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HIDEO TAMURA

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# Shadow scattering by magnetic fields in two dimensions

by

**Hideo TAMURA**

Department of Mathematics, Ubaraki University,
Mito Ibaraki 310, Japan.

ABSTRACT. – We study the semi-classical asymptotic behavior of total scattering cross sections for Schrödinger operators with magnetic fields compactly supported. It is known that the quantum cross sections double the classical ones in the semi-classical limit for Schrödinger operators with finite-range potentials. This fact is called the shadow scattering in the potential scattering theory. We here prove that the shadow scattering is not in general expected for the magnetic scattering case.

RÉSUMÉ. – Nous étudions le comportement asymptotique semi-classique de la section efficace totale de diffusion d'un opérateur de Schrödinger en champ magnétique à support compact. Il est bien connu que la section efficace quantique est deux fois plus grande que son analogue classique à la limite semi-classique pour un opérateur de Schrödinger avec potentiel de partie finie. Ce phénomène est appelé « shadow scattering » en théorie de la diffusion par un potentiel. Nous démontrons que ce phénomène ne se produit pas nécessairement en présence d'un champ magnétique.

## 1. INTRODUCTION

The quantum total scattering cross sections are known to double the classical ones in the semi-classical limit for Schrödinger operators with finite-range potentials ([2], [11], [15]). This fact is called the shadow scattering in the potential scattering theory and is one of interesting results in the semi-classical analysis for Schrödinger operators. We here consider a similar problem for scattering by magnetic fields with compact support. Stating our conclusion first, the shadow scattering is in general violated for the magnetic scattering case.

We shall explain the obtained results a little more presicely. We work in the two dimensional space $R^2$ with a generic point $x = (x_1, x_2)$. Let $A(x) = (a_1(x), a_2(x)) : R^2 \to R^2$ be a smooth magnetic vector potential. We sometimes identify $A$ with the one-form $A = a_1\, dx_1 + a_2\, dx_2$ and the magnetic field $b(x) = \partial_1 a_2 - \partial_2 a_1$, $\partial_j = \partial/\partial x_j$, with the two-form $dA = b(x)\, dx_1 \wedge dx_2$. Throughout the entire discussion, $b(x)$ is always assumed to satisfy that:

$b(x) \in C_0^\infty(R^2)$ is a real smooth function with compact support.

We do not make any decay assumptions on magnetic potentials $A$ with $b = dA$. Such magnetic potentials are not uniquely determined and cannot be expected to fall off rapidly at infinity, even if $b(x)$ is assumed to be compactly supported. In fact, we note that $A(x)$ does not decay faster than $O(|x|^{-1})$, if $b$ has non-vanishing flux $\int b(x)\, dx \neq 0$.

The motion of classical particle with unit mass is only determined by the magnetic field $b$ and it is governed by equation

$$x_1'' = b(x)\, x_2', \qquad x_2'' = -b(x)\, x_1'. \tag{1.1}$$

Hence the total cross section in classical mechanics is defined as

$$\sigma_{\mathrm{cl}}(\omega) = \mathrm{meas}\,(\Sigma_\omega)$$

for incident direction $\omega \in S^1$, $S^1$ being the unit circle, where $\Sigma_\omega = \mathrm{Proj}_\omega(G)$ denotes the projection of $G = \mathrm{supp}\, b$ onto the impact plane (straight line) $\Pi_\omega$ perpendicular to direction $\omega$.

Next we consider the total cross section in quantum mechanics. This is defined as a quantity invariant under gauge transformations. Thus we fix one of magnetic potentials $A(x) = (a_1(x), a_2(x))$ with $b = dA$ and define it as follows:

$$\begin{aligned} a_1(x) &= -(2\pi)^{-1}\, \partial_2 \int \log|x - y|\, b(y)\, dy, \\ a_2(x) &= (2\pi)^{-1}\, \partial_1 \int \log|x - y|\, b(y)\, dy, \end{aligned} \tag{1.2}$$

where the integrals with no domain attached are taken over the whole space. This abbreviation is used throughout. As is easily seen, $A(x)$ satisfies the relation $b = dA$. The quantum mechanics of particle with unit mass in the magnetic field $b$ is described by the Hamiltonian

$$H = H(A) = (-i\nabla - A)^2/2 = \sum_{j=1}^{2} (D_j - a_j)^2/2, \qquad D_j = -i\,\partial_j.$$

This operator admits a unique self-adjoint realization in the space $L^2(R^2)$ with domain $H^2(R^2)$ (Sobolev space of order two). We denote this realization by the same notation $H$. If we define the flux of $b$ as

$$\beta = (2\pi)^{-1} \int b(x)\,dx, \tag{1.3}$$

then $A(x)$ behaves like

$$\begin{aligned} a_1(x) &= -\beta x_2/|x|^2 + O(|x|^{-2}), \\ a_2(x) &= \beta x_1/|x|^2 + O(|x|^{-2}) \end{aligned} \tag{1.4}$$

as $|x| \to \infty$. Thus, as stated above, $A(x)$ does not fall off faster than $O(|x|^{-1})$ at infinity, when the flux $\beta \neq 0$ does not vanish. Hence the perturbation $H - H_0$ to the free Hamiltonian $H_0 = -\Delta/2$ is of long-range class. Nevertheless it is known ([8], [9]) that the ordinary wave operators

$$W_\pm(H, H_0) = s - \lim_{t \to \pm\infty} \exp(itH)\, \exp(-(itH_0) \tag{1.5}$$

exist and are asymptotically complete

$$\operatorname{Ran} W_\pm(H, H_0) = L^2(R^2). \tag{1.6}$$

It is also known [4] that $H$ has no bound states. Hence the scattering operator $S(H, H_0) = W_+^*(H, H_0)\, W_-(H, H_0)$ can be defined as a unitary operator on $L^2(R^2)$ and has the direct integral decomposition

$$S(H, H_0) = \int_0^\infty \oplus\, S(\lambda;\, H, H_0)\, d\lambda,$$

where the fiber $S(\lambda;\, H, H_0)$ is called a scattering matrix at energy $\lambda$ and acts as a unitary operator on $L^2(S^1)$. We should note that the scattering matrix does not necessarily admit the usual decomposition $Id + \{$Hilbert-Schmidt class$\}$, $Id$ being the identity operator, because of the long-range perturbation in (1.4).

