal{C}_\eta$ intertwines the representation of the Lie algebra L on $\mathbf{H}$, with a subrepresentation of L by right-invariant, essentially self-adjoint differential operators on any one of the reproducing kernel Hilbert spaces $L^2_\eta(G)$. To summarize, we found in section 4.1 that any square integrable representation U of G is unitarily equivalent to a subrepresentation of the left regular representation Λ on $L^2_\eta(G)$. Furthermore, the generators of G are represented by right invariant, essentially self-adjoint differential operators on $L^2_\eta(G)$.

Let $(\pi, \mathbf{H})$ be a representation of G, then we denote by $\mathcal{A}(\pi)$ the von Neumann algebra generated by the operators π_g, $g \in G$. By Proposition 5.6.4 in [7] there exists a projection operator P_I in the center of the von Neumann algebra $\mathcal{A}(\Lambda)$ such that the restriction Λ_I of Λ to the closed subspace $P_I[L^2(G)]$ of $L^2(G)$ is of type I, and such that the restriction of Λ to the orthogonal complement of $P_I[L^2(G)]$ has no type I part. Since G is separable and locally compact there exists by Theorem 5.1 in [8] a standard Borel measure ν on $\hat{G}$, the set of all inequivalent irreducible unitary representations of G, and a ν-measurable field $(U^\zeta, \mathbf{H}_\zeta)_{\zeta \in \hat{G}}$ of irreducible, square integrable, unitary representations of G, such that the type I part of Λ, Λ_I, can be decomposed into a direct integral,

$$\Lambda_I = \int_{\hat{G}} U^\zeta \otimes I_\zeta \, d\nu(\zeta),$$

where $U^\zeta \otimes I_\zeta$ is a representation of $G \times G$ on $\mathbf{H}_\zeta \otimes \mathbf{H}_\zeta$.

Denote by $\mathcal{H}(\overline{X}_k)$ the essentially self-adjoint Hamilton operator of a quantum system on $\mathbf{H}_\zeta$. Then the continuous representation of the solution to Schrödinger's equation, $\psi(t) = \exp[-i(t-t')\mathcal{H}(\overline{X}_k)]\psi(t')$, takes, on $L^2_\eta(G)$, the following form

$$\psi_\eta(l,t) = \int K_\eta(l,t;l',t')\psi_\eta(l',t')dg(l')$$

where,

$$\begin{aligned} K_\eta(l,t;l',t') &= \langle \eta(l), \exp[-i(t-t')\mathcal{H}(\overline{X}_k)]\eta(l')\rangle \\ &= [\mathcal{C}_\eta \exp[-i(t-t')\mathcal{H}(\overline{X}_k)]\eta(l')](l) \\ &= \mathcal{U}(t-t')[\mathcal{C}_\eta \eta(l')](l) \\ &= \mathcal{U}(t-t')\langle \eta, \overline{K^{1/2}U_{g^{-1}(l)g(l')}K^{1/2}\eta}\rangle \\ &= \mathcal{U}(t-t')\mathcal{K}_\eta(l;l'), \end{aligned} \tag{29}$$

where,

$$\mathcal{U}(t-t') = \exp[-i(t-t')\mathcal{H}(\tilde{x}_k(-i\nabla_l, l))].$$

Let $\alpha, \beta \in \mathcal{D}(G)$, then put

$$U(\alpha) = \int \alpha(g(l))\, U_{g(l)}\, dg(l),$$

$\alpha^*(g(l)) \equiv \Delta(g^{-1}(l))\alpha(g^{-1}(l))$, and define the map $\mathcal{D}(G) \times \mathcal{D}(G) \ni (\alpha, \beta) \to \alpha \star \beta \in \mathcal{D}(G)$ as follows:

$$(\alpha \star \beta)(g(l)) = \int \alpha(g(l'))\beta(g^{-1}(l')g(l))dg(l').$$

With these definitions we find that:

$$\begin{aligned}
&\mathcal{K}_\eta(\alpha,\beta)\\
&= \int\int \mathcal{K}_\eta(l;l')\alpha(g(l))\beta(g(l'))dg(l)dg(l')\\
&= \int\int \langle\eta, \overline{K^{1/2}U_{g^{-1}(l)g(l')}K^{1/2}}\eta\rangle\alpha(g(l))\beta(g(l'))dg(l)dg(l')\\
&= \int \langle\eta, \overline{K^{1/2}U_{g(l')}K^{1/2}}\eta\rangle\left[\int \alpha(g(l))\beta(g(l)g(l'))dg(l)\right]dg(l')\\
&= \int \langle\eta, \overline{K^{1/2}U_{g(l')}K^{1/2}}\eta\rangle(\alpha^*\star\beta)(g(l'))dg(l')\\
&= \langle\eta, \overline{K^{1/2}U(\alpha^*\star\beta)K^{1/2}}\eta\rangle.
\end{aligned}$$

Note that $\mathcal{K}_\eta(\alpha,\beta)$ is a bilinear, separately continuous form on $\mathcal{D}(G)\times\mathcal{D}(G)$. We call the bilinear separately continuous forms on $\mathcal{D}(G)\times\mathcal{D}(G)$ *kernels on* G. Also observe that $\mathcal{K}_\eta(\alpha,\beta)$ is a left invariant kernel, that is

$$\mathcal{K}_\eta(L_g\alpha, L_g\beta) = \mathcal{K}_\eta(\alpha,\beta),\ \text{for every } g\in G, \alpha,\beta\in\mathcal{D}(G).$$

Therefore, we can write (29) as

$$K_\eta(\alpha,t;\beta,t') = \mathcal{U}(t-t')\mathcal{K}_\eta(\alpha,\beta).$$

In the above construction $\eta\in\mathbf{D}(K^{1/2})$ was arbitrary, furthermore as shown elsewhere [8, Corollary 2] for $\alpha\in\mathcal{D}(G)$ the operator $\overline{K^{1/2}U(\alpha)K^{1/2}}$ is trace class. Therefore, we can choose any ONS $\{\phi_j\}_{j\in\mathbb{N}}$ in $\mathbf{D}(K^{1/2})$ and write

$$\mathcal{K}_{\mathbf{H}}(\alpha,\beta) = \sum_{j=1}^{\infty}\mathcal{K}_{\phi_j}(\alpha,\beta) = \mathrm{tr}[\overline{K^{1/2}U(\alpha^*\star\beta)K^{1/2}}].$$

Note that $\mathcal{K}_{\mathbf{H}}(\alpha,\beta)$ is a left invariant kernel on G, since each $\mathcal{K}_{\phi_j}(\alpha;\beta)$ is a left invariant kernel on G. Therefore, by Proposition VI.6.5 in [19] there exists a unique distribution S in $\mathcal{D}'(G)$ such that $\mathcal{K}_{\mathbf{H}}(\alpha,\beta) = S(\alpha^*\star\beta)$. In fact we see that $S(g^{-1}(l)g(l')) = \mathrm{tr}[\overline{K^{1/2}U_{g^{-1}(l)g(l')}K^{1/2}}]$. Therefore, we find the following propagator which is an element of $\mathcal{D}'(G)$:

$$K_{\mathbf{H}}(l,t;l',t') = \mathcal{U}(t-t')\mathrm{tr}[\overline{K^{1/2}U_{g^{-1}(l)g(l')}K^{1/2}}]. \tag{30}$$

Remark 4.2.1. – This propagator is clearly independent of η the fiducial vector that fixes a coherent state representation. However, this propagator is in general no longer a continuous function but a linear functional acting on $\mathcal{D}(G)$. We will see below that the elements of any reproducing kernel Hilbert space lie in the set of test functions for this propagator. $\diamondsuit$

LEMMA 4.2.1. – *The propagator* $K_{\mathbf{H}}(l,t;l',t')$ *given in (33) correctly propagates all elements of any reproducing kernel Hilbert space* $L^2_\eta(G)$, *associated with the irreducible, square integrable unitary representation* $U_{g(l)}$ *of the general Lie group* G.

Proof. – Let $\eta \in \mathbf{D}(K^{1/2})$ be arbitrary, then for $\psi_\eta(l',t') \in L^2_\eta(G)$ one can write

$$\begin{aligned}
&\int K_{\mathbf{H}}(l,t;l',t')\psi_\eta(l',t')dg(l')\\
&= \int \mathcal{U}(t-t')\mathrm{tr}[\overline{K^{1/2}U_{g^{-1}(l)g(l')}K^{1/2}}]\psi_\eta(l',t')dg(l')\\
&= \sum_{j=1}^{\infty}\mathcal{U}(t-t')\int \langle\phi_j, \overline{K^{1/2}U_{g^{-1}(l)g(l')}K^{1/2}\phi_j}\rangle\langle U_{g(l')}K^{1/2}\eta, \psi(t')\rangle dg(l')\\
&= \mathcal{U}(t-t')\left\langle K^{1/2}U_{g(l)}\sum_{j=1}^{\infty}\langle\phi_j,\eta\rangle\phi_j, \psi(t')\right\rangle\\
&= [\mathcal{C}_\eta \exp[-i(t-t')\mathcal{H}(\overline{X}_k)]\psi(t')](l)\\
&= \psi_\eta(l,t),
\end{aligned}$$

where the fourth equality holds by (23). Therefore,

$$\psi_\eta(l,t) = \int K_{\mathbf{H}}(l,t;l',t')\psi_\eta(l',t')dg(l'), \quad \forall\eta \in \mathbf{D}(K^{1/2}),$$

i.e. the propagator propagates the elements of any $L^2_\eta(G)$ correctly. $\square$

