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Remarks on intersection-equivalence and capacity-equivalence

by

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ABSTRACT. – Two random sets are intersection-equivalent if the probabilities that they hit any given set are comparable; two fixed sets are capacity-equivalent if they have positive capacity in the same (distance-dependent) kernels. We survey recent results on these equivalence relations, emphasizing their connections with Hausdorff dimension. We also describe an example which illustrates the role of probabilistic uniformity in capacity estimates for random sets.

Key words: capacity, Hausdorff dimension, Brownian motion, tree, percolation.

RÉSUMÉ. – Deux ensembles aléatoires sont appelés « intersection équivalents » si les probabilités qu'ils rencontrent un autre sous ensemble quelconque sont comparables, deux ensembles fixes sont appelés « capacité équivalents » s'ils ont tous deux une capacité positive par rapport à un même noyau (fonction de la distance). Nous passons en revue les résultats connus concernant ces deux relations d'équivalence en insistant sur leur relation avec la dimension de Hausdorff. Nous donnons aussi un exemple illustrant le rôle de l'uniformité probabiliste dans l'évaluation de la capacité d'ensembles aléatoires.

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1. INTERSECTION-EQUIVALENCE

Comparisons between different Markov processes have a long history. We focus here on comparisons between the ranges of Markov processes and random sets constructed by repeated "random cutouts". For example, it is classical that the zero-set of one-dimensional Brownian motion can be constructed by iteratively removing excursion intervals. (*See* [6] for an application of this to exact Hausdorff measures and [13] for some extensions to other "Renewal sets" on the line.) In higher dimensions, no such exact distributional identity is known, but the following weaker equivalence relation is useful.

DEFINITION 1. – *Two random sets A and B in a metric space (more precisely, their distributions) are* intersection-equivalent *in the open set U, if there exists a constant $C > 0$ such that for any closed set $\Lambda \subset U$, we have*

$$C^{-1}\mathbf{P}(A \cap \Lambda \neq \emptyset) \leq \mathbf{P}(B \cap \Lambda \neq \emptyset) \leq C\mathbf{P}(A \cap \Lambda \neq \emptyset). \qquad (1)$$

Using this terminology, we state a special case of a key theorem from Lyons [12].

THEOREM 1.1 (Lyons). – *Let T be a (finite or infinite) rooted tree where all vertices have finite degrees. Consider an independent percolation on T, in which each edge at distance k from the root is removed with probability $1/(k+1)$, and retained with probability $k/(k+1)$. Denote by $\mathcal{C}(\rho)$ the connected cluster containing the root. Denote by $\mathcal{R}$ the range of a simple random walk on T, started at the root and killed if and when it returns there. Then $\mathcal{C}(\rho)$ and $\mathcal{R}$ are intersection-equivalent. (In particular, simple random walk on T is transient iff the percolation cluster $\mathcal{C}(\rho)$ is infinite with positive probability.)*

In fact, [12] gives a general correspondence between edge-removal probabilities in percolation and transition probabilities for nearest-neighbor random walk, where the same equivalence is valid. This has implications in Euclidean space.

Given $d \geq 3$ and $0 < p < 1$, successive subdivisions of a cube into binary subcubes define a natural mapping from the regular tree of forward degree 2^d, to the unit cube in $\mathbb{R}^d$; the set $Q_d(p)$ is constructed by performing independent percolation with parameter p on this tree and considering the set of infinite paths emanating from the root in its percolation cluster. More formally, consider the natural tiling of the unit cube $[0,1]^d$ by 2^d closed cubes of side $1/2$. Let Z_1 be a random subcollection of these cubes, where each cube has probability p of belonging to Z_1, and these events are mutually independent. At the k'th stage, if Z_k is nonempty, tile each cube $Q \in Z_k$ by 2^d closed subcubes of side 2^{-k-1} (with disjoint interiors) and include each of these subcubes in Z_{k+1} with probability p (independently). Finally, define

$$Q_d(p) = \bigcap_{k=1}^{\infty} \bigcup_{Q \in Z_k} Q .$$

These sets were proposed in [14] as models which capture some features of turbulence.

THEOREM 1.2 ([19]). – *Let $B_d(t)$ denote d-dimensional Brownian motion, started according to any fixed distribution with a bounded density for $B_d(0)$. If $d \geq 3$ then the* range $[B_d] = \{B_d(t) : t \geq 0\}$ *is intersection-equivalent to $Q_d(2^{2-d})$ in the unit cube.*

This theorem is useful because the intersection of two Brownian paths is more complicated than a single path, while the intersection of two independent copies of $Q_d(p)$ has the same distribution as $Q_d(p^2)$. Applications to intersection properties of Brownian motion and other processes may be found in [19].