We shall discuss the above matter in some detail. Let

$$\gamma(x) = \tan^{-1}(x_2/x_1), \qquad 0 \le \gamma < 2\pi, \tag{1.7}$$

be the azimuth angle from the positive $x_1$ axis. Then $\nabla\gamma = (-x_2/|x|^2, x_1/|x|^2)$ and hence it follows from (1.4) that $A(x) = \beta\nabla\gamma + O(|x|^{-2})$. It should be noted that though $\gamma$ is smooth only in the plane split along the positive $x_1$ axis, $\nabla\gamma$ is smooth in $R^2\backslash\{0\}$. We here introduce the auxiliary Hamiltonian

$$H_\beta = H(B) = (-i\nabla - B)^2/2, \qquad B(x) = \beta\nabla\gamma(x),$$

for which the perturbation $H - H_\beta$ is of short-range class. By use of the chain rule of wave operators, we obtain the relation

$$W_\pm(H, H_0) = W_\pm(H, H_\beta)\, W_\pm(H_\beta, H_0)$$

and hence it follows that

$$S(H, H_0) = W_+^*(H_\beta, H_0)\, S(H, H_\beta)\, W_-(H_\beta, H_0)$$

with $S(H, H_\beta) = W_+^*(H, H_\beta) W_-(H, H_\beta)$. The magnetic potential $B(x)$ represents so-called magnetic string with strong singularity at the origin, so that the operator $H_\beta$ does not necessarily have the same domain as $H$ or $H_0$. However, if we take account of the fact that $H_\beta$ is rotationally invariant and admits the partial wave expansion, we can show [13] that the wave operators $W_\pm(H_\beta, H_0)$ exist and are asymptotically complete $\mathrm{Ran}\, W_\pm(H_\beta, H_0) = L^2(R^2)$ and hence the existence and completeness of wave operators $W_\pm(H, H_\beta)$ also follow from (1.5) and (1.6) at once. We can further calculate explicitly the integral kernel of scattering matrix $S(\lambda; H_\beta, H_0)$

$$(S(\lambda; H_\beta, H_0) f)(\theta) = \int_0^{2\pi} S_\beta(\theta - \theta')\, f(\theta')\, d\theta', \qquad f \in L^2(S^1),$$

where $\theta$ and $\theta'$ denote the azimuth angles from the positive $x_1$ axis. The kernel $S_\beta(\theta)$ is given by

$$S_\beta(\theta) = \cos\beta\pi\delta(\theta) - i\pi^{-1}\sin\beta\pi\, \exp(im\theta)\, \mathrm{v.p.}\, \frac{\exp(i\theta)}{\exp(i\theta) - 1}$$

with $m = [\beta]$, [ ] being the Gauss notation, where $\delta(\cdot)$ is the delta-function and v.p. denotes the principal value. As stated above, $H - H_\beta$ is a short-range perturbation. Hence the scattering matrix $S(\lambda; H, H_\beta)$ has the decomposition $Id - 2\pi i T(\lambda)$ with Hilbert-Schmidt operator $T(\lambda)$. Thus we can decompose $S(\lambda; H, H_0)$ into

$$S(\lambda; H, H_0) = S(\lambda; H_\beta, H_0) - 2\pi i T_b(\lambda), \qquad \lambda > 0, \qquad (1.8)$$

with Hilbert-Schmidt operator $T_b(\lambda)$. If, in particular, $\beta$ is an even integer, then

$$S(\lambda;\, H,\, H_0) = Id - 2\pi i T_b(\lambda)$$

and also if $\beta$ is an odd integer, then

$$S(\lambda;\, H,\, H_0) = -Id - 2\pi i T_b(\lambda).$$

We will give the explicit representation for the integral kernel $T_b(\theta,\, \omega;\, \lambda)$, $(\theta,\, \omega) \in S^1 \times S^1$, of the operator $T_b(\lambda)$ only for integer $\beta \in Z$ in section 3.

We move to the definition of quantum total cross section. The value of flux $\beta$ plays an important role. In fact, the total cross section becomes finite only for integer $\beta \in Z$. To see this, we begin by defining the scattering amplitude $f_b(\omega \to \theta;\, \lambda)$, $(\theta,\, \omega) \in S^1 \times S^1$, for scattering from incident direction $\omega$ to final one $\theta$ at energy $\lambda$ in the magnetic field $b$. It is given by

$$f_b(\omega \to \theta;\, \lambda) = (2\pi)^{1/2}(i\sqrt{2\lambda})^{-1/2}(S(\theta,\, \omega;\, \lambda;\, H,\, H_0) - \delta(\theta - \omega)),$$

where $S(\theta,\, \omega;\, \lambda;\, H,\, H_0)$ denotes the scattering kernel of $S(\lambda;\, H,\, H_0)$. Hence the differential scattering cross section reads $|\, f_b(\omega \to \theta;\, \lambda)\,|^2$ for $\theta \neq \omega$ and the total cross section $\sigma_b(\lambda,\, \omega)$ with incident direction $\omega$ at energy $\lambda$ is defined as

$$\sigma_b(\lambda,\, \omega) = \int |\, f_b(\omega \to \theta;\, \lambda)\,|^2\, d\theta. \tag{1.9}$$

If $\beta$ is an integer, then $\sigma_b(\lambda,\, \omega)$ is given by

$$\sigma_b(\lambda,\, \omega) = (2\pi)^3 (2\lambda)^{-1/2} \int |\, T_b(\theta,\, \omega;\, \lambda)\,|^2\, d\theta. \tag{1.10}$$

However, if $\beta$ is not an integer, the integral (1.9) is divergent because of strong sigularity of $f_b(\omega \to \theta;\, \lambda)$ near the forward direction $\theta = \omega$. Thus the quantum total cross section can be defined only for integer $\beta \in Z$ unlike the classical one. The explicit representation for $\sigma_b(\lambda,\, \omega)$ with $\beta \in Z$ is given in Theorem 3.5.

