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Tomita conjugations and transitivity of locality

by

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ABSTRACT. – The *Tomita conjugation*, S, associated with a von Neumann algebra, $\mathcal{M}$, and a cyclic and separating vector, Ω, is the closure of the map $X\Omega \to X^*\Omega$, $X \in \mathcal{M}$. This notion can be generalized to algebras (and even more general families) of *unbounded* operators that appear in quantum field theory. It is shown that generalizations of the classical results of Borchers on transitivity of locality and of Bisognano and Wichmann on duality in quantum field theory follow from properties of such Tomita conjugations. The basis is the simple observation that two (unbounded) operator algebras with the same Tomita conjugation have the same unbounded weak commutant, provided one algebra is contained in the other.

RÉSUMÉ. – La *conjugaison de Tomita*, S, associée à une algèbre de von Neumann, $\mathcal{M}$, et à un vecteur cyclique et séparateur, Ω, est la fermeture de l'application $X\Omega \to X^*\Omega$, $X \in \mathcal{M}$. On peut généraliser cette notion aux algèbres (et même à des familles plus générales) d'opérateurs *non bornés* qui apparaissent dans la théorie quantique des champs. Dans cet exposé on utilise de telles conjugaisons de Tomita pour déduire certaines généralisations des résultats classiques de Borchers sur la transitivité de la localité et de Bisognano et Wichmann sur la dualité. Ces généralisations sont basées sur l'observation que deux algèbres dont les conjugaisons de Tomita coïncident ont le même commutant faible non-borné, si l'une des algèbres est une sous-algèbre de l'autre.

1. INTRODUCTION

The results reported in this contribution have been obtained in collaboration with H.J. Borchers [1], [2]. They concern the relation between Wightman quantum fields and associated local nets of von Neumann algebras, in particular the connection between Tomita conjugations defined by the former and duality properties of the latter. Our framework is in fact more general than the standard Wightman axioms. We are concerned with nets of $*$-algebras $\mathcal{P}(\mathcal{O})$ of (in general unbounded) operators on a common dense domain $\mathcal{D}$ in a Hilbert space $\mathcal{H}$. Here $\mathcal{O}$ denotes an open region in Minkowski space $\mathbf{R}^d$. We also consider distinguished subalgebras $\mathcal{P}_0(\mathcal{O}) \subset \mathcal{P}(\mathcal{O})$ (which might sometimes simply coincide with the $\mathcal{P}(\mathcal{O})$). It is assumed that all these algebras contain the unit operator and satisfy the following conditions:

(I) If $\mathcal{O}_1 \subset \mathcal{O}_2$, then $\mathcal{P}(\mathcal{O}_1) \subset \mathcal{P}(\mathcal{O}_2)$ and $\mathcal{P}_0(\mathcal{O}_1) \subset \mathcal{P}_0(\mathcal{O}_2)$.

(L) If $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated, then $\mathcal{P}(\mathcal{O}_1)$ and $\mathcal{P}(\mathcal{O}_2)$ commute on $\mathcal{D}$.

(BZ) There is a distinguished vector $\Omega \in \mathcal{D}$ (vacuum) such that $\mathcal{P}_0(\mathcal{O})\Omega$ is dense in $\mathcal{P}_0(\mathbf{R}^d)\Omega$ in the topology defined by the graph norms $||\psi||_X = ||\psi|| + ||X\psi||$ with $X \in \mathcal{P}(\mathbf{R}^d)$, for all open $\mathcal{O}$.

In the usual Wightman framework, where $\mathcal{P}(\mathcal{O})$ is the algebra generated by field operators smeared with test functions with support in $\mathcal{O}$, property (BZ) [3] is a consequence of translational invariance and spectrum condition for any translationally covariant subnet $\mathcal{P}_0(\cdot)$ of $\mathcal{P}(\cdot)$. It is the only relic of these conditions we shall need. Lorentz covariance will not be assumed. Our framework applies also to field theories on curved space-time and theories with more general localization properties than Wightman fields, as considered, e.g., in [4].

Within the setting just decribed we consider the following questions:

Q1. When does there exist a local net of von Neumann algebras $\mathcal{M}(\mathcal{O})$ such that every $X \in \mathcal{P}(\mathcal{O})$ has an extension to a closed operator $\hat{X}$ that is affiliated with $\mathcal{M}(\mathcal{O})$?

Q2. When is there such a net satisfying *duality*, i.e.,

$$\mathcal{M}(\mathcal{O})' = \mathcal{M}(\bar{\mathcal{O}}') \tag{1}$$

for some specified class $\mathcal{C}$ of regions $\mathcal{O}$ (like, for instance, double cones)? Here $\bar{\mathcal{O}}'$ is the space like complement of the closure $\bar{\mathcal{O}}$.

We shall give answers in terms of conditions on the subnet $\mathcal{P}_0(\cdot)$. Extensions to Fermi fields are considered in [1].

Question Q1 has a rather long history, see, e.g., [3], [5], [6], [7], [8], [9], [2] and further references quoted therein. In the papers [9] and [2] we advocated the point of view that this question should be understood as a noncommutative version of a classical moment problem.