The following sufficient condition for intersection-equivalence in the discrete setting was given in Benjamini, Pemantle, and Peres [1].

PROPOSITION 1.3. – *Suppose the Green functions for two Markov chains on the same countable state space (with the same initial state) ρ are bounded by constant multiples of each other. (It suffices that this bounded ratio property holds for the corresponding Martin kernels $K(x, y) = G(x, y)/G(\rho, y)$ or for their symmetrizations $K(x, y) + K(y, x)$.) Then the ranges of the two chains are intersection-equivalent.*

It is easy to see that if W_1 and W_2 are intersection-equivalent then $\mathbf{P}[|W_1 \cap F| = \infty] > 0$ iff $\mathbf{P}[|W_2 \cap F| = \infty] > 0$ for all sets F. The above proposition implies Theorem 1.1, since left-to-right enumeration of the set of leaves of a tree which remain connected to the root after independent percolation defines a Markov chain (*see* [1] for details). Another example where Proposition 1.3 applies involves simple random walk on the lattice $\mathbf{Z}^3$: The set of epochs at which the random walk visits the z-axis, is intersection-equivalent to the set of positive n such that the point $(0, 0, n)$ is in the range of the walk.

2. APPLICATIONS TO HAUSDORFF DIMENSION

PROPOSITION 2.1. – *Suppose that* $\beta > 0$, *and* W *is a random closed set in* $\mathbb{R}^d$.

1. *If each closed set* $\Lambda \subset \mathbb{R}^d$ *with Hausdorff dimension* $\dim_H(\Lambda) > \beta$ *intersects* W *with positive probability, then each such set* Λ *satisfies*

$$\mathbf{P}[\dim_H(\Lambda \cap W) > \dim_H(\Lambda) - \beta - \epsilon] > 0 \quad \text{for every } \epsilon > 0.$$

2. *If each closed set* K *with* $\dim_H(K) < \beta$ *is disjoint from* W *with probability* 1, *then any closed set* Λ *of Hausdorff dimension at least* β *satisfies*

$$\dim_H(\Lambda \cap W) \leq \dim_H(\Lambda) - \beta \text{ almost surely.}$$

The proposition is a consequence of the following lemma due to J. Hawkes; *see* [11] for a simpler proof of this lemma.

LEMMA 2.2 (Hawkes [9]). – *Let* $\gamma > 0$. *For any closed set* $\Lambda \subset [0,1]^d$ *we have*

(i) *If* $\dim_H(\Lambda) < \gamma$ *then* $\Lambda \cap Q_d(2^{-\gamma})$ *is almost surely empty.*

(ii) *If* $\dim_H(\Lambda) > \gamma$ *then* Λ *intersects* $Q_d(2^{-\gamma})$ *with positive probability.*

(iii) *If* $\dim_H(\Lambda) \geq \gamma$ *then*

$$||\dim_H(\Lambda \cap Q_d(2^{-\gamma}))||_\infty = \dim_H(\Lambda) - \gamma,$$

where the L^∞ *norm is taken in the underlying probability space.*

We note that part (iii) follows easily from (i) and (ii), since the intersection $Q_d(2^{-\gamma}) \cap Q'_d(2^{-\delta})$ has the same distribution as $Q_d(2^{-\gamma-\delta})$ (where $Q'_d(2^{-\delta})$ is an independent copy of $Q_d(2^{-\delta})$).

Proof of Proposition 2.1. – It is easy to reduce to the case where Λ is a subset of the unit cube.

1. Given $\epsilon > 0$ and a closed set Λ of dimension greater than β, choose γ such that

$$\dim_H(\Lambda) - \beta - \epsilon < \gamma < \dim_H(\Lambda) - \beta.$$

Perform fractal percolation with parameter $p = 2^{-\gamma}$, independently of W. By part (iii) of the lemma, $\dim_H(\Lambda \cap Q_d(2^{-\gamma})) > \beta + \epsilon$ with positive probability. By the given property of W, the triple intersection $W \cap \Lambda \cap Q_d(2^{-\gamma})$ is nonempty with positive probability. By part (i) of the lemma, $W \cap \Lambda$ must have dimension at least γ with positive probability.