We proceed to the problem on the semi-classical asymptotic behavior of total cross section for magnetic Schrödinger operator

$$H(h) = H(h;\, A) = (-ih\nabla - A)^2/2, \qquad 0 < h \ll 1, \tag{1.11}$$

with magnetic potential $A(x)$ defined by (1.2). By the investigation above, we have to assume that $\beta/h \in Z$ in order that the total cross section

$\sigma_b(\lambda, \omega; h)$ for scattering involving the pair $(H_0(h), H(h))$, $H_0(h) = -h^2\Delta/2$, is well defined. Then the total cross section is given as

$$\sigma_b(\lambda, \omega; h) = \sigma_{b/h}(\lambda h^{-2}, \omega) \tag{1.12}$$

by use of the above notation. The main results, somewhat loosely speaking, are that: (1) If $\Sigma_\omega$ is connected, then

$$\sigma_b(\lambda, \omega; h) \to 2\,\sigma_{\mathrm{cl}}(\omega) = 2\,\mathrm{meas}(\Sigma_\omega), \qquad h \to 0,$$

where $h$ tends to zero under resriction $\beta/h \in Z$; (2) If $\Sigma_\omega$ is not connected, then such a convergence does not necessarily hold true. The precise statement is formulated as Theorem 4.1 together with some additional assumptions on the classical dynamics (1.1). The idea developed in the proof extends to the higher dimensional case. We will make a brief comment on the three dimensional case in the final section.

We end the section by making comments on the notations accepted in the present paper; (i) We write $\langle\, , \rangle$ for the scalar product in $R^2$. (ii) We denote by $(\, , \,)$ the $L^2$ scalar product in $L^2(R^2)$. (iii) The notation $L^2_s(R^2)$ denotes the weighted $L^2$ space $L^2(R^2; \langle x \rangle^{2s}\, dx)$ with weight $\langle x \rangle^s = (1 + |x|^2)^{s/2}$.

## 2. MAGNETIC POTENTIALS

We keep the same notations as in the previous section and, in particular, the magnetic field $b(x)$ is always assumed to be a smooth function with compact support. Let $A(x) = (a_1(x), a_2(x))$ be defined by (1.2). In this section, we mention several properties of magnetic potential $A(x)$ as a series of lemmas. These lemmas are used to derive the representation for the total cross section $\sigma_b(\lambda, \omega)$.

The next lemma is easy to prove. We skip the proof. In fact, the proof is done by repeated use of partial integration, if we take account of the relation

$$\partial_x^\alpha \log|x - y| = (-1)^{|\alpha|}\,\partial_y^\alpha \log|x - y|.$$

LEMMA 2.1. – *Let $\beta$ and $\gamma(x)$ be defined by* (1.3) *and* (1.7), *respectively. Then $A(x)$ has the following properties.*

(1) *$A(x)$ is smooth and obeys the bounds*

$$\partial_x^\alpha A(x) = O(|x|^{-1-|\alpha|}), \qquad |x| \to \infty.$$

(2) *Let $B(x)$ be again defined as $B(x) = \beta\nabla\gamma(x)$. Then*

$$\partial_x^\alpha (A(x) - B(x)) = O(|x|^{-2-|\alpha|}), \qquad |x| \to \infty,$$

*and, in particular, one has*

$$\partial_x^\alpha (x_1 a_1(x) + x_2 a_2(x)) = O(|x|^{-1-|\alpha|}).$$

By Lemma 2.1 (2), we can define $\zeta(x)$ as

$$\zeta(x) = a(x) + \beta\gamma(x), \qquad x \neq 0, \tag{2.1}$$

where

$$a(x) = -\int_1^\infty (x_1 a_1(sx) + x_2 a_2(sx))\, ds.$$

A similar function integrated from 0 to 1 has been used in the spectral analysis for magnetic Schrödinger operators ([6], [7], [8]). As is shown in the lemma below, $A(x) - \nabla\zeta(x)$ has compact support.

LEMMA 2.2. – *Let $\zeta(x)$ and $a(x)$ be as above. Then one has:*

(1) $a(x)$ *is smooth in* $R^2\backslash\{0\}$, *and it obeys the bounds*

$$\partial_x^\alpha a(x) = O(|x|^{-1-|\alpha|}), \qquad |x| \to \infty.$$

(2) $A(x)$ *is represented as*

$$A(x) = \nabla\zeta(x) + E(x), \qquad x \neq 0,$$

*where* $E(x) = (e_1(x), e_2(x))$ *has compact support and is given by*

$$e_1(x) = \int_1^\infty s x_2 b(sx)\, ds, \qquad e_2(x) = -\int_1^\infty s x_1 b(sx)\, ds. \tag{2.2}$$

*Proof.* – (1) follows from Lemma 2.1 (2). To prove (2), we set $b_{jk}(x) = \partial_j a_k(x) - \partial_k a_j(x)$, so that $b(x) = b_{12}(x) = -b_{21}(x)$. Then a simple calculation yields

$$\partial_j a(x) = -\int_1^\infty \{a_j(sx) + s(d/ds)a_j(sx) + s x_k b_{jk}(sx)\}\, ds, \qquad k \neq j,$$

and hence it follows by partial integration that

$$\partial_j a(x) = a_j(x) - \int_1^\infty s x_k b_{jk}(sx)\, ds - \lim_{R\to\infty} R a_j(Rx).$$