In the above construction the irreducible, square integrable, unitary representation $U_{g(l)}$ was arbitrary, hence we can introduce such a propagator for each inequivalent unitary, irreducible, square integrable representation of G. By Corollary 5.1 in [8] there exits a positive, σ-finite standard Borel measure ν on $\hat{G}$, a ν-measurable decomposition $(U^\zeta_{g(l)}, \mathbf{H}_\zeta)_{\zeta\in\hat{G}}$ of Λ_I, and a measurable field $(K_\zeta)_{\zeta\in\hat{G}}$ of nonzero, positive, self-adjoint operators such that K_ζ is a semi-invariant operator of weight $\Delta(g^{-1})$ in $\mathbf{H}_\zeta$ for ν-almost all $\zeta \in \hat{G}$ such that for $\alpha, \beta \in P_I[\mathcal{D}(G)]$

$$\delta_e(\alpha^* \star \beta) = \int_{\hat{G}} \mathrm{tr}[\overline{K_\zeta^{1/2}U^\zeta(\alpha^* \star \beta)K_\zeta^{1/2}}]d\nu(\zeta), \tag{31}$$

is well defined. Here,

$$\delta_e(\alpha^* \star \beta) = \int\int \delta_e(g^{-1}(l)g(l'))\alpha(g(l))\beta(g(l'))dg(l)dg(l'),$$

and $\delta_e(g^{-1}(l)g(l'))$ is given in the chosen parameterization by

$$\delta_e(g^{-1}(l)g(l')) = \frac{1}{\gamma(l)} \prod_{k=1}^{d} \delta(l^k - l'^k).$$

Hence, we can write down the following propagator for Λ_I of G on $L^2(G)$,

$$\begin{aligned} K(\alpha, t; \beta, t') &= \int_{\hat{G}} K_{\mathbf{H}_\zeta}(\alpha, t; \beta, t') d\nu(\zeta) \\ &= \mathcal{U}(t - t') \int_{\hat{G}} \mathcal{K}_{\mathbf{H}_\zeta}(\alpha; \beta) d\nu(\zeta) \\ &= \mathcal{U}(t - t') \int_{\hat{G}} \overline{\mathrm{tr}[K_\zeta^{1/2} U^\zeta(\alpha^* \star \beta) K_\zeta^{1/2}]} d\nu(\zeta) \\ &= \mathcal{U}(t - t') \delta_e(\alpha^* \star \beta). \end{aligned}$$

Therefore, we find the following propagator for the type I part of the left regular representation Λ_I:

$$K(l, t; l', t') = \exp[-i(t - t')\mathcal{H}(\tilde{x}_k(-i\nabla_l, l))] \delta_e(g^{-1}(l)g(l')). \qquad (32)$$

Remark 4.2.2. – Observe, that this propagator is clearly independent of the fiducial vector and the irreducible, square integrable unitary representation one has chosen for G. A sufficiently large set of test functions for this propagator is given by $C(G) \cap L^2(G)$, where $C(G)$ is the set of all continuous functions on G. Hence, the elements of any reproducing kernel Hilbert space $L^2_\eta(G)$ are allowed test functions for the propagator given by (32), and therefore, for the propagator given by (30). Therefore, we have shown the first part of the following Theorem:

THEOREM 4.2.2. – *The propagator $K(l, t; l', t')$ in (32) is a propagator for the type I part of the left regular representation of the general Lie group G which correctly propagates all elements of any reproducing kernel Hilbert space $L^2_\eta(G)$ associated with an arbitrary irreducible, square integrable, unitary representation $U^\zeta_{g(l)}$ of G, $\zeta \in \hat{G}$.*

Proof. – To prove the second part of Theorem 4.2.2, let $U^{\zeta'}_{g(l)}$ and $\eta \in \mathbf{D}(K_{\zeta'}^{1/2})$ be arbitrary. For any $\psi_\eta(l) \in L^2_\eta(G)$, associated with $U^{\zeta'}_{g(l)}$, we can write

$$\begin{aligned}
&\int_G K(l,t;l',t')\psi_\eta(l',t')dg(l') \\
&= \int_G \mathcal{U}(t-t')\delta_e(g^{-1}(l)g(l'))\langle U^{\zeta'}_{g(l')}K^{1/2}_{\zeta'}\eta, \psi(t')\rangle dg(l') \\
&= \mathcal{U}(t-t')[\mathcal{C}_\eta\psi(t')](l) \\
&= [\mathcal{C}_\eta \exp[-i(t-t')\mathcal{H}(\overline{X}_k)]\psi_{\zeta'}(t')](l) \\
&= \psi_\eta(l,t).
\end{aligned}$$

Therefore,

$$\psi_\eta(l,t) = \int K(l,t;l',t')\psi_\eta(l',t')dg(l'),$$

for all $\eta \in \mathbf{D}(K^{1/2}_{\zeta'})$' and any $\zeta' \in \hat{G}$, *i.e.* this propagator propagates all elements of any reproducing kernel Hilbert space $L^2_\eta(G)$ associated with an arbitrary irreducible representation $U^{\zeta'}_{g(l)}$ correctly. □

Hence, we have succeeded in constructing a *representation independent propagator* for a general Lie group.

4.3. Path Integral Formulation of the Representation Independent Propagator

From (32) it is easily seen that the representation independent propagator is a weak solution to Schrödinger's equation, *i.e.*

$$i\partial_t K(l,t;l',t') = \mathcal{H}(\tilde{x}_1(-i\nabla_l,l),...,\tilde{x}_d(-i\nabla_l,l))K(l,t;l',t'), \quad (33)$$

Taking in (32) the limit $t \to t'$ yields the following initial value problem

$$\begin{aligned}
&i\partial_t K(l,t;l',t') = \mathcal{H}(\tilde{x}_1(-i\nabla_l,l),...,\tilde{x}_d(-i\nabla_l,l))K(l,t;l',t'), \\
&\lim_{t\to t'} K(l,t;l',t') = \delta_e(g^{-1}(l)g(l')). \quad (34)
\end{aligned}$$

Remark 4.3.1. – Observe that the coherent state propagator given in (29) is also a weak solution to the Schrödinger equation (33). However, it satisfies the initial value problem

$$\begin{aligned}
&i\partial_t K_\eta(l,t;l',t') = \mathcal{H}(\tilde{x}_1(-i\nabla_l,l),...,\tilde{x}_d(-i\nabla_l,l))K_\eta(l,t;l',t'), \\
&\lim_{t\to t'} K_\eta(l,t;l',t') = \mathcal{K}_\eta(l;l'). \quad (35)
\end{aligned}$$

Therefore, we can write

$$i\partial_t K_{\#}(l,t;l',t') = \mathcal{H}(\tilde{x}_1(-i\nabla_l,l),...,\tilde{x}_d(-i\nabla_l,l))K_{\#}(l,t;l',t'), \quad (36)$$

where $K_{\#}$ denotes either K_η or K. Note that the initial conditions, *i.e.* either (34) or (35) determine which function is under consideration. $\diamondsuit$

We now interpret the Schrödinger equation (36) with the initial condition (34) as a Schrödinger equation appropriate to d *separate and independent canonical* degrees of freedom. Hence, $l^1,...,l^d$ are viewed as d "coordinates", and we are looking at the irreducible Schrödinger representation of a special class of d-variable Hamilton operators, ones where the classical Hamiltonian is restricted to have the form $\mathcal{H}(\check{x}_1(p,l),...,\check{x}_d(p,l))$, instead of the most general form $\mathcal{H}(p_1,...,p_d,l^1,...,l^d)$. In fact the differential operators given in Lemma 4.1.1 (i) are elements of the right invariant enveloping algebra of the d-dimensional Schrödinger representation on $L^2(G)$. Based on this interpretation one can give the representation independent propagator the following standard formal phase-space path integral formulation in which the integrand assumes the form appropriate to continuous and differentiable paths

$$\begin{aligned} &K(l'',t'';l',t') \\ &= \mathcal{M}\int \exp\left\{ i\int \left[\sum_{m=1}^{d} p_m \dot{l}^m - \mathcal{H}(\check{x}_1(p,l),\cdots,\check{x}_d(p,l)) \right] dt \right\} \\ &\times \prod_{t\in[t',t'']} dl(t)dp(t), \end{aligned} \quad (37)$$

where "p_1",...,"p_d" denote "momenta" conjugate to the "coordinates" "l^1",...,"l^d". Note that we have used the special form of the Hamiltonian and that its arguments are given by

$$\check{x}_k(p,l) = \sum_{m=1}^{d} {\rho^{-1}}_k{}^m(g(l))p_m, \quad k = 1,...,d.$$

The integration over the "coordinates" is restricted to the parameter space $\mathcal{G}$. If part of $\mathcal{G}$ is compact the momenta conjugate to the restricted range or periodic "coordinates" of this part of the group are discrete variables. For this class of momenta the notation $\int\prod dp(t)$ is then properly to be understood as sums rather than integrals. Before we can turn to a regularized lattice prescription for the representation independent propagator, we first have to spell out what we mean by a Schrödinger representation on $L^2(G)$. Let $\{\mathcal{L}^k, P_{\mathcal{L}^k}\}_{k=1}^d$ be a family of symmetric operators with a common dense invariant domain **D** on some Hilbert space **H** satisfying:

1. The following canonical commutation relations (CCR),

$$[\mathcal{L}^a, \mathcal{L}^b] = 0;\ [P_{\mathcal{L}^a}, P_{\mathcal{L}^b}] = 0;\ [\mathcal{L}^a, P_{\mathcal{L}^b}] = i\delta^a{}_b I, \qquad a, b = 1, \dots, d.$$