2. Let $\gamma > \dim_H(\Lambda) - \beta$. By part (iii) of the lemma $\dim_H(\Lambda \cap Q_d(2^{-\gamma})) < \beta$ a.s., so the assumption on W implies that $W \cap \Lambda \cap Q_d(2^{-\gamma}) = \emptyset$ a.s. Part (ii) of the lemma yields $\dim_H(W \cap \Lambda) \leq \gamma$, and since γ can take any value greater than $\dim_H(\Lambda) - \beta$, this concludes the proof. $\square$

Examples

1. **Random lines:** Let L be a random line, such that its direction is uniformly distributed on $[0, 2\pi)$, and its distance from the origin is independent of the direction and has a distribution mutually absolutely continuous with Lebesgue measure on $[0, \infty)$.

Marstrand's projection theorem [15] asserts that for any closed set Λ of dimension greater than 1 in the plane, the orthogonal projections of Λ to lines in almost all directions have positive Lebesgue measure. In particular, this implies that $\mathbf{P}[L \cap \Lambda \neq \emptyset] > 0$. Applying Proposition 1.2, we obtain a version of *Marstrand's intersection theorem* [15], which usually requires a seperate proof:

$$|| \dim_H(L \cap \Lambda)||_\infty = \dim_H(\Lambda) - 1\,.$$

Mattila [16] established analogues of Marstrand's theorems in higher dimensions, some of which can be proved in the same way.

2. **Super-Brownian motion.** Dawson, Iscoe, and Perkins [4] showed that the closed support W of the measure-valued diffusion called Super-Brownian motion in $\mathbb{R}^d$ has the following property for $d > 4$: The random set W intersects sets of dimension greater than $d-4$ with positive probability and a.s. misses sets of dimension less than $d-4$. Thus both parts of Proposition 2.1 may be applied.

3. CAPACITY-EQUIVALENCE

Let ν be a Borel measure on a metric space (X, η). Let $f : [0, \infty) \to [0, \infty]$ be a decreasing continuous kernel function. (We allow $f(0)$ to be infinite.) Define the **energy** of ν with respect to f by

$$\mathcal{E}_f(\nu) = \int_X \int_X f(\eta(x, y))\, d\nu(x)\, d\nu(y)$$

and the **capacity** of a Borel set $\Lambda \subset X$ with respect to f by

$$\mathrm{Cap}_f(\Lambda) = [\inf_{\nu(\Lambda)=1} \mathcal{E}_f(\nu)]^{-1}.$$

When $f(r) = r^{-\alpha}$, we write Cap_α for Cap_f, and then Frostman's theorem ensures that $\sup\{\alpha : \mathrm{Cap}_\alpha(\Lambda) > 0\} = \dim_H(\Lambda)$ for any closed set Λ. (*see*, e.g., Kahane [10], page 133).

In [17] the following equivalence relation was introduced in order to study Galton-Watson trees.

DEFINITION 2. – *The sets A, B in metric spaces X_1, X_2, are* **capacity-equivalent** *if there exist positive constants C_1, C_2 such that*

$$C_1 \mathrm{Cap}_f(B) \leq \mathrm{Cap}_f(A) \leq C_2 \mathrm{Cap}_f(B) \quad \textit{for all} \quad f.$$

Many Markov processes have capacity criteria for hitting sets, (*see* [3], [5], [8] and the references therein). Thus, loosely speaking, capacity-equivalence is a dual notion to intersection-equivalence.

The following two theorems on capacity-equivalence were proved in joint work with R. Pemantle and J. W. Shapiro.

THEOREM 3.1 ([18]). – *The trace $B[0,1]$ of Brownian motion in Euclidean space of dimension $d \geq 3$, is a.s. capacity-equivalent to the unit square $[0,1]^2$.*

THEOREM 3.2. – *The zero-set of one-dimensional Brownian motion, $Z = \{t \in [0,1] : B_t = 0\}$ is a.s. capacity-equivalent to the middle-1/2 Cantor set*

$$\mathcal{K}_{1/2} = \left\{ \sum_{n=1}^{\infty} b_n 4^{-n} : b_n = 0, 3 \right\}.$$

Hawkes ([7] Theorem 5) established that, for a fixed log-convex f, finiteness of the integral

$$\int_0^1 f(r) r^{-1/2} \, dr, \tag{2}$$

is necessary and sufficient for the Brownian zero set Z to a.s. have positive capacity with respect to f. (Sufficiency was proved earlier by Kahane in the first (1968) edition of [10]; *see* [10] page 236, Theorem 2.) Since it is easy to check that finiteness of the integral (2) is equivalent to $\mathrm{Cap}_f(\mathcal{K}_{1/2}) > 0$, Theorem 3.2 is a uniform version of this result of Kahane and Hawkes. Next, we describe a different random set that illustrates why the uniformity in the kernel is not automatic. This set will be the boundary of a random tree with at most four children for every vertex.

