

# ANNALES DE L'I. H. P., SECTION A

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*Annales de l'I. H. P., section A*, tome 64, n° 3 (1996), p. 309-323

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## **Controlling Griffiths' singularities**

by

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**ABSTRACT.** – We give new techniques for studying the smoothness of quenched correlation functions. A new proof for the Random Field Ising Model in the strong field regime is also given.

**RÉSUMÉ.** – Nous décrivons une nouvelle technique pour démontrer que la magnétisation (ou toute fonction de corrélation « trempée ») est infiniment différentiable.

Cette méthode est appliquée au cas du modèle d'Ising en champ aléatoire.

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### **1. INTRODUCTION**

In 1969 Robert Griffiths [8] discussed some novel features displayed by random ferromagnetic systems. He considered the statistical mechanics of

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\* Supported in part by the NSF Grant DMS-9208029.

† Supported in part by the CNPq and FAPESP.

a site dilute Ising model with energy function given by

$$H = - \sum_{\langle xy \rangle} J_{xy} \sigma_x \sigma_y + h \sum_x \sigma_x \quad (1.1)$$

with  $J_{xy} = \xi_x \xi_y$ , where the independent random variables  $\xi_x$  are 1 or 0 with probability  $p$  and  $1-p$  respectively. Here  $x \in \mathbb{Z}^d$ ,  $\langle xy \rangle$  denotes a pair of nearest neighbor sites in  $\mathbb{Z}^d$ ,  $\sigma_x = \pm 1$ ; we will use  $\mathbb{E}$  and  $\mathbb{P}$  to denote the expectation and probability measure in the underlying probability space of the random parameters and  $\langle \cdot \rangle$  to denote thermal averages. Griffiths showed that the quenched magnetization,

$$m(h) = \mathbb{E}(\langle \sigma_x \rangle), \quad (1.2)$$

considered as a function of  $z = e^{2\beta h}$ , exhibits non-analytical behavior at  $z = 1$  for any  $\beta > \beta_c$ , where  $\beta_c$  is the critical inverse temperature for the homogeneous system (*i.e.*,  $p = 1$ ), if  $p < p_c$ , the critical site percolation probability. Notice that in this case a typical realization of the system consists of finite noninteracting islands of spins, so clearly there is no spontaneous magnetization and no long range order. But those typical realizations also contain infinitely many arbitrarily large islands where the system is arbitrarily close to infinite homogeneous systems which display non-analytic behavior for  $\beta > \beta_c$ , resulting in the non-analytic behavior of the infinite volume quenched magnetization.

If  $\beta_c(p)$  denotes the critical inverse temperature for the dilute system, then  $\beta_c(p) > \beta_c \equiv \beta_c(1)$  for any  $0 < p < 1$ . Griffiths argued that there is non-analyticity of the quenched magnetization if  $\beta_c < \beta < \beta_c(p)$  for any  $0 < p < 1$ . Proofs of the last statement (based on Griffiths' original argument) were provided by Sütö [14] and Fröhlich [4].

Griffiths' arguments should apply to a large class of ferromagnetic models; in particular, if the couplings  $J_{xy} > 0$  are independent identically distributed random variables, which may assume with non zero probability arbitrarily large values, these singularities should occur for every value of the temperature. But the existing proofs [8], [14], [4] show non-analyticity of the quenched magnetization *only* for the (site or bond) dilute Ising ferromagnet.

The *raison d'être* of Griffiths' singularities is the fact that even if, with probability one, the infinite system is not ordered as whole, there are, also with probability one, infinitely many arbitrarily large regions inside which the system is strongly correlated. This phenomenon is now recognized to be a regular feature in the statistical mechanics of disordered systems not just of the type discussed above. It has the unpleasant consequence that

the usual high temperature or low activity expansions, the standard tools for obtaining exponential decay of correlation functions (and also existence and uniqueness of the thermodynamical limit), fail to converge.

In this review we consider a class of systems whose typical representative is an Ising model in  $\mathbb{Z}^d$  whose Hamiltonian is given in a finite volume  $\Lambda \subset \mathbb{Z}^d$  by

$$H_\Lambda = - \sum_{\langle xy \rangle \in \Lambda^*} J_{xy} \sigma_x \sigma_y + \mathcal{B} \sum_{x \in \Lambda} h_x \sigma_x + h \sum_{x \in \Lambda} \sigma_x \quad (1.3)$$

where the couplings  $\mathbf{J} = \{J_{xy}, \langle xy \rangle \in \mathbb{Z}^{d*}\}$  and the external fields  $\mathbf{h} = \{h_x, x \in \mathbb{Z}^d\}$  are independent families of independent identically distributed (within each family) random variables; we allow the random variable  $J_{xy}$  to take also the value  $+\infty$ . We use the notation  $\Lambda^* = \{\langle xy \rangle; x, y \in \Lambda\}$ . If  $\mathcal{B} = 0$ , the model may be used to describe a spin glass or a random ferromagnet; if the  $J_{xy} \equiv J > 0$ , we have the random field Ising model.

The study of such random models requires new techniques and expansions, capable of avoiding the problems associated with those infinitely many, but sparsely distributed, arbitrarily large regions inside which the system is strongly correlated, effectively controlling the ‘‘Griffiths’ singularities’’.

The first results of this kind were obtained by Olivieri, Perez and Rosa Jr. [12], who studied the Ising ferromagnet with random couplings ( $J_{xy} \geq 0$ ,  $h_x \equiv 0$ ,  $h = 0$ ), and showed exponential decay of two point functions in the presence of Griffiths’ singularities if  $E(J_{xy}) < \infty$ . This finite moment condition was later removed by Perez [13], who also treated general correlation functions.

