

# ANNALES DE L'I. H. P., SECTION A

UGO MOSCHELLA

## **New results on de Sitter quantum field theory**

*Annales de l'I. H. P., section A*, tome 63, n° 4 (1995), p. 411-426

[http://www.numdam.org/item?id=AIHPA\\_1995\\_\\_63\\_4\\_411\\_0](http://www.numdam.org/item?id=AIHPA_1995__63_4_411_0)

© Gauthier-Villars, 1995, tous droits réservés.

L'accès aux archives de la revue « Annales de l'I. H. P., section A » implique l'accord avec les conditions générales d'utilisation (<http://www.numdam.org/conditions>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM

Article numérisé dans le cadre du programme  
Numérisation de documents anciens mathématiques

<http://www.numdam.org/>

## **New results on de Sitter quantum field theory (\*)**

by

**Ugo MOSCHELLA**

Service de Physique Théorique, C.E. Saclay,  
91191 Gif-sur-Yvette, France.

---

**ABSTRACT.** – We describe a new approach to  $d$ -dimensional de Sitter quantum field theory. This approach allows a complete characterization of the preferred de Sitter vacua for Klein-Gordon field theories in terms of the analyticity properties of the two-point function, for which we provide a new integral representation. The latter relies on a natural basis of de Sitter plane-waves, which are holomorphic in tubular domains of the complexified de Sitter space-time. Finally we discuss a possible general approach to interacting de Sitter field theories, which, among other properties, justifies the “Wick rotation” to the “euclidean sphere”.

**RÉSUMÉ.** – Nous présentons une nouvelle méthode d’approche à la théorie quantique des champs sur l’espace-temps courbe de de Sitter en dimension  $d$  quelconque. Cette méthode permet une caractérisation complète des « états de vide préférentielles » des théories de Klein-Gordon sur cet espace-temps, au moyen des propriétés d’analyticité de la fonction à deux points, fonction dont nous donnons une nouvelle représentation intégrale. Cette dernière met en jeu une base naturelle d’ondes planes de sitteriennes holomorphes dans des domaines tubulaires de l’espace-temps de de Sitter complexifié. Nous présentons aussi un cadre axiomatique possible pour l’étude des théories de champs de sitteriennes en interaction, cadre qui justifie entre autres propriétés le passage à la « sphère euclidienne » (analogue de la « rotation de Wick »).

---

(\*) Based on an invited talk at the “Colloquium on New Problems in the General Theory of Quantized Fields”, Paris, July 1994.

Quantum field theory on de Sitter space-time is a subject which has been studied by many authors over the last thirty years. The first historical reason for this popularity is the fact that de Sitter space-time is the most symmetrical example of curved space-time manifold. Indeed, the de Sitter metric is a solution of the cosmological Einstein's equations which has the same degree of symmetry as the flat Minkowski solution. Actually, it can be seen as a one-parameter deformation of the latter which involves a fundamental length  $R$ . The space-time corresponding to this metric may then be visualised by a  $d$ -dimensional one-sheeted hyperboloid embedded in a Minkowski ambient space  $\mathbb{R}^{d+1}$ . The symmetry group inherent in the de Sitter space-time is the Lorentz group of the ambient space, and the very existence of this (maximal) symmetry group explains the popularity of this space-time as a convenient simple model to develop techniques of QFT on a gravitational background, in view of an application to more general space-times (perhaps with no symmetries).

The interest in the de Sitter metric increased tremendously in the last fifteen years on a much more physical ground, since it turned out that it plays a central role in the inflationary cosmological scenario (*see* [14] and references therein). According to the latter, the universe undergoes a phase of exponential expansion (quasi-de Sitter phase) in the very early epochs of its life. A possible explanation of phenomena occurring in the very early universe then relies on an interplay between space-time curvature and thermodynamics and a prominent role is played by the mechanisms of symmetry breaking and restoration in a de Sitter QFT.

These reasons explain the enormous amount of literature on de Sitter QFT. Several different approaches have been used for the task of quantizing fields on this space-time, but, they have left open questions either at the level of first principles or even at the more practical level of doing calculations.

Indeed, the peculiarities of de Sitter space-time have not been fully exploited in the past just because people have been looking at it as the paradigm of a more general curved space-time. Therefore the methods that have been applied are closely similar to the ones that may be used in other cases. In particular, the most popular approach to curved space-time QFT (*see e.g.* [2]) follows the lesson of the canonical quantization of fields in flat Minkowski space. In the simplest case of a bosonic linear field one considers the action

$$S = \int \frac{1}{2} \sqrt{-g(x)} \{g^{\mu\nu}(x) \phi_{,\mu} \phi_{,\nu} - (M^2 + \xi\rho(x)) \phi^2(x)\} d^d x \quad (1)$$

where  $\rho(x)$  is the Ricci scalar curvature. In this expression there is a coupling between the quantum field and the (unquantized) background given by the term  $\xi\rho(x)\phi^2$ , besides the gravitational effects due to the metric  $g^{\mu\nu}(x)$ . The field satisfies a Klein-Gordon type equation  $[\square_g + M^2 + \xi\rho(x)]\phi = 0$ , where  $\square_g = (-g)^{-\frac{1}{2}} \partial_\mu (-g)^{\frac{1}{2}} g^{\mu\nu} \partial_\nu$ .

One then introduces the scalar product

$$(\phi_1, \phi_2) = -i \int_\Sigma \bar{\phi}_1(x) \overleftrightarrow{\partial}_\mu \phi_2(x) d\Sigma^\mu,$$

where  $\Sigma$  is a spacelike hypersurface and  $d\Sigma$  is the associated volume element, and looks for a complete set (in the sense of the given scalar product) of mode solutions  $u_i(x)$  of the field equation. The field  $\phi$  is then given by mode expansion

$$\phi(x) = \sum_i [a_i u_i(x) + \alpha_i^\dagger \bar{u}_i(x)] \tag{2}$$

and canonical quantization is achieved by assuming the commutation rules (CCR)  $[a_i, \alpha_j^\dagger] = \delta_{ij}$ ,  $[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0$  and by choosing the corresponding vacuum. The ambiguities inherent the quantization of fields on a gravitational background appear here clearly: in fact the previous mode expansion is generally based on an arbitrary choice of local coordinates (which may or may not extend to the whole space). Moreover, since there is no such thing as a global energy operator, it is in general impossible to characterize the physically relevant vacuum states as the fundamental states for the energy in the usual sense; what is lacking is therefore the analogue of a spectral condition.

