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On the problem of defining a specific theory within the frame of local quantum physics

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ABSTRACT. – The notion and use of germs of states are discussed.

RÉSUMÉ. – La notion de germe d'état et son utilisation sont l'objet de cette discussion.

1. INTRODUCTION

The customary procedure of defining a specific theory or model in Quantum Field Theory starts by writing down a Lagrangean in terms of point-like fields. This defines first of all a classical field theory which is then subjected to a process called "quantization", either in the old fashioned form of replacing the numerical-valued fields by field operators with commutation relations or by a Feynman-Schwinger functional integral which should directly give the correlation functions of fields in the vacuum state. It appears that the physically most successful models (QED, QCD, standard model) are selected by a local gauge principle which strongly restricts the choice of the Lagrangean. This principle is well understood and very natural on the classical level. There it means that we deal with a fiber bundle. There appear two distinct kinds of fields. The first are the "matter fields" which give a coordination of the fiber above a point in space-time. There is no unique preferred coordinatization due to the existence of a symmetry group, the gauge group. The second kind is the connection form (gauge fields) which establishes the natural identification of points on neighboring fibers. The Lagrangean yields partial differential equations between these field quantities which serve to define those sections in the bundle which correspond to physically allowed field configurations. In the process of quantization the point-like fields become singular objects and, as a consequence, the non linear field equations cannot be carried over as they stand. Remedies to overcome these difficulties have been devised (infinite renormalization, etc.). They are pragmatically successful but even if one is willing to believe that one ultimately arrives at a well-defined theory in this way, the path by which one arrives at this is not very transparent. The formal Lagrangean is a heuristic crutch, but an indispensable one as long as no other way of defining the theory is known. So it seems desirable to develop an alternative approach which aims at defining the theory directly in the setting of relativistic quantum theory. We shall describe some first steps towards this goal; the solution of the problem mentioned in the title remains still far away.

We use the frame of Local Quantum Physics. It differs from conventional Quantum Field Theory in two respects. First, it does not start from quantities associated with points but with (open) space-time regions. To each such bounded region $\mathcal{O}$ there is an algebra $\mathcal{A}(\mathcal{O})$. This is natural, not only because we cannot assume *a priori* any algebraic structure for fields at a single point but also because it allows the possibility of more general generating elements than those derived from point-like fields, for instance gauge strings with end points marked by matter fields. Secondly we consider only the algebras generated by "observables"; in field theoretic language that means that its elements consist of gauge invariant quantities. There are good reasons for this too. For these elements we can assume Einstein causality: elements belonging to space-like separated regions commute. Further we can work in a positive definite Hilbert space (no ghost problems). Also all information is encoded in these algebras. We adopt the standard assumptions: Einstein causality, Poincaré invariance, positivity of the energy. The algebras $\mathcal{A}(\mathcal{O})$ will be taken as von Neumann algebras acting in a Hilbert space $\mathcal{H}$. We assume that space-time translations are implemented in $\mathcal{H}$ by unitary operators $U(a) = e^{i P_\mu a^\mu}$, the P_μ being interpreted as the global energy-momentum operators. For details *see* [1].

We take as the central message from Quantum Field Theory that all information characterizing the theory–apart from the two global features involving the translation operators and Einstein causality–is strictly local *i.e.* expressed in the structure of the theory in an arbitrarily small neighborhood of a point. If we take it serious that in quantum theory we always have to consider neighborhoods, we have to replace the notion of a fiber bundle by that of a sheaf. The needed information consists then of two parts: first the description of germs (of a presheaf), secondly the rules of joining the germs to obtain the theory in finite regions. We shall consider only the first part and can offer only a few remarks about the second.

2. GERMS OF STATES

As J. E. Roberts first pointed out [2], the notion of a presheef and its germs in state space is naturally related to the net structure of the algebras. Consider for each algebra $\mathcal{A}(\mathcal{O})$ the set of (normal) linear forms on it. It is a Banach space $\Sigma(\mathcal{O})$. The subset of positive, normalized linear forms are the states. The algebra $\mathcal{A}(\mathcal{O})$, considered as a Banach space, is the dual space of $\Sigma(\mathcal{O})$. To simplify the language let us denote by $\mathcal{A}$ the algebra of the largest region considered and by Σ the set of linear forms on it. In Σ we have a natural restriction map. If $\mathcal{O}_1 \subset \mathcal{O}$ then $\mathcal{A}(\mathcal{O}_1)$ is a subalgebra of $\mathcal{A}(\mathcal{O})$; hence a form $\varphi \in \Sigma(\mathcal{O})$ has a restriction to $\mathcal{A}(\mathcal{O}_1)$ thus yielding an element of $\Sigma(\mathcal{O}_1)$. We may also regard an element of $\Sigma(\mathcal{O})$ as an equivalence class of elements in Σ, namely the class of all forms on Σ which coincide on $\mathcal{A}(\mathcal{O})$. In this way we may pass to the limit of a point, defining two forms to be equivalent with respect to the point x