Exponential decay of truncated correlation functions and uniqueness of the Gibbs state for the class of models described by (1.3), for small  $\beta$  or large  $\mathcal{B}$ , were obtained by Berretti [2] with strong restrictions on the probability distributions of the random parameters ( $E(e^{a|J_{xy}|}) < \infty$  for all  $a > 0$ ;  $\mathbb{P}\{h_x = 0\} = 0$ ). Fröhlich and Imbrie [5], through an intricate analysis of partially resummed high temperature/low activity expansions were able to obtain these results under less restrictive assumptions on the probability distributions of the relevant random parameters ( $|J_{xy}| < \infty$  with a slowly decaying distribution, e.g. a Cauchy distribution, for small  $\beta$ , and  $\mathbb{P}\{h_x = 0\} = 0$  for large  $\mathcal{B}$ ). Bassalygo and Dobrushin [1] proved uniqueness of the Gibbs state for small  $\beta$  with no assumptions on the probability distributions if  $|J_{xy}| < \infty$ . The small  $\beta$  behavior of long range spin glasses has been studied by Fröhlich and Zegarliniski [6] and Zegarliniski [16].

In his original article, Griffiths [8] wondered whether the quenched magnetization, although non-analytic, was an infinitely differentiable function of the uniform external field  $h$ . We answered this question for the class of models described by (1.3) in joint work with von Dreifus [3], proving that at high temperature or at strong field  $\mathcal{B}$ , in spite of the non-analyticity pointed out by Griffiths, the magnetization, or more generally all quenched correlation functions, are infinitely differentiable functions of the uniform external field  $h$ . We also showed uniqueness of the Gibbs state and exponential decay of truncated correlation functions with probability one. Our results require no assumptions on the probability distributions of  $J_{xy}$  and  $h_x$ , except for the obvious requirement of no percolation of infinite couplings (e.g.,  $\mathbb{P}\{J_{xy} = +\infty\}$  small), and, in the strong field situation, for the also obvious requirement that zero magnetic fields do not percolate (e.g.,  $\mathbb{P}\{h_x = 0\}$  small). To prove these results, we developed a modified high temperature/low activity expansion whose convergence can be displayed through simple and elementary probabilistic arguments. Our methods can be applied to any lattice model in classical statistical mechanics. For models with finite range interaction, bounded spins and independence of the random parameters, the application is straightforward.

Another proof of uniqueness of the Gibbs measures at high temperature or strong magnetic fields and of exponential decay of the corresponding quenched correlation functions was given by Gielis and Maes [7], using an analysis based on the study of “disagreement percolation”.

This review is organized as follows. The next section contains precise statements of our results with von Dreifus [3]. In section 3 we sketch our (simple) proof for a model that has been extensively studied in the literature [2], [5]: the random field Ising model in the strong field regime.

## 2. STATEMENT OF RESULTS

Let us consider a system whose Hamiltonian is given by (1.3). Boundary conditions may be introduced in the usual way. Given  $\Lambda$ , we define its boundary  $\partial\Lambda$  and its external boundary  $\partial\Lambda^+$  by

$$\partial\Lambda = \{\langle xy \rangle \in \mathbb{Z}^{d*}; x \in \Lambda, y \notin \Lambda\}, \quad (2.1)$$

$$\partial\Lambda^+ = \{y \in \mathbb{Z}^d; \langle xy \rangle \in \partial\Lambda \text{ for some } x \in \Lambda\}. \quad (2.2)$$

A boundary condition on  $\Lambda$  is a map  $\chi: \partial\Lambda^+ \rightarrow [-1, 1]$ . It is an external boundary condition if it is a configuration on  $\partial\Lambda^+$ , i.e., a map

$\chi: \partial\Lambda^+ \rightarrow \{-1, 1\}$ . If  $\chi \equiv 0$  we have free boundary conditions. We set

$$H_\Lambda^\chi(\sigma) = H_\Lambda(\sigma) - \sum_{\langle xy \rangle \in \partial\Lambda} J_{xy} \sigma_x \chi_y. \tag{2.3}$$

Finite volume thermal averages of local observables (*i.e.*, functions of a finite number of spins) at fixed  $\mathbf{J}$  and  $\mathbf{h}$ , with boundary condition  $\chi$ , are defined by

$$\langle A \rangle_\Lambda^\chi = \frac{\sum_\sigma A(\sigma) e^{-\beta H_\Lambda^\chi(\sigma)}}{Z_\Lambda^\chi} \quad \text{with} \quad Z_\Lambda^\chi = \sum_\sigma e^{-\beta H_\Lambda^\chi(\sigma)}, \tag{2.4}$$

the sums running over all configurations  $\sigma$  in  $\Lambda$ ;  $\beta$  being the inverse temperature. If some  $J_{xy} = +\infty$  we take limits in (2.4). In the case of free boundary conditions we will simply write  $\langle A \rangle_\Lambda$ . When necessary we will make explicit the dependence on the uniform external field  $h$ .

The truncated or connected finite volume correlation function of two local observables  $A(\sigma)$  and  $B(\sigma)$ , with boundary condition  $\chi$ , is defined by :

$$\langle A; B \rangle_\Lambda^\chi = \langle AB \rangle_\Lambda^\chi - \langle A \rangle_\Lambda^\chi \langle B \rangle_\Lambda^\chi. \tag{2.5}$$

More generally, given a state  $\prec \succ$  on an algebra of local observables, we define the truncated correlation function (Ursell function) of  $n$  local observables  $A_1, \dots, A_n$  by (e.g., [15])

$$\prec A_1; A_2; \dots; A_n \succ = \sum_{\mathcal{P}} (-1)^{|\mathcal{P}|-1} (|\mathcal{P}| - 1)! \prod_{P \in \mathcal{P}} \prec \prod_{p \in P} A_p \succ, \tag{2.6}$$

where the sum runs over all partitions  $\mathcal{P}$  of  $\{1, \dots, n\}$ . We recall

$$\prec A_1; A_2; \dots; A_n \succ = \frac{\partial^n}{\partial s_1 \partial s_2 \dots \partial s_n} \ln \prec \exp \left( \sum_{i=1}^n s_i A_i \right) \succ \Big|_{s_1 = \dots = s_n = 0} \tag{2.7}$$

Given a local observable  $A$  we set  $\|A\| = \sup_\sigma |A(\sigma)|$ , and denote by  $\text{supp } A$  the support of  $A$ , that is, the (finite) set of  $x \in \mathbb{Z}^d$  such that  $A(\sigma)$  depends non-trivially on  $\sigma_x$ .