Question Q2 has been treated, together with the first question, in the fundamental papers by Bisognano and Wichmann [10], [11] who proved duality for local nets of von Neumann algebras generated by Poincaré covariant fields transforming with finite dimensional representations of the Lorentz-group. In this classical analysis of duality one may distinguish between two main steps. The first is the identification of the modular conjugation J and the modular group Δ^{is} associated with the wedge algebras $\mathcal{P}(W)$. Here W denotes a wedge-domain, bounded by two light-like half-planes. The modular objects J and Δ are obtained from the polar decomposition of the *Tomita conjugation* $S_{\mathcal{P}(W)}$ associated with $\mathcal{P}(W)$, which is the closure of the map $X\Omega \mapsto X^*\Omega$ with $X \in \mathcal{P}(W)$. This first step is closely related to the PCT theorem: In a theory of Wightman fields transforming covariantly with a finite dimensional representation of the Lorentz group the modular conjugation is the PCT operator combined with a rotation, and the modular group is given by Lorentz boosts. This geometrical description of the modular objects thus relies on the covariance with respect to the Lorentz group. It implies that the adjoints of the Tomita conjugations satisfy the equation $S^*_{\mathcal{P}(W)} = S_{\mathcal{P}(\bar{W}')}$. The second step, i.e., the derivation of duality from such properties of the Tomita conjugations, can be understood in a more general setting without reference to the geometrical interpretation, however. This will be discused below. One bonus of this general setting is that the intimate link of duality with the concept of equivalence classes of local fields [12] becomes quite transparent.

In summary, our assertions concerning the two questions above are as follows:

• The existence of a net of local von Neumann algebras $\mathcal{M}(\mathcal{O})$ associated with the unbounded operator algebras $\mathcal{P}(\mathcal{O})$ is a noncommutatative moment problem.

• Duality of the net $\mathcal{M}(\cdot)$ is simply related to a property of the Tomita conjugations associated with the net $\mathcal{P}(\cdot)$ of unbounded operator algebras.

• This property of the Tomita conjugations leads naturally to an *equivalence class* of nets of unbounded operator algebras that are relatively local to each other and associated with the same local net of von Neumann algebras.

2. TOMITA CONJUGATIONS AND UNBOUNDED COMMUTANTS

Let $\mathcal{M}$ and $\mathcal{N}$ be two von Neumann algebras with a common cyclic and separating vector Ω. Let $S_{\mathcal{M}}$ and $S_{\mathcal{N}}$ denote the corresponding Tomita conjugations, i.e., the closures of the mappings $A\Omega \mapsto A^*\Omega$ with $A \in \mathcal{M}$ and $A \in \mathcal{N}$, respectively. Let J and Δ denote the corresponding antiunitary involution and positive operator obtained by polar decomposition of the Tomita conjugations, i.e.,

$$S_{\mathcal{M}} = J_{\mathcal{M}}\Delta_{\mathcal{M}}^{1/2}, \qquad S_{\mathcal{N}} = J_{\mathcal{N}}\Delta_{\mathcal{N}}^{1/2}.$$

The following proposition is well known, cf., e.g., [13], Thm. 9.2.36.

PROPOSITION 1. – *If $\mathcal{N} \subset \mathcal{M}$, then the following are equivalent:*

$$(a)\ \mathcal{N} = \mathcal{M} \quad (b)\ S_{\mathcal{N}} = S_{\mathcal{M}} \quad (c)\ \Delta_{\mathcal{N}} = \Delta_{\mathcal{M}} \quad (d)\ J_{\mathcal{N}} = J_{\mathcal{M}} \tag{2}$$

Proof. – The implications $(a) \Rightarrow (b) \Rightarrow (c)$ and $(b) \Rightarrow (d)$ are obvious. To prove $(d) \Rightarrow (a)$ one may use Tomita's theorem: Since $\mathcal{N} \subset \mathcal{M}$ by assumption, one has

$$\mathcal{N}' = J_{\mathcal{N}}\mathcal{N}J_{\mathcal{N}} \subset J_{\mathcal{N}}\mathcal{M}J_{\mathcal{N}} = J_{\mathcal{M}}\mathcal{M}J_{\mathcal{M}}M = \mathcal{M}'$$

and hence $\mathcal{M} \subset \mathcal{N}$. For $(c) \Rightarrow (a)$ see [13], Thm. 9.2.36. QED

We now make the following observation: Although step $(d) \Rightarrow (a)$ in the proof above uses Tomita's theorem, the implication $(b) \Rightarrow (a)$ is in fact elementary, as seen by the following computation:

If $X \in \mathcal{M}$, $Y \in \mathcal{N}'$ and $Z_1, Z_2 \in \mathcal{N}$, then

$$\langle Z_1\Omega, XY\, Z_2\Omega\rangle = \langle Z_2^* X^* Z_1\Omega, Y\Omega\rangle. \tag{3}$$

Because $1 = S_{\mathcal{M}}^2 = S_{\mathcal{N}}S_{\mathcal{M}}$, and $S_{\mathcal{N}}$ is antilinear, this is in turn equal to

$$\langle S_{\mathcal{N}}^*(Y\Omega), S_{\mathcal{M}}(Z_2^* X^* Z_1\Omega)\rangle = \langle Y^*\Omega, Z_1^* X Z_2\Omega\rangle = \langle Z_1\Omega, YX\, Z_2\Omega\rangle. \tag{4}$$

Since $\mathcal{N}\Omega$ is dense by assumption, this shows that $\mathcal{N}' \subset \mathcal{M}'$ and thus $\mathcal{M} = \mathcal{N}$.

This elementary part of Proposition 1 can be generalized to *unbounded* operators. Before stating this generalization we introduce some concepts.

