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Phase space properties of local observables and structure of scaling limits

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ABSTRACT. – For any given algebra of local observables in relativistic quantum field theory there exists an associated scaling algebra which permits one to introduce renormalization group transformations and to construct the scaling (short distance) limit of the theory. On the basis of this result it is discussed how the phase space properties of a theory determine the structure of its scaling limit. Bounds on the number of local degrees of freedom appearing in the scaling limit are given which allow one to distinguish between theories with classical and quantum scaling limits. The results can also be used to establish physically significant algebraic properties of the scaling limit theories, such as the split property.

RÉSUMÉ. – A toute algèbre d'observables locales d'une théorie de champs relativistes, est associée une algèbre d'échelles permettant d'introduire des transformations de groupe de renormalisation et de construire la limite d'échelle à courte distance de la théorie. Sur la base de ce résultat, nous discutons comment les propriétés d'espace des phases de la théorie déterminent sa limite d'échelle. Nous donnons des bornes sur le nombre des degrés de liberté locaux apparaissant dans cette limite, lesquelles permettent de discerner entre théories ayant des limites d'échelle classiques et quantiques. Ces résultats peuvent aussi être utilisés pour établir les propriétés algébriques physiquement significatives des théories de limite d'échelle, comme par exemple la « propriété de séparation » (split property).

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1. INTRODUCTION

The general analysis of the structure of local observables at small spatio–temporal scales is in several respects an interesting issue. It is of relevance in the classification of the possible ultraviolet properties of local quantum field theories and a prerequisite for the proper description of the particle–like structures appearing at small scales, such as quarks and gluons. Moreover, the short distance analysis is crucial to the understanding of symmetries, such as colour, which come to light only at small scales, and it may be a key to the reconstruction of the local gauge groups from the (gauge invariant) observables.

A general framework for the systematic analysis of these problems has recently been proposed in [1]. It is based on ideas of renormalization group theory (*cf.* [2] and references quoted there) which are incorporated into the setting of local quantum physics by the novel concept of scaling algebra. Within this framework one can construct in a model independent way the scaling (short distance) limit of any given theory and analyze its structure.

It is the aim of the present article to review this approach and to study the relation between the structure of the scaling limit and the phase space properties of the underlying theory. This analysis is carried out in the algebraic Haag-Kastler framework of local quantum theory [3]. Since this setting may be less well-known, we briefly describe here in conventional field theoretic terms the basic ideas and main results of the present investigation.

Given any local quantum field theory, we consider the corresponding observable Wightman fields, currents etc. which act as operator–valued distributions on the physical Hilbert space. Since we are interested in the short distance properties of the theory, we have to study the effect of a change of the spatio–temporal scale on the observables, while keeping the scales c of velocity and $\hbar$ of action fixed. In the field theoretic setting such a change of scale induces transformations of the underlying observable fields. If $\phi(x)$ is such a (hermitean) field which, at the original scale, is localized at the space–time point x, then the corresponding field at other scales is obtained by setting

$$\phi_\lambda(x) \doteq N_\lambda \, \phi(\lambda x), \tag{1.1}$$

where $\lambda > 0$ is a scaling factor. We call $\phi_\lambda(x)$ the field at scale λ. Whereas the action of the scaling transformations on the argument of the field needs no explanation, its effect on the scale of field strength, given by the positive factor N_λ, is more subtle. The familiar idea is to adjust this factor in such a way that the expectation values of the fields at scale λ in some given reference state are of equal order of magnitude for all $\lambda > 0$. The precise way in which this idea is implemented is a matter of convention. One may integrate for example $\phi_\lambda(x)$ with a suitable (real) test function $f(x)$,

$$\phi_\lambda(f) \doteq \int d^4x f(x)\, \phi_\lambda(x), \tag{1.2}$$

and demand that

$$(\Omega,\, \phi_\lambda(f)\phi_\lambda(f)\, \Omega) = \text{const} \quad \text{for} \quad \lambda > 0, \tag{1.3}$$

where Ω is the vacuum vector. Yet one could impose such a constraint just as well on higher moments of $\phi_\lambda(f)$. By conditions of this type one can determine the factor N_λ and thereby adjust the scale of field strength. We denote the factor N_λ which has been fixed by such a renormalization condition by Z_λ.

For the analysis of the theory at small scales one has to consider the n–point correlation functions of the fields at scale λ and to proceed to the scaling limit $\lambda \searrow 0$. There appears, however, a problem. As is well known, the product of quantum fields at neighbouring space–time points is quite singular and consequently the renormalization factors Z_λ tend to 0 in this limit. One needs rather precise information on the way how Z_λ approaches 0 in order to be able to control the scaling limit of the fields. It is apparent that if one chooses in relation (1.1) a factor N_λ such that the quotient N_λ / Z_λ approaches 0 or ∞ in the scaling limit one ends up with a senseless result.

In the conventional setting of quantum field theory this problem can be solved under favourable circumstances (asymptotically free theories), since renormalization group equations and perturbative methods provide reliable information on the asymptotic behaviour of Z_λ. This approach works also in some renormalizable theories where the underlying renormalization group equations have a non–vanishing but small ultraviolet fixed point. Yet in the case of theories without a (small) ultraviolet fixed point the method does not lead to reliable results, nor can a rigorous treatment of non–renormalizable theories be even addressed. In view of this fact a model–independent approach to the short distance analysis of local quantum field theories would seem to be impossible.

A way out of this problem which is quite simple has been proposed in [1]. Within the field–theoretical setting it can be described as follows. In a first step one proceeds from the unbounded field operators

$\phi_\lambda(f)$ to corresponding bounded operators, such as the unitary operators $\exp(i\phi_\lambda(f))$. This has the effect that, irrespective of the choice of the factor N_λ in the definition of $\phi_\lambda(x)$, there do not appear any divergence problems: the resulting operators are bounded in norm, uniformly in λ.

The second crucial step is to restrict the four–momentum of these bounded operators in accord with the uncertainty principle. Roughly speaking, one considers only operators which can transfer to physical states at scale λ momentum proportional to λ^{-1}, hence they occupy for all $\lambda > 0$ the same phase space volume. The desired operators are obtained by suitable space–time averages,

$$A_\lambda \doteq \int d^4y\, g(y) \exp(i\phi_\lambda(f_y)), \quad \lambda > 0, \tag{1.4}$$

where $g(y)$ is any test function and $f_y(x) \doteq f(x-y)$. It turns out that this restriction on the momentum transfer has the following effect: if one chooses N_λ such that N_λ/Z_λ tends to ∞ in the limit of small λ, then all correlation functions involving the corresponding sequence of operators A_λ converge to 0. Similarly, if N_λ/Z_λ tends to 0, then A_λ converges (in the sense of correlation functions) to const $\cdot\, 1$. Thus in either case the operators A_λ tend in the scaling limit to multiples of the identity. Only in the special case where the asymptotic behaviour of N_λ coincides with that of Z_λ does it happen that the correlation functions retain a non-trivial operator content in the scaling limit.

In view of this fact one does not need to know the behaviour of Z_λ and may admit in the above construction all possible factors N_λ. The theory takes care by itself of those choices which are unreasonable and lets them disappear in the scaling limit. It is only if N_λ has the right asymptotic behaviour that the sequences A_λ give rise to non-trivial operator limits. One may view this method as an implicit way of introducing renormalization group transformations. It allows one to study the short distance properties of local quantum field theories in a model independent manner.

It is convenient in this analysis to regard the operators A_λ obtained by the above procedure as functions of the scaling parameter λ. These functions form in an obvious way an algebra, the scaling algebra. For the construction of the scaling limit of the theory one considers the expectation values of sums and products of the operator functions in the limit $\lambda \searrow 0$,

$$\lim_{\lambda \searrow 0} (\Omega, \sum A_\lambda A_\lambda^{'} \cdots A_\lambda^{''} \Omega) = (\Omega_0, \sum A_0 A_0^{'} \cdots A_0^{''} \Omega_0). \tag{1.5}$$

Actually, these limits may only exist for suitable subsequences of the scaling parameter λ. We disregard this problem for the moment but return to it in the main text. What is of interest here is the fact that the limits of the correlation functions determine, by an application of the reconstruction theorem to the scaling algebra, a local, covariant theory with unique vacuum vector Ω_0. This explains the notation on the right hand side of equation (1.5). By this universal method one can construct the scaling limit of any given theory.

On the basis of this result the possible structure of scaling limit theories has been classified in [1]. There are two extreme cases. The first possibility is that the scaling limit theory is "classical". This happens if the correlations between all observables disappear at small scales and the correlation functions in (1.5) factorize in the limit. Examples may be certain non–renormalizable theories, where the leading short distance singularities of the fields are not governed by the two point functions (*cf.* also the remarks in the conclusions). In the second, more familiar case the quantum correlations persist in the scaling limit. One then ends up with a full–fledged quantum field theory.

We want to clarify in this article the relation between the nature of the scaling limit and the phase space properties of the underlying theory and establish criteria for deciding which of the above cases is realized. Although phase space is a poorly defined concept in quantum field theory, one can introduce a measure of its size by appealing to a semiclassical picture and counting the number of states of limited energy which are localized in a fixed spacetime region. To illustrate this idea, let us consider for given spacetime region $\mathcal{O}$ and any $\lambda > 0$ the vectors

$$\int d^4y \, g(y) \, \exp\left(i\phi_\lambda(f_y)\right) \Omega, \quad \operatorname{supp} f \subset \mathcal{O}. \tag{1.6}$$

Here g is a fixed test function whose Fourier transform has support in a given region $\widetilde{\mathcal{O}}$ of momentum space. These vectors describe local excitations of the vacuum vector Ω in the region $\lambda\mathcal{O}$ whose energy momentum content is confined to $\lambda^{-1}\widetilde{\mathcal{O}}$. In the following we denote the set of all these vectors by $\mathcal{S}_\lambda$.