2. The operator $\Delta = \sum_{k=1}^d [(P_{\mathcal{L}^k})^2 + (\mathcal{L}^k)^2]$ is essentially self-adjoint.

Note that condition 2 above ensures that $\{\mathcal{L}^k, P_{\mathcal{L}^k}\}_{k=1}^d$ is a family of essentially self-adjoint operators (*cf.* [21]). Let $\mathbf{\Psi}$ be a dense set of analytic vectors for Δ and the family $\{\mathcal{L}^k, P_{\mathcal{L}^k}\}_{k=1}^d$, then we denote by $\mathbf{\Phi}$ the closure of $\mathbf{\Psi}$ in the nuclear topology on $\mathbf{\Psi}$ (e.g. [4]). In fact, one can show that the families of operators $\{P_{\mathcal{L}^k}\}_{k=1}^d$ and $\{\mathcal{L}_k\}_{k=1}^d$ form two separate complete commuting systems of operators, respectively (*cf.* [4]). Let us denote by $\mathbf{\Phi}'$ the dual space of $\mathbf{\Phi}$, that is the space of all $\mathcal{T}_\Phi$-continuous linear functionals acting on $\mathbf{\Phi}$. Then by the Nuclear Spectral Theorem there exist common generalized eigenvectors, $\langle l^1, \dots, l^d| \in \mathbf{\Phi}'$ and $\langle p_1, \dots, p_d| \in \mathbf{\Phi}'$, respectively, such that

$$\langle l^1, \dots, l^d|\mathcal{L}^{a\dagger} = l^a \langle l^1, \dots, l^d|, \quad l^a \in \mathrm{spec}(\overline{\mathcal{L}^a}), \quad a = 1, \dots, d,$$
$$\langle p_1, \dots, p_d|P^\dagger_{\mathcal{L}^a} = p_a \langle p_1, \dots, p_d|, \quad p_a \in \mathrm{spec}(\overline{P_{\mathcal{L}^a}}), \quad a = 1, \dots, d,$$

normalized such that

$$\langle l''^1, \dots, l''^d|l'^1, \dots, l'^d\rangle = \delta_e(g^{-1}(l'')g(l')),$$
$$\langle p''_1, \dots, p''_d|p'_1, \dots, p'_d\rangle = \prod_{k=1}^d \delta(p''_k, p'_k),$$

where

$$\delta(p''_k, p'_k) = \begin{cases} \delta_{p''_k p'_k} & \text{If the spectrum of } \overline{\mathrm{P}_{\mathcal{L}^k}} \text{ is discrete} \\ \delta(p''_k - p'_k) & \text{If the spectrum of } \overline{\mathrm{P}_{\mathcal{L}^k}} \text{ is continuous} \end{cases}$$

and giving rise to the resolutions of identity

$$\int_{\mathrm{spec}(\overline{\mathcal{L}^1})\times\dots\times\mathrm{spec}(\overline{\mathcal{L}^d})} |l\rangle\langle l|\, dg(l) = I, \tag{38}$$

$$\int_{\mathrm{spec}(\overline{P_{\mathcal{L}^1}})\times\dots\times\mathrm{spec}(\overline{P_{\mathcal{L}^d}})} |p\rangle\langle p| dp = I, \tag{39}$$

where $|l\rangle \equiv |l^1, \dots, l^d\rangle$ and $|p\rangle \equiv |p_1, \dots, p_d\rangle$.

Remark 4.3.2. – If the spectrum of $\overline{P_{\mathcal{L}^k}}$ is discrete then dp_k denotes a pure point measure such that the integration over p_k reduces to summation over spec$(\overline{P_{\mathcal{L}^k}})$. ◇

On $L^2(G)$ these operators can be represented as

$$\mathcal{L}^a \doteq l^a \quad \text{and} \quad P_{\mathcal{L}^a} \doteq -i\widehat{\partial}_{l^a} \equiv -i[\partial_{l^a} + \frac{1}{2}\Gamma^a(l)], \tag{40}$$

where $\mathbf{D}_S = \mathcal{S}(G)$, the set of functions of rapid decrease on G, is chosen as the common dense invariant domain of these operators. Here $\Gamma^a(l)$ is defined as $\Gamma^a(l) \equiv \partial_{l^a} \ln \gamma(l)$ and where $\gamma(l)$ is given in (24). It is easily seen that these operators satisfy the CCR, are symmetric on $L^2(G)$, and that $-i\widehat{\nabla}_l$ has the following generalized eigenfunctions

$$\langle l|p\rangle = \gamma^{-1/2}(l) \exp\left(i \sum_{k=1}^{d} p_k l^k\right),$$

where $\widehat{\nabla}_l = (\widehat{\partial}_{l^1}, \ldots, \widehat{\partial}_{l^d})$. We normalize these functions so that

$$\frac{1}{K\sqrt{\gamma(l'')\gamma(l')}} \int \exp\left[i \sum_{k=1}^{d} p_k (l''^k - l'^k)\right] dp_1 \ldots dp_d = \delta_e(g^{-1}(l'')g(l')),$$

where K denotes the normalization constant. Therefore, we find for the normalized generalized eigenfunctions of $-i\widehat{\nabla}_l$:

$$\langle l|p\rangle = \frac{1}{\sqrt{K\gamma(l)}} \exp\left(i \sum_{k=1}^{d} p_k l^k\right). \tag{41}$$

We call (40) a *d-dimensional Schrödinger representation on* $L^2(G)$. Moreover, the differential operators $\{\tilde{x}_k(-i\nabla_l, l)\}_{k=1}^d$ can be written as follows:

LEMMA 4.3.1. – *Using the differential operators* $\{-i\widehat{\partial}_{l^a}\}_{a=1}^d$ *given in (40) the right invariant differential operators* $\{\tilde{x}_k(-i\nabla_l, l)\}_{k=1}^d$ *defined in Lemma 4.1.1 (i) can be written as:*

$$\tilde{x}_k(-i\widehat{\nabla}_l, l) = \sum_{m=1}^{d} \frac{1}{2}[{\rho^{-1}}_k{}^m(g(l))(-i\widehat{\partial}_{l^m}) + (-i\widehat{\partial}_{l^m}){\rho^{-1}}_k{}^m(g(l))],$$
$$k = 1, \ldots, d, \tag{42}$$

where $\widehat{\nabla}_l = (\widehat{\partial}_{l^1}, \ldots, \widehat{\partial}_{l^d})$.

Proof. – Since $-i\partial_{l^a} = -i\widehat{\partial}_{l^a} + (i/2)\Gamma^a(l)$, $a = 1, \ldots, d$, the differential operators $\{\tilde{x}_k(-i\nabla_l, l)\}_{k=1}^d$ become after substitution of this expression

$$\begin{aligned} &\tilde{x}_k(-i\widehat{\partial}_{l^1} + (i/2)\Gamma^1(l), \ldots, -i\widehat{\partial}_{l^d} + (i/2)\Gamma^d(l), l^1, \ldots, l^d) \\ &= \sum_{m=1}^{d} {\rho^{-1}}_k^{\ m}(g(l))[-i\widehat{\partial}_{l^m} + \frac{i}{2}\Gamma^m(l)] \\ &= \sum_{m=1}^{d} 1/2[{\rho^{-1}}_k^{\ m}(g(l))(-i\widehat{\partial}_{l^m}) + {\rho^{-1}}_k^{\ m}(g(l))(-i\widehat{\partial}_{l^m})] \\ &+ \frac{i}{2}{\rho^{-1}}_k^{\ m}(g(l))\Gamma^m(l). \end{aligned}$$

Using $[{\rho^{-1}}_k^{\ m}(g(l)), -i\widehat{\partial}_{l^m}] = i\partial_{l^m}{\rho^{-1}}_k^{\ m}(g(l))$ and the definition of $\Gamma^m(l)$ yields

$$\begin{aligned} &\tilde{x}_k(-i\widehat{\partial}_{l^1} + (i/2)\Gamma^1(l), \ldots, -i\widehat{\partial}_{l^d} + (i/2)\Gamma^d(l), l^1, \ldots, l^d) \\ &= \sum_{m=1}^{d} \frac{1}{2}[{\rho^{-1}}_k^{\ m}(g(l))(-i\widehat{\partial}_{l^m}) + (-i\widehat{\partial}_{l^m}){\rho^{-1}}_k^{\ m}(g(l))] \\ &+ \frac{i}{2\gamma(l)}\sum_{m=1}^{d} \partial_{l^m}[{\rho^{-1}}_k^{\ m}(g(l))\gamma(l)]. \end{aligned}$$

Since the operators $\tilde{x}_k(-i\nabla_l, l)$ are essentially self-adjoint on any reproducing kernel Hilbert-space $L^2_\eta(G)$ (*cf.* Corollary 4.1.2) and since $\gamma(l) \neq 0$ one concludes that

$$\sum_{m=1}^{d} \partial_{l^m}[{\rho^{-1}}_k^{\ m}(g(l))\gamma(l)] = 0, \quad k = 1, \ldots, d,$$

and therefore,

$$\begin{aligned} &\tilde{x}_k(-i\widehat{\nabla}_l, l) = \sum_{m=1}^{d} \frac{1}{2}[{\rho^{-1}}_k^{\ m}(g(l))(-i\widehat{\partial}_{l^m}) + (-i\widehat{\partial}_{l^m}){\rho^{-1}}_k^{\ m}(g(l))], \\ &k = 1, \ldots, d. \quad \square \end{aligned}$$

Remark 4.3.3. – This Lemma shows that the differential operators $\{\tilde{x}_k(-i\widehat{\nabla}_l, l)\}_{k=1}^d$ are elements of the right invariant enveloping algebra of the d-dimensional Schrödinger representation on $L^2(G)$. $\diamondsuit$

Adapting methods used in [14] and [17] we can give the representation independent propagator the following regularized lattice prescription.