Notation. – An infinite self-avoiding path in a tree T, starting at the root of T, is called a **ray**. The set of all rays is called the **boundary** of T

and denoted ∂T. If two rays $x, y \in \partial T$ have exactly n edges in common, we let $\eta(x, y) = 4^{-n}$. Then $(\partial T, \eta)$ is a compact metric space provided T is locally finite. For every vertex v we write $|v|$ for its level, *i.e.*, the number of edges between it and the root. Neighbours of v at level $|v|+1$ are called *children* of v. The set of vertices at level n of T is denoted T_n, and its cardinality is denoted $|T_n|$.

We note that if every vertex of T has at most 4 children, then T corresponds via base 4 representation to a compact subset of the unit interval; the natural mapping from $(\partial T, \eta)$ to the unit interval is Lipshitz, and changes capacities only by a bounded multiplicative factor (*see* [2] and [17]).

Next, we recall an alternative representation of energy on the boundary of a tree T, which is obtained via summation by parts (*see* [11], [2] or [17]). Given a kernel f, write

$$\widetilde{f}(n) = f(4^{-n}) - f(4^{1-n}) \quad \text{for } n > 0, \quad \text{and} \quad \widetilde{f}(0) = f(1). \tag{3}$$

Then for any measure μ on ∂T,

$$\mathcal{E}_f(\mu) = \sum_{v \in T} \widetilde{f}(|v|)\mu(v)^2 , \tag{4}$$

where $\mu(v)$ is the measure of the set of rays going through v.

In particular, if the tree is *spherically symmetric* (*i.e.*, the number of children of any vertex v depends only on the level $|v|$) then the uniform measure has minimal energy among all probability measures, and

$$[\mathrm{Cap}_f(\partial T)]^{-1} = \sum_{n \geq 0} \widetilde{f}(n)|T_n|^{-1} . \tag{5}$$

PROPOSITION 3.3. – *Let $T^{(2)}$ denote an infinite binary tree. There is a tree-valued random variable Γ with the following properties:*

(i) *For any fixed kernel f, with probability one,* $\mathrm{Cap}_f(\partial\Gamma) > 0$ *if and only if* $\mathrm{Cap}_f(\partial T^{(2)}) > 0$.

(ii) *The boundary of Γ is* not *capacity-equivalent to the boundary of the binary tree $T^{(2)}$; indeed there exists a random kernel $h = h_\Gamma$ (depending on the sample Γ) that satisfies* $\mathrm{Cap}_h(\partial T^{(2)}) > 0$ *but gives* $\mathrm{Cap}_h(\partial\Gamma) = 0$ *almost surely.*

Proof. – Consider the random spherically-symmetric tree Γ, constructed as follows. For each $k \geq 1$, pick a random integer n_k uniformly in the interval $[3^k + k, 3^{k+1} - k]$, with all picks independent; define Γ to be the random tree where every vertex at level n has

$$\begin{cases} \text{one child} & \text{for } n \in [n_k - k, n_k) \\ \text{four children} & \text{for } n \in [n_k, n_k + k) \\ \text{and two children} & \text{otherwise.} \end{cases}$$

(i) Let f be any kernel. Since $|\Gamma_n| \leq 2^n$ for every n, the expression (5) for capacity implies that $\mathrm{Cap}_f(\partial\Gamma) \leq \mathrm{Cap}_f(\partial T^{(2)})$. For the converse, note that for $n \in [3^k, 3^{k+1})$, the cardinality of Γ_n is at least 2^{n-k}, and differs from 2^n with probability less than $k3^{-k}$. Therefore

$$\mathbf{E}[|\Gamma_n|^{-1}] \leq 2^{-n}(1 + k3^{-k}2^k). \tag{6}$$

assume that $\mathrm{Cap}_f(\partial T^{(2)}) > 0$, *i.e.*, $\sum_n \widetilde{f}(n)2^{-n} < \infty$. By (5) and (6), the expectation of $[\mathrm{Cap}_f(\partial\Gamma)]^{-1}$ is finite, so that $\mathrm{Cap}_f(\partial\Gamma) > 0$ almost surely.