By Lemma 2.1 (2) again, we have $\lim_{R\to\infty} RA(Rx) = B(x)$. This proves (2) and the proof is complete. □

Let $\chi_0 \in C_0^\infty([0, \infty))$ be a basic cut-off function such that $0 \leq \chi_0 \leq 1$ and

$$\chi_0(s) = 1 \quad \text{for } 0 \leq s \leq 1, \qquad = 0 \quad \text{for } s > 2 \tag{2.3}$$

and we set $\chi_\infty(s) = 1 - \chi_0(s)$. We define the multiplication operators

$$J = \exp(i\zeta(x))\times, \qquad J_R = \chi_\infty(|x|/R) \exp(i\zeta(x))\times \tag{2.4}$$

for $R \gg 1$ large enough. If $\beta$ is an integer, then the function $\exp(i\zeta(x))$ is smooth in $R^2 \backslash \{0\}$. Hence we can define

$$V = HJ_R - J_R H_0 = H(A) J_R - J_R H_0. \tag{2.5}$$

LEMMA 2.3. – *Assume that $\beta$ is an integer. Let $\chi_R(x) = \chi_0(|x|/R)$. If $R \gg 1$ is large enough, then*

$$V = J[\chi_R, H_0],$$

*where $[\ ,\ ]$ denotes the commutator notation. Hence the coefficients of $V$ are all compactly supported and vanish in a neighborhood of the origin.*

*Proof.* – We write $\chi_{\infty R}(x) = \chi_\infty(|x|/R)$, so that $\chi_R + \chi_{\infty R} = 1$. By the gauge transformation, $V$ is calculated as

$$V = J(H(A - \nabla\zeta)\chi_{\infty R} - \chi_{\infty R} H_0).$$

If $R \gg 1$ is taken large enough, then it follows from Lemma 2.2 (2) that $A - \nabla\zeta = 0$ on the support of $\chi_{\infty R}$. This completes the proof. □

## 3. TOTAL SCATTERING CROSS SECTIONS

Throughout the section, the flux $\beta$ is assumed to be an integer and $R \gg 1$ is fixed large enough. The aim here is to give the representation for the total cross section $\sigma_b(\lambda, \omega)$ with $\beta \in Z$.

LEMMA 3.1. – *Let $J_R$ be defined by* (2.4). *Then there exist the strong limits*

$$W_{\pm 0}(J_R^*) = s - \lim_{t \to \pm\infty} \exp(itH_0) J_R^* \exp(-itH_0)$$

*and one has*

$$\operatorname{Ran} W_{\pm 0}(J_R^*) = L^2(R^2). \tag{3.1}$$

*Proof.* – We write $\chi_\infty(x)$ for $\chi_\infty(|x|/R)$ and calculate

$$H_0 J_R^* - J_R^* H_0 = J^*(H(\nabla\zeta)\chi_\infty - \chi_\infty H_0).$$

The operator in brackets on the right side takes the form

$$H(\nabla\zeta)\,\chi_\infty - \chi_\infty\,H_0 = -\beta\chi_\infty\,\langle\nabla\gamma,\,\nabla\rangle + O(|x|^{-2})\,\nabla + O(|x|^{-2}), \qquad |x|\to\infty.$$

The second and third operators are of short-range class. On the other hand, the first operator can be rewritten as $\beta\chi_\infty\,\langle\nabla\gamma,\,\nabla\rangle = O(|x|^{-2})\,L_0$ with $L_0 = x_1\,D_2 - x_2\,D_1$. The operator $L_0$ commutes with $H_0$. Hence it can be shown that the strong limits exist. We can prove the existence of $W_{\pm 0}(J_R)$ in exactly the same way. This implies (3.1). □

LEMMA 3.2. – *There exist the strong limits*

$$W_\pm(J_R) = s - \lim_{t\to\pm\infty} \exp(itH)\,J_R\,\exp(-itH_0)$$

*and one has*

$$\operatorname{Ran} W_\pm(J_R) = L^2(R^2). \tag{3.2}$$

*Proof.* – We again write $\chi_\infty(x)$ for $\chi_\infty(|x|/R)$. By Lemma 2.3, $V = HJ_R - J_R\,H_0$ is a short-range perturbation and hence the existence of strong limits follows immediately. As is easily seen, $W_\pm(H,\,H_0) = W_\pm(J_R)\,W_{\pm 0}(J_R^*)$. This, together with (1.6) and (3.1), proves (3.2). □

LEMMA 3.3. – *Let $W_{\pm 0}(J_R^*)$ be as in Lemma 3.1. Then one has*

$$(W_{\pm 0}(J_R^*)\,f)\hat{\ }(\xi) = \exp(-i\beta\gamma(\pm\xi))\hat{f}(\xi), \qquad f\in L^2(R^2),$$

*where*

$$\hat{f}(\xi) = (2\pi)^{-1}\int \exp(-i\,\langle x,\,\xi\rangle)\,f(x)\,dx$$

*denotes the Fourier transformation of $f$.*

*Proof.* – We consider the + case only. As is well known, the free solution behaves like

$$(\exp(-itH_0)\,f)(x) = (it)^{-1}\exp(i|x|^2/2t)\hat{f}(x/t) + o(1), \qquad t\to\infty,$$

where $o(1)$ denotes terms convergent to zero strongly as $t\to\infty$. By Lemma 2.2 (1), $a(x)$ falls off at infinity and also the azimuth angle $\gamma(x)$ is homogeneous of degree zero. Thus we have

$$(J_R^*\exp(-itH_0)\,f)(x) = (it)^{-1}\exp(i|x|^2/2t) \times \exp(-i\beta\gamma(x/t))\,\hat{f}(x/t) + o(1).$$

This proves the lemma. □

We here summarize the spectral properties of $H$ for later requirements. By Lemma 2.1, $a_j$ satisfies $\partial_x^\alpha a_j = O(|x|^{-1-|\alpha|})$. Hence it follows from the results due to [4] that: (1) $H$ has no bound states; (2) For any $s > 1/2$, the boundary values of resolvents

$$R(\lambda \pm i0;\, H) = \lim_{\varepsilon \downarrow 0} (H - \lambda \mp i\varepsilon)^{-1} \,:\, L_s^2(R^2) \to L_{-s}^2(R^2), \qquad \lambda > 0,$$

exist in the operator topology, where the convergence is uniform locally in $\lambda$.