PROPOSITION 4.3.2. – *Let $\mathcal{H}_\epsilon = \mathcal{H}/(I+\epsilon\mathcal{H}^2)$ be a sequence of regularized bounded Hamilton operators on* **H**, *where $\epsilon = (t'' - t')/(N+1)$. Then provided the indicated integrals exist* (see *below) the representation independent propagator in (34) can be given the following d-dimensional lattice phase-space path integral representation:*

$$K(l'',t'';l',t') = \frac{1}{\sqrt{\gamma(l'')\gamma(l')}} \lim_{N\to\infty} \int \cdots \int \times \exp\left\{ i \sum_{j=0}^{N} [p_{j+1/2} \cdot (l_{j+1} - l_j) - \epsilon \mathcal{H}_\epsilon(\check{x}_k(p_{j+1/2}; l_{j+1}, l_j))] \right\} \times \prod_{j=1}^{N} dl_j^1 \dots dl_j^d \prod_{j+1/2=0}^{N} \frac{dp_{1^{j+1/2}} \dots dp_{d^{j+1/2}}}{K^d}, \tag{43}$$

where $l_{N+1} = l''$, $l_0 = l'$ and the arguments of the Hamiltonian are given by the following functions:

$$\check{x}_k(p_{j+1/2}; l_{j+1}, l_j) = \sum_{m=1}^{d} \frac{{\rho^{-1}}_k{}^m(g(l_{j+1})) + {\rho^{-1}}_k{}^m(g(l_j))}{2} p_{m^{j+1/2}}, \quad k = 1, \dots, d.$$

Remark 4.3.4. – If part of the parameter space $\mathcal{G}$ is compact then we denote by $\mathcal{R}$ the class of momenta conjugate to the restricted range or periodic "coordinates". If $p_k \in \mathcal{R}$ then dp_k denotes a pure point measure such that the integration over $p_{k^{j+1/2}}$ reduces to summation over the discrete spectrum of $\overline{P_{\mathcal{L}^k}}$. $\diamondsuit$

Proof. – Since the Hamilton operator $\mathcal{H}$ is in general an unbounded operator, we introduce the following sequence of regularized bounded Hamilton operators on **H**

$$\mathcal{H}_\delta = \frac{\mathcal{H}}{I + \delta\mathcal{H}^2}, \quad \delta > 0.$$

Then it is straightforward to show, by using the Spectral Theorem and the Monotone Convergence Theorem, that for all $\psi \in \mathbf{D}(\mathcal{H}) \subset \mathbf{H}$ one has

$$\text{s-}\lim_{\delta\to 0} \mathcal{H}_\delta = \mathcal{H},$$

and that on all of **H** one has

$$\text{s-}\lim_{N\to\infty} [I - i\epsilon\mathcal{H}_\epsilon]^{N+1} = \exp[-i(t'' - t')\mathcal{H}], \tag{44}$$

where $\epsilon \equiv (t'' - t')/(N+1)$. Now in order to obtain the lattice phase-space path integral in (43) one can proceed as follows. Let $\{\phi_j\}_{j=1}^{\infty}$ be an arbitrary ONS in $\mathbf{\Phi} \subset \mathbf{H}$, then

$$\begin{aligned} K(l'',t'';l',t') &= \exp[-i(t''-t')\mathcal{H}(\tilde{x}_k(-i\widehat{\partial}_{l^a}, l^a))]\langle l''|l'\rangle \\ &= \langle l''|\exp[-i(t''-t')\mathcal{H}(\tilde{x}_k(P_{\mathcal{L}^a}, \mathcal{L}^a))]|l'\rangle \\ &= \sum_{j,k=1}^{\infty} \langle l''|\phi_k\rangle\langle\phi_k, \exp[-i(t''-t')\mathcal{H}(\tilde{x}_k(P_{\mathcal{L}^a}, \mathcal{L}^a))]\phi_j\rangle\langle\phi_j|l'\rangle \\ &= \lim_{N\to\infty} \sum_{j,k=1}^{\infty} \langle l''|\phi_k\rangle\langle\phi_k, [1 - i\epsilon\mathcal{H}_\epsilon(\tilde{x}_k(P_{\mathcal{L}^a}, \mathcal{L}^a))]^{N+1}\phi_j\rangle\langle\phi_j|l'\rangle \\ &= \lim_{N\to\infty} \langle l''|[1 - i\epsilon\mathcal{H}_\epsilon(\tilde{x}_k(P_{\mathcal{L}^a}, \mathcal{L}^a))]^{N+1}|l'\rangle \end{aligned}$$

where $\langle\cdot|\cdot\rangle$ denotes the generalized inner product. Note that the second line holds true since each $\phi \in \mathbf{\Phi}$ gives rise to a linear functional acting on $\mathbf{\Phi}$ in the following manner $L_\phi(\psi) = \langle\phi|\psi\rangle \equiv \langle\phi, \psi\rangle$ for all $\psi \in \mathbf{\Phi}$. Hence, one has that $\langle\phi_k|\exp[-i(t''-t')\mathcal{H}]|\phi_j\rangle = L_{\phi_k}(\exp[-i(t''-t')\mathcal{H}]\phi_j) = \langle\phi_k, \exp[-i(t''-t')\mathcal{H}]\phi_j\rangle$. In the third line we have used (44) and Moore's Interchange of Limits Theorem (*see* [9, Lemma I.7.6]) to justify the interchange of the limit with the infinite sum. Hence, we find the following expression for $K(l'',t'';l',t')$:

$$K(l'',t'';l',t') = \lim_{N\to\infty} \langle l''|[1 - i\epsilon\mathcal{H}_\epsilon(\tilde{x}_k(P_{\mathcal{L}^a}, \mathcal{L}^a))]^{N+1}|l'\rangle.$$

Inserting the resolution of identity (38) N-times this becomes

$$K(l'',t'';l',t') = \lim_{N\to\infty} \int\ldots\int \prod_{j=0}^{N} \langle l_{j+1}|[1 - i\epsilon\mathcal{H}_\epsilon(\tilde{x}_k(P_{\mathcal{L}^a}, \mathcal{L}^a))]|l_j\rangle \times \prod_{j=1}^{N} \gamma(l_j)dl_j^1 \ldots dl_j^d, \tag{45}$$

where $l'' = l_{N+1}$, $l' = l_0$. Therefore, we have to evaluate $\langle l_{j+1}|[1 - i\epsilon\mathcal{H}_\epsilon]|l_j\rangle$. This can be done as follows:

$$\begin{aligned} &\langle l_{j+1}|[1 - i\epsilon\mathcal{H}_\epsilon(\tilde{x}_k(P_{\mathcal{L}^a}, \mathcal{L}^a))]|l_j\rangle \\ &= \langle l_{j+1}|[1 - i\epsilon\mathcal{H}_\epsilon(\tilde{x}_k(P_{\mathcal{L}^a}; l_{j+1}, l_j))]|l_j\rangle + o(\epsilon^2) \\ &= \int \langle l_{j+1}|p_{j+1/2}\rangle\langle p_{j+1/2}|[1 - i\epsilon\mathcal{H}_\epsilon(\tilde{x}_k(P_{\mathcal{L}^a}; l_{j+1}, l_j))]|l_j\rangle dp_{j+1/2} + o(\epsilon^2) \\ &= \int \langle l_{j+1}|p_{j+1/2}\rangle\overline{\langle l_j|p_{j+1/2}\rangle}\, [1 - i\epsilon\mathcal{H}_\epsilon(\check{x}_k(p_{j+1/2}; l_{j+1}, l_j))]dp_{j+1/2} \\ &\quad + o(\epsilon^2), \end{aligned}$$

where

$$\check{x}_k(p_{j+1/2}; l_{j+1}, l_j) = \sum_{m=1}^{d} \frac{{\rho^{-1}}_k{}^m(g(l_{j+1})) + {\rho^{-1}}_k{}^m(g(l_j))}{2}\, p_{m^{j+1/2}},$$
$k = 1, \ldots, d.$

Substituting the right hand side of (41) into the above expression yields

$$\begin{aligned}&\langle l_{j+1}|1 - i\epsilon\mathcal{H}_\epsilon|l_j\rangle\\ &= \frac{1}{\sqrt{\gamma(l_{j+1})\gamma(l_j)}} \int e^{ip_{j+1/2}\cdot(l_{j+1}-l_j)}[1 - i\epsilon\mathcal{H}_\epsilon(\check{x}_k(p_{j+1/2}; l_{j+1}, l_j))]\\ &\times \frac{dp_{j+1/2}}{K} + o(\epsilon^2)\end{aligned} \quad (46)$$

Now inserting (46) into (45) yields

$$\begin{aligned}K(l'', t''; l', t') &= \frac{1}{\sqrt{\gamma(l'')\gamma(l')}} \lim_{N\to\infty} \int \cdots \int\\ &\times \exp\left\{ i \sum_{j=0}^{N} [p_{j+1/2} \cdot (l_{j+1} - l_j) \right\} \prod_{j=0}^{N} [1 - i\epsilon\mathcal{H}_\epsilon(\check{x}_k(p_{j+1/2}; l_{j+1}, l_j))]\\ &\times \prod_{j=1}^{N} dl_j^1 \ldots dl_j^d \prod_{j+1/2=0}^{N} \frac{dp_{1^{j+1/2}} \ldots dp_{d^{j+1/2}}}{K^d},\end{aligned} \quad (47)$$

Equation (47) represents a valid lattice phase-space path integral representation of the propagator $K(l'', t''; l', t')$. One can now interpret the term $1 - i\epsilon\mathcal{H}_\epsilon(\check{x}_k)$ as the first order approximation of $\exp[-i\epsilon\mathcal{H}_\epsilon(\check{x}_k)]$ for small ϵ. Hence, provided the indicated integrals (or sums as necessary) exist one may replace (47) by the more suggestive expression:

$$\begin{aligned}K(l'', t''; l', t') &= \frac{1}{\sqrt{\gamma(l'')\gamma(l')}} \lim_{N\to\infty} \int \cdots \int\\ &\times \exp\left\{ i \sum_{j=0}^{N} [p_{j+1/2} \cdot (l_{j+1} - l_j) - \epsilon\mathcal{H}_\epsilon(\check{x}_k(p_{j+1/2}; l_{j+1}, l_j))] \right\}\\ &\times \prod_{j=1}^{N} dl_j^1 \ldots dl_j^d \prod_{j+1/2=0}^{N} \frac{dp_{1^{j+1/2}} \ldots dp_{d^{j+1/2}}}{K^d},\end{aligned}$$

which is the desired expression. □

Remark 4.3.5. – Observe that even though the group manifold is a *curved* manifold the regularized lattice expression for the representation independent propagator - save for the prefactor $1/\sqrt{\gamma(l'')\gamma(l')}$ - has the conventional form of a lattice phase-space path integral on a d-dimensional *flat* manifold. Also note that the lattice expression for the representation independent propagator exhibits the correct time reversal symmetry.

Furthermore, we have made no assumptions about the nature of the physical systems we are considering, other than that their Hamilton operators be essentially self-adjoint. Hence, one can use (43) in principle to describe the motion of a general physical system, not just that of a free particle, on the group manifold of a general Lie group G. In addition, there are no $\hbar^2$ corrections present in the Lagrangian. Therefore, we have arrived at an extremely natural path integral formulation for the motion of a *general physical system* on the group manifold of a general Lie group. ◇

5. EXAMPLE: A REPRESENTATION INDEPENDENT PROPAGATOR FOR THE AFFINE GROUP

We now introduce a representation independent propagator for the affine group. The affine group is the group of linear transformations without reflections on the real line, $\mathbb{R} \ni x \rightarrow p^{-1}x - q$, where $0 < p < \infty$ and $-\infty < q < \infty$. This group has been used by Klauder [15] for the coherent state path integral quantization of one-dimensional systems for which the canonical momentum p is restricted to be positive for all times. For further applications of the affine group in quantum physics the reader is referred to [15] and references there in.