(ii) Given the random sequence $\{n_k\}$ in the construction of Γ, we define $\widetilde{h}$ on the nonnegative integers by $\widetilde{h}(n_k) = 2^{n_k - k}$ and $\widetilde{h}(n) = 0$ if $n \notin \{n_k\}_{k\geq 1}$. Letting $h(4^{-n}) = \sum_{j=0}^{n} \widetilde{h}(j)$, we obtain from (5) that $\mathrm{Cap}_h(\partial T^{(2)}) = [\sum_k 2^{n_k - k}2^{-n_k}]^{-1} > 0$ but on the other hand $|\Gamma_{n_k}| = 2^{n_k - k}$ so that $\mathrm{Cap}_h(\partial\Gamma) = [\sum_k 2^{n_k - k}2^{k - n_k}]^{-1} = 0$. $\square$

REFERENCES

[1] I. Benjamini, R. Pemantle and Y. Peres, Martin capacity for Markov chains, *Ann. Probab.*, Vol. **23**, 1994, pp. 1332-1346.

[2] I. Benjamini and Y. Peres, Random walks on a tree and capacity in the interval, *Ann. Inst. Henri Poincaré*, Vol. **28**, 1992, pp. 557-592.

[3] K. L. Chung, Probabilistic approach in potential theory to the equilibrium problem, *Ann. Inst. Fourier, Grenoble*, Vol. **23**, 1973, pp. 313-322.

[4] D. Dawson, I. Iscoe and E. A. Perkins, Super-Brownian motion: Path properties and hitting probabilities, *Probab. Th. Rel. Fields*, Vol. **83**, 1989, pp. 135-206.

[5] P. J. Fitzsimmons and T. Salisbury, Capacity and energy for multiparameter Markov processes, *Ann. Inst. Henri Poincarè, Probab.*, Vol. **25**, 1989, pp. 325-350.

[6] S. Graf, R. D. Mauldin and S. C. Williams, The exact Hausdorff dimension in random recursive constructions, *Memoirs*, the American Mathematical Society, Providence, Rhode Island.

[7] J. Hawkes, Local properties of some Gaussian processes, *Zeits. Wahr. verw. Geb.*, Vol. **40**, 1977, pp. 309-315.

[8] J. Hawkes, Potential theory of Lévy processes, *Proc. London Math. Soc.*, (3), Vol. **38**, 1979, pp. 335-352.

[9] J. Hawkes, Trees generated by a simple branching process, *J. London Math. Soc.*, Vol. **24**, 1981, pp. 373-384.

[10] J. P. Kahane, *Some random series of functions*, Second edition, Cambridge University Press, 1985.

[11] R. Lyons, Random walks and percolation on trees, *Ann. Probab.*, Vol. **18**, 1990, pp. 931-958.

[12] R. LYONS, Random walks, capacity and percolation on trees, *Ann. Probab.*, Vol. **20**, 1992, pp. 2043-2088.
[13] B. B. MANDELBROT, Renewal sets and random cutouts, *Zeits. Wahr. verw. Geb.*, Vol. **22**, 1972, pp. 145-157.
[14] B. B. MANDELBROT, Intermittent turbulence in self-similar cascades: divergence of high moments and dimension of the carrier, *J. of Fluid Mechanics*, Vol. **62**, 1974, pp. 331-358.
[15] J. M. MARSTRAND, Some fundamental geometrical properties of plane sets of fractional dimensions, *Proc. London Math. Soc.*, Vol. **4**, 1954, pp. 257-302.
[16] P. MATTILA, Hausdorff dimension, orthogonal projections and intersections with planes, *Ann. Acad. Sci. Fenn., Ser. A, Math.*, Vol. **1**, 1975, pp. 227-244.
[17] R. PEMANTLE and Y. PERES, Galton-Watson trees with the same mean have the same polar sets, *Ann. Probab.*, Vol. **23**, 1995, pp. 1102-1124.
[18] R. PEMANTLE, Y. PERES and J. W. SHAPIRO, The trace of spatial Brownian motion is capacity-equivalent to the unit square, to appear in *Probab. Th. Rel. Fields*, 1996.
[19] Y. PERES, Intersection-equivalence of Brownian paths and certain branching processes, to appear in *Comm. Math. Phys.*, 1996.

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