The precise statements of our results are presented in the two theorems below which consider separately the two situations, high temperature or strong field, to which our methods apply. We will use  $p_c^b(d)$  and  $p_c^s(d)$

to denote the critical probabilities for bond and site percolation in  $\mathbb{Z}^d$ , respectively. Recall (e.g., [9])

$$\begin{aligned} p_c^b(1) &= p_c^s(1) = 1, \\ \frac{1}{2} &= p_c^b(2) < p_c^s(2) < 1, \\ 0 < \frac{1}{2d-1} &< p_c^b(d) \leq p_c^s(d) < 1 \quad \text{for } d \geq 3. \end{aligned} \quad (2.8)$$

We will use the  $\ell^1$  norm in  $\mathbb{Z}^d$ :

$$\|x\|_1 = \sum_{i=1}^d |x_i|;$$

distances in  $\mathbb{Z}^d$  will be measured with respect to this norm. Given  $X, Y \subset \mathbb{Z}^d$ ,  $d(X, Y)$  will denote the distance between  $X$  and  $Y$ ; notice that in the  $\ell^1$  norm

$$d(X, Y) = \min\{|G|; G \subset \mathbb{Z}^{d*} \text{ connecting } X \text{ and } Y\}. \quad (2.9)$$

More generally, if  $X_1, \dots, X_n \subset \mathbb{Z}^d$ , we set

$$d(X_1, \dots, X_n) = \min\{|G|; G \subset \mathbb{Z}^{d*} \text{ connecting } X_1, \dots, X_n\}. \quad (2.10)$$

Here by  $G \subset \mathbb{Z}^{d*}$  connecting  $X_1, \dots, X_n$  we mean that for each  $i, j \in \{1, \dots, n\}$ ,  $i \neq j$ , we can find  $\langle x_1 y_1 \rangle, \dots, \langle x_\ell y_\ell \rangle \in G$  with  $x_1 \in X_i$  and  $y_\ell \in X_j$ , such that for each  $k = 1, \dots, \ell - 1$  we have either  $x_{k+1} = y_k$  or we can find  $t \in \{1, \dots, n\}$  so that  $x_{k+1}, y_k \in X_t$ .

If  $A$  and  $B$  are local observables, we will write  $d(A, B)$  for the distance between the supports of  $A$  and  $B$ , i.e.,  $d(\text{supp } A, \text{supp } B)$ . We will also write  $d(A, x_1, \dots, x_n)$  for  $d(\text{supp } A, x_1, \dots, x_n)$ .

We start with the strong field case. We will denote by  $p_c^{(2)}(d)$  the critical probability for site percolation on the lattice  $\mathbb{Z}_2^d$ , which has for vertices the subset of  $\mathbb{Z}^d$  (also denoted by  $\mathbb{Z}_2^d$ ) consisting of all sites  $x \in \mathbb{Z}^d$  with  $\|x\|_1$  an even integer, and for edges the collection

$$\mathbb{Z}_2^{d*} = \{[xy]; x, y \in \mathbb{Z}_2^d \text{ with } \|x - y\|_1 = 2\}; \quad (2.11)$$

notice that each site in  $\mathbb{Z}_2^d$  has  $2d^2$  nearest neighbors, i.e., it belongs to  $2d^2$  edges. It is easy to see that  $p_c^{(2)}(1) = 1$  and, if  $d \geq 2$ ,

$$\frac{1}{2d^2 - 1} < p_c^{(2)}(d) \leq p_c^s(d). \quad (2.12)$$

For each  $x \in \mathbb{Z}^d$  and  $\mathcal{B} > 0$  (we can take  $\mathcal{B} > 0$  in (1.3) without loss of generality) we define

$$Y_{\mathcal{B},x} = Y_{\mathcal{B},x}(\mathbf{J}, \mathbf{h}) = \mathcal{B}|h_x| - 6 \sum_{y: \|y-x\|_1=1} |J_{xy}|, \tag{2.13}$$

and for  $\delta \geq 0$  we set  $Y_{\mathcal{B},x}^{(\delta)} = Y_{\mathcal{B},x} \mathbf{1}_{\{Y_{\mathcal{B},x} > \delta\}}$  and  $q_{\mathcal{B},\delta} = \mathbb{P}\{Y_{\mathcal{B},x} \leq \delta\}$ . We have  $\lim_{\mathcal{B} \rightarrow \infty} q_{\mathcal{B},\delta} = q_\infty$  for any  $\delta \geq 0$ , where

$$\begin{aligned} q_\infty &= \mathbb{P}\{\{h_x = 0\} \cup (\cup_{y: \|y-x\|_1=1} \{J_{xy} = +\infty\})\} \\ &\leq \mathbb{P}\{h_x = 0\} + 2d p_\infty, \end{aligned} \tag{2.14}$$

where  $p_\infty = \mathbb{P}\{J_{xy} = +\infty\}$ . Notice that  $q_\infty < p_c^{(2)}(d)$  implies  $p_\infty < p_c^b(d)$ .