DEFINITION. – *Let $\mathcal{D}$ be a linear subspace of a Hilbert space $\mathcal{H}$. A* linear $*$-operator family *$\mathcal{F}$ on $\mathcal{D}$ is a linear space of closable, linear operators*

$\mathcal{D} \to \mathcal{H}$ such that for each $X \in \mathcal{F}$ the restriction $X^\dagger := X^*|\mathcal{D}$ of the adjoint operator X^* to $\mathcal{D}$ belongs also to $\mathcal{F}$.

DEFINITION. – *Let $\mathcal{F}$ be a linear $*$-operator family on $\mathcal{D}$ and $\mathcal{D}_0 \subset \mathcal{D}$. The* unbounded weak commutant *$(\mathcal{F}, \mathcal{D}_0)^{uw}$ is the set of all linear operators Y with $\mathcal{D}_0 \subset D(Y) \cap D(Y^*)$ such that*

$$\langle X^*\phi, Y\psi\rangle = \langle Y^*\phi, X\psi\rangle \tag{5}$$

for all $X \in \mathcal{F}$, $\phi, \psi \in \mathcal{D}_0$. Here $D(Y)$ and $D(Y^*)$ denote respectively the domain of definition of Y and its adjoint. The bounded weak commutant *$(\mathcal{F}, \mathcal{D}_0)^w$ is the bounded part of $(\mathcal{F}, \mathcal{D}_0)^{uw}$.*

Let $\mathcal{F}$ be a linear $*$-operator family on $\mathcal{D}$ containing the unit opertator 1. Suppose $\Omega \in \mathcal{D}$ is a vector such that $\mathcal{F}\Omega \subset \mathcal{D}$ is dense in $\mathcal{H}$ and Ω is also cyclic for $(\mathcal{F}, \mathcal{F}\Omega)^{uw}$. We define an operator $S_{\mathcal{F}}^{(0)} : \mathcal{F}\Omega \to \mathcal{H}$ by

$$S_{\mathcal{F}}^{(0)} X\Omega = X^*\Omega. \tag{6}$$

Since

$$\langle Y\Omega, X^*\Omega\rangle = \langle X\Omega, Y^*\Omega\rangle \tag{7}$$

for $X \in \mathcal{F}$, $Y \in (\mathcal{F}, \mathcal{F}\Omega)^{uw}$, and Ω is cyclic for the unbounded weak commutant, $S_{\mathcal{F}}^{(0)}$ is closable with

$$S_{\mathcal{F}}^{(0)*} Y\Omega = Y^*\Omega \tag{8}$$

for $Y \in (\mathcal{F}, \mathcal{F}\Omega)^{uw}$. The *Tomita conjugation* $S_{\mathcal{F}}$ associated with $\mathcal{F}$ and Ω is the closure of $S_{\mathcal{F}}^{(0)}$. We note that $\{Y\Omega \,|\, Y \in (\mathcal{F}, \mathcal{F}\Omega)^{uw}\} \subset D(S_{\mathcal{F}}^*)$ and

$$S_{\mathcal{F}}^* Y\Omega = Y^*\Omega \tag{9}$$

for $Y \in (\mathcal{F}, \mathcal{F}\Omega)^{uw}$.

The annonunced generalization of Proposition 1 is

PROPOSITION 2. – *Let $\mathcal{A}$ be a $*$-operator algebra on a dense domain $\mathcal{D}$ and $\mathcal{F} \subset \mathcal{A}$ a linear $*$-operator family on $\mathcal{D}$ containing 1. Let $\Omega \in \mathcal{D}$ be cyclic for $\mathcal{F}$ and $(\mathcal{A}, \mathcal{A}\Omega)^{uw}$ (and hence also for $\mathcal{A}$ and $(\mathcal{F}, \mathcal{A}\Omega)^{uw}$). Then the equality of Tomita conjugations,*

$$S_{\mathcal{F}} = S_{\mathcal{A}} \tag{10}$$

implies

$$(\mathcal{F}, \mathcal{A}\Omega)^{uw} \subset (\mathcal{A}, \mathcal{F}\Omega)^{uw}. \tag{11}$$

If in addition $\mathcal{F}\Omega$ is a core for each $X \in \mathcal{A}$, then

$$(\mathcal{F}, \mathcal{A}\Omega)^{uw} = (\mathcal{A}, \mathcal{F}\Omega)^{uw}. \tag{12}$$

Proof. – The first part is essentially a repetition of $(b) \Rightarrow (a)$ above with $\mathcal{F}$ playing the role of $\mathcal{N}$ and $\mathcal{A}$ of $\mathcal{M}$. The difference is that the product XY with $X \in \mathcal{A}$ and $Y \in (\mathcal{F}, \mathcal{A}\Omega)^{uw}$ is in general not defined. Instead one considers, with $Z_1, Z_2 \in \mathcal{F}$:

$$\begin{aligned}\langle X^* Z_1 \Omega, Y Z_2 \Omega\rangle &= \langle Z_2^* X^* Z_1 \Omega, Y\Omega\rangle = \langle S_{\mathcal{F}}^*(Y\Omega), S_{\mathcal{A}}(Z_2^* X^* Z_1 \Omega)\rangle \\ &= \langle Y^*\Omega, Z_1^* X Z_2 \Omega\rangle = \langle Y^* Z_1 \Omega, X Z_2 \Omega\rangle.\end{aligned}$$