5.1. Affine Coherent States

Let us denote by X_1 and X_2 a representation of the basis of the Lie algebra associated with the affine group by self-adjoint operators with common dense invariant domain $\widetilde{\mathbf{D}}$ on some Hilbert space $\mathbf{H}$. Since X_1 and X_2 are a representation of the basis of the Lie algebra associated with the affine group, it follows that these operators satisfy the commutation relations

$$[X_1, X_1] = 0, \quad [X_2, X_2] = 0, \quad \text{and} \quad [X_1, X_2] = -iX_1.$$

Since X_1 and X_2 are chosen to be self-adjoint they can be exponentiated to one-parameter unitary subgroups of the affine group. Since the affine group is a connected solvable Lie group every group element can be written as the product of these one-parameter unitary subgroups (*cf.* [3, Theorem 3.5.1]). With the above parameterization the map:

$$g(p,q) \rightarrow U_{g(p,q)} = \exp(-iqX_1)\exp(i\ln pX_2)$$

provides a unitary representation of the affine group on $\mathbf{H}$, for all $(p,q) \in \mathbf{P}^+$, where $\mathbf{P}^+ = \{(p,q) : 0 < p < \infty,\ -\infty < q < \infty\}$. The unitary representations of the affine group have been studied by Aslaksen and Klauder [1] and Gel'fand and Neumark [11] and it is known that there exist only two (faithful) inequivalent irreducible unitary representations for this group, one for which X_1 is a positive self-adjoint operator and one for which X_1 is a negative self-adjoint operator. We denote the irreducible unitary representation of the affine group corresponding to X_1 positive by $U^1_{g(p,q)}$ and to X_1 negative by $U^2_{g(p,q)}$, respectively.

The continuous representation theory using the affine group has been investigated by Aslaksen and Klauder [2] where it was shown that for $\xi, \phi \in \mathbf{H}$, $\phi \neq 0$ the factor $\langle U^\zeta_{g(p,q)}\xi, \phi\rangle$, $\zeta = 1, 2$, is square integrable if and only if $\xi \in \mathbf{D}(C^{-1/2})$, where the operator C is given by $C = \frac{1}{2\pi}|X_1|$. Hence, the irreducible unitary representations of the affine group are square integrable for a dense set of vectors in $\mathbf{H}$. Moreover, in [2] the following orthogonality relations have been established for the irreducible unitary representations of the affine group:

$$\int \langle \chi, U^\zeta_{g(p,q)}\xi\rangle\langle U^\zeta_{g(p,q)}\xi', \chi'\rangle dpdq = \langle \chi', \chi\rangle\langle C^{-1/2}\xi', C^{-1/2}\xi\rangle, \quad \zeta = 1, 2$$

where $\chi, \chi' \in \mathbf{H}$, and $\xi, \xi' \in \mathbf{D}(C^{-1/2})$. Hence, each of the irreducible unitary representations can be used to define a set of coherent states:

$$\eta(p,q) = U^\zeta_{g(p,q)} C^{1/2}\eta, \quad \zeta = 1, 2,$$

where $\eta \in \mathbf{D}(C^{1/2})$ and $\|\eta\| = 1$. These states give rise to a resolution of identity and a continuous representation of the Hilbert space $\mathbf{H}$ on any one of the reproducing kernel Hilbert spaces $L^2_\eta(\mathbf{P}^+) \subset L^2(\mathbf{P}^+)$.

5.2. The Representation Independent Propagator

Using Theorem 2.1 (ii) in [23] we find:

$$idU^\zeta_{g(p,q)} U^{\zeta *}_{g(p,q)} = X_1 dq + \left(\frac{q}{p}X_1 - \frac{1}{p}X_2\right)dp, \quad \zeta = 1, 2$$

from which we identify the following 2×2 coefficient matrix $[\rho_m{}^k(g(p,q))]$:

$$[\rho_m{}^k(g(p,q))] = \begin{pmatrix} 1 & 0 \\ \frac{q}{p} & -\frac{1}{p} \end{pmatrix}.$$

Inverting this 2×2 matrix we find:

$$[\rho^{-1}{}_m{}^k(g(p,q))] = \begin{pmatrix} 1 & 0 \\ q & -p \end{pmatrix}.$$

With these coefficients we find by Lemma 4.1.1 for the differential operators that describe the action of the affine operators X_1 and X_2 on any reproducing kernel Hilbert space $L^2_\eta(\mathbf{P}^+)$ the following:

$$\begin{aligned} \tilde{x}_1 &= -i\partial_q, \\ \tilde{x}_2 &= ip\partial_p - iq\partial_q. \end{aligned}$$

Thus, if we denote by $\mathcal{H}(X_1, X_2)$ the essentially self-adjoint Hamilton operator of a quantum mechanical system on $\mathbf{H}$ then by Theorem 4.2.2 the representation independent propagator for the affine group is given by:

$$K(p'', q'', t''; p', q', t') = \exp[-i(t'' - t')\mathcal{H}(\tilde{x}_1, \tilde{x}_2)]\, \delta(p'' - p')\delta(q'' - q').$$

By Proposition 4.3.2 we can give the representation independent propagator for the affine group the following regularized lattice phase-space path integral representation:

$$\begin{aligned} &K(p'', q'', t''; p', q', t') \\ &= \lim_{N\to\infty} \int \ldots \int \exp\Bigg\{ i \sum_{j=0}^{N} \Bigg[x_{j+1/2}(p_{j+1} - p_j) + k_{j+1/2}(q_{j+1} - q_j) \\ &-\epsilon\mathcal{H}_\epsilon\left(k_{j+1/2}, \frac{1}{2}[k_{j+1/2}(q_{j+1} + q_j) - x_{j+1/2}(p_{j+1} + p_j)] \right) \Bigg] \Bigg\} \\ &\times \prod_{j=1}^{N} dp_j dq_j \prod_{j=0}^{N} \frac{dk_{j+1/2} dx_{j+1/2}}{(2\pi)^2}, \end{aligned}$$

where $(p_{N+1}, q_{N+1}) = (p'', q'')$, $(p_0, q_0) = (p', q')$, and $\epsilon = (t'' - t')/(N + 1)$. In this expression one can preform the following three consecutive variable changes. For all j, one first lets $x_{j+1/2} \to x_{j+1/2} + (q_{j+1} + q_j)k_{j+1/2}/(p_{j+1} + p_j)$, followed by the substitution $x_{j+1/2} \to$

$-2x_{j+1/2}/(p_{j+1}+p_j)$, and finally one lets $k_{j+1/2} \to \frac{1}{2}(p_{j+1}+p_j)k_{j+1/2}$. Then the resulting regularized phase-space path integral is given, by

$$
\begin{aligned}
&K(p'',q'',t'';p',q',t')\\
&= \lim_{N\to\infty} \int \ldots \int \exp\left(i \sum_{j=0}^{N} \left\{ \frac{1}{2} k_{j+1/2}[(q_{j+1}+q_j)(p_{j+1}-p_j) \right.\right.\\
&+ (p_{j+1}+p_j)(q_{j+1}-q_j)]\\
&\left.\left. - x_{j+1/2}\frac{2(p_{j+1}-p_j)}{(p_{j+1}+p_j)} - \epsilon \mathcal{H}_\epsilon\left(\frac{1}{2}(p_{j+1}+p_j)k_{j+1/2}, x_{j+1/2}\right)\right\}\right)\\
&\times \prod_{j=1}^{N} dp_j dq_j \prod_{j=0}^{N} \frac{dk_{j+1/2}dx_{j+1/2}}{(2\pi)^2}.
\end{aligned}
$$

Therefore, taking an improper limit by interchanging the operation of integration with the limit with respect to N we find the following formal phase-space path integral representation for the representation independent propagator for the affine group:

$$
\begin{aligned}
&K(p'',q'',t'';p',q',t')\\
&= \mathcal{M} \int \exp\left\{ i \int [k(\dot{\overline{qp}}) - x(\dot{\overline{\ln p}}) - \mathcal{H}(pk, x)]dt \right\} \mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x,
\end{aligned}
$$

where $(\dot{\overline{\cdot}})$ denotes $d/dt(\ \cdot\ )$. This expression agrees with the one found in [18] up to a numerical factor M, given by $M = \langle \eta, |X_1|^{-1}\eta \rangle$, which is used in the normalization of the resolution of identity in the definition of coherent states for the affine group due to Aslaksen and Klauder [2]. We now formally evaluate the representation independent propagator for two soluble examples.

5.2.1. The Free Particle

Our first example is that of the free particle where $\mathcal{H}(X_1, X_2) = X_1^2/(2m)$. This Hamilton operator is clearly essentially self-adjoint. In this case the representation independent propagator becomes

$$
\begin{aligned}
&K(p'',q'',t'';p',q',t')\\
&= \mathcal{M} \int \exp\left\{ i \int \left[k(\dot{\overline{pq}}) - x(\dot{\overline{\ln p}}) - \frac{(pk)^2}{2m} \right] dt \right\} \mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x\\
&= \bar{\mathcal{N}} \int \exp\left\{ i \int \left[\frac{m}{2}\left(\frac{\dot{\overline{pq}}}{p}\right)^2 - x\left(\frac{\dot{p}}{p}\right)\right] dt \right\} \mathcal{D}q \prod_{t\in[t',t'']} \frac{dp(t)}{p(t)} \mathcal{D}x\\
&= \mathcal{N} \int \exp\left\{ i \int \left[\frac{m}{2}\left(\dot{q} + q\frac{\dot{p}}{p}\right)^2 \right] dt \right\} \delta\{\dot{p}\}\mathcal{D}q\mathcal{D}p
\end{aligned}
$$

Carrying out the remaining two integrations we obtain as our final result,

$$K(p'',q'',t'';p',q',t) = \sqrt{\frac{m}{2\pi i(t''-t')}}\ \delta(p''-p')\ \exp\left[\frac{im}{2(t''-t')}(q''-q')^2\right].$$

Observe that, up to the presence of the delta function $\delta(p''-p')$, this result is in perfect agreement with the usual result for the free particle, even though we only consider the positive or negative half of phase-space, *i.e.* p is constrained to be either positive or negative.