We denote the generalized eigenfunction of the free Hamiltonian $H_0$ as

$$\varphi_0(x;\, \lambda,\, \omega) = \exp(i\sqrt{2\lambda}\,\langle x,\, \omega\rangle), \qquad (\lambda,\, \omega) \in (0,\, \infty) \times S^1,$$

and define the unitary mapping $F \,:\, L^2(R_x^2) \to L^2((0,\, \infty)) \otimes L^2(S^1)$ by

$$(Ff)(\lambda,\, \omega) = (2\pi)^{-1} \int \bar{\varphi}_0(x;\, \lambda, \omega)\, f(x)\, dx = \hat{f}(\sqrt{2\lambda}\omega).$$

The mapping $F$ yields the spectral representation for $H_0$ in the sense that $H_0$ is transformed into the multiplication operator by $\lambda$ in the space $L^2((0, \infty)) \otimes L^2(S^1)$. We also define the trace operator $F(\lambda) \,:\, L_s^2(R^2) \to L^2(S^1)$, $s > 1/2$, by

$$(F(\lambda)\, f)(\omega) = \hat{f}(\xi)\,|_{\xi = \sqrt{2\lambda}\omega}.$$

By Lemmas 3.1 and 3.2, we obtain the relation

$$S(H,\, H_0) = W_{+0}^*(J_R^*)\, S(J_R)\, W_{-0}(J_R^*) \tag{3.3}$$

with $S(J_R) = W_+^*(J_R)\, W_-(J_R)$. The operator $S(J_R)$ commutes with $H_0$ and hence it is represented as a decomposable operator

$$S(J_R) = \int_0^\infty \oplus\, S(\lambda;\, J_R)\, d\lambda,$$

where $S(\lambda;\, J_R) \,:\, L^2(S^1) \to L^2(S^1)$ is unitary and acts as

$$(F(\lambda)\, S(J_R)\, f)(\omega) = (S(\lambda;\, J_R)\, F(\lambda)\, f)(\omega)$$

for $f \in L_s^2(R^2)$ with $s > 1/2$. The lemma below follows as a special case of general results in abstract scattering theory (*see*, for example, Chap. 5 of [14]).

LEMMA 3.4. – *Let $V$ be defined by* (2.5). *Then $S(\lambda;\, J_R)$ is represented as*

$$S(\lambda;\, J_R) = Id - 2\pi i T(\lambda;\, J_R)$$

*where* $T(\lambda;\ J_R)$ *is given by*

$$T(\lambda;\ J_R) = F(\lambda)\,(J_R^* V - V^* R(\lambda + i0;\ H)\, V)\, F^*(\lambda).$$

By Lemmas 3.3 and 3.4, we obtain from (3.3) that

$$S(\lambda;\ H,\ H_0) = \exp(i\beta\pi)\, Id - 2\pi i M_+^* T(\lambda;\ J_R)\, M_-,$$

where $M_\pm : L^2(S^1) \to L^2(S^1)$ is the multiplication operators associated with $\exp(-i\beta\gamma(\pm\omega))$. The integral kernel of $T(\lambda;\ J_R)$ is given by

$$T(\theta,\omega;\ \lambda;\ J_R) = (2\pi)^{-2}\,((J_R^* - V^* R(\lambda + i0;\ H))\, V\varphi_0(\lambda,\omega),\ \varphi_0(\lambda,\theta)),$$

where $\varphi_0(\lambda,\omega) = \varphi_0(x;\ \lambda,\omega)$. This yields the kernel representation for the operator $T_b(\lambda)$ in (1.8) and also it follows from (1.10) that

$$\sigma_b(\lambda;\ \omega) = (2\pi)^3 (2\lambda)^{-1/2} \int |T(\theta,\omega;\ \lambda;\ J_R)|^2\, d\theta. \qquad (3.4)$$

This representation for $\sigma_b(\lambda,\omega)$ is obviously independent of the choice $R \gg 1$.

THEOREM 3.5. – *Let* $V$ *be again defined in* (2.5). *Then one has*

$$\sigma_b(\lambda,\omega) = 2(2\lambda)^{-1/2}\, \mathrm{Im}(R(\lambda + i0;\ H)\, V\varphi_0(\lambda,\omega),\ V\varphi_0(\lambda,\omega)).$$

*Proof.* – We write $T(\lambda)$ for $T(\lambda;\ J_R)$. The functions on both the side of the above relation are continuous in $(\lambda,\omega) \in (0,\infty) \times S^1$. Hence it suffices to prove this relation in the weak form. Let $f(\lambda,\omega)$ be a real smooth function with compact support. We denote by $f(\lambda)$ the multiplication operator with $f(\lambda,\omega)$ acting on $L^2(S^1)$ for each $\lambda > 0$. Then it follows from (3.4) that

$$\int\int \sigma_b(\lambda,\omega)\,|f(\lambda,\omega)|^2\, d\omega\, d\lambda$$
$$= (2\pi)^3 \int (2\lambda)^{-1/2}\, \mathrm{Tr}(f(\lambda)\, T^*(\lambda)\, T(\lambda)\, f(\lambda))\, d\lambda.$$