**THEOREM 2.1 (Strong Field Regime).** – *If  $q_\infty < p_c^{(2)}(d)$ , then for each  $\beta > 0$  we can find  $\mathcal{B}_1(\beta, d) < \infty$ , monotonically decreasing in  $\beta$ , and  $\epsilon(\beta, d) > 0$ , such that:*

(i) *For any  $\beta > 0$  and  $\mathcal{B} > \mathcal{B}_1(\beta, d)$ , we can find  $C = C(\beta, \mathcal{B}) < \infty$  and  $m = m(\beta, \mathcal{B}) > 0$ , such that for any two local observables  $A$  and  $B$  and any finite  $\Lambda$  containing their supports, we have*

$$\mathbb{E}(|\langle A; B \rangle_\Lambda^\chi|) \leq C |\text{supp } A| \|A\| \|B\| e^{-m d(A,B)}, \tag{2.15}$$

for any  $|h| < \epsilon(\beta, d)$  and any boundary condition  $\chi$  on  $\Lambda$ .

(ii) *there exists a set  $\Omega$  of realizations of the random parameters  $(\mathbf{J}, \mathbf{h})$ , with  $\mathbb{P}\{(\mathbf{J}, \mathbf{h}) \in \Omega\} = 1$ , and for each  $\beta > 0$  and  $\mathcal{B} > \mathcal{B}_1(\beta, d)$  we can choose  $\mu = \mu(\beta, \mathcal{B}) > 0$ , with  $\lim_{\mathcal{B} \rightarrow \infty} \mu(\beta, \mathcal{B}) = \infty$ , such that if  $(\mathbf{J}, \mathbf{h}) \in \Omega$ ,  $\beta > 0$ ,  $\mathcal{B} > \mathcal{B}_1(\beta, d)$  and  $|h| < \epsilon(\beta, d)$ , then:*

(a) *For any two local observables  $A$  and  $B$ , any finite  $\Lambda$  containing their supports, and any boundary condition  $\chi$  on  $\Lambda$ , we have*

$$|\langle A; B \rangle_\Lambda^\chi| \leq D_A \|A\| \|B\| e^{-\mu d(A,B)}, \tag{2.16}$$

for some  $D_A = D(\text{supp } A, \mathbf{J}, \mathbf{h}, \beta, \mathcal{B}) < \infty$ .

(b) *For every local observable  $A$ , the thermodynamical limit*

$$\langle A \rangle \equiv \lim_{\Lambda \rightarrow \mathbb{Z}^d} \langle A \rangle_\Lambda^{\chi_\Lambda} \tag{2.17}$$

*exists and is independent of the boundary condition  $\chi_\Lambda$  used in each finite volume  $\Lambda$ . In particular, there is a unique Gibbs state.*



(iii) For any  $\beta > 0$ ,  $\mathcal{B} > \mathcal{B}_1(\beta, d)$  and  $|h| < \epsilon(\beta, d)$  the quenched expectation  $\mathbb{E}(\langle A \rangle(h))$  of a local observable  $A$  is an infinitely differentiable function of the uniform external field  $h$ . In particular, for each  $n = 1, 2, \dots$  there exists a constant  $C_n < \infty$ , depending only on  $C$ ,  $m$  and  $n$ , such that

$$\begin{aligned} & \mathbb{E}(|\langle A; \sigma_{x_1}; \dots; \sigma_{x_n} \rangle|) \\ & \leq C_n \|\text{supp } A\| \|A\| \exp \left\{ -\frac{2m}{(n+1)!} d(A, x_1, \dots, x_n) \right\} \end{aligned} \quad (2.18)$$

for all local observables  $A$  and  $x_1, \dots, x_n \in \mathbb{Z}^d$ , and

$$\frac{\partial^n}{\partial h^n} \mathbb{E}\langle A \rangle = (-\beta)^n \sum_{x_1, \dots, x_n \in \mathbb{Z}^d} \mathbb{E}\langle A; \sigma_{x_1}; \dots; \sigma_{x_n} \rangle. \quad (2.19)$$

*Remark 2.2.* – If  $q_\infty < \frac{1}{2(2d^2-1)}$  and we pick  $\mathcal{B} > 0$  such that  $q_{\mathcal{B},0} < \frac{1}{2(2d^2-1)}$ , then for any inverse temperature  $\beta$  and  $h \in \mathbb{R}$  such that

$$\bar{\theta}_{\mathcal{B},\beta,h} \equiv 3\mathbb{E}(e^{-2\beta(Y_{\mathcal{B},x}^{(0)} - 2|h|)}) + q_{\mathcal{B},0} < \frac{1}{2(2d^2-1)}, \quad (2.20)$$

we can take

$$m = -\frac{1}{2} \log(2(2d^2-1)\bar{\theta}_{\mathcal{B},\beta,h}) \quad (2.21)$$

and

$$C = 8d^3(2d+1)^2[(2d^2-1)^{\frac{3}{2}}(2\bar{\theta}_{\mathcal{B},\beta,h})^{\frac{1}{2}}(1-2(2d^2-1)\bar{\theta}_{\mathcal{B},\beta,h})]^{-1} \quad (2.22)$$

in (2.15).

*Remark 2.3.* – If the  $J_{xy}$  are bounded, say  $|J_{xy}| \leq M < \infty$ , then we only need  $q_\infty < p_c^s(d)$  in Theorem 2.1. In this case, if  $q_\infty < \frac{1}{2d-1}$  we can take

$$m = -\log\left((2d-1)\hat{\theta}_{\mathcal{B},\beta,h}\right) \quad (2.23)$$

and

$$C = 4d \left[ (2d-1) \left( 1 - (2d-1)\hat{\theta}_{\mathcal{B},\beta,h} \right) \right]^{-1} \hat{\theta}_{\mathcal{B},\beta,h}, \quad (2.24)$$

if  $\hat{\theta}_{\mathcal{B},\beta,h} < \frac{1}{2d-1}$ , where  $\hat{\theta}_{\mathcal{B},\beta,h}$  is defined as in (2.20), but with  $Y_{\mathcal{B},x}$  replaced by  $\hat{Y}_{\mathcal{B},x} = \mathcal{B}|h_x| - 6M$ .