$$\varphi_1 \underset{x}{\sim} \varphi_2 \tag{1}$$

if there exists *any* neighborhood of x on which the restrictions of φ_1 and φ_2 coincide. Such an equivalence class will be called a germ at x and the set of all germs at x is a linear space Σ_x. It is, however, no longer a Banach space. Let us choose a 1-parameter family of neighborhoods $\mathcal{O}_r$ of x, e.g., standard double cones (diamonds) with base radius r centered at x. Denote the norm of the restriction of a form φ to $\mathcal{A}(\mathcal{O}_r)$ by $\|\varphi\|(r)$. This is a bounded function of r, non-increasing as $r \to 0$. The equivalence class $[\varphi]_x$ of forms with respect to the point x (the germ of φ) has as a characteristic, common to all its elements, the germ of the function $\|\varphi\|(r)$.

The next question is how to obtain a tractable description of Σ_x by which the distinction between different theories shows up. Here we remember one requirement on the theory whose relevance for various aspects has come to the foreground in recent years: the compactness or, in sharper form, the nuclearity property. Roughly speaking this property means that finite parts of phase space should correspond to finite dimensional subspaces in $\mathcal{H}$. Here phase space is understood as arising from a simultaneous restriction of the total energy and the space-time volume considered (*see* [1], chapter V.5 and the literature quoted there, especially [3] and [4]). Let Σ_E denote the subspace of forms which arise from matrix elements of the observables between state vectors in $\mathcal{H}_E$, the subspace of $\mathcal{H}$ with energy below E. Correspondingly we have $\Sigma_E(\mathcal{O})$, the restriction of these forms to $\mathcal{A}(\mathcal{O})$ and $\Sigma_E(x)$ their germs at x. The essential substance of the compactness requirement is that for any chosen accuracy ε there is a finite dimensional subspace $\Sigma_E^{(n)}(\mathcal{O})$, with its dimensionality n increasing with E and r, such that for any $\varphi \in \Sigma_E(\mathcal{O})$ there is an approximate $\hat{\varphi} \in \Sigma_E^{(n)}(\mathcal{O})$ satisfying

$$\|\varphi - \hat{\varphi}\| \leq \varepsilon \|\varphi\|. \tag{2}$$

As ε decreases n will increase. Since n is an integer, however, this will happen in discrete steps. For our purposes we shall adopt the following version of the compactness requirement whose relation to other formulation will not be discussed here.

For fixed E there is a sequence of positive, smooth functions $\varepsilon_k(r)$, $k = 0, 1, 2, \ldots$ with $\varepsilon_0 = 1$ and $\varepsilon_{k+1}/\varepsilon_k$ vanishing at $r = 0$. For each k there is a subspace $\mathcal{N}_k$ of $\Sigma_E(x)$ consisting of those germs for which $\|\varphi\|(r)$ vanishes stronger than ε_k at $r = 0$. There is a natural number n_k giving the dimension of the quotient space $\Sigma_E(x)/\mathcal{N}_k$. The n_k and the germs of the functions ε_k are characteristic of the theory.

We take the ε_k to be optimally chosen so that $\|\varphi\|(r) = \varepsilon_k$ is attained for some $\varphi \in \mathcal{N}_{k-1}$. To coordinatize the germs in Σ_E we choose an ordered basis $A_{k,l}(r) \in \mathcal{A}(\mathcal{O}_r)$ with $\|A_{k,l}\| = 1$ such that $|\varphi(A_{k,l}(r)| = \varepsilon_k(r)\|\varphi\|$ for $\varphi \in \Sigma_E/\mathcal{N}_k$, $l = 1, \ldots, (n_k - n_{k-1})$. It follows that $\mathrm{limit}_{r\to 0} A_{k,l}(r)/\varepsilon_k(r)$ exists as a bounded sesquilinear form on $\mathcal{H}_E$ which might be called a point-like field. However, since we know so far nothing about the dependence of the structure on the energy bound E this object could change with increasing E.