It has been pointed out by Haag and Swieca [4] that, disregarding vectors of small norm, the subsets $\mathcal{S}_\lambda$ of the physical Hilbert space should be finite dimensional in physically reasonable theories. A convenient measure of the size of these sets is provided by the notion of epsilon content. This is the number $N_\lambda(\varepsilon)$ of vectors in $\mathcal{S}_\lambda$ whose mutual distance is larger than a given $\varepsilon > 0$. Thus the epsilon content $N_\lambda(\varepsilon)$ provides information on the number of states which are affiliated with the regions $\lambda\mathcal{O}$, $\lambda^{-1}\widetilde{\mathcal{O}}$ of configuration and momentum space. We mention as an aside that the dependence of these numbers on ε and the given regions is intimately related to thermal properties of the underlying theory [5, 6].

As we shall see, the structure of the scaling limit of a theory depends crucially on the size of the epsilon contents $N_\lambda(\varepsilon)$ in the limit $\lambda \searrow 0$. Disregarding fine points, there are the following possibilities: if the limit $N_0(\varepsilon)$ of the epsilon contents behaves for small ε like ε^{-p} for some $p > 0$, then the scaling limit is classical. Otherwise it is a quantum field theory. Moreover, if $N_0(\varepsilon)$ behaves like $\exp(\varepsilon^{-q})$ for some sufficiently small $q > 0$, then the scaling limit theory satisfies a nuclearity condition, proposed in [5], which is a sharpened version of the Haag–Swieca criterion. This result can be used to establish a strong form of causal independence (split property [7]) in the scaling limit theories. On the other hand, if the epsilon contents $N_\lambda(\varepsilon)$ diverge in the limit of small λ, then the scaling limit theory no longer complies with the condition of Haag and Swieca.

Further results of a similar nature which reveal an intimate connection between the number of degrees of freedom affiliated with specific regions of phase space and the short distance properties of a theory will be given in the main text. One may hope that these results provide the basis for an extensive analysis and classification of the possible ultraviolet properties of local observables.

We conclude this introduction with a brief summary. In the subsequent section we recall the basic notions used in the algebraic approach to local quantum physics and compile some results which enter in our discussion of phase space properties. Section 3 contains a review of the construction of scaling algebras and scaling limits. The heart of the paper is Section 4, where the analysis of the relation between phase space and short distance properties is given. The paper closes with a brief discussion of examples and an outlook on further developments of the theory.

2. PHASE SPACE PROPERTIES OF LOCAL OBSERVABLES

As mentioned in the Introduction, we make use in this investigation of the algebraic framework of local quantum physics [3]. This allows us to discuss the short distance properties of local observables in full generality, including also gauge theories, where one considers besides point like fields other observables, such as Wilson loops and string operators. In order to establish our terminology we list in the first part of this section the basic assumptions which ought to be satisfied by the observables in any physically reasonable theory. In the second part we recall some mathematical concepts and results which are of relevance in our discussion of phase space properties.

1. *(Locality)* The observables of the underlying theory generate a *local net* over Minkowski space $\mathbb{R}^4$, that is an inclusion preserving map $\mathcal{O} \rightarrow \mathfrak{A}(\mathcal{O})$ from the set of open, bounded regions $\mathcal{O} \subset \mathbb{R}^4$ to unital C*-algebras $\mathfrak{A}(\mathcal{O})$ on the physical Hilbert space $\mathcal{H}$. Thus each $\mathfrak{A}(\mathcal{O})$ is a norm closed subalgebra of the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ which is stable under taking adjoints and contains the unit operator, and there holds

$$\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2. \tag{2.1}$$

The net complies with the principle of locality (Einstein causality) according to which observables in spacelike separated regions commute,

$$\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2') \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2'. \tag{2.2}$$

Here $\mathcal{O}'$ denotes the spacelike complement of $\mathcal{O}$ and $\mathfrak{A}(\mathcal{O})'$ the set of operators in $\mathcal{B}(\mathcal{H})$ which commute with all operators in $\mathfrak{A}(\mathcal{O})$. One may think of $\mathfrak{A}(\mathcal{O})$ as the algebra generated by all observables which can be measured in the spacetime region $\mathcal{O}$. The global algebra $\mathfrak{A}$ of observables is generated by all local algebras $\mathfrak{A}(\mathcal{O})$ (as inductive limit in the norm topology). We recall that for the interpretation of a theory it is not really necessary to assign a specific physical meaning to individual operators. All what matters is the information about the localization properties of the operators, which is encoded in the net structure [3].

2. *(Covariance)* On the Hilbert space $\mathcal{H}$ there exists a continuous unitary representation U of the space–time translations $\mathbb{R}^4$ which induces automorphisms of the given net of observables[1]. Thus for each $x \in \mathbb{R}^4$ there is an $\alpha_x \in \mathrm{Aut}\mathfrak{A}$ given by

$$\alpha_x(A) \doteq U(x)AU(x)^{-1}, \quad A \in \mathfrak{A}, \tag{2.3}$$

and, in an obvious notation,

$$\alpha_x(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}(\mathcal{O} + x) \tag{2.4}$$

for any region $\mathcal{O}$. In addition to this fundamental postulate we assume that the operator valued functions

$$x \rightarrow \alpha_x(A), \qquad A \in \mathfrak{A} \tag{2.5}$$

[1] We make no assumptions with regard to Lorentz transformations. Thus the present framework is slightly more general than the one used in [1].

are continuous in the norm topology. This condition, which is crucial in the present investigation, is always satisfied by a sufficiently rich set of local observables and does not impose any essential restrictions of generality.

3. *(Spectrum condition)* The joint spectrum of the energy–momentum operators, *i.e.*, the generators of the unitary representation U of the translations, is contained in the closed forward lightcone $\overline{V}_+ = \{p \in \mathbb{R}^4 : p_0 \geq |\boldsymbol{p}|\}$. Moreover, there is an (up to a phase unique) unit vector $\Omega \in \mathcal{H}$, representing the vacuum, which is invariant under the action of the representation U,

$$U(x)\Omega = \Omega, \quad x \in \mathbb{R}^4. \tag{2.6}$$

Let us turn now to the description of the phase space properties of the observables. There we have to rely on concepts from the theory of compact linear maps between Banach spaces. We recall here some basic definitions and useful results.

Let $\mathcal{E}$ be any Banach space with norm $\|\cdot\|_{\mathcal{E}}$. The unit ball of $\mathcal{E}$ is denoted by $\mathcal{E}_{\mathrm{I}}$ and the space of continuous linear functionals on $\mathcal{E}$ by $\mathcal{E}^*$. Given another Banach space $\mathcal{F}$, we denote the space of continuous linear maps L from $\mathcal{E}$ to $\mathcal{F}$ by $\mathcal{L}(\mathcal{E}, \mathcal{F})$. The latter space is again a Banach space with norm given by

$$\|L\| \doteq \sup\{\|L(E)\|_{\mathcal{F}} : E \in \mathcal{E}_{\mathrm{I}}\}. \tag{2.7}$$

A map $L \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ is said to be compact if the image of $\mathcal{E}_{\mathrm{I}}$ under the action of L has compact closure in $\mathcal{F}$. A convenient measure which provides more detailed information on the size of the range of compact maps is the notion of epsilon content.

DEFINITION. – *Let $L \in \mathcal{L}(\mathcal{E}, \mathcal{F})$ and let, for given $\varepsilon > 0$, $N_L(\varepsilon)$ be the maximal number of elements $E_i \in \mathcal{E}_{\mathrm{I}}$, $i = 1, \ldots N_L(\varepsilon)$, such that $\|L(E_i - E_j)\| > \varepsilon$ for $i \neq j$. The number $N_L(\varepsilon)$ is called epsilon content of L. It is finite for all $\varepsilon > 0$ if and only if L is compact [8].*

It is apparent that the epsilon content of L increases if ε decreases, and it tends to infinity if ε approaches 0 (unless L is the zero map). If L is of finite rank, then the epsilon content $N_L(\varepsilon)$ behaves for small ε like ε^{-p} for some positive number p. The converse statement is also true [8].

We are now in a position to formulate the condition proposed by Haag and Swieca [4] to characterize physically significant theories with decent phase space properties. We state this condition here in a slightly modified but mathematically equivalent form.

4. *(Compactness)* For given $\beta > 0$ and spacetime region $\mathcal{O}$, let $\Theta_{\beta,\mathcal{O}}$ be the map from $\mathfrak{A}(\mathcal{O})$ into $\mathcal{H}$, defined by

$$\Theta_{\beta,\mathcal{O}}(A) \doteq e^{-\beta H} A\Omega, \quad A \in \mathfrak{A}(\mathcal{O}), \tag{2.8}$$

where H is the (positive) generator of the time translations. The maps $\Theta_{\beta,\mathcal{O}}$ are compact for any $\beta > 0$ and any bounded $\mathcal{O}$.