We calculate the trace in the integrand on the right side. Since $S(\lambda;\ J_R)$: $L^2(S^1) \to L^2(S^1)$ is unitary, we obtain

$$T^*(\lambda)\, T(\lambda) = i(2\pi)^{-1}(T(\lambda) - T^*(\lambda)).$$

Hence the trace under consideration is decomposed into the sum $I_1(\lambda) + I_2(\lambda)$, where

$$I_1(\lambda) = -2(2\pi)^{-3} \int \mathrm{Im}(J_R^* V\varphi_0(\lambda,\omega),\ \varphi_0(\lambda,\omega))\,|f(\lambda,\omega)|^2\, d\omega,$$

$$I_2(\lambda) = 2(2\pi)^{-3} \int \mathrm{Im}(R(\lambda + i0;\ H)\, V\varphi_0(\lambda,\omega),$$
$$V\varphi_0(\lambda,\omega))\,|f(\lambda,\omega)|^2\, d\omega.$$

We assert that $I_1(\lambda) = 0$. To see this, we compute

$$J_R^* V - V^* J_R = [1 - \chi_{\infty R}^2, H_0]$$

with $\chi_{\infty R} = 1 - \chi_0(|x|/R)$. Since $H_0\varphi_0 = \lambda\varphi_0$, the assertion above follows at once and the proof is complete. □

We end the section by giving the representation for total scattering cross sections in the semi-classical case. Let $H_0(h) = -h^2\Delta/2$, $0 < h \ll 1$, and let $H(h) = H(h; A)$ be defined by (1.11). We assume that $\beta/h \in Z$ and denote again by $\sigma_b(\lambda, \omega; h)$ the total scattering cross section for the pair $(H_0(h), H(h))$. The representation for $\sigma_b(\lambda, \omega; h)$ can be easily derived from Theorem 3.5. We use the following notations:

$$J(h) = \exp(i\zeta(x)/h)\times, \qquad J_R(h) = \chi_\infty(|x|/R)\exp(i\zeta(x)/h)\times,$$
$$V(h) = H(h)J_R(h) - J_R(h)H_0(h) = J(h)[\chi_R, H_0(h)],$$
$$\chi_R = \chi_0(|x|/R).$$

Under these notations, we obtain from (1.12) that

$$\sigma_b(\lambda, \omega; h) = 2(2\lambda)^{-1/2}h^{-1}\mathrm{Im}\,(R(\lambda+i0; H(h))V(h)\varphi,\ V(h)\varphi) \quad (3.5)$$

with $\varphi = \exp(i\sqrt{2\lambda}\,h^{-1}\langle x, \omega\rangle)$.

## 4. SHADOW SCATTERING BY MAGNETIC FIELDS

In this section we formulate the main theorem. We set ourselves in the following situation. Let $G = \mathrm{supp}\, b$ again denote the support of magnetic field $b$. We assume that $G$ consists of a finite number of connected components. We decompose the projection $\Sigma_\omega = \mathrm{Proj}_\omega(G)$ of $G$ onto the impact plane $\Pi_\omega$ into a disjoint union of non-empty compact intervals

$$\Sigma_\omega = \bigcup_{j=1}^{n} \Sigma_{j\omega}, \qquad \Sigma_{j\omega} \cap \Sigma_{k\omega} = \emptyset \qquad \text{for} \quad j \neq k. \quad (4.1)$$

For notational brevity, we write each interval as $\Sigma_{j\omega} = [c_j, d_j]$, $1 \le j \le n$, and assume without loss of generality that $c_1 < d_1 < c_2 < \cdots < d_{n-1} < c_n < d_n$, where the positive direction in $\Pi_\omega$ is taken in such a way that the azimuth angle from $\omega$ equals $\pi/2$. We further denote by $\Lambda_{j\omega} = [d_j, c_{j+1}]$, $1 \le j \le n-1$, the gap interval between the adjoining intervals $\Sigma_j$ and $\Sigma_{j+1}$. According to decomposition (4.1), the support $G$ is also decomposed as $G = \bigcup_{j=1}^{n} G_j$ with $\Sigma_{j\omega} = \mathrm{Proj}_\omega(G_j)$ and the flux $\beta$ is represented as the sum $\beta = \sum_{j=1}^{n} \beta_j$, where

$$\beta_j = (2\pi)^{-1} \int_{G_j} b(x)\, dx, \qquad 1 \le j \le n.$$

Since the total cross section is invariant under translation, we may assume that the origin $x = 0$ is contained in $G_1$ as an interior point.

Before stating the main theorem, we make two assumptions on classical system (1.1). Let $x(t) = (x_1(t), x_2(t))$ be a solution to (1.1). We say that $\lambda\,(= |x'(t)|^2/2)$ is a non-trapping energy, if all solutions $x(t)$ with energy $\lambda$ escape to infinity as $t \to \pm\infty$. As is easily seen, the totality of non-trapping energies is open in $(0, \infty)$. Let $\lambda > 0$ be in a non-trapping energy range. Then we denote by $x(t;\, \lambda,\, \omega,\, z)$, $z \in \Pi_\omega$, the solution to (1.1) behaving like

$$x(t;\, \lambda,\, \omega,\, z) = \sqrt{2\lambda}\,\omega t + z, \qquad t \ll -1.$$

This solution describes the motion of classical particle with the incident momentum $\sqrt{2\lambda}\,\omega$ and the impact parameter $z$. As the second assumption, such a particle is assumed to have a momentum different from the incident one after scattering by the magnetic field $b$

$$x'(t;\, \lambda,\, \omega,\, z) \ne \sqrt{2\lambda}\,\omega, \qquad t \gg 1, \qquad \text{for} \quad z \in \mathrm{Int}\, \Sigma_\omega, \tag{4.2}$$

where $\mathrm{Int}\, \Sigma_\omega$ denotes the interior of $\Sigma_\omega$. It is enough for the discussion below to assume (4.2) only for *a. e.* $z \in \Sigma_\omega$. For later reference, we here note that the solution $x(t)$ to (1.1) is connected with the solution $(q(t),\, p(t))$ to the Hamilton system

$$q' = \nabla_p H(q,\, p), \qquad p' = -\nabla_q H(q,\, p) \tag{4.3}$$

through the relation

$$q(t) = x(t), \qquad p(t) = x'(t) + A(x(t)), \tag{4.4}$$

where $H(q,\, p) = (p - A(q))^2/2$ and $p(t)$ is called the canonical momentum.