5.2.2. The Hamilton Operator $\mathcal{H}(X_1, X_2) = \frac{1}{2m}X_1^2 + \omega X_2$

The second example we consider is that of the Hamilton operator $\mathcal{H}(X_1, X_2) = X_1^2/2m + \omega X_2$. One easily shows that this Hamilton operator is essentially self-adjoint. The representation independent propagator takes the following form

$$\begin{aligned}
&K(p'',q'',t'';p',q',t') \\
&= \mathcal{M}\int \exp\left\{i\int_0^T \left[k(\dot{\overline{pq}}) - x(\dot{\overline{\ln p}}) - \frac{(pk)^2}{2m} - \omega x\right] dt\right\}\mathcal{D}p\mathcal{D}q\mathcal{D}k\mathcal{D}x \\
&= \bar{\mathcal{N}}\int \exp\left\{i\int_0^T \left[\frac{m}{2}\left(\frac{\dot{\overline{pq}}}{p}\right)^2 - x\left(\frac{\dot{p}}{p}+\omega\right)\right] dt\right\}\mathcal{D}q \prod_{t\in[t',t'']}\frac{dp(t)}{p(t)}\mathcal{D}x \\
&= \bar{\mathcal{N}}\int \exp\left\{i\int_0^T \left[\frac{m}{2}\left(\dot{q} + q\frac{\dot{p}}{p}\right)^2 - x\left(\frac{\dot{p}}{p}+\omega\right)\right] dt\right\}\mathcal{D}q \prod_{t\in[t',t'']}\frac{dp(t)}{p(t)}\mathcal{D}x \\
&= \mathcal{N}\int \exp\left\{i\int_0^T \left[\frac{m}{2}\left(\dot{q} + q\frac{\dot{p}}{p}\right)^2\right] dt\right\}\delta\{\dot{p}+\omega p\}\mathcal{D}q\mathcal{D}p \\
&= \delta\left(e^{\omega T/2}p'' - e^{-\omega T/2}p'\right)\mathcal{N}\int \exp\left\{i\int_0^T \left[\frac{m}{2}(\dot{q}-\omega q)^2\right] dt\right\}\mathcal{D}q,
\end{aligned}$$

where $T \equiv t'' - t'$. The final path integral we have to solve is a Lagrangian path integral for a quadratic Lagrangian which can be done using extremal methods; *see* [22]. The action for this Lagrangian path integral is given by

$$I_{cl} = \frac{m}{2}\int_0^T (\dot{q}-\omega q)^2 dt$$

variation of which yields the equation of motion

$$\ddot{q} = \omega^2 q,$$

which has the general solution

$$q(t) = A\sinh(\omega t) + B\cosh(\omega t).$$

After imposing the proper boundary conditions, one finds the evaluated classical action to be

$$S_{cl} = \frac{m\omega}{2\sinh(\omega T)}\{[(q'')^2 + (q')^2]\cosh(\omega T) - 2q''q'\} - \frac{m\omega}{2}[(q'')^2 - (q')^2].$$

So that our final result for the representation independent propagator with this Lagrangian becomes:

$$\begin{aligned} &K(p'', q'', t''; p', q', t') \\ &= \sqrt{\frac{m\omega}{2\pi i \sinh(\omega T)}}\,\delta\left(e^{\omega T/2}p'' - e^{-\omega T/2}p'\right) \\ &\times \exp\left(\frac{im\omega}{2\sinh(\omega T)}\{[(q'')^2 + (q')^2]\cosh(\omega T) - 2q''q'\}\right. \\ &\left. - \frac{im\omega}{2}[(q'')^2 - (q')^2]\right). \end{aligned}$$

Observe that the evaluated action functional in the exponent of this propagator, save for the term $-(m\omega/2)[(q'')^2 - (q')^2]$, agrees with the evaluated action functional one obtains for the propagator of the harmonic oscillator in imaginary time formulation although it is not of the same "physical origin".

6. CLASSICAL LIMIT OF THE REPRESENTATION INDEPENDENT PROPAGATOR

Even though the regularized lattice phase-space path integral representation for the representation independent propagator has been constructed by interpreting the appropriate Schrödinger equation (36) as a Schrödinger equation for *d separate and independent canonical* degrees of freedom, it should, nevertheless, be true that the classical limit for the representation independent propagator refers to the degree(s) of freedom associated with the Lie group G. In particular we will show that this is true for a general Lie group since the classical equations of motion obtained from the action functional for the representation independent propagator imply the classical equations of motion obtained from the most general classical action functional of the coherent state propagator for G.

6.1. Classical Limit of Non-Compact Lie groups

The discussion of the classical limit of compact semisimple Lie groups presented in [23] cannot be generalized to non-compact semisimple Lie groups, since they do not admit faithful unitary finite-dimensional representations (*cf.* [3, Corollary 8.1.4]). Hence, we must follow a different route to achieve a well defined classical limit of the most general action functional appropriate to a general non-compact Lie group G, given by

$$I = \|K^{1/2}\xi\|^{-2} \int \Big[i\hbar\Big\langle \xi(l), \frac{d}{dt}\xi(l)\Big\rangle - \langle\xi(l), \mathcal{H}(\overline{X}_1, \ldots, \overline{X}_d)\xi(l)\Big\rangle\Big]dt,$$

where $\xi(l) = U_{g(l)}K^{1/2}\xi$ and it is assumed that $K^{1/2}\xi \in \widetilde{\mathbf{D}}$ (*cf.* Eq. (15)). Without loss in generality we can set $\eta = K^{1/2}\xi/\|K^{1/2}\xi\|$, then our most general action functional becomes

$$I = \int \Big[i\hbar\Big\langle \eta(l), \frac{d}{dt}\eta(l)\Big\rangle - \langle\eta(l), \mathcal{H}(\overline{X}_1, \ldots, \overline{X}_d)\eta(l)\Big\rangle\Big]dt, \qquad (48)$$

where $\eta(l) = U_{g(l)}\eta$ and where it is assumed that $\eta \in \widetilde{\mathbf{D}} \cap \mathbf{D}(K^{-1/2})$.

In our discussion of the classical limit of the action functional given in (48) we use an abstract formalism for taking the $\hbar \to 0$ limit developed by Yaffe [27]. Yaffe [27] considers a family of quantum theories characterized by some parameter χ, such as $\hbar$, and studies the limit of these theories as χ approaches zero. It is assumed that *each* theory is defined on some Hilbert space $\mathbf{H}_\chi$ with some Hamilton operator $\mathcal{H}_\chi$. Furthermore, it is assumed that there exists a Lie group G, with associated Lie algebra L, that has on each Hilbert space $\mathbf{H}_\chi$ a unitary representation U^χ. We assume for definiteness that U^χ is parameterized as

$$U^\chi_{g(l)} = \prod_{k=1}^{d} \exp\left(\frac{-i}{\chi} l^k \overline{X}_k\right),$$

up to some ordering. Then the first assumption, which restricts the choice of the group, is

Assumption 1. – *Each unitary representation of G on $\mathbf{H}_\chi$ is irreducible.*
Hence, on each Hilbert space $\mathbf{H}_\chi$ one can define a set of coherent states $\eta_\chi(l) = U^\chi_{g(l)}\eta_\chi$. For any operator O acting on $\mathbf{H}_\chi$, we define the *upper symbol* $O_{\eta_\chi}(l)$ by

$$O_{\eta_\chi}(l) = \langle \eta_\chi(l), O\eta_\chi(l)\rangle \text{ for all } l \in \mathcal{L},$$

i.e. the upper symbol is a set of coherent state expectation values. The second assumption restricts the possible fiducial vectors η_χ one can choose. For each value of χ we require

Assumption 2. – *Zero is the only observable whose upper symbol identically vanishes.*

By an observable we mean a family of self-adjoint operators consisting of one self-adjoint operator acting in each Hilbert space $\mathbf{H}_\chi$. An example in which assumption 2 is not valid is given by SU(2) coherent states based on a fiducial vector that is not the highest (or lowest) weight vector, in this case a unique specification of any observable by its upper symbol may not be possible (*cf.* [17, p. 34]). Note that assumption 2 implies that two different operators cannot have the same upper symbol. Hence, one can uniquely recover any operator from its symbol. As pointed out in [27, p. 411], "this means that it is sufficient to study the behavior of the symbols of various operators in order to characterize the theory completely."

Observe that the $\chi \to 0$ limit of an arbitrary observable does not have to exist. In order to have some control over the $\chi \to 0$ limit one introduces the concept of a *classical observable.* According to Yaffe [27] an observable O is called a *classical observable* if the limits of its coherent state matrix elements exist,

$$\lim_{\chi \to 0} \frac{\langle \eta_\chi(l), O\eta_\chi(l') \rangle}{\langle \eta_\chi(l), \eta_\chi(l') \rangle},$$

and are finite for all $l, l' \in \mathcal{L}$. The set of all classical observables is denoted by $\mathcal{O}_c$. Clearly the set $\mathcal{O}_c$ is a subset of all possible observables, hence it is possible that measurements using only observables of $\mathcal{O}_c$ may fail to distinguish between different coherent states. Therefore, two different coherent states, $\eta_\chi(l)$ and $\eta_\chi(l')$ are called classically equivalent if for all $O \in \mathcal{O}_c$ one has

$$\lim_{\chi \to 0} \langle \eta_\chi(l), O\eta_\chi(l) \rangle = \lim_{\chi \to 0} \langle \eta_\chi(l'), O\eta_\chi(l') \rangle.$$

The third assumption states that classically inequivalent coherent states become orthogonal in the $\chi \to 0$ limit. In particular,

Assumption 3. – *The limit* $\Phi[\eta_\chi(l), \eta_\chi(l')] \equiv -\lim_{\chi \to 0} \chi \ln \langle \eta_\chi(l), \eta_\chi(l') \rangle$ *exists for all* $l, l' \in \mathcal{L}$ *and* $\Phi[\eta_\chi(l), \eta_\chi(l')]$ *satisfies the conditions* *(i)* $\Re\mathrm{e}\{\Phi[\eta_\chi(l), \eta_\chi(l')]\} > 0$ *if* $\eta_\chi(l)$ *and* $\eta_\chi(l')$ *are classically inequivalent.*

(ii) $\Re e\{\Phi[\eta_\chi(l), \eta_\chi(l')]\} = 0$ *if* $\eta_\chi(l)$ *and* $\eta_\chi(l')$ *are classically equivalent, and*

$$i\partial_t \left\{ \Phi\left[\eta_\chi(l), \exp\left(\frac{-i}{\chi} tX\right)\eta_\chi(l)\right] - \Phi\left[\eta_\chi(l), \exp\left(\frac{-i}{\chi} tX\right)\eta_\chi(l')\right] \right\}|_{t=0} = 0$$

$\forall X \in L$.