For physical motivations of this condition, *cf.* [4] and [5]. The compactness condition has rigorously been established in several models. Examples are massive [4] and massless [9] free field theories and certain interacting theories in two spacetime dimensions, such as the $P(\phi)_2$ models, the Yukawa theory Y_2 and theories exhibiting solitons, *cf.* [10, Sec. 4] and references quoted there. In fact, any theory which has the so–called split property also satisfies the Haag–Swieca compactness condition [10]. On the other hand one knows that the compactness condition is violated in theories with an unreasonably large number of local degrees of freedom, such as generalized free fields with continuous mass spectrum [4]. Thus the maps $\Theta_{\beta,\mathcal{O}}$ are a convenient tool to characterize the phase space properties of a theory. For a discussion of related concepts and their comparison with the compactness condition, *cf.* [10] and [11].

An important class of compact maps which enter in quantitative versions of the compactness condition, proposed in [5] and [10], are the so–called nuclear maps. They are defined as follows.

DEFINITION. – *Let $L \in L(\mathcal{E}, \mathcal{F})$ be a map such that for suitable sequences $e_n \in \mathcal{E}^*$ and $F_n \in \mathcal{F}$, $n \in \mathbb{N}$, there holds (in the sense of strong convergence in $\mathcal{F}$)*

$$L(E) = \sum_n e_n(E) F_n, \quad E \in \mathcal{E}.$$

If there holds in addition $\sum_n \|e_n\|^p_{\mathcal{E}^*} \|F_n\|^p_{\mathcal{F}} < \infty$ for some $p > 0$ the map L is said to be *p–nuclear*. The p–nuclear maps form a vector space which is equipped with the (quasi) norm [8]

$$\|L\|_p = \inf \left(\sum_n \|e_n\|^p_{\mathcal{E}^*} \|F_n\|^p_{\mathcal{F}} \right)^{1/p},$$

where the infimum is to be taken with respect to all possible decompositions of L. A map which is p–nuclear for all $p > 0$ is said to be of *type s*.

It has been argued in [10] that, disregarding theories with a maximal (Hagedorn) temperature, the maps $\Theta_{\beta,\mathcal{O}}$ in the compactness condition ought to be of type s, *i.e.*,

$$\|\Theta_{\beta,\mathcal{O}}\|_p < \infty \quad \text{for} \quad p > 0. \tag{2.9}$$

In the present investigation we will extract from the dependence of the p–norms on β and $\mathcal{O}$ information on the nature of the scaling limit of the theory. In the argument we make use of the fact that the epsilon content and the p–norms of nuclear maps are closely related. We quote in this context the following useful result.

LEMMA 2.1. – *Let $L \in \mathcal{L}(\mathcal{E}, \mathcal{F})$, where $\mathcal{F}$ is a Hilbert space.*

(i) If L is p–nuclear for some $0 < p < 1$, its epsilon content satisfies for any $q > p/(1-p)$

$$N_L(\varepsilon) \leq \exp\left(c\,\|L\|_p^q/\varepsilon^q\right), \quad \varepsilon > 0,$$

where the constant c depends on p, q, but not on L.

(ii) Conversely, if for some $0 < p \leq 1$ there is a sequence $\varepsilon_m > 0, m \in \mathbb{N}$, such that

$$\sum_m \left(m^{1/2}\varepsilon_m N_L(\varepsilon_m)^{1/m}\right)^p < \infty,$$

the map L is p–nuclear and

$$\|L\|_p \leq (2\pi)^{1/2}\Big(\sum_m \left(m^{1/2}\varepsilon_m N_L(\varepsilon_m)^{1/m}\right)^p\Big)^{1/p}.$$

(iii) If, for some $0 < q < 2/3$, there holds

$$\sup_{\varepsilon>0} \varepsilon\big(\ln N_L(\varepsilon)\big)^{1/q} < \infty,$$

the map L is p–nuclear for $p > 2q/(2-q)$ and

$$\|L\|_p \leq c \sup_{\varepsilon>0} \varepsilon\big(\ln N_L(\varepsilon)\big)^{1/q},$$

where c depends on p, q, but not on L.

The arguments for the proof of these statements can be extracted from Proposition 2.5 and Lemma 2.2 in [13]. We refrain from giving here the straightforward details.

3. SCALING ALGEBRAS AND SCALING LIMITS

We review in this section the construction of the scaling algebra associated with any given local net of observables which complies with the first three conditions given in Sec. 2. As has been shown in [1], the scaling algebra provides a convenient tool for the definition and analysis of the scaling limit of a theory.

The elements of the scaling algebra are bounded functions of the scaling parameter $\lambda > 0$ with values in the algebra of observables $\mathfrak{A}$,

$$\underline{A} : \mathbb{R}^+ \to \mathfrak{A}. \tag{3.1}$$

We mark these functions in the following by underlining. A simple example of such a function has been given in the Introduction, *cf.* relation (1.4). As has been discussed in [1], the values $\underline{A}_\lambda$ of the functions $\underline{A}$ are, for given $\lambda > 0$, to be interpreted as elements of the theory at scale λ. One therefore defines for these functions the following algebraic relations: for any $\underline{A}, \underline{B}$ and $a, b \in \mathbb{C}$ one puts for $\lambda > 0$

$$\begin{aligned} (a\underline{A} + b\underline{B})_\lambda &\doteq a\underline{A}_\lambda + b\underline{B}_\lambda \\ (\underline{A} \cdot \underline{B})_\lambda &\doteq \underline{A}_\lambda \cdot \underline{B}_\lambda \\ (\underline{A}^*)_\lambda &\doteq \underline{A}_\lambda^{\,*}. \end{aligned} \tag{3.2}$$

In this way the functions $\underline{A}$ acquire the structure of a *-algebra with unit given by $\underline{1}_\lambda = 1$. A norm on this algebra (which in fact is a C*–norm) is obtained by setting

$$||\underline{A}|| \doteq \sup_{\lambda > 0} ||\underline{A}_\lambda||. \tag{3.3}$$

The translations $x \in \mathbb{R}^4$ induce an action $\underline{\alpha}_x$ on the functions $\underline{A}$, given by

$$(\underline{\alpha}_x(\underline{A}))_\lambda \doteq \alpha_{\lambda x}(\underline{A}_\lambda). \tag{3.4}$$

One considers only functions $\underline{A}$ which satisfy with respect to this action the continuity condition

$$||\underline{\alpha}_x(\underline{A}) - \underline{A}|| \to 0 \quad \text{for} \quad x \to 0. \tag{3.5}$$

This crucial requirement amounts to specific restrictions on the four-momentum of the operators $\underline{A}_\lambda, \lambda > 0$, which are suggested by the basic ideas of renormalization group theory [1]. The effect of this restriction has been described in the Introduction in the case of a simple example (*cf.* the remarks after relation (1.4)). In [1] it was assumed that also the Lorentz transformations act norm–continuously on $\underline{A}$. But we do not make such an assumption here and consider the larger class of functions satisfying only condition (3.5).

The local structure of the underlying net of observables induces a corresponding local structure on the functions $\underline{A}$. One defines for any open, bounded spacetime region $\mathcal{O} \subset \mathbb{R}^4$ the subset $\underline{\mathfrak{A}}(\mathcal{O})$ of continuous (in the sense of condition (3.5)) functions given by

$$\underline{\mathfrak{A}}(\mathcal{O}) \doteq \{\underline{A} : \underline{A}_\lambda \in \mathfrak{A}(\lambda\mathcal{O}) \quad \text{for} \quad \lambda > 0\}. \tag{3.6}$$

Since each $\mathfrak{A}(\lambda\mathcal{O})$ is a C*-algebra, it follows that $\underline{\mathfrak{A}}(\mathcal{O})$ is a C*-algebra as well. Moreover, there holds

$$\underline{\mathfrak{A}}(\mathcal{O}_1) \subset \underline{\mathfrak{A}}(\mathcal{O}_2) \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2, \tag{3.7}$$

thus the assignment $\mathcal{O} \to \underline{\mathfrak{A}}(\mathcal{O})$ defines a net of C*–algebras over Minkowski space. The C*–inductive limit of this net is denoted by $\underline{\mathfrak{A}}$ and called *scaling algebra*. It is a straightforward consequence of conditions (2.2) and (2.4) that the net $\mathcal{O} \to \underline{\mathfrak{A}}(\mathcal{O})$ is local,

$$\underline{\mathfrak{A}}(\mathcal{O}_1) \subset \underline{\mathfrak{A}}(\mathcal{O}_2)' \quad \text{if} \quad \mathcal{O}_1 \subset \mathcal{O}_2{}', \tag{3.8}$$

(where, by abuse of notation, we have used the symbol $\underline{\mathfrak{A}}(\mathcal{O})'$ for the relative commutant of $\underline{\mathfrak{A}}(\mathcal{O})$ in $\underline{\mathfrak{A}}$) and covariant,

$$\underline{\alpha}_x(\underline{\mathfrak{A}}(\mathcal{O})) = \underline{\mathfrak{A}}(\mathcal{O} + x). \tag{3.9}$$

The local, covariant net[2] $\underline{\mathfrak{A}}, \underline{\alpha}_{\mathbb{R}^4}$ is called *scaling net* of the underlying theory.

In this general formalism one can describe changes of the spatio–temporal scale by an automorphic action $\underline{\sigma}_{\mathbb{R}^+}$ of the multiplicative group $\mathbb{R}^+$ on the scaling algebra $\underline{\mathfrak{A}}$. It is given for any $\mu > 0$ by

$$(\underline{\sigma}_\mu(\underline{A}))_\lambda \doteq \underline{A}_{\lambda\mu}, \quad \lambda > 0. \tag{3.10}$$

As is easily verified, there hold the relations

$$\begin{gathered} \underline{\sigma}_\mu \circ \underline{\alpha}_x = \underline{\alpha}_{\mu x} \circ \underline{\sigma}_\mu \\ \underline{\sigma}_\mu(\underline{\mathfrak{A}}(\mathcal{O})) = \underline{\mathfrak{A}}(\mu\mathcal{O}), \end{gathered} \tag{3.11}$$

[2] As is common practice, we denote the net and its inductive limit by the same symbol.

which reveal the geometrical significance of the scaling transformations $\underline{\sigma}_{\mathbb{R}+}$. These automorphisms may be understood as the algebraic version of renormalization group transformations [1].