The first equality uses that $X^* Z_1 \in \mathcal{A}$, $Z_2 \in \mathcal{F}$ and $Y \in (\mathcal{F}, \mathcal{A}\Omega)^{uw}$, the second one that $1 = S_{\mathcal{A}}^2 = S_{\mathcal{F}} S_{\mathcal{A}}$, the third that $\mathcal{A}$ is a $*$-algebra, and the last that $X Z_2 \in \mathcal{A}$, $Z_1 \in \mathcal{F}$ and $Y \in (\mathcal{F}, \mathcal{A}\Omega)^{uw}$. This proves that $(\mathcal{F}, \mathcal{A}\Omega)^{uw} \subset (\mathcal{A}, \mathcal{F}\Omega)^{uw}$. The proof of equality, given that $\mathcal{F}\Omega$ is a core for each $X \in \mathcal{A}$, is based on the observation that under this condition, $[Y, (X^\dagger)^*]\Omega = 0$ for all $X \in \mathcal{A}$, $Y \in (\mathcal{F}, \mathcal{A}\Omega)^{uw}$. See [1], Prop. 2.7 for the details. QED

Proposition 2 is the basis for our discussion of transitivity of locality and duality.

3. UNBOUNDED DUALITY AND TRANSITIVITY OF LOCALITY

Let $\mathcal{P}_0(\cdot)$, $\mathcal{P}_1(\cdot)$ and $\mathcal{P}_2(\cdot)$ be nets of $*$-algebras of (unbounded) operators, i.e., for each open set $\mathcal{O}$ in Minkowski space $\mathcal{P}_i(\mathcal{O})$ is a $*$-algebra of operators, containing 1, on a common invariant domain $\mathcal{D}$ and $\mathcal{P}_i(\mathcal{O}_1) \subset \mathcal{P}_i(\mathcal{O}_2)$ if $\mathcal{O}_1 \subset \mathcal{O}_2$, $i = 0, 1, 2$. If

$$[\mathcal{P}_i(\mathcal{O}_1), \mathcal{P}_j(\mathcal{O}_2)] = \{0\} \tag{13}$$

for space-like separated $\mathcal{O}_1$, $\mathcal{O}_2$ we say that $\mathcal{P}_i(\cdot)$ is *relatively local* to $\mathcal{P}_j(\cdot)$. Within the framework of Lorentz covariant Wightman fields it is a classical result of Borchers [12] that two fields that are relatively local to an irreducible field are relatively local to each other. We shall now show how this result can be understood as a property of the Tomita conjugations associated with the irreducible field.

We start by singling out two classes of open subsets of Minkowski space $\mathbf{R}^d$, denoted by $\mathcal{K}$ and $\mathcal{W}$, respectively. In this section we deal essentially only with one of these classes, $\mathcal{W}$, but the other class, $\mathcal{K}$, will become important in the last section and we prefer to discuss them both at once. To have a concrete picture in mind, one might think of the sets in $\mathcal{K}$ as double cones and those in $\mathcal{W}$ as space-like wedges, but all that matters are the following general properties of these classes:

(K) For each pair $x, y \in \mathbf{R}^d$ of space-like separated points there exist space like separated sets $K_1, K_2 \in \mathcal{K}$ with $x \in K_1$, $y \in K_2$. Moreover, if $K \in \mathcal{K}$, then $\bar{K}'$ contains some $K_1 \in \mathcal{K}$.

(KW) Each $K \in \mathcal{K}$ is contained in some $W \in \mathcal{W}$ and if $K_1, K_2 \in \mathcal{K}$ are space-like separated, then there are space-like separated $W_1, W_2 \in \mathcal{W}$, such that $K_1 \subset W_1$ and $K_2 \subset W_2$. Moreover, each $W \in \mathcal{W}$ contains some $K \in \mathcal{K}$, and the same holds for $\bar{W}'$.

We also assume that the following additivity property holds for the nets $\mathcal{P}(\cdot)$ under consideration:

(A) For all open subsets $\mathcal{O}$ of Minkowski space, the algebra $\mathcal{P}(\mathcal{O})$ is generated by the algebras $\mathcal{P}(K)$ with $K \subset \mathcal{O}$, $K \in \mathcal{K}$.

The understanding is that $\mathcal{P}(\mathcal{O}) = \mathbf{C} \cdot 1$ if $\mathcal{O}$ does not contain any set in $\mathcal{K}$, which can e.g. happen if $\mathcal{K}$ contains only sets of some minimal size. Since the operators in $\mathcal{P}(\mathcal{O})$ are in general unbounded, we prefer not to close the algebras in any particular topology and (A) should be understood in a purely algebraic sense.

Instead of regarding (A) as an axiom, we might to begin with require that the algebras $\mathcal{P}(\mathcal{O})$ are given for $\mathcal{O} \in \mathcal{K}$ only and use (A) to define the algebras for more general regions. Note that by (K) and (KW), locality of the whole net is equivalent to locality restricted to either of the classes $\mathcal{K}$ or $\mathcal{W}$.

Let $\mathcal{P}(\cdot)$ be a net satisfying the locality condition (L) and Ω a distinguished vector in $\mathcal{D}$ such that $\mathcal{P}(K)\Omega$ is dense in $\mathcal{H}$ for all $K \in \mathcal{K}$. In the following we regard Ω as fixed. From condition (KW) it follow that Ω is cyclic for all $\mathcal{P}(W)$ and $\mathcal{P}(\bar{W}')$ and hence (by locality) also for the corresponding unbounded weak commutants. Hence the Tomita conjugations $S_{\mathcal{P}(\mathcal{O})}$ associated with Ω and $\mathcal{P}(\mathcal{O})$ are well defined for $\mathcal{O} = W$ or $\bar{W}'$, $W \in \mathcal{W}$.