We are now in a position to formulate the main theorem.

THEOREM 4.1. – *Let the notations be as above. Assume that* $\lambda$ *is fixed in a non-trapping energy range and* (4.2) *is fulfilled. Then one has*

$$\sigma_b(\lambda, \omega; h) = 2\,\mathrm{meas}(\Sigma_\omega) + 4\sum_{j=1}^{n-1} \sin^2(\nu_j \pi/h)\,\mathrm{meas}(\Lambda_{j\omega}) + o(1), \qquad h \to 0,$$

*where* $h$ *tends to zero under restriction* $\beta/h \in Z$ *and*

$$\nu_j = \sum_{k=1}^{j} \beta_k, \qquad 1 \le j \le n-1. \tag{4.5}$$

*Remark.* – The theorem above implies that if $\Sigma_\omega$ is connected, then the quantum total cross section doubles the classical one under the two assumptions on classical system (1.1) and hence the shadow scattering is established. However, if $\Sigma_\omega$ is not connected, the shadow scattering is not in general expected for scattering by magnetic fields with compact support.

We will prove Theorem 4.1 in the next section. The remaining part of the section is devoted to preparing several basic lemmas which are used in proving the theorem.

Let $S_0$ be the set of all symbols $r(x, \xi)$ such that $r(x, \xi)$ is smooth in $R^2 \times R^2$ and satisfies the estimate

$$|\partial_x^\alpha \partial_\xi^\beta r(x, \xi)| \le C_{\alpha\beta} \langle x \rangle^{-|\alpha|} \langle \xi \rangle^{-|\beta|}.$$

We denote by $OPS_0$ the class of pseudodifferential operators

$$r(x, hD)\,u = (2\pi h)^{-2} \int\!\!\int \exp(ih^{-1}\langle x - y, \xi \rangle)\, r(x, \xi)\, u(y)\, dy\, d\xi$$

with symbol $r(x, \xi) \in S_0$.

LEMMA 4.2. – *Assume that* $\lambda > 0$ *is in a non-trapping energy range. Then one has:*

(1) *The resolvent* $R(\lambda + i0;\, H(h))$ *obeys the bound*

$$\| \langle x \rangle^{-s} R(\lambda + i0;\, H(h)) \langle x \rangle^{-s} \| = O(h^{-1}), \qquad h \to 0,$$

*for* $s > 1/2$ *as a bounded operator from* $L^2(R^2)$ *into itself.*

(2) *Let* $Q(x)$ *be a bounded function with compact support. Let* $r(x, hD)$ *be of class* $OPS_0$. *Assume that the symbol* $r(x, \xi)$ *is supported in the outgoing region*

$$\Gamma_+ = \{ (x, \xi) : |x| > M,\ c^{-1} < |\xi| < c,\ \langle x, \xi \rangle > 0 \}$$

*for some* $c > 1$. *If* $M \gg 1$ *is chosen large enough, then*

$$\| QR(\lambda + i0;\, H(h))\, r(x,\, hD) \| = O(h^N)$$

*for any* $N \gg 1$.

LEMMA 4.3. – *Let* $(q(t;\, q_0,\, p_0),\, p(t;\, q_0,\, p_0))$ *be the solution to the Hamilton system* (4.3) *with initial state* $(q(0),\, p(0)) = (q_0,\, p_0)$. *Denote by* $\Theta_t$ *the mapping*

$$\Theta_t : (q_0,\, p_0) \to (q(t;\, q_0,\, p_0),\, p(t;\, q_0,\, p_0)).$$

*Assume that* $r_0(x,\, \xi) \in S_0$ *is of compact support. If* $r_t(x,\, \xi) \in S_0$ *vanishes in a small neighborhood of*

$$\Gamma_t = \{ (x,\, \xi) : (x,\, \xi) = (p(t),\, q(t)) = \Theta_t(q_0,\, p_0),\, (q_0,\, p_0) \in \operatorname{supp} r_0 \},$$

*then one has*

$$\| \langle x \rangle^N r_t(x,\, hD) \exp(-ih^{-1} tH(h))\, r_0(x,\, hD) \| = O(h^N)$$

*for any* $N \gg 1$, *where the order estimate is locally uniform in* $t$.

The semi-classical resolvent estimate (Lemma 4.2) has been already proved by [3], [11] in the potential scattering case. The argument there extends to the case of magnetic Schrödinger operators without any essential changes. In particular, statement (2) is a special form of microlocal resolvent estimates ([5], [12]). This can be intuitively understood from the fact that solutions to the Hamilton system (4.3) with initial states in the outgoing region $\Gamma_+$ never pass over the support of $Q$ for time $t > 0$. We skip the proof of Lemma 4.2. Lemma 4.3 is also a special case of the result (Theorem IV-10) obtained by [10]. The result corresponds to the Egorov theorem on the propagation of singularities for hyperbolic equations.