In the case of a dilatation invariant theory the situation is much simpler. On the one hand there are energy independent "scaling orbits" $A_{k,l}(r)$ provided by the dilatations. Further, the ε_k (leading to the same numbers n_k for all energies) are powers

$$\varepsilon_k(E, r) = (Er)^{\gamma_k}. \tag{3}$$

Therefore point-like fields

$$\Phi_{k,\,l} = \lim A_{k,\,l}\,(r)\,r^{-\gamma_k} \tag{4}$$

can be defined as sesquilinear forms on any $\mathcal{H}_E$. Their bound in $\mathcal{H}_E$ increases in proportion to E^{γ_k}. So the analysis of Fredenhagen and Hertel [5] applies which shows that the $\Phi_{k,\,l}$ are Wightman fields which furthermore have specific dimensions γ_k. The set of these is the Borchers class of observable fields in the theory and γ_k gives an ordering in this set. Of course, with Φ also the derivatives $\partial_\mu\,\Phi$ will appear in this set and one can also define a product of such fields as (the dual of) the germs of

$$r^{-(\gamma_1+\gamma_2)}\,\varphi\,(A_1\,(r)\,A_2\,(r)), \tag{5}$$

where the indices 1 and 2 stand for k_1, l_1 and k_2, l_2. Since $A_1\,A_2 \in \mathcal{A}\,(\mathcal{O}_r)$ and has norm bounded by 1 only the values of φ on the basis elements $A_{k,\,l}$ for low k will be important and give a contribution proportional to $r^{\gamma_k-\gamma_1-\gamma_2}$. Symbolically one can express this as an operator product expansion

$$\Phi_1\,\Phi_2 = \sum c_{k,\,l}\,(r)\,\Phi_k \tag{6}$$

where the coefficient functions $c_{k,\,l}$ behave like $r^{\gamma_k-\gamma_1-\gamma_2}$.

In the physically interesting case of asymptotic dilatation invariance it appears reasonable to expect that the essential aspects of this structure remain valid, at least for low k-values. Recently Buchholz and Verch [6] have formulated ideas of a renormalization group analysis in the algebraic language. Perhaps their approach can provide a starting point for a rigorous discussion of the germs in a theory with asymptotic dilatation invariance.

Summing up we can see that the n_k, γ_k together with the operator product expansion give strongly restrictive information about the theory. However we cannot expect that this defines the theory, in particular not in the case of a gauge theory where the set of point-like gauge invariant fields may not be sufficiently rich.

3. THE EXAMPLE OF A FREE FIELD

It is desirable to verify and illustrate the described structure in the case of a free field theory. Various versions of the compactness and nuclearity properties have been studied for this case in [4]. It turns out to be a considerably harder task to verify the version from which our discussion in the last section started. Buchholz recognized that the computation of the functions ε_k and the dimensionalities n_k amounts to the determination of the eigenvalues and degeneracies of a complicated operator in the single particle space of the theory and he succeeded in obtaining good estimates allowing to verify the claims made in the last section [7].

Here we shall confine ourselves to a few heuristic remarks which may serve to form an intuitive picture. Consider the theory of a free, scalar, neutral field Φ. The Hilbert space $\mathcal{H}$ is the Fock space of multiparticle states. We write $\Psi_n = |g_1, \ldots, g_n\rangle$ for the state vector of an n-particle state with wave functions g_i (irrespective of whether some of them are identical). The g_i are described as functions of the 3-momentum $\boldsymbol{p}$; we use the canonical scalar product

$$\langle g_1, g_2 \rangle = \int \overline{g_1(\boldsymbol{p})}\, g_2(\boldsymbol{p})\, d^3\boldsymbol{p}.$$

A (weakly) dense set in $\mathcal{A}(\mathcal{O})$ is given by linear combinations of Weyl operators $W(f) = e^{i\,\Phi(f)}$ where f runs through the test functions having support in $\mathcal{O}$. For $\mathcal{O} = \mathcal{O}_r$ centered at the origin we can represent the Weyl operators by their Cauchy data at $x^0 = 0$

$$\begin{aligned} W_1(f_1) &= \exp i \int \Phi(\boldsymbol{x})\, f_1(\boldsymbol{x})\, d^3\boldsymbol{x}; \\ W_2(f_2) &= \exp i \int \pi(\boldsymbol{x})\, f_2(\boldsymbol{x})\, d^3\boldsymbol{x} \end{aligned} \tag{7}$$

with $\pi = \partial_0 \Phi$ and f_α test functions in 3-space with support in the ball with radius r around the origin. The forms in $\Sigma(\mathcal{O})$ are linear combinations of matrix elements

$$\varphi_{nm}(A) = \langle \Psi_n |A| \Psi_m \rangle. \tag{8}$$

One has $(\alpha = 1, 2)$

$$\begin{aligned} &\langle g'_1, \ldots, g'_n |W_\alpha(f)| g_1, \ldots, g_m \rangle \\ &\quad = \sum e^{-\frac{1}{2}\langle f_\alpha | f_\alpha \rangle} \prod_j \langle g'_j | f_\alpha \rangle \prod_l \langle f_\alpha | g_l \rangle \prod_{i \neq j,\, k \neq l} \langle g'_i | g_k \rangle \end{aligned} \tag{9}$$

with

$$f_\alpha(\boldsymbol{p}) = \begin{cases} \tilde{f}(\boldsymbol{p})\, |\boldsymbol{p}|^{-\frac{1}{2}} & \text{for } \alpha = 1 \\ \tilde{f}(\boldsymbol{p})\, |\boldsymbol{p}|^{\frac{1}{2}} & \text{for } \alpha = 2 \end{cases} \tag{10}$$

where

$$\tilde{f}(\boldsymbol{p}) = \int f(\boldsymbol{x})\, e^{-i\boldsymbol{p}\boldsymbol{x}}\, d^3\boldsymbol{x}.$$