DEFINITION. – *The net* $\mathcal{P}(\cdot)$ *satisfies* $\mathcal{W}$-duality *if*

$$S^*_{\mathcal{P}(W)} = S_{\mathcal{P}(\bar{W}')} \tag{14}$$

for all $W \in \mathcal{W}$.

Because of Proposition 1 one may regard this definition as a natural generalization to unbounded operators of the usual duality condition (1) (with $\mathcal{O} = W \in \mathcal{W}$) for nets of von Neumann algebras. If $\mathcal{W}$ is the class of space-like wedges and Ω the vacuum vector, the duality condition (3) holds for nets of *unbounded* operators generated by Poincaré covariant Wightman fields transforming with finite dimensional representations of the Lorentz group. This is a consequence of the Bisognano-Wichmann analysis [10], [11].

As will soon become clear the duality condition (14) is the crucial ingredient in establishing transitivity of locality, but there is one technical point concerning domains of the unbounded operators we have to deal with first. We would like (13) to hold as an operator equation on the common invariant domain $\mathcal{D}$ and not only in the weak sense as equality of matrix elements on some subdomain of $\mathcal{D}$. These two versions of relative locality need not always be equivalent, but a simple regularity property of generators of the nets under consideration enables one to derive the former from the latter.

DEFINITION. – *Let X be a closable operator and H a self adjoint operator on a Hilbert space $\mathcal{H}$. Let E_I be the spectral projector of H for a subset $I \subset \mathcal{R}$. We say that X obeys* compact H-bounds, *if $E_I\mathcal{H} \subset D(\bar{X})$, and $\bar{X}E_I$ is a bounded operator for all bounded intervals I.*

We remark that this definition is equivalent to the following: For *some* measurable function F that vanishes nowhere on the spectrum of H, it holds that $F(H)\mathcal{H} \subset D(\bar{X})$ and $\bar{X}F(H)$ is a bounded operator. In the next section we shall consider bounds were the growth properties of F are restricted.

LEMMA 1. – *Let $\mathcal{G}$ be a family of operators on a dense domain $\mathcal{D} \subset \mathcal{H}$ and let H be a self adjoint operator, densely defined in $\mathcal{H}$. Suppose that*

(i) *The operators in $\mathcal{G}$ obey compact H-bounds.*

(ii) *$\mathcal{D}$ is invariant under e^{itH}, $t \in \mathcal{R}$, and for each $\psi \in \mathcal{D}$, $X \in \mathcal{G}$ the function $t \mapsto Xe^{itH}\psi$ is continuous and polynomially bounded.*

Then every dense domain $\mathcal{D}_0 \subset \mathcal{D}$ that is invariant under the unitary group e^{itH}, is dense in $\mathcal{D}$ in the graph topology induced by $\mathcal{G}$.

For a proof of the lemma see the appendix of [1]. We note that condition (ii) is always satisfied for the algebras generated by tempered Wightman fields if H is the Hamiltonian. We can now state our version of transitivity of locality:

THEOREM 1. – *Let $\mathcal{P}_j(\cdot)$, $j = 0, 1, 2$ be three nets of $*$-algebras on a common dense domain $\mathcal{D}$ containing a vector Ω, such that $\mathcal{D}_0 = \mathcal{P}_0(\mathbf{R}^d)\Omega$ is dense and condition (BZ) holds with $\mathcal{P}(\cdot)$ defined as the net generated by*

$\mathcal{P}_0(\cdot)$, $\mathcal{P}_1(\cdot)$ and $\mathcal{P}_2(\cdot)$. Moreover, assume that for $j = 0, 1, 2$, the algebras $\mathcal{P}_j(W)$ with $W \in \mathcal{W}$, contain sets $\mathcal{G}_i(W)$ of generators satisfying the conditions of Lemma 1, and that $\mathcal{D}_0$ is invariant under $\exp(itH)$. *If now*

- *the net $\mathcal{P}_0(\cdot)$ is local and satisfies $\mathcal{W}$-duality*
- *$\mathcal{P}_i(\cdot)$ is relatively local to $\mathcal{P}_0(\cdot)$, $i = 1, 2$,*

then $\mathcal{P}_1(\cdot)$ and $\mathcal{P}_2(\cdot)$ are relatively local to each other.

Proof. – We use Proposition 2 with $\mathcal{F} = \mathcal{P}_0(W)$ and $\mathcal{A}$ the algebra generated by $\mathcal{P}_0(W)$ and $\mathcal{P}_2(W)$. Note first that $\mathcal{P}_0(\bar{W}') \subset (\mathcal{A}, \mathcal{A}\Omega)^{uw}$ implies that $S_{\mathcal{P}_0(\bar{W}')} \subset S_{\mathcal{A}}^*$ and hence

$$S_{\mathcal{A}} \subset S_{\mathcal{P}_0(\bar{W}')}^* = S_{\mathcal{P}_0(W)} = S_{\mathcal{F}} \tag{15}$$

where the duality condition (14) has been used. Since $\mathcal{F} \subset \mathcal{A}$ it follows that $S_{\mathcal{F}} = S_{\mathcal{A}}$, and by Proposition 2 thus

$$(\mathcal{F}, \mathcal{A}\Omega)^{uw} \subset (\mathcal{A}, \mathcal{F}\Omega)^{uw}. \tag{16}$$

Since $\mathcal{P}_1(\bar{W}')$ commutes with $\mathcal{F}$ and $\mathcal{P}_2(W) \subset \mathcal{A}$ we thus have

$$\mathcal{P}_1(\bar{W}') \subset (\mathcal{P}_2(W), \mathcal{P}_0(W)\Omega)^{uw}. \tag{17}$$