As shown in [27] assumption 3 implies that classical observables cannot "move" the coherent states. Hence, any fixed $U^\chi_{g(l)}$ cannot be a classical observable except $U^\chi_e = I_{\mathbf{H}_\chi}$. However, as shown in [27] assumption 3 implies that any $X \in L$ is an acceptable classical observable. Moreover, as pointed out in [27], assumption 3 implies that if $\eta_\chi(l)$ and $\eta_\chi(l')$ are classically equivalent then

$$\lim_{\chi\to 0} \frac{\langle \eta_\chi(l), O\eta_\chi(l')\rangle}{\langle \eta_\chi(l), \eta_\chi(l')\rangle} = \lim_{\chi\to 0} O_{\eta_\chi}(l) \text{ for all } O \in \mathcal{O}_c.$$

As shown in [27], this fact together with assumption 3 allows one to establish the following factorization for any pair of classical observables O and O':

$$\lim_{\chi\to 0}[(OO')_{\eta_\chi}(l) - O_{\eta_\chi}(l)O'_{\eta_\chi}(l)] = 0 \tag{49}$$

With these three assumptions one gains some control over the $\chi \to 0$ limit, however the quantum dynamics is left completely unrestricted. In order to gain complete control over the $\chi \to 0$ limit one has to require

Assumption 4. – $\mathcal{H}_\chi$ *is a classical observable.*
As shown in [27] this set of assumptions is sufficient to show that a quantum theory reduces to a classical theory as $\chi \to 0$.

We now discuss the classical limit of the action functional in (48). Since we are working with Lie groups that have irreducible square integrable representations assumption 1 is automatically satisfied. We assume that we have selected the fiducial vector η such that assumption 2 is satisfied and we also assume that assumption 3 is satisfied. To satisfy assumption 4 we restrict ourselves to Hamilton operators that are arbitrary polynomials of the generators $\{\overline{X}_k\}^d_{k=1}$. Then the most general classical action functional appropriate to the coherent state propagator for G is given by

$$I_{cl} = \lim_{\hbar\to 0} \int \left[i\hbar\left\langle \eta(l), \frac{d}{dt}\eta(l)\right\rangle - \langle \eta(l), \mathcal{H}(\overline{X}_1, ..., \overline{X}_d)\eta(l)\rangle \right] dt$$
$$= \int \left[\sum_{k,m=1}^{d} \lambda_m{}^k(g(l)) \dot{l}^m v_k - \mathcal{H}(\overline{X}_{k\eta}(l)) \right] dt$$

$$= \int \left[\sum_{k,m=1}^{d} \lambda_m{}^k(g(l))\dot{l}^m v_k - \mathcal{H}\left(\sum_{b=1}^{d} U_1{}^b(l)v_b, \ldots, \sum_{b=1}^{d} U_d{}^b(l)v_b \right) \right] dt, \tag{50}$$

where we have used (49) and (20). The $v_k \equiv \lim_{\hbar\to 0}\langle \eta, \overline{X}_k \eta\rangle$, $k = 1, \ldots, d$, are real constants.

Extremal variation of this action functional, with respect to the independent labels l^b, holding the end points fixed, yields the equations of motion

$$\sum_{b,s=1}^{d} v_s\{\partial_{l^c}\lambda_b{}^s(g(l)) - \partial_{l^b}\lambda_c{}^s(g(l))\}\dot{l}^b = \sum_{a,f=1}^{d} \mathcal{H}^a \partial_{l^c}[U_a{}^f(l)]v_f, \tag{51}$$

where $\mathcal{H}^a$ denotes the partial derivative of $\mathcal{H}$ with respect to the a-th argument; $a = 1, \ldots, d$.

Remark 6.1.1. – Generally the constants $v_1, ..., v_d$ are nonzero and are the *vestiges of the coherent state representation induced by* η *that remain even after the limit* $\hbar \to 0$ *has been taken.* ◇

6.2. Classical Limit of the Representation Independent Propagator

In the case of the representation independent propagator one identifies the classical action functional as (*see* Proposition 4.3.2)

$$\begin{aligned} I_{cl} &= \int \left[\sum_{j=1}^{d} p_j \dot{l}^j - \mathcal{H}(\check{x}_1(p,l), \ldots, \check{x}_d(p,l)) \right] dt \\ &= \int \left[\sum_{j=1}^{d} p_j \dot{l}^j - \mathcal{H}\left(\sum_{j=1}^{d} \rho^{-1}{}_1{}^j(g(l))p_j, \ldots, \sum_{j=1}^{d} \rho^{-1}{}_d{}^j(g(l))p_j \right) \right] dt. \end{aligned} \tag{52}$$

Varying this action functional holding the end points fixed yields the following set of equations of motion

$$\dot{l}^b = \sum_{a=1}^{d} \mathcal{H}^a \rho^{-1}{}_a{}^b(g(l)), \tag{53}$$

$$\dot{p}_c = -\sum_{a,j=1}^{d} \mathcal{H}^a \partial_{l^c}[\rho^{-1}{}_a{}^j(g(l))]p_j. \tag{54}$$

Substitution of $\mathcal{H}^s = \sum_{f=1}^d \rho_f{}^s(g(l))\dot{l}^f$ into (54), and contraction of both sides of the resulting relation with $\lambda^{-1}{}_h{}^c(g(l))$ yields:

$$\sum_{c=1}^{d} \lambda^{-1}{}_h{}^c(g(l))\dot{p}_c = \sum_{c,s=1}^{d} \sum_{f,j=1}^{d} \dot{l}^f \lambda^{-1}{}_h{}^c(g(l))\partial_{l^c}[\rho_f{}^s(g(l))]\rho^{-1}{}_s{}^j(g(l))\, p_j, \tag{55}$$

where

$$\sum_{s=1}^{d} \rho_f{}^s(g(l))\partial_{l^c}[\rho^{-1}{}_s{}^j(g(l))] = \sum_{s=1}^{d} -\partial_{l^c}[\rho_f{}^s(g(l))]\rho^{-1}{}_s{}^j(g(l))$$

has been used.

Claim. – *The following relation holds:*

$$\sum_{c,f,j,s=1}^{d} \dot{l}^f \lambda^{-1}{}_h{}^c(g(l))\partial_{l^c}[\rho_f{}^s(g(l))]\rho^{-1}{}_s{}^j(g(l))p_j = -\sum_{j,m=1}^{d} \partial_{l^m}[\lambda^{-1}{}_h{}^j(g(l))]\dot{l}^m p_j. \tag{56}$$

Proof. – To establish equation (56) it is sufficient to show that

$$\partial_{l^m}\lambda^{-1}{}_h{}^b(g(l)) = \sum_{f,s=1}^{d} \lambda^{-1}{}_h{}^f(g(l))\rho_m{}^s(g(l))\partial_{l^f}[\rho^{-1}{}_s{}^b(g(l))] \tag{57}$$

holds. Equation (57) can be rewritten as

$$\sum_{s=1}^{d} \lambda_n{}^s(g(l))\partial_{l^m}[\lambda^{-1}{}_s{}^b(g(l))] = \sum_{s=1}^{d} \rho_m{}^s(g(l))\partial_{l^n}[\rho^{-1}{}_s{}^b(g(l))],$$

which is the expression given in Lemma 3.2.2 (iii), and therefore, establishes (56). □

If one inserts (56) into (55) one finds

$$\frac{d}{dt}\left[\sum_{j=1}^{d} \lambda^{-1}{}_h{}^j(g(l))p_j\right] = 0. \tag{58}$$

Therefore, we can choose a set of integration constants, $c_1, ..., c_d$, such that

$$p_j = \sum_{m=1}^{d} \lambda_j{}^m(g(l))c_m. \tag{59}$$

Substitution of this form of p_j into (53) and (54), yields the following set of $2d$ equations

$$\dot{l}^b = \sum_{a=1}^{d} \mathcal{H}^a \left(\sum_{s=1}^{d} U_k{}^s(l) c_s \right) \rho^{-1}{}_a{}^b(g(l)),$$

$$\sum_{s=1}^{d} \partial_t [\lambda_c{}^s(g(l))] c_s$$

$$= - \sum_{a=1}^{d} \sum_{j,m=1}^{d} \mathcal{H}^a \left(\sum_{s=1}^{d} U_k{}^s(l) c_s \right) \partial_{l^c} [\rho^{-1}{}_a{}^j(g(l))] \lambda_j{}^m(g(l)) c_m.$$

After differentiation with respect to time these equations take the form

$$\dot{l}^b = \sum_{a=1}^{d} \mathcal{H}^a \rho^{-1}{}_a{}^b(g(l)), \tag{60}$$

$$\sum_{b,s=1}^{d} \partial_{l^b} [\lambda_c{}^s(g(l))] \dot{l}^b c_s$$

$$= - \sum_{a=1}^{d} \sum_{j,m=1}^{d} \mathcal{H}^a \partial_{l^c} [\rho^{-1}{}_a{}^j(g(l))] \lambda_j{}^m(g(l)) c_m. \tag{61}$$