For moderate particle numbers the terms in (9) can be estimated by the following simple argument. If the total energies of the state vectors are below E then *a forteriori* the wave functions g, g' have to vanish for $|\boldsymbol{p}| > E$. We can therefore replace the f_α in the contractions with g, g' by f_E, resulting from f by cutting off the parts for $|\boldsymbol{p}| > E$. Due to the uncertainty relation $\tilde{f}(\boldsymbol{p})$ has essential support of at least extension r^{-1}. We consider the regime where $Er \ll 1$. Then the ratio $\|f_E\|/\|f\|$ will be largest if f is spherically symmetric and does not oscillate in x-space, implying

$$\tilde{f}(\boldsymbol{p}) \sim \text{constant} = a \qquad \text{for} \quad |\boldsymbol{p}| < r^{-1}; \qquad a = \int f(\boldsymbol{x})\, d^3\boldsymbol{x}.$$

With this we get

$$\|f_{\alpha,E}\|/\|f_\alpha\| = \begin{cases} E\,r & \text{for} \quad \alpha = 1 \\ (E\,r)^2 & \text{for} \quad \alpha = 2. \end{cases} \tag{11}$$

The term with k_1 contractions with f_1 and k_2 contractions with f_2 is thus bounded by

$$C\,(E\,r)^{k_1+2k_2}, \qquad C = \sup \|f\|^{k_1+k_2}\, e^{-\frac{1}{2}\|f\|^2}. \tag{12}$$

For moderate values of k the constant C is of order 1. The power in (12) gets higher if the 1-particle wave functions g are orthogonal to $g_0(\boldsymbol{p}) = |\boldsymbol{p}|^{-\frac{1}{2}}$ in the case of W_1, respectively to $h(\boldsymbol{p}) = |\boldsymbol{p}|^{\frac{1}{2}}$ in the case of W_2. This implies that if we require an accuracy $\varepsilon = (E\,r)^\gamma$ the approximate dimensionality of $\Sigma_E(\mathcal{O})$ is

for $\gamma < \gamma_1 = 1$ $n = 1$. All states are equivalent to the vacuum;

for $\gamma < \gamma_2 = 2$ $n = 2$. Besides the vacuum expectation value we need the matrix element between the vacuum and the 1-particle state with wave function g_0. We do not have to distinguish between $\langle g_0| \cdot |0\rangle$ and $\langle 0| \cdot |g_0\rangle$. They give the same contribution because $\tilde{f}$ is real;

for $\gamma < \gamma_3 = 3$ $n = 7$. We need the additional matrix elements $\langle h| \cdot |0\rangle$, $\langle g_0| \cdot |g_0\rangle$, and $\langle g_i| \cdot |0\rangle$ where $g_i = p_i\,|\boldsymbol{p}|^{-\frac{1}{2}}$, $i = 1, 2, 3$.

This agrees with the known "field content". As a dual basis we can choose on the lowest level the identity operator, on the next level the field Φ, then the derivatives $\partial_\mu \Phi$ and the Wick product: Φ^2. The levels are distinguished by the "canonical dimensions" of the various elements in the Borchers class.

Of course these heuristic remarks are far from being a proof. The level of non triviality of [7] may be assessed by noting that we have to consider not only a single Weyl operator but need a uniform estimate for all elements in $\mathcal{A}(\mathcal{O}_r)$ and a uniform estimate for all forms including those with high particle numbers.

4. CONCLUSIONS

The accuracy functions ε_k and corresponding dimensionalities give some restrictive information about the theory. If, in addition, one can define point-like fields and an operator product decompostion (which one might hope for in the case of an asymptotically dilatation invariant theory, at least for low k-values) then one may also have some relations between the elements in the dual basis, corresponding to field equations. However, these relations cannot be expected to contain all the information of the field equations in a gauge theory since, for good reasons mentioned in the introduction, the point-like fields arising are only the gauge invariant ones. In other words, the analogue of the connections in the classical case has not yet been incorporated. But even for a simple model in which only observable fields occur in the Lagrangean there remains the untackled problem of going over from the germs to the sections of a sheaf which describe the theory in finite regions. One idea concerning this problem has been suggested in the last section of [8]. It is, however, rather abstract and probably insufficient. So the major part in solving the problem stated in the title remains to be done.

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