We now turn to the high temperature case. *In this case we fix arbitrary  $\mathcal{B} \in \mathcal{R}$ ,  $\mathbf{h} \in \mathbb{R}^{\mathbb{Z}^d}$  and  $h \in \mathbb{R}$  in (1.3); only  $\mathbf{J}$  is random (all our estimates will be uniform in  $\mathcal{B}$ ,  $\mathbf{h}$  and  $h$ ).* For a given  $\delta > 0$  we set  $p_\delta = \mathbb{P}\{|J_{xy}| > \delta\}$  and  $J_{xy}^{(\delta)} = J_{xy} \mathbf{1}_{\{|J_{xy}| \leq \delta\}}$ . Notice  $\lim_{\delta \rightarrow \infty} p_\delta = p_\infty$ .

**THEOREM 2.4 (High Temperature Regime).** – *If  $p_\infty < p_c^b(d)$  there exists  $\beta_1 = \beta_1(d) > 0$ , such that:*

(i) *For all  $0 < \beta < \beta_1$  we can find  $C = C(\beta) < \infty$  and  $m = m(\beta) > 0$ , such that for any two local observables  $A$  and  $B$  and any finite  $\Lambda$  containing their supports, we have*

$$\mathbb{E}(|\langle A; B \rangle_\Lambda^\chi|) \leq C |\text{supp } A| \|A\| \|B\| e^{-m d(A,B)}, \tag{2.25}$$

for all  $\mathcal{B} \in \mathbb{R}$ ,  $\mathbf{h} \in \mathbb{R}^{\mathbb{Z}^d}$ ,  $h \in \mathbb{R}$  and any boundary condition  $\chi$  on  $\Lambda$ .

(ii) *There exists a set  $\mathcal{J}$  of realizations of the random couplings with  $\mathbb{P}\{\mathbf{J} \in \mathcal{J}\} = 1$ , and for each  $0 < \beta < \beta_1$  we can choose  $\mu = \mu(\beta) > 0$  with  $\lim_{\beta \rightarrow 0} \mu(\beta) = \infty$ , such that if  $\mathbf{J} \in \mathcal{J}$  and  $0 < \beta < \beta_1$ , then for all  $\mathcal{B} \in \mathbb{R}$ ,  $\mathbf{h} \in \mathbb{R}^{\mathbb{Z}^d}$  and  $h \in \mathbb{R}$ :*

(a) *For any two local observables  $A$  and  $B$ , any finite  $\Lambda$  containing their supports, and any boundary condition  $\chi$  on  $\Lambda$ , we have*

$$|\langle A; B \rangle_\Lambda^\chi| \leq D_A \|A\| \|B\| e^{-\mu d(A,B)}, \tag{2.26}$$

for some  $D_A = D(\text{supp } A, \mathbf{J}, \beta) < \infty$ .

(b) *For every local observable  $A$ , the thermodynamical limit*

$$\langle A \rangle \equiv \lim_{\Lambda \rightarrow \mathbb{Z}^d} \langle A \rangle_\Lambda^{\chi_\Lambda} \tag{2.27}$$

*exists and is independent of the boundary condition  $\chi_\Lambda$  used in each finite volume  $\Lambda$ . In particular, there is a unique Gibbs state.*

(iii) *For all  $0 < \beta < \beta_1$ ,  $\mathcal{B} \in \mathbb{R}$ ,  $\mathbf{h} \in \mathbb{R}^{\mathbb{Z}^d}$  and  $h \in \mathbb{R}$ , the quenched expectation  $\mathbb{E}(\langle A \rangle(h))$  of a local observable  $A$  is an infinitely differentiable function of the uniform external field  $h$ . In particular, for each  $n = 1, 2, \dots$  there exists a constant  $C_n < \infty$ , depending only on  $C$ ,  $m$  and  $n$ , such that*

$$\begin{aligned} &\mathbb{E}(|\langle A; \sigma_{x_1}; \dots; \sigma_{x_n} \rangle|) \\ &\leq C_n |\text{supp } A| \|A\| \exp\left\{-\frac{2m}{(n+1)!} d(A, x_1, \dots, x_n)\right\} \end{aligned} \tag{2.28}$$

for all local observables  $A$  and  $x_1, \dots, x_n \in \mathbb{Z}^d$ , and

$$\frac{\partial^n}{\partial h^n} \mathbb{E}\langle A \rangle = (-\beta)^n \sum_{x_1, \dots, x_n \in \mathbb{Z}^d} \mathbb{E}\langle A; \sigma_{x_1}; \dots; \sigma_{x_n} \rangle. \tag{2.29}$$

*Remark 2.5.* – If  $p_\infty < \frac{1}{2d-1}$  and we pick  $\delta > 0$  so  $p_\delta < \frac{1}{2d-1}$ , then for any inverse temperature  $\beta$  such that

$$\bar{\rho}_{\delta,\beta} \equiv (\mathbb{E}(e^{4\beta|J_{xy}^{(\delta)}|}) - 1) + p_\delta < \frac{1}{2d-1}, \quad (2.30)$$

we can take

$$m = -\log((2d-1)\bar{\rho}_{\delta,\beta}) \quad (2.31)$$

and

$$C = 4d[(2d-1)(1 - (2d-1)\bar{\rho}_{\delta,\beta})]^{-1}. \quad (2.32)$$

in (2.25).