These facts have pushed several authors to formulate various alternative prescriptions to select, among the possible vacua of a quantum field theory, those which can have a meaningful physical interpretation; let us quote, in view of their importance, *the adiabatic prescription*, *the local Hadamard condition*, and *the conformal criterion*, each of these having their own range of applications (see [2], [13] and references therein). In a more general conceptual framework, one must also mention the *local definiteness criterion* (see [12] and references therein).

Let us briefly recall the specially interesting case of Robertson-Walker spacetimes. For spatially flat space-times the metric is written as  $ds^2 = dt^2 - a^2(t) \sum_i (dx^i)^2 = C(\eta) [d\eta^2 - \sum_i (dx^i)^2]$ , where  $\eta$  is the conformal time. With this choice of coordinates, mode solutions are separable into factors:  $u_{\vec{k}} = (2\pi)^{\frac{1-d}{2}} C^{\frac{d-2}{4}}(\eta) e^{i\vec{k}\cdot\vec{x}} \chi_k(\eta)$  and the function  $\chi_k$  satisfies

the following equation:  $\frac{d^2 \chi_k}{d\eta^2} + [k^2 + C(M^2 + (\xi - \xi(d))\rho)] \chi_k(\eta) = 0$ ,

where  $\xi(d) = \frac{d-2}{4(d-1)}$ . The mode normalization is obtained by a

Wronskian condition:  $\bar{\chi}_k \partial_\eta \chi_k - \chi_k \partial_\eta \bar{\chi}_k = i$ . Going back to de Sitter, by using the following coordinate system one can look at de Sitter space-time as an exponentially expanding spatially flat Robertson-Walker space-time:

$$x = x_R(t, \vec{x}) \rightarrow \begin{cases} x^{(0)} = R \sinh \frac{t}{R} + \frac{1}{2R} \vec{x}^2 e^{\frac{t}{R}} \\ \mathbf{x} = (x^{(1)}, \dots, x^{(d-1)}) = \vec{x} e^{\frac{t}{R}} \\ x^{(d)} = R \cosh \frac{t}{R} + \frac{1}{2R} \vec{x}^2 e^{\frac{t}{R}} \end{cases} \quad (3)$$

(this parametrization is the most often used in inflationary cosmology; there are other parametrizations which exhibit parts of the Sitter manifold as spatially closed or open Robertson-Walker space-times, *see e.g.* [14]).

In these coordinates the metric is written  $ds^2 = dt^2 - e^{\frac{2t}{R}} \sum_i (dx^i)^2 =$

$R^2 \eta^{-2} [d\eta^2 - \sum_i (dx^i)^2]$ ,  $\eta = -R e^{\frac{t}{R}}$ . Mode solutions are then obtained

in the form of a linear combination of Hankel functions. For, instance in the four dimensional case we have  $\chi_k(\eta) = \frac{1}{2R} \sqrt{\frac{\pi}{2}} \eta^{\frac{3}{2}} [c_1(k) H_\nu^{(1)}(k\eta) + c_2(k) H_\nu^{(2)}(k\eta)]$  with normalization  $|c_2|^2 - |c_1|^2 = 1$ . The *adiabatic prescription* (*see e.g.* [2]) may then be invoked to set  $c_1 = 0$ ,  $c_2 = -1$ . Finally the two-point vacuum expectation value of the field may be calculated as a mode sum and identified to a precise hypergeometric function. The corresponding QFT is most frequently referred to as ‘‘Bunch and Davies’’ [7] or ‘‘Euclidean’’ [11] vacuum (representation) and has played a central role in the applications of de Sitter field theory to cosmology, especially for the aim of computing the spectrum of density fluctuations in the observed universe.

Very roughly speaking, this is the common way to think of de Sitter QFT, and after the papers that have appeared on the subject in the seventies no deeper investigations of the situation at the global level have been proposed, even though many mathematical aspects of these field theories have been studied, in particular, in connection with ideas of quantum field theory in general curved space-time manifolds ([12], [13]). For the sake of completeness one has to mention the functional integral approach to de Sitter QFT that has been used by Gibbons and Hawking [11], which gives

the name “euclidean” to the preferred vacua described above. As we will see, this approach is fully justified only in a globally analytical approach and this is what we want to introduce.

In the following we will describe a new and more global approach to de Sitter QFT which has been elaborated in a continuing collaboration with Bros ([3], [4], [5], [16]). This approach allows to fully characterize the preferred family of de Sitter invariant Klein-Gordon quantum field theories and also opens the way to a general treatment of interacting theories on this space-time. Our idea deals with the possibility of developing a general approach based on a set of fundamental principles, which should be completely similar to the Wightman approach for minkowskian fields. This is by no means obvious, since in the latter many important concepts are based on the Fourier representation of space-time, namely the energy-momentum space. Therefore one major objective to be reached would be to dispose of a global de Sitter-Fourier calculus. The possibility of such an approach relies on the properties of the geometry of the de Sitter space-time and of its complexification, which make this space-time so similar (although with many important differences) to the complex Minkowski space-time.