It is simple to prove that the conditions of Lemma 1 and (BZ) allow one to pass from this weak commutativity of $\mathcal{P}_1(\bar{W}')$ and $\mathcal{P}_2(W)\Omega$ on $\mathcal{P}_0(W)\Omega$ to strong commutativity on $\mathcal{D}$; see [1], Lemma 3.1 for details. QED

4. FROM UNBOUNDED TO BOUNDED LOCALITY

We turn now to question Q1 of the introduction, i.e., the the question when a net $\mathcal{P}(\cdot)$ of unbounded operator algebras can be associated with a local net $\mathcal{M}(\cdot)$ of von Neumann algebras. The problem is greatly simplified if it is known that the *bounded weak commutants* $\mathcal{P}(\mathcal{O})^w$ of the unbounded operator algebras $\mathcal{P}(\mathcal{O})$ are also *algebras.* (Here and in the sequel we write for short $\mathcal{P}(\mathcal{O})^w$ instead of $(\mathcal{P}(\mathcal{O}), \mathcal{D})^w$.) As shown in [DSW] this is the case if the algebras $\mathcal{P}(\mathcal{O})$ have generators that satisfy H-bounds of a certain type (called *generalized H-bounds* in [7]), namely, if there is an index $\alpha < 1$ such that $\bar{X}\exp(-|H|^\alpha)$ is a bounded operator, for each generator X. Such bounds are thus more stringent than the compact H-bounds of the last section, where it was only required that $\bar{X}F(H)$ is bounded for *some* function F that is strictly positive on the spectrum of

H, but there seems to be no reason to doubt that they are satisfied in most cases of interest.

The weak commutants $\mathcal{P}(\mathcal{O})^w$ are always weakly closed, so if they are algebras, they are von Neumann algebras. It is then simple to show (see [7]) that every $X \in \mathcal{P}(\mathcal{O})$ has an extension affiliated with the von Neumann algebra

$$\mathcal{M}_{\min}(\mathcal{O}) = \mathcal{P}(\mathcal{O})^{w\prime}. \tag{18}$$

This von Neumann algebra is minimal in the sense that it is contained in any von Neumann algbra with the property that every operator in $\mathcal{P}(\mathcal{O})$ has a closed extension affiliated to it. Hence the answer to question Q1 amounts to finding conditions for $\mathcal{M}_{\min}(\cdot)$ to be a *local* net.

The condition we give is in terms of a certain positivity property of the state defined by the cyclic vector Ω. The following definition is adapted from [14]:

DEFINITION. – *Let ω be a state (i.e., a positive, linear functional) on a $*$-algebra $\mathcal{A}$. The state is said to be* centrally positive *with respect to a hermitian element $X \in \mathcal{A}$ if $\omega(Z) \geq 0$ for all Z of the form $Z = \sum_n X^n Y_n$, where the $Y_n \in \mathcal{A}$ are such that $\sum_n \lambda^n Y_n$ is a sum of squares in the algebra for each $\lambda \in \mathbf{R}$, i.e., $\sum_n \lambda^n Y_n = \sum_i Z_i(\lambda)^* Z_i(\lambda)$ for some $Z_i(\lambda) \in \mathcal{A}$.*

It is easy to see that if ω is centrally positive with respect to X, then $\pi_\omega(X)$ necesarily commutes with $\pi_\omega(\mathcal{A})$, where π_ω denotes the GNS representation defined by ω. Central positivity is in general stronger than simple positivity, i.e., positivity on elements of the form $\sum_j Z_j^* Z_j$. This can be seen from examples of polynomials in two or more variables that are positive as functions but cannot be written as a sum of squares. Central positivity of a state on an algebra of polynomials is precisely the condition that guarantees that the state can be represented by a postitve measure, i.e., the solution to the corresponding *moment problem.*

We can now formulate our answer to question Q1:

THEOREM 2. – *Assume the weak commutants $\mathcal{P}(K)^w$ with $K \in \mathcal{K}$ are algebras. The minimal net $\mathcal{M}_{\min}(\mathcal{O}) = \mathcal{P}(\mathcal{O})^{w\prime}$ satisfies locality if and only if for each $K \in \mathcal{K}$ and every X in some set of hermitean generators for $\mathcal{P}(K)$ the state defined by Ω on the algebra generated by X and $\mathcal{P}(\bar{K}')$ is centrally positive with respect to X.*

For the proof of this theorem see [2], Theorem 4.6. An essential ingredient is Powers' extension theorem [14] for centrally positive states.

5. FROM UNBOUNDED TO BOUNDED DUALITY

We now assume that the condition of Theorem 2 is satisfied for a net $\mathcal{P}(\cdot)$ of unbounded operator algebras with cyclic vector Ω and consider the relation between duality properties of the unbounded and bounded operator algebras. Our first assertion is

PROPOSITION 3. – *$\mathcal{W}$-duality of $\mathcal{P}(\cdot)$ in the sense of* (14) *implies $\mathcal{W}$-duality of $\mathcal{M}_{\min}(\cdot)$, i.e.,*

$$\mathcal{M}_{\min}(W)' = \mathcal{M}_{\min}(\bar{W}'). \tag{19}$$

Proof. – We have $\mathcal{M}_{\min}(W) = \mathcal{P}(W)^{w\prime}$ and $\mathcal{M}_{\min}(\bar{W}') = \mathcal{P}(\bar{W}')^{w\prime}$, and hence

$$S_{\mathcal{P}(W)} \subset S^*_{\mathcal{P}(W)^w} = S_{\mathcal{P}(W)^{w\prime}} = S_{\mathcal{M}_{\min}(W)}. \tag{20}$$