The scaling net comprises in a comprehensive manner information about the underlying theory at all spatio–temporal scales. Moreover, it allows one to *compare* the properties of the theory at different scales because of the connection between the respective observables established by the functions $\underline{A}$. This connection is not as rigid as in the conventional approach to the renormalization group, *cf.* relation (1.1). But it contains sufficient information for the physical interpretation of the theory at arbitrarily small spatio–temporal scales.

Within the setting of the scaling algebra the short distance analysis of physical states is performed as follows: if ω is any given physical state on the algebra of observables $\mathfrak{A}$ (e.g., the vacuum state $\omega(\cdot) = (\Omega, \cdot\, \Omega)$) one defines its lift $\underline{\omega}_\lambda$ to the scaling algebra $\underline{\mathfrak{A}}$ at given scale $\lambda > 0$ according to

$$\underline{\omega}_\lambda(\underline{A}) \doteq \omega(\underline{A}_\lambda), \quad \underline{A} \in \underline{\mathfrak{A}}. \tag{3.12}$$

The functionals $\underline{\omega}_\lambda$ are states on the net $\underline{\mathfrak{A}}, \underline{\alpha}_{\mathbb{R}^4}$, from which one can recover the properties of the given state ω at scale $\lambda > 0$, *cf.* [1, Prop. 3.4]. Moreover the formalism allows one to proceed to the scaling limit $\lambda \searrow 0$. To this end one has to regard the family of states $\underline{\omega}_\lambda, \lambda > 0$, as a net directed towards $\lambda = 0$ and to study its limit behaviour.

As has been discussed in [1], there appears a minor technical problem: the net $\underline{\omega}_\lambda, \lambda > 0$, does not converge since the scaling algebra comprises the orbits of local observables arising from an abundance of admissible renormalization group transformations (*cf.* the freedom of choosing N_λ in the example (1.4)). But, being a bounded set of functionals on the Banach space $\underline{\mathfrak{A}}$, the net always contains subnets which converge in the weak-*-topology according to the Banach-Alaoglu theorem. We recall that the latter statement means that there exist states $\underline{\omega}_0$ on $\underline{\mathfrak{A}}$ such that for any given finite set of elements $\underline{A}^{(n)} \in \underline{\mathfrak{A}}, n = 1, \dots N$, one can find some sequence $\lambda_m, m \in \mathbb{N}$, tending to 0, such that

$$\lim_{m \to \infty} \underline{\omega}_{\lambda_m}(\underline{A}^{(n)}) = \underline{\omega}_0(\underline{A}^{(n)}) \quad \text{for} \quad n = 1, \dots N. \tag{3.13}$$

The set of all scaling limit states $\underline{\omega}_0$ on $\underline{\mathfrak{A}}$ which arises from the given state ω on $\mathfrak{A}$ by this construction, is denoted by $\mathrm{SL}(\omega)$. We mention as an aside that $\mathrm{SL}(\omega)$ does not depend on the choice of ω within the class of physically admissible states [1, Cor. 4.2].

Although the set $\mathrm{SL}(\omega)$ contains many elements, the apparent ambiguities in the definition of the scaling limit disappear in general if one takes into account the proper interpretation of the states $\underline{\omega}_0$, inherited from the states $\underline{\omega}_\lambda$ at finite scales $\lambda > 0$. The procedure is as follows. Given $\underline{\omega}_0 \in \mathrm{SL}(\omega)$ one applies the GNS–reconstruction theorem, yielding a representation π_0 of $\underline{\mathfrak{A}}$ on some Hilbert space $\mathcal{H}_0$ and a cyclic vector Ω_0 such that

$$(3.14) \qquad \underline{\omega}_0(\underline{A}) = (\Omega_0, \pi_0(\underline{A})\,\Omega_0), \quad \underline{A} \in \underline{\mathfrak{A}}.$$

It has been shown in [1, Lem. 4.3] that any $\underline{\omega}_0 \in \mathrm{SL}(\omega)$ is a pure vacuum state on $\underline{\mathfrak{A}}$. Hence there exists on $\mathcal{H}_0$ a continuous unitary representation U_0 of the space-time translations $\mathbb{R}^4$ such that for $\underline{A} \in \underline{\mathfrak{A}}$ there holds

$$(3.15) \qquad U_0(x)\pi_0(\underline{A})U_0(x)^{-1} = \pi_0(\underline{\alpha}_x(\underline{A})), \quad x \in \mathbb{R}^4.$$

Moreover, the joint spectrum of the generators of U_0 is contained in the closed forward lightcone $\overline{V}_+$ and Ω_0 is the (up to a phase) unique unit vector in $\mathcal{H}_0$ which is invariant under the action of U_0,

$$(3.16) \qquad U_0(x)\Omega_0 = \Omega_0.$$

From the latter fact it follows that the algebra $\pi_0(\underline{\mathfrak{A}})$ is irreducible.

For the physical interpretation of $\underline{\omega}_0$ one proceeds to the corresponding net

$$(3.17) \qquad \mathcal{O} \to \mathfrak{A}_0(\mathcal{O}) \doteq \pi_0(\underline{\mathfrak{A}}(\mathcal{O}))$$

which is local and covariant with respect to the automorphic action $\alpha^{(0)}_{\mathbb{R}^4}$ of the space-time translations, given by

$$(3.18) \qquad \alpha^{(0)}_x(\cdot) \doteq U_0(x) \,\cdot\, U_0(x)^{-1}, \quad x \in \mathbb{R}^4.$$

It has been argued in [1] that the nets $\mathfrak{A}_0, \alpha^{(0)}_{\mathbb{R}^4}$ obtained in this way for different choices of $\underline{\omega}_0 \in \mathrm{SL}(\omega)$ ought to describe the same physics in generic cases, *i.e.*, they should be isomorphic. One can then regard any one of these nets as the *unique* scaling limit of the underlying theory. But it may also happen that the nets $\mathfrak{A}_0, \alpha^{(0)}_{\mathbb{R}^4}$ are non–isomorphic in certain theories for different choices of $\underline{\omega}_0 \in \mathrm{SL}(\omega)$. This situation will occur if the underlying theory cannot be described at small scales in terms of a single theory since its structure varies continually if one approaches $\lambda = 0$. The theory is then said to have a *degenerate* scaling limit.

In either case the states $\underline{\omega}_0 \in \mathrm{SL}(\omega)$ give rise to pure vacuum states ω_0 on the corresponding nets $\mathfrak{A}_0$, $\alpha^{(0)}_{\mathbb{R}^4}$, which are given by

$$\omega_0(\cdot) = (\Omega_0, \cdot\ \Omega_0). \tag{3.19}$$

They describe the properties of the underlying state ω on $\mathfrak{A}$ in the scaling limit. Thus each triple $\mathfrak{A}_0, \alpha^{(0)}_{\mathbb{R}^4}, \Omega_0$ complies with the general conditions imposed on a net of observables in Sec. 2, with the possible exception of the compactness condition.

We are interested here in the general structure of the scaling limit theories. As has been mentioned in the Introduction, there exist two clear-cut alternatives which follow from the fact that the scaling limit states are pure vacuum states. Given $\underline{\omega}_0 \in \mathrm{SL}(\omega)$ one has as a first possibility:

(i) The net $\mathfrak{A}_0$ fixed by $\underline{\omega}_0$ consists only of multiples of the identity.

If this case is at hand for every choice of $\underline{\omega}_0 \in \mathrm{SL}(\omega)$ the theory is said to have a *classical* scaling limit. For in such theories all correlations between observables disappear at small scales. In theories with a degenerate scaling limit it may happen, however, that only some of the states in $\mathrm{SL}(\omega)$ lead to nets which belong to the preceding class. The second possibility is:

(ii) The net $\mathfrak{A}_0$ associated with $\underline{\omega}_0$ is non-trivial in the sense that the algebras corresponding to (sufficiently large) bounded regions are infinite dimensional and non-commutative.

That the latter case is the only alternative to the former can be seen in many ways. It follows for example from the following lemma which we quote for later reference. Its proof is based on standard arguments.

LEMMA 3.1. – *Let U be a continuous unitary representation of $\mathbb{R}^4$ on a Hilbert space $\mathcal{H}$ which satisfies the relativistic spectrum condition, let $\Omega \in \mathcal{H}$ be an (up to a phase unique) unit vector which is invariant under the action of U and let A be a bounded operator on $\mathcal{H}$ such that $[A^*, U(x)AU(x)^{-1}] = 0$ for x varying in some open set of $\mathbb{R}^4$. There are the following two alternatives.*

(i) $A\Omega = a\,\Omega$ for some $a \in \mathbb{C}$.