Let us briefly recall some facts. The de Sitter space-time may be represented by a  $d$ -dimensional one-sheeted hyperboloid

$$X_d(R) = X_d = \{x \in \mathbb{R}^{d+1} : x^{(0)2} - x^{(1)2} - \dots - x^{(d)2} = -R^2\} \quad (4)$$

embedded in a Minkowski ambient space  $\mathbb{R}^{d+1}$  whose scalar product is denoted by  $x \cdot y = x^{(0)} y^{(0)} - x^{(1)} y^{(1)} - \dots - x^{(d)} y^{(d)}$  with, as usual,  $x^2 = x \cdot x$ ;  $X_d$  is then equipped with a *causal* ordering relation which is induced by that of  $\mathbb{R}^{d+1}$ : let  $\bar{V}^+ = \{x \in \mathbb{R}^{d+1} : x^{(0)} \geq \sqrt{x^{(1)2} + \dots + x^{(d)2}}\}$  be the future cone of the origin in the ambient space; then, for  $x, y \in X_d$ ,  $x \geq y \leftrightarrow x - y \in \bar{V}^+$ . The future and past cone of a given event  $x$  in  $X_d$  are given by  $\Gamma^\pm(x) = \{y \in X_d : y \geq x (x \geq y)\}$ . The boundary set of  $\Gamma^+(x) \cup \Gamma^-(x)$  is the “light-cone”  $\partial\Gamma(x)$  and it is the union of all linear generatrices of  $X_d$  containing the point  $x$ :  $\partial\Gamma(x) = \{y \in X_d : (x - y)^2 = 0\}$ . Two events  $x$  or  $y$  of  $X_d$  are in “acausal relation”, or “space-like separated” if  $y \notin \Gamma^+(x) \cup \Gamma^-(x)$ , *i.e.* if  $x \cdot y > -R^2$ . The relativity group of the de Sitter space-time is the pseudo-orthogonal group  $G_d = SO_0(1, d)$  leaving invariant each of the sheets of the cone  $C = C^+ \cup C^-$ :

$$C^\pm = \{x \in \mathbb{R}^{d+1} : x^{(0)2} - x^{(1)2} - \dots - x^{(d)2} = 0, \text{sgn } x^{(0)} = \pm\}. \quad (5)$$

The  $G_d$ -invariant volume on  $X_d$  will be denoted by  $d\sigma(x)$ . Since the group  $G_d$  acts in a transitive way on  $X_d$ , it is convenient to distinguish

a *base point*  $x_0$  in  $X_d$  which plays the role of the origin in Minkowski space-time; we choose the point  $x_0 = (0, \dots, 0, R)$ , and consider the tangent space  $\Pi_d$  to  $X_d$ , namely the hyperplane  $\Pi_d = \{x \in \mathbb{R}^{d+1} : x^{(d)} = R\}$  as the  $d$ -dimensional Minkowski space-time (with pseudo-metric  $dx^{(0)^2} - dx^{(1)^2} - \dots - dx^{(d-1)^2}$ ) onto which the de Sitter space-time can be contracted in the zero-curvature limit.

Let us introduce also some crucial geometrical notions concerning the complex hyperboloid

$$X_d^{(c)}(R) = X_d^{(c)} = \{z = x + iy \in \mathbb{C}^{d+1} : z^{(0)^2} - z^{(1)^2} - \dots - z^{(d)^2} = -R^2\}, \quad (6)$$

equivalently characterized as the set  $X_d^{(c)} = \{(x, y) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1} : x^2 - y^2 = -R^2, x \cdot y = 0\}$ . We define the following open subsets of  $X_d^{(c)}$ :

$$\mathcal{T}^+ = T^+ \cap X_d^{(c)}, \quad \mathcal{T}^- = T^- \cap X_d^{(c)}, \quad (7)$$

where  $T^\pm = \mathbb{R}^{d+1} + iV^\pm$  are the so-called forward and backward tubes in  $\mathbb{C}^{d+1}$  which are the (minimal) analyticity domains of the Fourier-Laplace transforms of tempered distributions  $\tilde{f}(p)$  with support contained in  $\bar{V}^+$  or in  $\bar{V}^-$ , obtained in connection with the spectral condition in Minkowski QFT in  $\mathbb{R}^{d+1}$  [18]. In the same way as  $\bar{T}^+ \cup \bar{T}^-$  contains the “euclidean subspace”  $\mathcal{E}_{d+1} = \{z = (iy^{(0)}, x^{(1)}, \dots, x^{(d)}) : (y^{(0)}, x^{(1)}, \dots, x^{(d)}) \in \mathbb{R}^{d+1}\}$  of the complex Minkowski space-time  $\mathbb{C}^{d+1}$ , one easily checks that  $\bar{\mathcal{T}}^+ \cup \bar{\mathcal{T}}^-$  contains the sphere  $S_d = \{z = (iy^{(0)}, x^{(1)}, \dots, x^{(d)}) : y^{(0)^2} + x^{(1)^2} + \dots + x^{(d)^2} = R^2\}$ . We call  $\mathcal{T}^+$ ,  $\mathcal{T}^-$  and  $S_d$  the “forward” and “backward tubes” and the “euclidean sphere” of  $X_d^{(c)}$ .

Armed with these geometrical notions we now follow the historical pattern and reconsider first the de Sitter Klein-Gordon theories. In the following, we will neither make use of any particular coordinate system on de Sitter space-time nor treat space and time variables on a different footing. In this respect, we stay closer to the spirit of general relativity. Since  $\rho(x) = \rho = 12/R^2$  the de Sitter Klein-Gordon field equation may be rewritten as

$$\square_d \phi + \mu^2 \phi = 0, \quad (8)$$

where  $\mu^2 = M^2 + \xi\rho$  is a mass parameter, and we look for an operator-valued distributional [18] solution for this equation. Instead of looking for mode solutions which are factorized in some suitable coordinate system we will make use of an important class of solutions of the equation  $(\square_d + \lambda)\psi = 0$ , where now  $\lambda$  is real or complex. These solutions have appeared long ago in the mathematical literature ([8], [15]) and are going

to play the same role of *plane wave basis* as the exponentials in the Minkowski case; by consequence, they will be the essential ingredients of a de Sitter Fourier analysis, one of the issues which we want to discuss. There is however an important difference: in contrast with the minkowskian exponentials, these waves are singular on  $(d - 1)$ -dimensional light-like manifolds and can at first instance be defined only on suitable halves of the hyperboloid. We will need an appropriate  $i\varepsilon$ -prescription to obtain global waves. Here is the relevant definition: let  $\xi \in C^+$  and consider the function

$$\psi_+^\xi(x, s) = \left( \frac{|x \cdot \xi|}{m R} \right)^s, \quad s \in \mathbb{C}, \tag{9}$$

defined for those  $x \in X_d$  such that  $x \cdot \xi > 0$  (same definition for  $\psi_-^\xi(x, s)$  for those  $x \in X_d$  such that  $x \cdot \xi < 0$ );  $m$  is an auxiliary mass parameter introduced here for dimensional reasons, but it will have also a minkowskian physical interpretation. One has that

$$\square_d \psi_+^\xi(x, s) = \frac{s(d - 1 + s)}{R^2} \psi_+^\xi(x, s) \tag{10}$$