The simpler case of remark 2.5 was discussed by Klein [11].

### 3. THE RANDOM FIELD ISING MODEL IN THE STRONG FIELD REGIME

The random field Ising model Hamiltonian is given in a finite volume  $\Lambda \subset \mathbb{Z}^d$  by the special case of (1.3):

$$H_\Lambda = -J \sum_{\langle xy \rangle \in \Lambda^*} \sigma_x \sigma_y + \mathcal{B} \sum_{x \in \Lambda} h_x \sigma_x + h \sum_{x \in \Lambda} \sigma_x. \quad (3.33)$$

We will discuss the proof of Theorem 2.1 for this Hamiltonian for the case when  $\mathbb{P}\{h_x = 0\} < \frac{1}{2d-1}$ . (Notice that we have  $q_\infty = \mathbb{P}\{h_x = 0\}$  so we are in the situation of Remark 2.3.) Our strategy may be summarized as follows: given a realization  $\mathbf{h}$  of the random magnetic field and a set  $\mathcal{S}$  of sites, we perform a low activity expansion outside  $\mathcal{S}$ . We then show that if  $\mathcal{S}$  is taken to be the appropriate *singular set*, characterized by  $\mathcal{B}h_x$  being small, and we are in the situation when a site has low probability of belonging to  $\mathcal{S}$  (*i.e.*, large  $\mathcal{B}$ ), we get decay after either averaging in  $\mathbf{h}$  (part (i) of Theorem 2.1) or by picking  $\mathbf{h}$  in a set of probability one by a Borel-Cantelli argument (part (ii)).

To deal with truncated correlation functions we use the duplication trick. We thus consider two non-interacting copies of the original system, *i.e.*, a new spin system with configurations  $\tilde{\sigma} = \{\tilde{\sigma}_x = (\sigma_x, \sigma'_x); x \in \mathbb{Z}^d\}$ ,  $\sigma_x, \sigma'_x \in \{-1, +1\}$ , and Hamiltonian  $\tilde{H}_\Lambda(\tilde{\sigma})$ , where for any function  $F(\sigma)$  we set

$$\tilde{F}_\Lambda(\tilde{\sigma}) = F_\Lambda(\sigma) + F_\Lambda(\sigma') \quad (3.34)$$

The set of all configurations of the duplicated system in a given region  $\Lambda \subset \mathbb{Z}^d$  will be denoted by  $\mathcal{C}(\Lambda)$ . Finite volume thermal averages of an observable  $C(\tilde{\sigma})$  in the duplicated system, with boundary condition  $\chi$  (same for both copies), are given by

$$\langle\langle C \rangle\rangle_{\Lambda}^{\chi} = \frac{\sum_{\tilde{\sigma} \in \mathcal{C}(\Lambda)} C(\tilde{\sigma}) e^{-\beta \tilde{H}_{\Lambda}^{\chi}(\tilde{\sigma})}}{\tilde{Z}_{\Lambda}^{\chi}} \quad \text{with} \quad \tilde{Z}_{\Lambda}^{\chi} = \sum_{\tilde{\sigma} \in \mathcal{C}(\Lambda)} e^{-\beta \tilde{H}_{\Lambda}^{\chi}(\tilde{\sigma})}. \quad (3.35)$$

Truncated correlation functions of the original system may be expressed as ordinary correlation functions of the duplicated system through the identity

$$\langle A; B \rangle_{\Lambda}^{\chi} = \frac{1}{2} \langle\langle \hat{A} \hat{B} \rangle\rangle_{\Lambda}^{\chi}, \quad (3.36)$$

where to every observable  $A$  of the original system we associate an observable  $\hat{A}$  of the duplicated system by setting

$$\hat{A}(\tilde{\sigma}) = A(\sigma) - A(\sigma'). \quad (3.37)$$

We now define a self-avoiding *site* walk  $\nu$  from a site  $x$  to another site  $y$ , written  $\nu : x \rightsquigarrow y$ , as a finite sequence  $x_1, x_2, \dots, x_n$  of sites in  $\mathbb{Z}^d$ , such that:

1.  $x_1 = x$  and  $x_n = y$ .
2.  $\|x_{i+1} - x_i\|_1 = 1$  for  $i = 1, \dots, n$ .
3.  $x_i \neq x_j$  if  $i \neq j$ .

For such  $\nu$  we set  $|\nu| = n$ . We define  $\mathcal{N}_{xy} = \{\nu : x \rightsquigarrow y\}$  and set  $\mathcal{N}_x = \bigcup_{y \in \mathbb{Z}^d} \mathcal{N}_{xy}$ . In addition, given two local observables  $A$  and  $B$ , we write  $\mathcal{N}_{AB} = \{\nu : x \rightsquigarrow y : x \in \text{supp } A, y \in \text{supp } B\}$ .

The following low activity expansion for fixed  $\mathbf{h}, h, J$  and  $B$  in (3.3) is the crucial ingredient in the proof of Theorem 2.1.