Next contract (60) with $\sum_{s=1}^{d} \partial_{l^c} [\lambda_b{}^s(g(l))] c_s$ and find

$$\sum_{b,s=1}^{d} \partial_{l^c} [\lambda_b{}^s(g(l))] \dot{l}^b c_s$$

$$= \sum_{a=1}^{d} \sum_{b,s=1}^{d} \mathcal{H}^a \rho^{-1}{}_a{}^b(g(l)) \partial_{l^c} [\lambda_b{}^s(g(l))] c_s, \tag{62}$$

$$\sum_{b,s=1}^{d} \partial_{l^b} [\lambda_c{}^s(g(l))] \dot{l}^b c_s$$

$$= - \sum_{a=1}^{d} \sum_{j,m=1}^{d} \mathcal{H}^a \partial_{l^c} [\rho^{-1}{}_a{}^j(g(l))] \lambda_j{}^m(g(l)) c_m. \tag{63}$$

Subtracting (63) from (62) yields

$$\sum_{b,s=1}^{d} c_s \{ \partial_{l^c} \lambda_b{}^s(g(l)) - \partial_{l^b} \lambda_c{}^s(g(l)) \} \dot{l}^b = \sum_{a,f=1}^{d} \mathcal{H}^a \partial_{l^c} [U_a{}^f(l)] c_f, \tag{64}$$

where Lemma 3.2.1 has been used. Among all possible allowed values of $c_1, \ldots, c_d$ are those that coincide with $v_1, \ldots, v_d$ for an arbitrary fiducial vector. Hence, for this choice of $c_1, \ldots, c_d$ the above equations coincide with the equations of motion obtained from the most general classical action functional for the coherent propagator for G [*see* Eq. (51)]. Therefore, the set of classical equations of motion obtained from the classical action functional of the representation independent propagator implies the set of classical equations of motion obtained from the most general classical action functional of the coherent state propagator for G. Thus, we find that the set of solutions of the representation independent classical equations of motion appropriate to the representation independent propagator for a general Lie group G with square integrable, irreducible representations includes *every possible solution* of the classical equations of motion appropriate to the most general coherent state propagator for G. We summarize all this in the following Proposition:

PROPOSITION 6.2.1. – *Let G be a real, separable, locally compact, connected and simply connected Lie group whose unitary irreducible representations are square integrable. If the fiducial vector satisfies Assumption 2 then the equations of motion obtained from the action functional of the representation independent propagator imply the equations of motion obtained from the most general classical action functional for the coherent state propagator for G.*

7. CONCLUSION AND OUTLOOK

In this section, as before, we mean by a general Lie group a real, separable, locally compact, connected and simply connected Lie group with irreducible, square integrable unitary representations, unless we explicitly state otherwise. We have focused our attention in this paper on general Lie groups since for this case the existence of a resolution of identity is guarantied in general by (23) and we were able to construct a representation independent propagator rigorously. It would be interesting to see if the construction of the representation independent propagator presented in section 4.4 can be extended to Lie groups that do not posses square integrable, irreducible representations, such as the Euclidian group. The obstacle one has to overcome when one considers such groups is the introduction of a resolution of identity. This problem has recently been solved by Isham and Klauder [13] for the n-dimensional Euclidian group $E(n)$. In [13] attention is focused on *reducible*, square integrable representations of $E(n)$. In this case it becomes possible to introduce a set of coherent states, *i.e.* to establish a resolution of identity, and to introduce a coherent state propagator.

Therefore, if the problem of introducing coherent states for these groups can be solved, then one can use the following argument to introduce a *fiducial vector independent propagator* for these groups. Denote by U a generic, continuous, unitary representation of a Lie group G on some Hilbert space **H**, which does not need to be a square integrable, irreducible representation. For definiteness let us assume that we have parameterized the Lie group G such that the representation U is given by:

$$U_{g(l)} = \exp(-il^1\overline{X}_1)\dots\exp(-il^d\overline{X}_d),$$

for some ordering, where the $\overline{X}_1, \dots, \overline{X}_d$ form an integrable, representation of the associated Lie algebra L of G by essentially self-adjoint operators on some common dense invariant domain $\widetilde{\mathbf{D}} \in \mathbf{H}$ and where l is an element of a d-dimensional parameter space $\mathcal{G}$. Let us denote by $\eta(l)$ the coherent states associated with the representation $U_{g(l)}$ of G, where $\eta \in \mathbf{H}$ is the fixed, normalized fiducial vector. Let us furthermore assume, that these states give rise to a resolution of identity

$$I_{\mathbf{H}} = \int_{\mathcal{G}} \eta(l)\langle\eta(l), \cdot\rangle d\mu(l),$$

where $d\mu(l)$ denotes the normalized, left invariant group measure given by

$$d\mu(l) = \frac{1}{|G|}dg(l),$$

where $1/|G|$ denotes the normalization; for the definition of $dg(l)$ *see* (26). We can now use this set of coherent states to give a continuous representation of **H**. We define the map

$$\begin{aligned} \mathcal{C}_\eta : \mathbf{H} &\to L^2(G, d\mu(l)) \\ \psi &\mapsto [\mathcal{C}_\eta\psi](l) = \psi_\eta(l) \equiv \langle\eta(l), \psi\rangle. \end{aligned}$$

Which as we know yields a representation of **H** by bounded, continuous, square integrable functions on the closed subspace $L^2_\eta(G, d\mu(l))$ of $L^2(G, d\mu(l))$.

We now introduce the *fiducial vector independent propagator* $K_{\mathbf{H}}(l'', t''; l', t')$ as follows, it is a single, (possibly generalized) function that is independent of any particular choice of the fiducial vector, which, nevertheless, propagates the ψ_η correctly, *i.e.*,

$$\psi_\eta(l,t) = \int_{\mathcal{G}} K_{\mathbf{H}}(l,t;l',t')\psi_\eta(l',t')d\mu(l') \tag{65}$$

If equation (65) is to hold for arbitrary η, we must require that

$$\lim_{t\longrightarrow t'} K_{\mathbf{H}}(l,t;l',t') = |G|\delta_e(g^{-1}(l)g(l')), \tag{66}$$

where $\delta_e(g^{-1}(l)g(l'))$ is defined in (31).

An analysis of our results presented in section 4 shows that Lemma 4.1.1, Corollary 4.1.2 hold for *reducible*, square integrable representations. Even though we have stated Lemma 4.1.1 and Corollary 4.1.2 for irreducible representations, this property of the representation is not used in the proofs of these results (*see* [23]), hence these results also apply to the case when one considers reducible representations. Therefore, it is a direct consequence of Lemma 4.1.1 (i) that for any $\psi \in \widetilde{\mathbf{D}}$

$$\tilde{x}_k(-i\nabla_l, l)[\mathcal{C}_\eta\psi](l) = [\mathcal{C}_\eta\overline{X}_k\psi](l), \qquad k = 1,\dots,d,$$

holds independently of η. Hence, we find that $\mathcal{C}_\eta$ intertwines the representation of the Lie algebra L associated with G on $\mathbf{H}$, with a subrepresentation of L by right invariant, essentially self-adjoint differential operators on any one of the reproducing kernel Hilbert spaces $L^2_\eta(G, d\mu(l))$.

Denote by $\mathcal{H}(\overline{X}_k)$ the essentially self-adjoint Hamilton operator of a quantum system on $\mathbf{H}$. Then the continuous representation of Schrödinger's equation on $\mathbf{H}$, $i\partial_t\psi(t) = \mathcal{H}\psi(t)$, takes on $L^2_\eta(G, d\mu(l))$, the following form

$$\begin{aligned} i\partial_t\psi_\eta(l,t) &= [\mathcal{C}_\eta\mathcal{H}(\overline{X}_k)\psi(t)](l) \\ &= \mathcal{H}(\tilde{x}_k(-i\nabla_l, l))\psi_\eta(l,t). \end{aligned}$$

Using (65) we find that the fiducial vector independent propagator $K_{\mathbf{H}}$ is a solution to this Schrödinger equation, *i.e.*

$$i\partial_t K_{\mathbf{H}}(l,t;l',t') = \mathcal{H}(\tilde{x}_k(-i\nabla_l, l))K_{\mathbf{H}}(l,t;l',t').$$

Therefore, together with equation (66), we find the following initial value problem

$$\begin{aligned} i\partial_t K_{\mathbf{H}}(l,t;l',t') &= \mathcal{H}(\tilde{x}_k(-i\nabla_l, l))K_{\mathbf{H}}(l,t;l',t'), \\ \lim_{t\longrightarrow t'} K_{\mathbf{H}}(l,t;l',t') &= |G|\delta_e(g^{-1}(l)g(l')). \end{aligned} \tag{67}$$

It was such an initial value problem that we have taken as our starting point for the path integral formulation of the representation independent propagator. Hence, we find that one can, using Proposition 4.3.2, introduce path integral representations for general Lie groups that have reducible square integrable representations. However, observe that we can only introduce a *fiducial vector independent propagator* in this case. This program of has been explicitly carried out for the case of $E(2)$ by Tulsian and Klauder [26].

Observe that if one considers groups with *reducible*, square integrable representations one has to proceed on a case by case basis since a general theory in this case is lacking. It would therefore, be of some interest to see if the theory developed by Duflo and Moore [8] for locally compact groups with irreducible, square integrable, unitary representations can be extended to locally compact groups with *reducible*, square integrable, unitary representations.

In our opinion another interesting avenue to achieve the path integral quantization of the form (43) for general Lie groups that do not have square integrable, irreducible representations would be to start form the classical mechanics associated with the particular Lie group one considers and to try to derive the form of the action functional we have arrived at in Proposition 4.3.2. The quantization would then be achieved by postulating (43) as the path integral quantization for these kinds of Lie groups.

Furthermore, we believe that the representation independent propagator holds considerable interest for quantum field theory. We have used in this paper the word representation independent in a dual meaning, its first meaning pertained to the fact that the representation independent propagator is independent of the choice of the fiducial vector and its second meaning to the fact that this propagator is also independent of the choice of the unitary, irreducible representation of the Lie group G. In the case of quantum field theory these two meanings of the word representation independent are inextricably related, since the *dynamics* chooses a representation for the basic kinematical variables (*cf.* [12, pp. 56-57] and [17, pp. 82-83]). We therefore, believe that it would be a worthwhile task to extend our concept of a representation independent propagator into the realm of quantum field theory.

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