Physical values of the parameter  $s$  are given by

$$s = -\frac{d - 1}{2} + i\nu, \quad \text{with} \quad \begin{cases} \nu \in \mathbb{R} \\ i\nu \in \mathbb{R}, \quad |\nu| \leq \frac{d - 1}{2} \end{cases} \tag{11}$$

corresponding in the first case to  $-s(d - 1 + s) = \left(\frac{d - 1}{2}\right)^2 + \nu^2 = \mu^2 R^2 = M^2 R^2 + 12\xi$  and in the latter case to  $-s(d - 1 + s) = \left(\frac{d - 1}{2}\right)^2 - \nu^2 = \mu^2 R^2 = M^2 R^2 + 12\xi$ . The interpretation of the functions  $\psi_+^\xi(x, s)$  as plane waves in the de Sitter space-time is also supported by their large  $R$  behaviour; in fact, parametrizing  $\xi$  by the wave-vector of a (minkowskian) particle of mass  $m$ , *i.e.*  $\xi = [k^0, \vec{k}, -m]$ , with  $k^0 = \sqrt{\vec{k}^2 + m^2}$ , gives that

$$\lim_{R \rightarrow \infty} \left( \frac{x_R(x) \cdot \xi}{m R} \right)^{-\frac{d-1}{2} + imR} = \exp(ik \cdot x); \tag{12}$$

in this equation, points in the de Sitter universe must be described using the minkowskian space-time variable measured in units of the de Sitter radius  $R$ ; one can choose any parametrization which reproduces, in a neighbourhood of the base point  $x_0$ , the cartesian parametrization of the corresponding tangent plane to  $X_d$ ; note that also the dimensionless parameter  $\nu$  has to be taken proportional to the radius  $R$  to get the desired limit.

Let us see how to use the previous plane waves to construct the two-point vacuum expectation value of a Klein- Gordon quantum field. The first



problem is that of extending the waves to the whole (real) hyperboloid. In particular, we need to specify the correct phase factors to glue together  $\psi_+^\xi$  and  $\psi_-^\xi$ . The relevant solution for this problem is found by making use of the geometry of the complex hyperboloid  $X_d^{(c)}$ ; in particular the tuboids  $T^+$  and  $T^-$  play here a crucial role.

To prepare the ground let us examine what happens in the flat case. The two-point function for the Klein-Gordon can be written most simply by using its Fourier representation:

$$\mathcal{W}_m(x-y) = \frac{1}{2(2\pi)^{\frac{d}{2}}} \int e^{-ik \cdot y} e^{ik \cdot x} \theta(k^0) \delta(k^2 - m^2) dk. \quad (13)$$

We see from this formula that the two-point function is obtained as a superposition of the plane waves  $\exp(-ik \cdot x)$  and  $\exp(ik \cdot y)$  and the momentum variable  $k$  is integrated w.r. to the measure  $d\mu = \theta(k^0) \delta(k^2 - m^2) dk$ ; this measure is chosen to solve the wave equation and to satisfy the spectral condition [18]. As a consequence of the spectral condition (or by direct inspection from the convergence properties of the integral at the r.h.s. of equation (13)) we have that the distribution  $\mathcal{W}_m$  is the boundary value of an analytic function  $W(z)$ , which is analytic in the backward tube  $T^-$ , where  $\Re z = x - y$ , and the boundary value is taken from  $T^-$ . Actually the spectral condition is equivalent to these analyticity properties.

Having these facts in mind we now consider the de Sitter case. First of all we note that we can extend the plane waves  $\psi^\xi(x, s)$  to analytic functions in  $T^+$  or  $T^-$ . Indeed, when  $\xi \in C^+$  is fixed and  $z$  varies in  $T^+$  or in  $T^-$  then the functions

$$\psi^\xi(z, s) = \left( \frac{z \cdot \xi}{mR} \right)^s, \quad s \in \mathbb{C} \quad (14)$$

are globally defined (uniquely up to a phase) and holomorphic in  $z$ , because  $\Im(z \cdot \xi)$  has a fixed sign. These global waves allow a spectral analysis of the two-point functions very similar to the previous one. Let us introduce the following function:

$$W_\nu(z_1, z_2) = c_{d,\nu} \int_\gamma (z \cdot \xi_\gamma)^{-\frac{d-1}{2} + i\nu} (\xi_\gamma \cdot z_2)^{-\frac{d-1}{2} - i\nu} d\mu_\gamma(\xi_\gamma), \quad (15)$$

where  $z_1, z_2 \in X_d^{(c)}$  and such that  $z_1 \in T^-$ ,  $z_2 \in T^+$ . The integration is performed along any basis submanifold  $\gamma$  of the cone  $C^+$  (i.e. a submanifold intersecting almost all the generatrices of the cone) w.r. to a corresponding measure  $d\mu_\gamma$  induced by the invariant measure on the cone (the integrand

in (15) appearing as the restriction to  $\gamma$  of a closed differential form). For instance, by choosing

$$\gamma_d = \gamma_d^+ \cup \gamma_d^- = \{\xi \in C^+ : \xi^{(d)} = m\} \cup \{\xi \in C^+ : \xi^{(d)} = -m\}$$

as integration manifold we obtain that  $d\mu_\gamma$  is the corresponding Lorentz invariant measure while for the spherical basis  $\gamma_0 = \{\xi \in C^+ : \xi^{(0)} = m\}$   $d\mu_\gamma$  is the rotation invariant measure.