*(ii) For any $r > 0$ the linear span of the vectors $U(x)A\Omega$, $|x| < r$, is infinite dimensional. Then the same holds true for the *-algebra generated by the operators $U(x)AU(x)^{-1}$, $|x| < r$, and this algebra is non–commutative if r is sufficiently large.*

Proof. – It follows from the spectrum condition by an argument of the Reeh–Schlieder type that the closure $\mathcal{K}$ of the linear span of vectors $U(x)A\Omega$, $|x| < r$, coincides with that of the vectors $U(x)A\Omega$, $x \in \mathbb{R}^4$. Thus

$\mathcal{K}$ is invariant under the action of U. If $\mathcal{K}$ is finite dimensional one can see by an application of the spectral theorem to the unitary group $U(\mathbb{R}^4)|\mathcal{K}$ that

$$(3.20) \qquad (A\Omega,\, U(x)A\Omega) = \sum_n a_n e^{ip_n x}, \quad x \in \mathbb{R}^4.$$

Here the sum is finite and $p_n \in \overline{V}_+$ because of the spectrum condition. By the commutation properties of A and the invariance of Ω under the action of U, this function coincides with $(A^*\Omega,\, U(-x)A^*\Omega)$ for x varying in some open set. The latter function extends, because of the spectrum condition, to an analytic function on the backward tube $\mathbb{R}^4 - iV_+$ and it is bounded there. Since the former function is entire analytic the edge of the wedge theorem implies that the two functions coincide on the backward tube. Hence, in view of their boundedness, there holds $p_n = 0$, i.e., the functions are constant. Consequently $U(x)A\Omega = A\Omega$ for $x \in \mathbb{R}^4$, and taking into account the uniqueness of the invariant vector Ω one arrives at the conclusion that $A\Omega = a\,\Omega$ for some $a \in \mathbb{C}$. This is case (i).

The alternative is that $\mathcal{K}$ is infinite dimensional. Then the *-algebra generated by the operators $U(x)AU(x)^{-1}$, $|x| < r$, is infinite dimensional as well since the subspace obtained by applying this algebra to Ω contains $\mathcal{K}$. Moreover, if this algebra were commutative for every choice of $r > 0$, the commutator $[A^*,\, U(x)AU(x)^{-1}]$ would vanish for all $x \in \mathbb{R}^4$ and consequently the functions $x \to (A\Omega,\, U(x)A\Omega)$ and $x \to (A^*\Omega,\, U(-x)A^*\Omega)$ would coincide. Since the Fourier transforms of these functions have support in $\overline{V}_+$ and $-\overline{V}_+$, respectively, they would have to be constant and consequently $A\Omega = a\,\Omega$ for some $a \in \mathbb{C}$. Thus $\mathcal{K}$ would be one–dimensional, which is a contradiction. □

Since any local operator A complies with the assumption in this lemma and since the scaling limit states are pure and hence (in their superselection sector) unique vacuum states, we see that there are only the two general possibilities for the nets $\mathfrak{A}_0$, mentioned above. If the second case is at hand for *every* choice of $\underline{\omega}_0 \in \mathrm{SL}(\omega)$ we say the theory has a *pure quantum* scaling limit (which may be degenerate, though).

The preceding classification of the possible structure of the scaling limit arises as a logical alternative within the general setting of the theory of local observables. Yet it does not shed any light on the question as to which case is at hand in a given theory. This point will be clarified in the subsequent section.

4. PHASE SPACE AND SCALING LIMIT

We turn now to the analysis of the relation between the phase–space properties of a theory and the nature of its scaling limit. As was explained in Sec. 3, the phase space properties can be described in terms of the maps $\Theta_{\beta,\mathcal{O}}$, defined in relation (2.8), which depend on the choice of a parameter $\beta > 0$ and a bounded spacetime region $\mathcal{O} \subset \mathbb{R}^4$. Throughout this section we assume that these maps are compact and denote their epsilon contents by $N_{\beta,\mathcal{O}}(\varepsilon), \varepsilon > 0$.

We also consider the analogous maps in the scaling limit theories: let $\underline{\omega}_0 \in \mathrm{SL}(\omega)$, let $\mathfrak{A}_0, \alpha^{(0)}_{\mathbb{R}^4}$ be the corresponding covariant net and let $\Omega_0 \in \mathcal{H}_0$ be the corresponding vacuum vector. Given $\beta, \mathcal{O}$ we define a map $\Theta^{(0)}_{\beta,\mathcal{O}}$ from $\mathfrak{A}_0(\mathcal{O})$ into $\mathcal{H}_0$, setting

$$\Theta^{(0)}_{\beta,\mathcal{O}}(A) \doteq e^{-\beta H_0} A\Omega_0, \quad A \in \mathfrak{A}_0(\mathcal{O}), \tag{4.1}$$

where H_0 denotes the generator of the time translations in the scaling limit theory, *cf.* relation (3.15). The epsilon content of this map is denoted by $N^{(0)}_{\beta,\mathcal{O}}(\varepsilon), \varepsilon > 0$, provided it is finite.

It is our aim to derive information on the properties of the maps $\Theta^{(0)}_{\beta,\mathcal{O}}$ from the structure of the maps $\Theta_{\beta,\mathcal{O}}$ in the underlying theory. In a first preparatory step we pick a test function f_β on $\mathbb{R}$ whose Fourier transform $\widetilde{f_\beta}$ is equal to $(2\pi)^{-1/2} e^{-\beta\omega}$ for $\omega \geq 0$ and arbitrary otherwise. Since the time translations $\underline{\alpha}_t$ act norm continuously on the scaling algebra $\underline{\mathfrak{A}}$ it follows that the integrals (in the sense of Bochner)

$$\underline{\alpha}_{f_\beta}(\underline{A}) \doteq \int dt\, f_\beta(t)\, \underline{\alpha}_t(\underline{A}), \quad \underline{A} \in \underline{\mathfrak{A}} \tag{4.2}$$

are elements of the scaling algebra $\underline{\mathfrak{A}}$. Each $\underline{\omega}_0 \in \mathrm{SL}(\omega)$ is a weak-*-limit point of the family of states $\underline{\omega}_\lambda, \lambda > 0$, which are the lifts of the vacuum state $\omega(\cdot) = (\Omega, \cdot\, \Omega)$ on $\mathfrak{A}$ to the scaling algebra. Hence, recalling relation (3.13), there exists for each *finite* set of elements $\underline{A} \in \underline{\mathfrak{A}}$ some sequence $\lambda_m, m \in \mathbb{N}$, tending to 0, such that

$$\begin{aligned} ||e^{-\beta H_0}\pi_0(\underline{A})\Omega_0||^2 &= ||\pi_0(\underline{\alpha}_{f_\beta}(\underline{A}))\Omega_0||^2 \\ &= \underline{\omega}_0(\underline{\alpha}_{f_\beta}(\underline{A})^*\underline{\alpha}_{f_\beta}(\underline{A})) = \lim_m \underline{\omega}_{\lambda_m}(\underline{\alpha}_{f_\beta}(\underline{A})^*\underline{\alpha}_{f_\beta}(\underline{A})) \\ &= \lim_m ||\underline{\alpha}_{f_\beta}(\underline{A})_{\lambda_m}\Omega||^2 = \lim_m ||e^{-\lambda_m\beta H}\underline{A}_{\lambda_m}\Omega||^2, \end{aligned} \tag{4.3}$$

where in the first and last equality we made use of the spectrum condition and the specific form of f_β.

Relation (4.3) will be the key to the proof of the subsequent lemmas. In the argument we make also use of the following well–known fact in the theory of C*-algebras, *cf.* for example [14, Ch. I.8]: if $\mathfrak{B}$ is a unital C*-algebra and π some representation of $\mathfrak{B}$ there holds

$$\pi(\mathfrak{B})_{\mathrm{I}} = \pi(\mathfrak{B}_{\mathrm{I}}), \tag{4.4}$$

where the subscript I denotes the unit ball in the respective algebra.

LEMMA 4.1. – *The epsilon contents of the maps* $\Theta_{\beta,\mathcal{O}}$ *and* $\Theta^{(0)}_{\beta,\mathcal{O}}$ *satisfy*

$$N^{(0)}_{\beta,\mathcal{O}}(\varepsilon) \leq \limsup_{\lambda \searrow 0} N_{\lambda\beta,\lambda\mathcal{O}}(\varepsilon), \quad \varepsilon > 0.$$

Proof – Let $A^{(n)} \in \mathfrak{A}_0(\mathcal{O})_{\mathrm{I}}, n = 1,\dots N$, be such that there holds $||\Theta^{(0)}_{\beta,\mathcal{O}}(A^{(n')} - A^{(n'')})|| > \varepsilon$ for $n' \neq n''$. As $\mathfrak{A}_0(\mathcal{O})_{\mathrm{I}} = \pi_0(\underline{\mathfrak{A}}(\mathcal{O}))_{\mathrm{I}} = \pi_0(\underline{\mathfrak{A}}(\mathcal{O})_{\mathrm{I}})$, there exist N elements $\underline{A}^{(n)} \in \underline{\mathfrak{A}}(\mathcal{O})_{\mathrm{I}}$, such that $A^{(n)} = \pi_0(\underline{A}^{(n)}), n = 1,\dots N$, and consequently

$$||e^{-\beta H_0}\pi_0(\underline{A}^{(n')} - \underline{A}^{(n'')})\Omega_0|| > \varepsilon \quad \text{if} \quad n' \neq n''.$$

Since we are only dealing with a finite number of elements of $\underline{\mathfrak{A}}(\mathcal{O})$ we can apply relation (4.3) and find that there is some number m_0 such that for all $m \geq m_0$ and $n', n'' = 1,\dots N$ there holds

$$||e^{-\lambda_m \beta H}(\underline{A}^{(n')}_{\lambda_m} - \underline{A}^{(n'')}_{\lambda_m})\Omega|| > \varepsilon \quad \text{if} \quad n' \neq n''.$$

But $\underline{A}^{(n)}_\lambda \in \mathfrak{A}(\lambda\mathcal{O})_{\mathrm{I}}, \lambda > 0$, hence we see from the latter inequality that N cannot be larger than the epsilon content of the maps $\Theta_{\lambda_m\beta,\lambda_m\mathcal{O}}, m \geq m_0$. The statement then follows. □

In the next step we establish a lower bound on the epsilon content of the maps $\Theta^{(0)}_{\beta,\mathcal{O}}$ in the scaling limit theories.