In the same way

$$S_{\mathcal{P}(\bar{W}')} \subset S_{\mathcal{M}_{\min}(\bar{W}')}. \tag{21}$$

Now

$$S^*_{\mathcal{P}(W)^w} = S_{\mathcal{P}(\bar{W}')} \tag{22}$$

by $\mathcal{W}$-duality of $\mathcal{P}(\cdot)$, so

$$S^*_{\mathcal{M}_{\min}(W)} \subset S_{\mathcal{M}_{\min}(\bar{W}')}. \tag{23}$$

Locality of $\mathcal{M}_{\min}(\cdot)$ gives the other inclusion, so

$$S^*_{\mathcal{M}_{\min}(W)} = S_{\mathcal{M}_{\min}(\bar{W}')}, \tag{24}$$

which by Proposition 1 (or Proposition 2) implies (19). QED

Next we consider the class $\mathcal{K}$ ("double cones"), related to $\mathcal{W}$ ("wedges") by the conditions (K) and (KW). For a local net $\mathcal{M}(\cdot)$ of von Neumann algebras *$\mathcal{K}$-duality* means that

$$\mathcal{M}(K)' = \mathcal{M}(\bar{K}') \tag{25}$$

where $\mathcal{M}(\bar{K}')$ is the von Neumann algebra generated by all $\mathcal{M}(K_1)$ with $K_1 \subset \bar{K}'$. By the same computation as above it is clear that if the net $\mathcal{P}(\cdot)$ satisfies $\mathcal{K}$-duality in the sense that that (14) holds with $W \in \mathcal{W}$ replaced by $K \in \mathcal{K}$, then $\mathcal{M}_{\min}(\cdot))$ satisfies (19). However, even if this not the case, $\mathcal{W}$-duality of $\mathcal{P}(\cdot)$ implies at least that $\mathcal{M}_{\min}(\cdot)$ has an *extension* satisfying $\mathcal{K}$-duality. In fact we have

THEOREM 3. – *Suppose $\mathcal{P}(\cdot)$ satisfies the conditions of Theorem 2 so that $\mathcal{M}_{\min}(K) = \mathcal{P}(K)^{w\prime}$, $K \in \mathcal{K}$, is a local net of von Neumann algebras. If $\mathcal{P}(\cdot)$ also satisfies $\mathcal{W}$-duality, then $\mathcal{M}_{\min}(\cdot)$ has a unique extension satisfying $\mathcal{K}$-duality, namely*

$$\mathcal{M}_{\max}(K) = \mathcal{P}(K')^{w} \tag{26}$$

for $K \in \mathcal{K}$. All local nets of von Neumann algebras to which $\mathcal{P}(\cdot)$ is associated lie between $\mathcal{M}_{\min}(\cdot)$ and $\mathcal{M}_{\max}(\cdot)$.

Proof. – The key point is that $\mathcal{W}$-duality of $\mathcal{M}_{\min}(\cdot)$ (which follows from $\mathcal{W}$-duality of $\mathcal{P}(\cdot)$ by Proposition 3) implies

$$\bigcap_{W \supset K, W \in \mathcal{W}} \mathcal{M}_{\min}(W) = \mathcal{M}_{\min}(K')'. \tag{27}$$

This is a simple computation, cf. [1], Lemma 4.1. The left side of (27) is clearly a local net, hence the right side, $\mathcal{M}_{\max}(K)$, is also local, i.e., $\mathcal{M}_{\max}(K) \subset \mathcal{M}_{\max}(K')'$. On the other hand, by locality of $\mathcal{M}_{\min}(\cdot)$, $\mathcal{M}_{\min}(K') \subset \mathcal{M}_{\max}(K')$ and hence $\mathcal{M}_{\max}(K')' \subset \mathcal{M}_{\min}(K')' = \mathcal{M}_{\max}(K)$. Thus $\mathcal{M}_{\max}(\cdot)$ satisfies $\mathcal{K}$-duality. The other assertions are obvious, since duality of a net implies that it has no proper local extension. QED

Remark. – Theorem 4 can also be stated as the assertion that $\mathcal{W}$-duality of $\mathcal{P}(\cdot)$ implies *essential duality* of $\mathcal{M}_{\min}(\cdot)$. Essential duality of a local net $\mathcal{M}(\cdot)$ is by definition the property that $\mathcal{M}^{d}(K) := \mathcal{M}(K')'$ is also a local net [15].

Our last result states that the net $\mathcal{M}_{\max}(\cdot)$ can be generated already by a subnet $\mathcal{P}_0(\cdot)$ of $\mathcal{P}(\cdot)$, provided $\mathcal{P}_0(\cdot)$ satisfies $\mathcal{W}$-duality.

THEOREM 4. – *Let $\mathcal{P}(\cdot)$ satisfy the premises of Theorem 2 and let $\mathcal{P}_0(\cdot)$ be a subnet satisfying the same conditions. Suppose furthermore that Ω is cyclic for $\mathcal{P}_0(W)$ and $\mathcal{P}_0(\bar{W}')$, $W \in \mathcal{W}$, and that $\mathcal{P}_0(\cdot)$ satisfies $\mathcal{W}$-duality. Then the conclusions of Theorem 3 hold, and moreover*

$$\mathcal{M}_{\max}(K) = \mathcal{P}_0(K')^{w}, \tag{28}$$

i.e., $\mathcal{P}(\cdot)$ is associated with a net of von Neumann algebras that is generated by $\mathcal{P}_0(\cdot)$ and satisfies $\mathcal{K}$-duality.