The function  $W_\nu$  is a solution in both variables of the (complex) de Sitter Klein-Gordon equation which is analytic in the domain  $\mathcal{T}_{12} = \{(z_1, z_2) \in X_d^{(c)} \times X_d^{(c)} : z_1 \in \mathcal{T}^-, z_2 \in \mathcal{T}^+\}$ . This is clear by construction. A little more work is demanded to show that it is actually a function of the single de Sitter invariant variable  $(z_1 - z_2)^2 = -2R^2 - 2z_1 \cdot z_2$ . This property permits an explicit computation, by fixing one of the two points; this is most easily done by choosing the following points

$$z_1 = \begin{bmatrix} -iR\lambda \\ -iR\sqrt{\lambda^2 - 1} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad z_2 = \begin{bmatrix} iR \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \lambda \in (0, 1); \quad (16)$$

and by integrating on the spherical basis of the cone. It follows that

$$W_\nu(z_1, z_2) = \frac{2c_{d,\nu} e^{\pi\nu} \pi^{\frac{d}{2}}}{R^{d-1} \Gamma\left(\frac{d}{2}\right)} P_{-\frac{d-1}{2}+i\nu}^{(d+1)}(\lambda), \quad (17)$$

where  $P_{-\frac{d-1}{2}+i\nu}^{(d+1)}(\lambda)$  is given by:

$$P_{-\frac{d-1}{2}+i\nu}^{(d+1)}(\lambda) = \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{d-1}{2}\right)} \times \int_0^\pi (\lambda + \sqrt{\lambda^2 - 1} \cos \theta)^{-\frac{d-1}{2}+i\nu} \sin^{d-2} \theta d\theta \quad (18)$$

$P_{-\frac{d-1}{2}+i\nu}^{(d+1)}(w)$  is proportional to the Gegenbauer (hypergeometric) function of the first kind  $C_{-\frac{d-1}{2}+i\nu}^{\left(\frac{d-1}{2}\right)}(w)$  [1]. The constant  $c_{d,\nu}$  may be fixed by

imposing the CCR's and we obtain

$$c_{d,\nu} = \frac{\Gamma\left(\frac{d-1}{2} + i\nu\right) \Gamma\left(\frac{d-1}{2} - i\nu\right) \Gamma(d-1) e^{-\pi\nu} R}{(2\sqrt{\pi})^{2d-1} \Gamma\left(\frac{d-1}{2}\right) \Gamma\left(\frac{d}{2}\right)}. \tag{19}$$

One can see here how this analytical and geometrical intrinsic (no particular coordinate system is used) approach is also useful to perform calculations: to derive the integral representation of the hypergeometric function we have made no use of sum theorems for special functions. We had only to substitute in the Fourier-type representation of the two-point function (15) a suitably chosen pair of points of the complex hyperboloid and the relevant hypergeometric function came out automatically. Equations (15), (17) and (18) show the key property of the two-point function:

Maximal analyticity property:

$W_\nu(z_1, z_2)$  is maximally analytic, i.e. can be analytically continued in the “cut-domain”  $\Delta = X_d^{(c)} \times X_d^{(c)} \setminus \{(z_1, z_2) \in X_d^{(c)} \times X_d^{(c)} : (z_1 - z_2)^2 = \rho \geq 0\}$ . Furthermore,  $W_\nu(z_1, z_2)$  satisfies in  $\Delta$  the complex covariance condition:  $W_\nu(gz_1, gz_2) = W_\nu(z_1, z_2)$  for all  $g \in SO_0(1, d)^c$ , the complexified of the group  $SO_0(1, d)$ .

These properties characterize  $W_\nu$  as being an invariant perikernel ([3], [6]) on  $X_d^{(c)}$  with domain  $\Delta$ . The two-point Wightman function  $\mathcal{W}_\nu(x_1, x_2) = \langle \Omega, \phi(x_1) \phi(x_2) \Omega \rangle$  is the boundary value of  $W_\nu(z_1, z_2)$  from  $\mathcal{T}_{12}$ ; we obtain the following representation:

$$\begin{aligned} & \mathcal{W}_\nu(x_1, x_2) \\ &= c_{d,\nu} \int_\gamma \left( (x \cdot \xi_\gamma)_+^{-\frac{d-1}{2} + i\nu} + e^{-i\pi\left(-\frac{d-1}{2} + i\nu\right)} (x_1 \cdot \xi_\gamma)_-^{-\frac{d-1}{2} + i\nu} \right) \\ & \quad \left( (x_2 \cdot \xi_\gamma)_+^{-\frac{d-1}{2} - i\nu} + e^{i\pi\left(-\frac{d-1}{2} - i\nu\right)} (x_2 \cdot \xi_\gamma)_-^{-\frac{d-1}{2} - i\nu} \right) d\mu_\gamma \tag{20} \end{aligned}$$

where we have used a notation introduced in [10]. The Fourier representation for the two-point function of the Minkowski free field of mass  $m$  is obtained as the limit of equation (20) for  $R \rightarrow \infty$ ,  $\nu = mR$ . The “permuted Wightman function”  $\mathcal{W}_\nu(x_2, x_1)$  is the boundary value of  $W_\nu(z_1, z_2)$  from the domain  $\mathcal{T}_{21} = \{(z_1, z_2) : z_1 \in \mathcal{T}^+, z_2 \in \mathcal{T}^-\}$ . This allows the explicit construction of the commutator and the Green functions.

An important consequence of the representation (20) is the introduction of a natural Fourier transform on the hyperboloid. Given a function  $f \in \mathcal{D}(X_d)$

we define its Fourier transform as the following pair of homogeneous complex functions on the cone  $C^+$ :

$$\begin{aligned} \tilde{f}_{\pm} \left( \xi, -\frac{d-1}{2} - i\nu \right) \\ = \int_{X_d} \left( (x \cdot \xi)_+^{-\frac{d-1}{2} - i\nu} + e^{\pm i\pi} \left( -\frac{d-1}{2} - i\nu \right) (x \cdot \xi)_-^{-\frac{d-1}{2} - i\nu} \right) \\ \times f(x) d\sigma(x) \end{aligned} \quad (21)$$

More generally, we may introduce a Fourier transform depending on a complex parameter  $s$ :

$$\tilde{f}_{\pm}(\xi, s) = \int_{X_d} \left( (x \cdot \xi)_+^s + e^{\pm i\pi s} (x \cdot \xi)_-^s \right) f(x) d\sigma(x) \quad (22)$$

By using this Fourier transform one can show that the two-point function is positive-definite, *i.e.*

$$\int_{X_d \times X_d} \mathcal{W}_{\nu}(x_1, x_2) \bar{f}(x_1) f(x_2) d\sigma(x_1) d\sigma(x_2) \geq 0,$$

for any  $f \in C_0^{\infty}(X_d)$ . This is straightforward for real  $\nu$  and a little more difficult for imaginary  $\nu$ .