LEMMA 3.1. – Given  $\mathcal{S} \subset \mathbb{Z}^d$  let

$$\theta_x = \theta_x(\mathcal{S}, J, \mathbf{h}, h, \mathcal{B}, \beta) = \begin{cases} \zeta_x & \text{if } x \in \mathbb{Z}^d \setminus \mathcal{S}, \\ 1 & \text{if } x \in \mathcal{S}, \end{cases} \quad (3.38)$$

where

$$\zeta_x = \zeta_x(J, \mathbf{h}, h, \mathcal{B}, \beta) = 3 e^{-2\beta(W_{\mathcal{B},x} - 2|h|)}, \quad (3.39)$$

where  $W_{\mathcal{B},x} = \mathcal{B}|h_x| - 6J$ . Then for any local observables  $A$  and  $B$ , any finite  $\Lambda$  containing their supports, and any boundary condition  $\chi$  on  $\Lambda$ , we have

$$|\langle A; B \rangle_{\Lambda}^{\chi}| \leq 2 \|A\| \|B\| \sum_{\nu \in \mathcal{N}_{AB}} \prod_{x \in \nu} \theta_x. \quad (3.40)$$

*Proof.* – We start by rewriting the Hamiltonian (3.33) in terms of new variables

$$\eta_x = \frac{(\operatorname{sgn} h_x) \sigma_x + 1}{2} \in \{0, 1\}; \quad x \in \mathbb{Z}^d, \quad (3.41)$$

where  $\operatorname{sgn} u = 1$  if  $u \geq 0$  and  $\operatorname{sgn} u = -1$  otherwise. After a subtraction of an overall constant, we get :

$$\begin{aligned} H_\Lambda(\eta) = & -4 \sum_{\langle xy \rangle \in \Lambda^*} K_{xy} \eta_x \eta_y \\ & + 2 \sum_{x \in \Lambda} (\mathcal{B}|h_x| + (\operatorname{sgn} h_x) h) + \sum_{y \in \Lambda: \langle xy \rangle \in \Lambda^*} K_{xy} \eta_x, \end{aligned} \quad (3.42)$$

where  $K_{xy} = (\operatorname{sgn} h_x)(\operatorname{sgn} h_y)J$ . If  $\chi$  is a boundary condition on  $\Lambda$ , we have (after subtracting a harmless boundary term)

$$H_\Lambda^\chi(\eta) = H_\Lambda(\eta) + 2 \sum_{\langle xy \rangle \in \partial\Lambda} K_{xy} \eta_x |\chi_y|. \quad (3.43)$$

Given a configuration  $\tilde{\eta} = (\eta, \eta')$  of the duplicated system, we set

$$G_{\tilde{\eta}} = \{x \in \mathbb{Z}^d; \eta_x + \eta'_x > 0\}; \quad (3.44)$$

and say that a configuration  $\tilde{\eta}$  is compatible with  $G \subset \mathbb{Z}^d$ , and write  $\tilde{\eta} \prec G$ , if  $G_{\tilde{\eta}} = G$ . We rewrite  $\langle\langle \hat{A}\hat{B} \rangle\rangle_\Lambda^x$  as

$$\begin{aligned} \langle\langle \hat{A}\hat{B} \rangle\rangle_\Lambda^x &= \frac{1}{\tilde{Z}_\Lambda^x} \sum_{\tilde{\eta}} \hat{A} \hat{B} e^{-\beta \tilde{H}_\Lambda^x(\tilde{\eta})} \\ &= \frac{1}{\tilde{Z}_\Lambda^x} \sum_{G \subset \Lambda} \sum_{\tilde{\eta} \prec G} \hat{A} \hat{B} e^{-\beta \tilde{H}_G^{\Lambda, x}(\tilde{\eta})}, \end{aligned} \quad (3.45)$$

where

$$\begin{aligned} H_G^{\Lambda, x}(\eta) = & -4 \sum_{\langle xy \rangle \in G^*} K_{xy} \eta_x \eta_y \\ & + 2 \sum_{x \in G} (\mathcal{B}|h_x| + (\operatorname{sgn} h_x) h) + \sum_{y \in \Lambda: \langle xy \rangle \in \Lambda^*} K_{xy} + \sum_{y \in \partial\Lambda^+: \langle xy \rangle \in \partial\Lambda} K_{xy} |\chi_y| \eta_x. \end{aligned} \quad (3.46)$$

We now perform a low activity expansion in  $\Lambda \setminus S$  only. Again, due to the invariance of the Hamiltonian of the duplicated system under the

exchange  $\eta \leftrightarrow \eta'$ , we can restrict the sum in (3.45) to those to those  $G$  of the form  $G = \nu_S \cup G'$ , where  $\nu \in \mathcal{N}_{\Lambda, AB} = \{\nu' \in \mathcal{N}_{AB}; \nu' \subset \Lambda\}$ ,  $\nu_S = \nu \setminus \mathcal{S}$  and  $G' \subset \mathcal{G}_{S, \nu} = \Lambda \setminus \nu_S$ . Thus

$$\begin{aligned} \left| \left\langle \left\langle \hat{A}\hat{B} \right\rangle \right\rangle_{\Lambda}^x \right| &\leq 4 \|A\| \|B\| \frac{1}{\tilde{Z}_{\Lambda}^x} \sum_{\nu \in \mathcal{N}_{\Lambda, AB}} \sum_{G' \subset \mathcal{G}_{S, \nu}} \sum_{\tilde{\eta} \prec \nu_S \cup G'} e^{-\beta \tilde{H}_{\nu_S \cup G'}^{\Lambda, x}(\tilde{\eta})} \\ &= 4 \|A\| \|B\| \frac{1}{\tilde{Z}_{\Lambda}^x} \sum_{\nu \in \mathcal{N}_{\Lambda, AB}} \sum_{G' \subset \mathcal{G}_{S, \nu}} \sum_{\tilde{\eta} \prec \nu_S \cup G'} \\ &\quad \times \prod_{x \in \nu_S} e^{-\beta \tilde{\Gamma}_x^{\Lambda, x}(\tilde{\eta})} e^{-\beta \tilde{H}_{G'}^{\Lambda, x}(\tilde{\eta})} \end{aligned} \tag{3.47}$$

where

$$\begin{aligned} \Gamma_x^{\Lambda, x}(\eta) &= -4 \sum_{y \in \Lambda: \langle xy \rangle \in \Lambda^*} K_{xy} \eta_x \eta_y \\ &+ 2(\mathcal{B}|h_x| + (\text{sgn } h_x) h + \sum_{y \in \Lambda: \langle xy \rangle \in \Lambda^*} K_{xy} + \sum_{y \in \partial \Lambda^+: \langle xy \rangle \in \partial \Lambda} K_{xy} |\chi_y|) \eta_x. \end{aligned} \tag{3.48}$$