LEMMA 4.2. – *Let* $\mathcal{O}_0$ *be any spacetime region whose closure is contained in the interior of* $\mathcal{O}$ *and let* $\beta_0 > \beta$. *The epsilon contents of the maps* $\Theta^{(0)}_{\beta,\mathcal{O}}$ *and* $\Theta_{\beta,\mathcal{O}}$ *satisfy*

$$\liminf_{\lambda \searrow 0} N_{\lambda\beta_0,\lambda\mathcal{O}_0}(\varepsilon) \leq N^{(0)}_{\beta,\mathcal{O}}(\varepsilon), \quad \varepsilon > 0.$$

Remark. – It is an open problem whether this result persists if one requires that the elements of the scaling algebra are also norm–continuous with respect to the action of Lorentz transformations.

Proof. – Let $\varepsilon > 0$ and let N be any natural number which is less than or equal to the limes inferior of the numbers $N_{\lambda\beta_0,\lambda\mathcal{O}_0}(\varepsilon)$ for λ tending to 0. Accordingly there exist a $\lambda_0 > 0$ and, for any given $0 < \lambda < \lambda_0$, N operators $A_\lambda^{(n)} \in \mathfrak{A}(\lambda\mathcal{O}_0)_1, n = 1, \ldots N$, such that

$$||e^{-\lambda\beta H}(A_\lambda^{(n')} - A_\lambda^{(n'')})\Omega|| > \varepsilon \quad \text{if} \quad n' \neq n''.$$

We pick now a suitable non-negative, integrable function f on $\mathbb{R}^4$ which has support in a sufficiently small neighbourhood of the origin 0 such that $\mathcal{O}_0 + \text{supp} f \subset \mathcal{O}$, in an obvious notation. Moreover, $\int d^4x\, f(x) = 1$, and the Fourier transform of f has to satisfy the lower bound

$$|\widetilde{f}(p)| \geq (2\pi)^{-2}\, e^{-(\beta_0-\beta)p_0} \quad \text{for} \quad p \in \overline{V}_+. \tag{4.5}$$

(Such functions f can be obtained from the elementary function $\mathbb{R} \ni u \to l_r(u)$, where $l_r(u) = (1/2r)\ln(r/|u|)$ for $|u| < r$ and $l_r(u) = 0$ for $|u| \geq r$, setting $f(x) \doteq \prod_{\nu=0}^3 l_r(x_\nu)$ and choosing r sufficiently small.) With the help of this function and the operators $A_\lambda^{(n)}$ we can define elements $\underline{A}^{(n)}, n = 1, \ldots N$, of the scaling algebra $\underline{\mathfrak{A}}$ according to

$$\underline{A}_\lambda^{(n)} \doteq \int d^4x\, f(x)\alpha_{\lambda x}(A_\lambda^{(n)}) \quad \text{for} \quad 0 < \lambda \leq \lambda_0,$$

and $\underline{A}_\lambda^{(n)} = 0$ for $\lambda > \lambda_0$. Since f is absolutely integrable and the operators $A_\lambda^{(n)}$ are bounded in norm by 1 each $\underline{A}^{(n)}$ satisfies the continuity requirement (3.5). In fact,

$$||\underline{\alpha}_x(\underline{A}^{(n)}) - \underline{A}^{(n)}|| \leq \int d^4x'\, |f(x'-x) - f(x')|.$$

Moreover, because of the localization and normalization properties of f as well as of the operators $A_\lambda^{(n)}$, there holds $\underline{A}_\lambda^{(n)} \in \mathfrak{A}(\lambda\mathcal{O})_1, \lambda > 0$, and consequently $\underline{A}^{(n)} \in \underline{\mathfrak{A}}(\mathcal{O})_1, n = 1, \ldots N$. We apply once again relation (4.3), giving for $n', n'' = 1, \ldots N, n' \neq n''$, and some suitable sequence $\lambda_m, m \in \mathbb{N}$, tending to 0,

$$\begin{aligned}
&||e^{-\beta H_0}\pi_0(\underline{A}^{(n')} - \underline{A}^{(n'')})\Omega_0|| \\
&= \lim_m ||e^{-\lambda_m\beta H}(\underline{A}_{\lambda_m}^{(n')} - \underline{A}_{\lambda_m}^{(n'')})\Omega|| \\
&= \lim_m ||(2\pi)^2\, \widetilde{f}(\lambda_m P)e^{-\lambda_m\beta H}(A_{\lambda_m}^{(n')} - A_{\lambda_m}^{(n'')})\Omega|| \\
&\geq \liminf_{\lambda\searrow 0} ||e^{-\lambda\beta_0 H}(A_\lambda^{(n')} - A_\lambda^{(n'')})\Omega|| > \varepsilon.
\end{aligned}$$

Here P is the four–momentum operator and we made use of the lower bound (4.5) imposed on $\widetilde{f}$ on the forward lightcone $\overline{V}_+$, which contains the spectrum of P. From the above estimate and the fact that $\pi(\underline{A}^{(n)}) \in \mathfrak{A}_0(\mathcal{O})_1, n = 1, \dots N$, we see that $N \leq N^{(0)}_{\beta,\mathcal{O}}(\varepsilon)$. This completes the proof of the statement. □

Making use of these results we will establish now conditions in terms of the epsilon contents $N_{\beta,\mathcal{O}}(\varepsilon), \varepsilon > 0$, which provide information on the nature of the scaling limit. In order to simplify the notation we set for $\varepsilon > 0$

$$
\begin{aligned}
\underline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon) &\doteq \liminf_{\lambda \searrow 0} N_{\lambda\beta,\lambda\mathcal{O}}(\varepsilon) \\
\overline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon) &\doteq \limsup_{\lambda \searrow 0} N_{\lambda\beta,\lambda\mathcal{O}}(\varepsilon),
\end{aligned} \tag{4.6}
$$

provided the respective limits exist. In these definitions we made use of the obvious fact that the limits depend on $\beta, \mathcal{O}$ in the combination $\beta^{-1}\mathcal{O}$. The information contained in the preceding two lemmas can thus be summarized in the inequalities

$$
\underline{N}_{\beta_0^{-1}\mathcal{O}_0} \leq N^{(0)}_{\beta,\mathcal{O}}(\varepsilon) \leq \overline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon), \quad \varepsilon > 0, \tag{4.7}
$$

which will be used in the proof of the following proposition. For the sake of simplicity we do not aim here at an optimal result. (*Cf.* the proof of Proposition 4.5 for certain refinements.)

PROPOSITION 4.3. – *The following conditions are necessary, respectively sufficient, for the underlying theory to have a scaling limit of the given type.*

(i) Pure quantum scaling limit: it is necessary that there is some bounded region $\mathcal{O}$ such that for any $p > 0$ there holds $\varepsilon^p\, \overline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon) \to \infty$ as $\varepsilon \searrow 0$. It is sufficient that $\varepsilon^2\, \underline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon) \to \infty$ as $\varepsilon \searrow 0$.

(ii) Scaling limit of Haag-Swieca type (all triples $\mathfrak{A}_0, \alpha^{(0)}_{\mathbb{R}^4}, \Omega_0$ arising from states in $SL(\omega)$ satisfy the Haag-Swieca compactness condition): it is necessary that $\underline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon) < \infty, \varepsilon > 0$, and sufficient that $\overline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon) < \infty, \varepsilon > 0$, for all bounded regions $\mathcal{O}$.

(iii) Classical scaling limit: it is necessary that, for each bounded spacetime region $\mathcal{O}$, $\varepsilon^2\, \underline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon) <$ const as $\varepsilon \searrow 0$. It is sufficient that there is some $p > 0$ such that $\varepsilon^p\, \overline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon) <$ const as $\varepsilon \searrow 0$.

Proof. – In the proof of these statements we make use of the fact that if L is a (non–zero) compact linear map between Banach spaces with epsilon content $N_L(\varepsilon), \varepsilon > 0$, then $\varepsilon^p\, N_L(\varepsilon) <$ const as $\varepsilon \searrow 0$ for some $p > 0$

if and only if L has finite rank. In particular, $\varepsilon^2 N_L(\varepsilon) < \text{const}$ as $\varepsilon \searrow 0$ if and only if L has rank 1 as a complex linear map. The proof of the if–part of these statements is straightforward, the only–if–part is an easy consequence of the results in [8, Sec. 9.6].