The maximal analyticity property also sheds light on the euclidean properties of the fields considered; in fact, by taking the restriction of  $W_{\nu}$  to the euclidean sphere  $S_d$  and obtain the Schwinger function  $S_{\nu}(z_1, z_2)$ ;

This is permitted since  $S_d \times S_d$  minus the set of coinciding points  $z_1 = z_2$ , is a subset of  $\Delta$ . It is perhaps worthwhile to stress that properties of analytic continuation *in all the variables* must be at the basis of every treatment of de Sitter field theories on the functional integral on the euclidean sphere; this includes the constructive approach to QFT on de Sitter space-time (in this connection *see* [9]) or the application that de Sitter theories may find in minkowskian constructive field theory, since the de Sitter radius  $R$  provides a natural infrared cutoff (the euclidean space is compact). Without the appropriate analyticity properties all results inferred by euclidean methods would not be relevant for the real de Sitter universe.

The maximal analyticity property can also be taken as the starting point for a general analysis of the status of two-point functions of de Sitter quantum fields (we mean free or interacting). This point of view has been initiated in [3] and fully developed in [5].

One could also ask for a physical basis for postulating these maximal analyticity properties. It turns out that the thermal properties of de Sitter generalised free fields (in the sense of the Gibbons-Hawking

temperature [11]) can be proven easily in our analytic framework and the maximal analyticity property of the two-point function is equivalent to those thermal properties (plus an *antipodal condition*). Let us see briefly how the things go. As in [11] we adopt the viewpoint of an observer sitting on the geodesic  $h(x_0)$  of  $x_0$ , contained in the  $(x^{(0)}, x^{(d)})$ -plane. The set of all events of  $X_d$  which can be connected with the observer by the reception and the emission of light-signals is the region  $\mathcal{U} = \{x \in X : x^{(d)} > |x^{(0)}|\}$ ; it is bordered by the “future” and “past” event horizons of the observer  $x^{(0)} = \pm x^{(d)}$ ,  $x^{(d)} > 0$ . The region  $\mathcal{U}$  is foliated by hyperbolic trajectories  $h_{\vec{x}}(x_0)$  parallel to the geodesic  $h(x_0) = h_{\vec{0}}(x_0)$ , according to the following parametrization of  $\mathcal{U} : x(\tau, \vec{x}) = \left( x^{(0)} = \sqrt{R^2 - \vec{x}^2} \sinh \frac{\tau}{R}, \vec{x} = \vec{x}, x^{(d)} = \sqrt{R^2 - \vec{x}^2} \cosh \frac{\tau}{R} \right)$ ;  $\tau$  is the proper time of our observer.

The curves  $h_{\vec{x}}(x_0)$  are the orbits of the one-parameter group  $T_{h(x_0)}$  of isometries of  $\mathcal{U}$  (see [13], [17] for a general discussion of this kind of structure):  $T_{h(x_0)}(t)[x(\tau, \vec{x})] = x(t + \tau, \vec{x}) \equiv x^t$ ,  $t \in \mathbb{R}$ . The complexified orbits of  $T_{h(x_0)}$ , namely the complex hyperbolae  $h_{\vec{x}}^{(c)}(x_0)$  have  $(2i\pi R)$ -periodicity in  $t$  and all their non-real points in  $\mathcal{T}^+$  and  $\mathcal{T}^-$ . This entails a remarkable property of the time-translated correlation functions  $\langle \Omega, \phi(x_1) \phi(x_2^t) \Omega \rangle = \mathcal{W}_\nu(x_1, x_2^t)$  and  $\langle \Omega, \phi(x_2^t) \phi(x_1) \Omega \rangle = \mathcal{W}_\nu(x_2^t, x_1)$ , where  $x_1$  and  $x_2$  are arbitrary events in  $\mathcal{U}$ . In fact, from the above stated *maximal analyticity property* of  $W_\nu(z_1, z_2)$  we deduce that  $W_\nu(x_1, x_2^t)$  defines a  $2i\pi R$ -periodic analytic function of  $t$ , whose domain is the periodic cut-plane

$$\begin{aligned} \mathbb{C}_{x_1, x_2}^{\text{cut}} &= \{t \in \mathbb{C}; \Im t \neq 2n\pi R, n \in \mathbb{Z}\} \\ &\cup \{t; t - 2in\pi R \in I_{(x_1, x_2)}, n \in \mathbb{Z}\}, \end{aligned}$$

where  $I_{x_1, x_2}$  is the real interval on which  $(x_1 - x_2^t)^2 < 0$ . One also checks that the boundary values of  $W_\nu(x_1, x_2^t)$  on  $\mathbb{R}$  coincide with the previous correlation functions (the jumps across the cuts being the retarded and advanced commutators); these properties imply that  $W_\nu(x_1, x_2^t)$  is analytic in the strip  $\{t \in \mathbb{C}, 0 < \Im t < 2i\pi R\}$  and satisfies the following K.M.S. relation at temperature  $T = 1/2\pi R$ :

$$\mathcal{W}_\nu(x_2^t, x_1) = \lim_{\varepsilon \rightarrow 0^+} \mathcal{W}_\nu(x_1, x_2^{t + \frac{2i\pi R}{c} - i\varepsilon}), \quad t \in \mathbb{R}. \quad (23)$$

The “energy operator”  $\mathcal{E}_{h(x_0)}$  associated with the geodesic  $h(x_0)$  is obtained by the spectral decomposition of the unitary representation of the time translation group  $T_{h(x_0)}$  in the Hilbert space  $\mathcal{H}$

of the theory, namely  $U_h^t(x_0) = \int_{-\infty}^{\infty} e^{i\omega t} dE_{h(x_0)}(\omega)$ , which yields  $\mathcal{E}_{h(x_0)} = \int_{-\infty}^{\infty} \omega dE_{h(x_0)}(\omega)$ . The previous K.M.S. condition is then equivalent to the fact that energy measurements performed by an observer at rest at the origin on states localized in  $\mathcal{U}$  are exponentially damped by a factor  $\exp(-2\pi R\omega)$  in the range of negative energies. In the limit of flat space-time this factor will kill all negative energies, so that one recovers the usual spectral condition of "positivity of the energy".