Since we have

$$\tilde{\Gamma}_x^{\Lambda, x}(\tilde{\eta}) \geq 2(W_{\mathcal{B}, x} - 2|h|) \text{ if } \eta_x + \eta'_x > 0, \tag{3.49}$$

we get

$$\begin{aligned} \left| \left\langle \left\langle \hat{A}\hat{B} \right\rangle \right\rangle_{\Lambda}^x \right| &\leq 4 \|A\| \|B\| \frac{1}{\tilde{Z}_{\Lambda}^x} \sum_{\nu \in \mathcal{N}_{\Lambda, AB}} \prod_{x \in \nu_S} 3e^{-2\beta(W_{\mathcal{B}, x} - 2|h|)} \sum_{G' \subset \mathcal{G}_{S, \nu}} \sum_{\tilde{\eta} \prec G'} e^{-\beta \tilde{H}_{G'}^{\Lambda, x}(\tilde{\eta})} \\ &\leq 4 \|A\| \|B\| \sum_{\nu \in \mathcal{N}_{\Lambda, AB}} \prod_{x \in \nu_S} 3e^{-2\beta(W_{\mathcal{B}, x} - 2|h|)}, \end{aligned} \tag{3.50}$$

since

$$\tilde{Z}_{\Lambda}^x = \sum_{G \subset \Lambda} \sum_{\tilde{\eta} \prec G} e^{-\beta \tilde{H}_G^{\Lambda, x}(\tilde{\eta})} \geq \sum_{G' \subset \mathcal{G}_{S, \nu}} \sum_{\tilde{\eta} \prec G'} e^{-\beta \tilde{H}_{G'}^{\Lambda, x}(\tilde{\eta})} \tag{3.51}$$

and

$$\sum_{\tilde{\eta} \prec \nu_S} 1 = 3^{|\nu_S|}. \tag{3.52}$$

(3.40) now follows from (3.50), (3.39) and (3.38). ■

*Proof of Theorem 2.1 (i).* – Let us fix  $\beta > 0$ . Since  $q_\infty < \frac{1}{(2d-1)}$ , there exists  $\mathcal{B}_1(\beta)$  such that for  $\mathcal{B} > \mathcal{B}_1(\beta)$  we have

$$\hat{q}_{\mathcal{B},0} \equiv \mathbb{P}(W_{\mathcal{B},x} \leq 0) < \frac{1}{(2d-1)}. \tag{3.53}$$

We set

$$\mathcal{S} = \{x \in \mathbb{Z}^d; W_{\mathcal{B},x} \leq 0\}. \tag{3.54}$$

Thus, we can choose  $\mathcal{B}_2(\beta)$  such that

$$\hat{\theta} \equiv \mathbb{E}(\theta_x) \leq 3\mathbb{E}(e^{-2\beta(W_{\mathcal{B},x}-2|h|)}) + \hat{q}_{\mathcal{B},0} < \frac{1}{(2d-1)} \tag{3.55}$$

for all  $\mathcal{B} > \mathcal{B}_2(\beta)$  and  $|h| \leq 1$ . (Notice that  $\mathcal{B}_2(\beta)$  can be taken decreasing in  $\beta$ .)

Since the  $\{\tilde{W}_{\mathcal{B},x}; x \in \mathbb{Z}^d\}$  are independent random variables, so are the  $\{\theta_x; x \in \mathbb{Z}^d\}$ ; it follows from (3.40) that for any local observables  $A$  and  $B$ , any finite  $\Lambda$  containing their supports, and any boundary condition  $\chi$  on  $\Lambda$ , we have

$$\mathbb{E}(|\langle A; B \rangle_\Lambda^\chi|) \leq 2 \|A\| \|B\| \sum_{\nu \in \mathcal{N}_{AB}} \hat{\theta}^{|\nu|}. \tag{3.56}$$

Theorem 2.1(i) (with (2.23) and (2.24)) now follows immediately from (3.56) and (3.55). ■

A weaker version of Theorem 2.1(ii)(a) now follows by a Borel-Cantelli argument. The full statement requires a more delicate analysis using a result of Kesten [10]. Theorem 2.1(ii)(b) follows from Theorem 2.1(ii)(a) plus some general considerations. Theorem 2.1(iii) follows from parts (i) and (ii) plus the following result about truncated correlation functions.

LEMMA 3.2. – Let  $\langle \cdot \rangle_\Lambda$  be a random state on the algebra of local observables with support in the set  $\Lambda \subset \mathbb{Z}^d$ , such that there exist  $C < \infty$  and  $m > 0$  for which

$$\mathbb{E}(|\langle A; B \rangle_\Lambda|) \leq C |\text{supp } A| \|A\| \|B\| e^{-m d(A,B)} \tag{3.57}$$

for any two local observables  $A$  and  $B$  with support in  $\Lambda$ . Then there exist constants  $C_n < \infty$ ,  $n = 1, 2, \dots$ , depending only on  $C$ ,  $m$  and  $n$ , such that

$$\mathbb{E}(|\langle A; \sigma_{x_1}; \dots; \sigma_{x_n} \rangle_\Lambda|) \leq C_n |\text{supp } A| \|A\| \exp\{-m_n d(A, x_1, \dots, x_n)\} \tag{3.58}$$

for all local observables  $A$  with support in  $\Lambda$  and all  $x_1, \dots, x_n \in \Lambda$ , where

$$m_n = \frac{2m}{(n+1)!}. \tag{3.59}$$

We refer to [3] for full details.

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(Manuscript received January 17, 1995.)