(i) If the theory has a pure quantum scaling limit there exists for each triple $\mathfrak{A}_0, \alpha_{\mathbb{R}^4}^{(0)}, \Omega_0$ a bounded spacetime region $\mathcal{O}$ such that the space of vectors $\mathfrak{A}_0(\mathcal{O})\Omega$ is infinite dimensional, *cf.* Lemma 3.1. Since $e^{-\beta H_0}$ is invertible, the space $e^{-\beta H_0}\mathfrak{A}_0(\mathcal{O})\Omega$ is then infinite dimensional as well, thus the map $\Theta_{\beta,\mathcal{O}}^{(0)}$ has infinite rank. Consequently $\varepsilon^p N_{\beta,\mathcal{O}}^{(0)}(\varepsilon) \to \infty$ as $\varepsilon \searrow 0$ for any finite $p > 0$. Because of relation (4.7) this proves the necessity of the given condition. On the other hand, if for some region $\mathcal{O}_0$ there holds $\varepsilon^2 \underline{N}_{\beta_0^{-1}\mathcal{O}_0}(\varepsilon) \to \infty$ as $\varepsilon \searrow 0$, relation (4.7) implies that $\varepsilon^2 N_{\beta,\mathcal{O}}^{(0)}(\varepsilon) \to \infty$ as $\varepsilon \searrow 0$, provided $0 < \beta < \beta_0$ and $\mathcal{O}$ contains the closure of $\mathcal{O}_0$ in its interior. Thus the map $\Theta_{\beta,\mathcal{O}}^{(0)}$ has rank larger than 1. From Lemma 3.1 and the covariance of the net $\mathfrak{A}_0, \alpha_{\mathbb{R}^4}^{(0)}$ it then follows that the maps $\Theta_{\beta,\mathcal{O}_1}^{(0)}$ have infinite rank for any region $\mathcal{O}_1$ containing the closure of $\mathcal{O}$ in its interior. This proves the sufficiency of the given condition.

(ii) The theory has a scaling limit of Haag–Swieca type if and only if the maps $\Theta_{\beta,\mathcal{O}}^{(0)}$ are compact for all $\beta > 0$ and bounded regions $\mathcal{O}$, *i.e.*, if and only if $N_{\beta,\mathcal{O}}^{(0)}(\varepsilon) < \infty, \varepsilon > 0$. The statement therefore follows from relation (4.7).

(iii) If the theory has a classical scaling limit all maps $\Theta_{\beta,\mathcal{O}}^{(0)}$ are of rank 1 and consequently $\varepsilon^2 N_{\beta,\mathcal{O}}^{(0)}(\varepsilon) < \text{const}$ as $\varepsilon \searrow 0$. The necessity of the given condition then follows from relation (4.7). On the other hand, if for some $p > 0, \varepsilon^p \overline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon) < \text{const}$ as $\varepsilon \searrow 0$, there holds also $\varepsilon^p N_{\beta,\mathcal{O}}^{(0)}(\varepsilon) < \text{const}$ as $\varepsilon \searrow 0$ and consequently the map $\Theta_{\beta,\mathcal{O}}^{(0)}$ has finite rank. But in view of Lemma 3.1 this is impossible for arbitrary bounded regions $\mathcal{O}$ unless all maps $\Theta_{\beta,\mathcal{O}}^{(0)}$ are of rank 1. This proves the sufficiency of the stated condition. □

We mention as an aside that in the interesting case (ii) of this proposition (scaling limit of Haag–Swieca type) the representation spaces $\mathcal{H}_0$ of the scaling limit theories $\mathfrak{A}_0, \alpha_{\mathbb{R}^4}^{(0)}, \Omega_0$ are separable. This can be seen from the fact that the countable union of compact sets

$$\bigcup_{n\in\mathbb{N}} n\, e^{-(\beta/n)H_0}\, \mathfrak{A}_0(\mathcal{O}_n)_1\, \Omega_0 \tag{4.8}$$

is dense in $\mathcal{H}_0$ if the bounded regions $\mathcal{O}_n$ tend to $\mathbb{R}^4$ in the limit of large n.

Another class of criteria characterizing the nature of the scaling limit of a theory is obtained by looking at the dependence of $\underline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon)$ and

$\overline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon)$ on the spacetime region $\mathcal{O}$. As is apparent from the definition, these quantities depend, for fixed ε and β, monotonically on $\mathcal{O}$. We make use of this fact in the formulation of the subsequent result.

PROPOSITION 4.4. – *The underlying theory has a*

(i) pure quantum scaling limit if, for some $\varepsilon > 0$, $\underline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon) \rightarrow \infty$ *as* $\mathcal{O} \nearrow \mathbb{R}^4$.

(ii) classical scaling limit if, for some $0 < \varepsilon < 2^{1/2}$, $\overline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon) < const$ *as* $\mathcal{O} \nearrow \mathbb{R}^4$.

Proof. – (i) If L is a complex linear map of rank 1 between two Banach spaces and if $||L|| \leq 1$, its epsilon content satisfies $N_L(\varepsilon) \leq (1+\varepsilon^{-1})^2$, $\varepsilon > 0$. This is a straightforward consequence of the fact that the image of the unit ball under the action of L can be identified with a circle of radius $||L||$ in the complex plane.

The defining relation (4.1) and the spectrum condition imply that $||\Theta^{(0)}_{\beta,\mathcal{O}}|| \leq 1$. Hence if for some triple $\mathfrak{A}_0, \alpha^{(0)}_{\mathbb{R}^4}, \Omega_0$ the maps $\Theta^{(0)}_{\beta,\mathcal{O}}$ would be of rank 1 for any $\beta > 0$ and any bounded region $\mathcal{O}$, it would follow that $N^{(0)}_{\beta,\mathcal{O}}(\varepsilon) \leq (1+\varepsilon^{-1})^2$ and consequently $\lim_{\mathcal{O}\nearrow\mathbb{R}^4} \underline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon) < \infty, \varepsilon > 0$, because of relation (4.7). This shows that if the condition in the first part of the statement is satisfied, not all of the maps $\Theta^{(0)}_{\beta,\mathcal{O}}$ can be of rank 1. Thus, as was explained before, the underlying theory has a pure quantum scaling limit.

(ii) If the scaling limit theory is not classical, there is a non–trivial scaling limit net $\mathfrak{A}_0, \alpha^{(0)}_{\mathbb{R}^4}$ acting on an infinite dimensional Hilbert space $\mathcal{H}_0$. Whence, given $0 < \varepsilon < 2^{1/2}$ and any finite number N, there exist a $\beta_0 > 0$ and N unit vectors $\Phi_n \in \mathcal{H}_0, n = 1, \ldots N$, such that

$$||e^{-\beta_0 H_0}(\Phi_{n'} - \Phi_{n''})|| > \varepsilon \quad \text{if} \quad n' \neq n''.$$

This assertion is a simple consequence of the facts that $e^{-\beta H_0}$ tends to 1 in the strong operator topology if β tends to 0 and the norm distance of orthogonal unit vectors is equal to $2^{1/2}$. Now since $\mathfrak{A}_0$ acts irreducibly on $\mathcal{H}_0$ there exist by Kaplansky's density theorem [15] some bounded region $\mathcal{O}_0$ and N operators $A^{(n)} \in \mathfrak{A}_0(\mathcal{O}_0)_1$ such that the norm distances $||e^{-\beta_0 H_0}(\Phi_n - A^{(n)}\Omega_0)||$ are so small that

$$||e^{-\beta_0 H_0}(A^{(n')} - A^{(n'')})\Omega_0|| > \varepsilon \quad \text{if} \quad n' \neq n''.$$

It follows that $N^{(0)}_{\beta_0,\mathcal{O}_0}(\varepsilon) \geq N$ and, by relation (4.7), $\overline{N}_{\beta_0^{-1}\mathcal{O}_0} \geq N$. Consequently $\overline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon) \geq N$ as $\mathcal{O} \nearrow \mathbb{R}^4$. Since N was arbitrary,

$\overline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon)$ diverges as $\mathcal{O} \nearrow \mathbb{R}^4$ for any choice of $0 < \varepsilon < 2^{1/2}$. Thus we conclude that the condition in the second part of the statement can only be satisfied if the theory has a classical scaling limit. $\square$

Since in physically relevant theories the maps $\Theta_{\beta,\mathcal{O}}$ are expected to be not only compact but also nuclear [5, 10], the following results involving the nuclear p–norms $||\Theta_{\beta,\mathcal{O}}||_p$ of these maps are of interest.

THEOREM 4.5. – *Given a theory where the maps $\Theta_{\beta,\mathcal{O}}$, defined in (2.8), are p–nuclear for some $0 < p < 1/3$ and $\limsup_{\lambda\searrow 0} ||\Theta_{\lambda\beta,\lambda\mathcal{O}}||_p < \infty$. The theory has a classical scaling limit if and only if there exists a constant c such that*

$$\limsup_{\lambda\searrow 0} ||\Theta_{\lambda\beta,\lambda\mathcal{O}}||_{2p} < c$$

uniformly for all bounded regions $\mathcal{O}$.

Proof. – For the proof of the if–part of the statement we make use of the first part of Lemma 2.1 which, under the given conditions, implies that for $q > 2p/(1 - 2p)$

$$\overline{N}_{\beta^{-1}\mathcal{O}}(\varepsilon) \leq \exp(\mathrm{const}/\,\varepsilon^q),$$

uniformly in $\mathcal{O}$. Hence the theory has a classical scaling limit according to the second part of Proposition 4.4.