In the general interacting case one cannot expect maximal analyticity properties to hold for  $n$ -point functions ( $n > 2$ ); we are left with the task of finding a suitable general approach to de Sitter QFT. Let us consider the case of a theory that can be characterized by a sequence of Wightman distributions  $\mathcal{W}_n(x_1, \dots, x_n) = \langle \Omega, \phi(x_1) \dots \phi(x_n) \Omega \rangle$  on  $[X_d]^n$ . The properties required for these Wightman functions can be summarized in a set of four axioms. If the first three of these axioms are straightforward adaptations to the de Sitter space-time of the corresponding axioms of Minkowskian Q.F.T., the fourth one which plays the role of the Spectral Condition is not so. In fact, due to the absence of a global energy-momentum interpretation on the curved space-time  $X_d$ , our fourth axiom will be formulated in terms of analytic continuation properties of the distributions  $\mathcal{W}_n$  in the complexified manifolds  $[X_d^{(c)}]^n$  corresponding to  $[X_d]^n$ .

Of course, the choice of such global analyticity properties may certainly be done in a non-unique way, and the corresponding properties of the GNS "vacuum"  $\Omega$  of the considered theories will of course depend on the postulated analyticity properties. In the general case, the thermal interpretation of the theory might provide a criterion of selection between several *a priori possible* global analytic structures. Another feature that this approach should assure is the possibility of going to the euclidean sphere.

*The axioms*

The set of distributions  $\mathcal{W}_n (n \geq 1)$  is assumed to satisfy the following properties (we limit ourselves here to the case of a boson field):

1. (Covariance). Each  $\mathcal{W}_n$  is de Sitter invariant, *i.e.*

$$\mathcal{W}_n(f_{n\{g\}}) = \mathcal{W}_n(f_n) \tag{24}$$

for all de Sitter transformations  $g$ , where

$$f_{n\{g\}}(x_1, \dots, x_n) = f_n(g^{-1}x_1, \dots, g^{-1}x_n) \tag{25}$$

2. (Locality)

$$\mathcal{W}_n(x_1, \dots, x_j, (x_{j+1}, \dots, x_n)) = \mathcal{W}_n(x_1, \dots, x_{j+1}, (x_j, \dots, x_n)) \quad (26)$$

if  $(x_j - x_{j+1})^2 < 0$ .

3. (Positive Definiteness). Given

$$f_0 \in \mathbb{C}, \quad f_1 \in \mathcal{D}(X_d), \dots, \quad f_k \in \mathcal{D}([X_d]^k),$$

then

$$\sum_{n, m=0}^k \mathcal{W}_{n+m}(\bar{f}_n \otimes g_m) \geq 0 \quad (27)$$

where  $(f_n \otimes g_m), (x_1, \dots, x_{n+m}) = f_n(x_1, \dots, x_n) g_m(x_{n+1}, \dots, x_{n+m})$ .

We shall now give a substitute for the usual spectral condition of axiomatic field theory in Minkowski space, which will be called “weak spectral condition”. We propose under this name analyticity properties of the Wightman functions which reproduce as closely as possible those implied by the usual spectral condition in Minkowskian QFT. In the latter, it is known that the spectral condition can be equivalently expressed by the following analyticity properties of the Wightman functions  $\mathcal{W}_n$  [18] resulting from the Laplace transform theorem in  $\mathbb{C}^{dn}$ : for each  $n$  ( $n \geq 2$ ) the distribution  $\mathcal{W}_n(x_1, \dots, x_n)$  is the boundary value of an analytic function  $W_n(z_1, \dots, z_n)$  defined in the tube

$$\begin{aligned} \mathbb{T}_{n-1}^d &= \{z = (z_1, \dots, z_n); \\ & z_j = x_j + iy_j \in \mathbb{C}^d, y_{j+1} - y_j \in V^+, 1 \leq j \leq n-1\} \quad (28) \end{aligned}$$

When the Minkowski space is replaced by the de Sitter space  $X_d$  embedded in  $\mathbb{R}^{d+1}$ , natural substitute for this property can be proposed by replacing (for each  $n$ ) the tube  $\mathbb{T}_{n-1}^d \subset \mathbb{C}^{dn}$  by the corresponding open subset  $\mathcal{T}_{n-1} = [X_d^{(c)}]^n \cap \mathbb{T}_{n-1}^{d+1}$  of  $[X_d^{(c)}]^n$ .

$$\begin{aligned} \mathcal{T}_{n-1} &= \{z = (z_1, \dots, z_n); \\ & z_j = x_j + iy_j \in X_d^{(c)}, y_{j+1} - y_j \in V^+, 1 \leq j \leq n-1\} \quad (29) \end{aligned}$$

is a domain of  $[X_d^{(c)}]^n$  which is moreover a tuboid (see [5] for a precise definition) above  $[X_d]^n$  we can state the following axiom:

4. (Weak Spectral Condition). For each  $n$  the distribution  $\mathcal{W}_n(x_1, \dots, x_n)$  is the boundary value of an analytic function  $W_n(z_1, \dots, z_n)$ , defined in the tuboid  $\mathcal{T}_{n-1}$  of the complex manifold  $[X_d^{(c)}]^n$ .