Proof. – $\mathcal{W}$-duality for $\mathcal{P}_0(\cdot)$ means that $S^*_{\mathcal{P}_0(W)} = S_{\mathcal{P}_0(\bar{W}')}$ for all $W \in \mathcal{W}$. Since $\mathcal{P}_0(W) \subset \mathcal{P}(W)$ and $\mathcal{P}_0(\bar{W}') \subset (\mathcal{P}(W), \mathcal{D})^{uw}$, we have $S_{\mathcal{P}_0(W)} \subset S_{\mathcal{P}(W)}$ and $S_{\mathcal{P}_0(\bar{W}')} \subset S^*_{\mathcal{P}(W)}$. Hence also $S^*_{\mathcal{P}_0(W)} \subset S^*_{\mathcal{P}(W)}$ which implies $S_{\mathcal{P}_0(W)} = S_{\mathcal{P}(W)}$. In the same way, $S_{\mathcal{P}_0(\bar{W}')} = S_{\mathcal{P}(\bar{W}')}$, so $\mathcal{P}(\cdot)$ also satisfies $\mathcal{W}$-duality. Equality of the Tomita conjugations implies that $\mathcal{P}(W)^w = \mathcal{P}_0(W)^w$ by Proposition 2. As in the proof of Theorem 3 this implies that the maximal nets generated by $\mathcal{P}_0(\cdot)$ and $\mathcal{P}(\cdot)$ coincide. QED

Note. – By transitivity of locality, Theorem 1, we know that the (unbounded) net $\mathcal{P}(\cdot)$ generated by all nets that are relatively local to a fixed net $\mathcal{P}_0(\cdot)$ is local, provided $\mathcal{P}_0(\cdot)$ satisfies $\mathcal{W}$-duality. Thus the last theorem states that under this condition the whole equivalence class of (unbounded) nets that are relatively local to $\mathcal{P}_0(\cdot)$ can be associated with a unique local net $\mathcal{M}_{\max}(\cdot)$ of von Neumann algebras that satisfies $\mathcal{K}$-duality and is generated by $\mathcal{P}_0(\cdot)$. In [7] the point is stressed that $\mathcal{P}_0(\cdot)$ can be quite "small"; it might even be generated by a *single* field operator, smeared with a test function whose Fourier transform wanishes nowhere, and its Poincaré transforms.

REFERENCES

[1] H. J. BORCHERS and J. YNGVASON, Transitivity of locality and duality in quantum field theory. Some modular aspect, *Rev. Math. Phys.*, Vol. **6**, 1994, pp. 597-619.

[2] H. J. BORCHERS and J. YNGVASON From Quantum Fields to Local von Neumann Algebras, *Rev. Math. Phys.*, Special Issue, 1992, pp. 15-47.

[3] H. J. BORCHERS and W. ZIMMERMANN, On the Self-Adjointness of Field Operators, *Nuovo Cimento*, Vol. **31**, 1964, pp. 1047-1059.

[4] D. BUCHHOLZ and K. FREDENHAGEN, Locality and the structure of particle states, *Commun. Math. Phys.*, Vol. **84**, 1982, pp. 1-54.

[5] W. DRIESSLER and J. FRÖHLICH, The reconstruction of local observable algebras from Euclidean Green's functions of relativistic quantum field theory, *Ann. Inst. Henri Poincaré*, Vol. **27**, 1977, pp. 221-236.

[6] J. GLIMM and A. JAFFE A. *Quantum Physics* , Berlin-Heidelberg-New York: Springer, 1981.

[7] W. DRIESSLER, S. SUMMERS and E. H. WICHMANN, On the Connection between Quantum Fields and von Neumann Algebras of Local Observables, *Commun. Math. Phys.*, Vol. **105**, 1986, pp. 49-84.

[8] D. BUCHHOLZ, On Quantum Fields that Generate Local Algebras, *J. Math. Phys.*, Vol. **31**, 1990, pp. 1839-1846.

[9] H. J. BORCHERS and J. YNGVASON, Positivity of Wightman Functionals and the Existence of Local Nets, *Commun. Math. Phys.*, Vol. **127**, 1990, pp. 607-615.

[10] J. BISOGNANO and E. H. WICHMANN, On the duality condition for a Hermitean scalar field, *J. Math. Phys.*, Vol. **16**, 1975, pp. 985-1007.

[11] J. BISOGNANO and E. H. WICHMANN, On the duality condition for quantum fields, *J. Math. Phys.*, Vol. **17**, 1976, pp. 303-321.

[12] H.-J. BORCHERS, Über die Mannigfaltigkeit der interpolierenden Felder zu einer kausalen S-Matrix, *Il Nuovo Cimento*, Vol. **15**, 1960, pp. 784-794.

[13] R. V. KADISON and J. R. RINGROSE, *Fundamentals of the Theory of Operator Algebras* II, New York: Academic press, 1986.
[14] R. T. POWERS, Self-Adjoint Algebras of Unbounded Operators, II, *Trans. Am. Math. Soc.*, Vol. **187**, 1974, pp. 261-293.
[15] J.ROBERTS, *Spontanously broken gauge symmetries and superselection rules*, in *International School of Mathematical Physics*, Universita' di Camerino, 1974.

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