For the proof of the only–if–part we have to rely on the following more refined version of Lemma 4.2: let $\beta_0, \mathcal{O}_0, \varepsilon_0$ be fixed and let $\lambda_m, m \in \mathbb{N}$, be some sequence, tending to 0 such that the sequence of epsilon contents $N_{\lambda_m\beta_0,\lambda_m\mathcal{O}_0}(\varepsilon_0), m \in \mathbb{N}$, converges or tends to $+\infty$. The corresponding subnet of lifted and scaled vacuum states $\underline{\omega}_{\lambda_m}, m \in \mathbb{N}$, still has limit points $\underline{\omega}_0 \in \mathrm{SL}(\omega)$. For the epsilon contents of the resulting maps $\Theta^{(0)}_{\beta,\mathcal{O}}$ one obtains the estimate (using the same arguments and notation as in the proof of Lemma 4.2)

$$\lim_m N_{\lambda_m\beta_0,\lambda_m\mathcal{O}_0}(\varepsilon_0) \leq N^{(0)}_{\beta,\mathcal{O}}(\varepsilon_0).$$

Hence if there exist $\beta_0, \mathcal{O}_0, \varepsilon_0$ and a sequence $\lambda_m, m \in \mathbb{N}$, as just described, for which

$$\lim_m N_{\lambda_m\beta_0,\lambda_m\mathcal{O}_0}(\varepsilon_0) > (1 + \varepsilon_0^{-1})^2,$$

the maps $\Theta^{(0)}_{\beta,\mathcal{O}}$ cannot be of rank 1 for all $\beta, \mathcal{O}$. Assuming that the theory has a classical scaling limit it follows from this remark and Lemma 3.1 that

$$\limsup_{\lambda\searrow 0} N_{\lambda\beta,\lambda\mathcal{O}}(\varepsilon) \leq (1 + \varepsilon^{-1})^2 \tag{4.9}$$

for all $\beta, \mathcal{O}$ and $\varepsilon > 0$.

Now according to the second part of Lemma 2.1 there holds

(4.10)
$$\limsup_{\lambda\searrow 0} ||\Theta_{\lambda\beta,\lambda\mathcal{O}}||_r \leq \limsup_{\lambda\searrow 0} (2\pi)^{1/2}\Big(\sum_m (m^{1/2}\varepsilon_m N_{\lambda\beta,\lambda\mathcal{O}}(\varepsilon_m)^{1/m})^r\Big)^{1/r},$$

provided the right hand side of this inequality exists for some $0 < r \leq 1$ and a suitable sequence $\varepsilon_m, m \in \mathbb{N}$. Since all maps $\Theta_{\lambda\beta,\lambda\mathcal{O}}$ are p–nuclear for some $0 < p < 1/3$ this condition is satisfied. In fact, putting $\varepsilon_m = m^{-1/q}$, where $q > p/(1-p)$, it follows from the assumptions and the first part of Lemma 2.1 that

$$\varepsilon_m N_{\lambda\beta,\lambda\mathcal{O}}(\varepsilon_m)^{1/m} \leq c\, m^{-1/q}, \quad m \in \mathbb{N},$$

where the constant c does not depend on λ, m. Putting $q = 3p/2, r = 2p$, we conclude that the limes superior on the right hand side of the estimate (4.10) can be pulled under the sum, thereby leading to a larger upper bound on the left hand side. Hence, taking into account relation (4.9), we arrive at

$$\limsup_{\lambda\searrow 0} ||\Theta_{\lambda\beta,\lambda\mathcal{O}}||_{2p} \leq (2\pi)^{1/2}\Big(\sum_m m^{p-4/3}(1+m^{2/3p})^{4p/m}\Big)^{1/2p}.$$

Since the right hand side of this inequality is finite and does not depend on $\beta, \mathcal{O}$, the only–if–part of the statement follows. □

As has been pointed out in [10, Sec. 5], the quantities $||\Theta_{\beta,\mathcal{O}}||_p^p$ are a certain substitute for the partition function of the Gibbs canonical ensemble at temperature $(p\beta)^{-1}$ in a container of size proportional to $\mathcal{O}$. Thus the nature of the scaling limit is intimately related to thermal properties of the underlying theory. It would be desirable to clarify this relation further.

We conclude this section with a result pertaining to the nuclearity properties of the maps $\Theta^{(0)}_{\beta,\mathcal{O}}$.

THEOREM 4.6. – *Consider a theory where the maps $\Theta_{\beta,\mathcal{O}}$ are p–nuclear for some $0 < p < 1/3$ and $\limsup_{\lambda\searrow 0} ||\Theta_{\lambda\beta,\lambda\mathcal{O}}||_p < \infty$. Then the maps $\Theta^{(0)}_{\beta,\mathcal{O}}$, defined in relation (4.1), are q-nuclear for $q > 2p/(2-3p)$, and there holds*

$$||\Theta^{(0)}_{\beta,\mathcal{O}}||_q \leq c\, \limsup_{\lambda\searrow 0} ||\Theta_{\lambda\beta,\lambda\mathcal{O}}||_p,$$

where c depends only on p, q.

Proof. – Given $0 < p < 1/3$, we pick q, r such that there holds $1 > q > 2r/(2-r) > 2p/(2-3p)$. Then $r > p/(1-p)$, so it follows from relation (4.7) and the first part of Lemma 2.1 that

$$N^{(0)}_{\beta,\mathcal{O}}(\varepsilon) \leq \limsup_{\lambda\searrow 0} \exp\left(c\, ||\Theta_{\lambda\beta,\lambda\mathcal{O}}||_p^r / \varepsilon^r\right), \quad \varepsilon > 0,$$

and consequently

$$\sup_{\varepsilon>0} \varepsilon \, (\ln N^{(0)}_{\beta,\mathcal{O}}(\varepsilon))^{1/r} \leq c^{1/r} \limsup_{\lambda \searrow 0} ||\Theta_{\lambda\beta,\lambda\mathcal{O}}||_p .$$

Since $r < 2/3$, the statement then follows from the last part of Lemma 2.1. □

The preceding proposition provides the basis for a more detailed investigation of the properties of the scaling limit theories. It can be used, for example, to establish the so-called *(distal) split-property* [7] in the scaling limit, provided the underlying theory has decent phase-space properties. The crucial step is the demonstration that the maps $\Theta^{(0)}_{\beta,\mathcal{O}}$ have certain specific nuclearity properties which can be expressed in various ways [5, 10, 16, 17]. In view of the preceding proposition these properties follow from corresponding properties of the maps $\Theta_{\beta,\mathcal{O}}$ in the underlying theory. We refrain from stating here the pertinent conditions and refer to the quoted publications.

5. CONCLUDING REMARKS

Making use of the novel concept of scaling algebra, introduced in [1], we have established an interesting relation between the phase space properties of a theory and the nature of its scaling limit. It turned out that some rough information on the number of degrees of freedom affiliated with certain specific regions of phase space is sufficient to distinguish between theories with a classical and (pure) quantum scaling limit. Moreover, one can deduce rather precise information on the phase space properties of the scaling limit from corresponding properties of the underlying theory.

The present results are a promising step towards the general understanding and the classification of the short distance properties of local nets of observables. But there are still many open problems. It is, for example, an intriguing question under which circumstances a theory has a *unique* quantum scaling limit, *cf.* Sec. 3. Phase space properties of the theory seem to matter also in this context, but there do not yet exist any definitive results. The uniqueness of the scaling limit has been established so far only in certain models and in dilation invariant theories [1].

It would be instructive to have a supply of examples illustrating also the various other possibilities appearing in the general classification of the structure of scaling limits, such as theories with a classical or degenerate scaling limit. A simple example of the classical type ought to be the net which is obtained from a generalized free field $\phi(x)$ with continuous mass spectrum according to the following procedure: the local algebras corresponding to regions of diameter λ are generated by the fields $\Box^{n_\lambda}\phi(x)$, where the numbers n_λ tend to infinity as λ tends to 0. In this way one obtains a local, Poincaré–covariant and weakly additive net. But, as is apparent from the construction, the algebras corresponding to shrinking regions contain, apart from multiples of the identity, operators with rapidly worsening ultraviolet properties. As has been pointed out in [1, Sec. 4] one may expect that such nets have a classical scaling limit. The model resembles to a certain extent the situation in field theories without ultraviolet fixed point, where one cannot remove the cutoff and proceed to point-like fields.

A model with a degenerate scaling limit ought to be the infinite tensor product theory constructed from free scalar fields with masses $m \in 2^{\mathbb{Z}}\, m_0$, where $m_0 > 0$ is fixed. This theory is invariant under the subgroup $2^{\mathbb{Z}}$ of dilations and consequently should coincide with one of its scaling limits. But the set of scaling limit states of a theory is invariant under arbitrary scaling transformations, hence there should appear in the scaling limit also the theories with scaled mass spectrum $\mu\, 2^{\mathbb{Z}}\, m_0$, which are in general non-isomorphic for different $\mu > 0$. The details of these simple but instructive examples will be worked out elsewhere.

It would also be desirable to understand better the relation between the present algebraic approach to the renormalization group [1] and the conventional field theoretic treatment [2]. By a combination of these methods one may hope to gain new insights into the possible ultraviolet properties of local field theories. To this end it would be necessary to identify in the algebraic setting invariants, in analogy to the beta function and the anomalous dimensions of local operators in field theory, which are apt to describe in a more quantitative manner the ultraviolet properties of a local net. A certain step in this direction is the purely algebraic characterization of asymptotically free theories, proposed in [1, Sec. 4].

Another interesting problem is the analysis of the superselection and particle structure emerging in the scaling limit. As has been pointed out in [12], *cf.* also [1], one can identify the *ultraparticles*, *i.e.*, the particle-like structures appearing at small scales, such as quarks, gluons and leptons, with the particle content (in the sense of Wigner) of the scaling limit theory. Similarly, the *ultrasymmetries* of a theory, *i.e.*, the symmetries which are visible at small scales, such as colour and flavour, can be identified with the global gauge group of the scaling limit theory. The discussion of interesting issues, such as the reconstruction of the local gauge group from the local observables or the confinement problem can then be based on a comparison of the particle and symmetry content of a theory with the ultraparticle and ultrasymmetry content of its scaling limit. Thus the concept of scaling algebra seems also a useful tool for the investigation of these more conceptual problems.

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