In the case of the two-point functions these axioms are equivalent to the maximal analyticity property. In the general case, in view of the locality axiom the  $n!$  permuted Wightman functions  $W_n^\pi$ , respectively analytic in the permuted tuboids  $\mathcal{T}_{n-1}^\pi$  have boundary values  $\mathcal{W}_n^\pi$  which coincide on the region  $\mathcal{R}_n$  of space-like configurations

$$\mathcal{R}_n = \{x = (x_1, \dots, x_n) \in [X_d]^n; (x_i - x_j)^2 < 0, 1 \leq i, j \leq n\}. \quad (30)$$

By making use of the edge-of-the-wedge theorem for tuboids [5] one sees that for each  $n$  there is an analytic function  $\mathfrak{W}_n(z_1, \dots, z_n)$  which is the common analytic continuation of all the functions  $W_n^\pi$  in the union of the corresponding tuboids  $\mathcal{T}_{n-1}^\pi$  together with a complex neighbourhood of  $\mathcal{R}_n$ . Finally we remark that this scheme includes the possibility of going to the Euclidean sphere by analytic continuation.

A point  $z = (z_1, \dots, z_n) \in [X_d^{(c)}]^n$  is called *euclidean* if  $\Re z_j^{(0)} = 0$  and  $\Im z_j^{(i)} = 0, 1 \leq i \leq d$ , for all  $j, 1 \leq j \leq n$ . Euclidean points in  $[X_d^{(c)}]^n$  are parametrized by the points of the manifold  $[S_d]^n$ , *i.e.* each  $z_j$  is associated to a point on the sphere  $S_R = \{(s, \mathbf{x}) \in \mathbb{R}^{(1+d)}, \mathbf{x} = x^{(1)}, \dots, x^{(d)} : s^2 + x^{(1)^2} + \dots + x^{(d)^2} = R^2\}$  so that  $z_j^{(0)} = is_j$  and  $z_j^{(i)} = x_j^{(i)}, 1 \leq i \leq d$ , for all  $j, 1 \leq j \leq n$ . The set of euclidean points  $z = (z_1, \dots, z_n) \in [X_d^{(c)}]^n$  such that  $z_j \neq z_k$  for all  $j, 1 \leq j, k \leq n$ , is called non-coincident euclidean region  $\mathcal{E}_n$ . It is possible show that indeed the non-coincident euclidean region  $\mathcal{E}_n$  is contained in the extended primitive domain  $D_n^{\text{ext}}$ .

By analogy with the Minkowski case, the restriction of the function  $W_n(z_1, \dots, z_n)$  to the non-coincident euclidean region is called the (non-coincident)  $n$ -point Schwinger function.

Therefore, our set of axioms may be one starting point for a full euclidean formulation of de Sitter quantum field theory, and, eventually, for a constructive approach. As already said, de Sitter Euclidean approach is also of interest for the construction of interacting Minkowskian quantum field theories, giving a natural more symmetrical (rotational covariant) infrared cutoff. The path to construct a QFT model by means of this procedure then would be: introduce an elementary length in the model at hand (curvature); solve the model in the euclidean de Sitter approach (*i.e.* construct an euclidean QFT on a sphere; the associated Schwinger functions will be rotationally covariant); reconstruct a de Sitter QFT by means of an Osterwalder-Schrader type theorem; recover eventually a Minkowski QFT by contraction.



## ACKNOWLEDGEMENTS

This work was supported by EEC-human capital and mobility program—contract n. ERBCHBICT930675. Many discussions with R. Schaeffer and J. Y. Ollitrault on the role of de Sitter QFT's in cosmology are gratefully acknowledged.

## REFERENCES

- [1] H. BATEMAN, *Higher Transcendental Functions*, Vol. I, Mc Graw-Hill, New York, 1954.
- [2] N. D. BIRRELL and P. C. W. DAVIES, *Quantum fields in curved space*, Cambridge University Press, Cambridge, 1982.
- [3] J. BROS, *Nucl. Phys.*, (Proc. Suppl.), Vol. **18B**, 1990, p. 22.
- [4] J. BROS, J.-P. GAZEAU and U. MOSCHELLA, *Phys. Rev. Lett.*, Vol. **73**, 1994, p. 1746.
- [5] J. BROS and U. MOSCHELLA, Two-point functions and quantum fields in de Sitter universe, Saclay preprint T94/160. An axiomatic approach to de Sitter Quantum Field Theory, in preparation. Fourier Transform on a one-sheeted hyperboloid, in preparation.
- [6] J. BROS and G. A. VIANO, *Forum Mathematicum*, to appear.
- [7] T. S. BUNCH and P. C. W. DAVIES, *Proc. R. Soc. Lond. A*, Vol. **360**, 1978, p. 117.
- [8] J. FARAUT, Noyaux sphériques sur un hyperboloïde à une nappe in *Lect. Notes in Math.*, Vol. **497**, Springer-Verlag, 1975.
- [9] R. FIGARI, R. HÖEGH-KROHN and C. R. NAPPI, *Commun. Math. Phys.*, Vol. **44**, 1975, p. 265.
- [10] I. M. GEL'FAND and E. G. SHILOV, *Generalized functions*, Vol. I, Academic Press, New York, 1964.
- [11] G. W. GIBBONS and S. W. HAWKING, *Phys. Rev. S.*, Vol. **10**, 1977, p. 2378.
- [12] R. HAAG, *Local Quantum Physics*, Field, Particles, Algebras, Springer, Berlin, 1992.
- [13] B. S. KAY and R. WALD, *Phys. Rep.*, Vol. **207**, 1991, p. 49.
- [14] A. LINDE, *Particle Physics and Inflationary Cosmology*, Harwood Academic Publishers, Chur, 1990.
- [15] S. MOLCHANOV, *Math. Dokl.*, Vol. **7**, 1966, p. 1553.
- [16] U. MOSCHELLA, Quantization curvature and temperature: the de Sitter space-time, in *Quantization and Infinite-Dimensional System*, J.-P. ANTOINE, S. TWAREQUE ALI, W. LISIECKI, I. M. MLADENOV and A. ODZIEWICZ eds., Plenum, New York, 1994, p. 183.
- [17] G. L. SEWELL, *Ann. Phys.*, Vol. **141**, 1982, p. 201.
- [18] R. F. STREATER and A. S. WIGHTMAN, *PCT, Spin and Statistics, and all that* W. A. BENJAMIN, New York, 1964.

(Manuscript received April 30, 1995.)