LEMMA 4.4. – *Let* $\Omega_j = \{ x = y\omega + z : z \in \Lambda_{j\omega} \}$ *and let* $E(x) = (e_1(x),\, e_2(x))$, $x \neq 0$, *be defined by* (2.2). *Define*

$$\psi_j(x) = \psi_j(y,\, z) = \int_{-\infty}^{y} \langle \omega,\, E(s\omega + z) \rangle\, ds$$

*for* $x \in \Omega_j$. *Then* $\psi_j$ *has the following properties:* (1) $\nabla \psi_j = E$, (2) $\psi_j = 0$ *for* $y < -L$ *and*

$$\psi_j = 2\pi(\beta - \nu_j), \qquad y > L, \tag{4.6}$$

*for* $L \gg 1$ *large enough, where* $\nu_j$ *is defined by* (4.5).

*Proof.* – By assumption, the origin is contained in $G_1$. Hence $E(x)$ is smooth in $\Omega_j$ and satisfies $dE = 0$ there, which follows from Lemma 2.2 at once. Thus (1) can be easily verified. By definition, it is trival that $\psi_j$ vanishes for $y < -L$ and (4.6) is obtained as an simple application of the Stokes formula. □

## 5. PROOF OR THEOREM 4.1

The proof is rather lengthy. We divide it into several steps. Throughout the proof, we work in the coordinate system $x = y\omega + z$, $z \in \Pi_\omega$.

(1) We begin by recalling the representation (3.5) for $\sigma_b(\lambda, \omega; h)$

$$\sigma_b(\lambda, \omega; h) = \alpha \operatorname{Im}(R(\lambda + i0; H(h)) V(h) \varphi, V(h) \varphi)$$

with $\alpha = 2(2\lambda)^{-1/2} h^{-1}$, where $\varphi = \exp(i\sqrt{2\lambda}\, h^{-1} \langle x, \omega \rangle)$,

$$V(h) = J(h) [\chi_R, H_0(h)], \qquad \chi_R = \chi_0(|x|/R), \qquad R \gg 1,$$

and $J(h)$ is the multiplication operator with function $\exp(i\zeta(x)/h)$, $\zeta(x)$ being defined by (2.1). All the coefficients of first order differential operator $V(h)$ are supported in $B_R = \{x : R < |x| < 2R\}$. We now consider a smooth partition of unity over the impact plane $\Pi_\omega$. It can be easily seen from Lemma 4.2 (1) that

$$\| V^*(h) R(\lambda + i0; H(h)) V(h) \| = O(h) \tag{5.1}$$

as a bounded operator from $L^2(R^2)$ into itself. Hence the contribution from partitions with support in a neighborhood of the boundary $\partial\Sigma_\omega$ can be made as small as desired uniformly in $h$ because of the boundedness (5.1). Thus it suffices to evaluate such a term as

$$I(f, g) = \alpha \operatorname{Im}(R(\lambda + i0; H(h)) fV(h) \varphi, gV(h) \varphi),$$

where $f, g \in C_0^\infty(\Pi_\omega)$ are non-negative functions supported away from the boundary $\partial\Sigma_\omega$.

(2) We represent the resolvent $R(\lambda + i0; H(h))$ by the time-dependent formula

$$ih^{-1} \int_0^T \exp(-ih^{-1} t (H(h) - \lambda))\, dt$$
$$+ R(\lambda + i0; H(h)) \exp(-ih^{-1} T (H(h) - \lambda))$$

for $T > 0$. If $T \gg 1$ is taken large enough, then the contribution from the second operator can be neglected. In fact, by the non-trapping condition, solutions to Hamilton system (4.3) starting from $B_R$ with energy $\lambda$ lie in the outgoing region $\Gamma_+$ as in Lemma 4.2 at time $T \gg 1$. Hence it follows from Lemma 4.3 that

$$\| \langle x \rangle^N r(x, hD) \exp(-ih^{-1} TH(h)) fV(h) \Psi \| = O(h^N), \qquad N \gg 1,$$

for symbol $r(x, \xi) \in S_0$ vanishing on $\Gamma_+$. This, together with Lemma 4.2 (2), implies that

$$I(f, g) = \alpha h^{-1} \int_0^T \operatorname{Re}(\exp(-ih^{-1} t (H(h) - \lambda)) fV(h) \varphi, \\ gV(h) \varphi) \, dt + O(h)$$

for $T \gg R$. Thus the proof is reduced to evaluating the integral on the right side.

(3) Let $c_1 < d_1 < \cdots < c_n < d_n$ be as in the previous section. We first deal with the case

$$\operatorname{supp} f \subset (-\infty, c_1) \qquad \text{or} \qquad (d_n, \infty). \tag{5.2}$$

We consider only the case $\operatorname{supp} f \subset (d_n, \infty)$. We compute

$$fV(h) \varphi = i (2\lambda)^{1/2} h \exp(i\zeta(x)/h) (\partial_y \chi_R) f\varphi + O(h^2) \tag{5.3}$$

and construct an approximate solution to

$$u(t) = \exp(-ih^{-1} t (H(h) - \lambda)) \exp(i\zeta(x)/h) (\partial_y \chi_R) f\varphi. \tag{5.4}$$

By Lemma 2.2, $A(x) = \nabla\zeta(x)$ for $z > d_n$. Hence, by the gauge transformation, $u(t)$ can be approximated by a free solution

$$\exp(i\zeta(x)/h) \exp(-ih^{-1} t (H_0(h) - \lambda)) (\partial_y \chi_R) f\varphi \\ \sim \exp(i\zeta(x)/h) (\partial_y \chi_R) (y - \sqrt{2\lambda} t, z) f(z) \varphi.$$

The approximate solution is still supported in $(d_n, \infty)$ as a function of $z$. The construction can be rigorously justified by the Duhamel principle and we can prove that

$$u(t) = \exp(i\zeta(x)/h) (\partial_y \chi_R) (y - \sqrt{2\lambda} t, z) f(z) \varphi + O(h) \tag{5.5}$$

uniformly in $t$, $0 \le t \le T$. Since $gV(h)\varphi$ takes a form similar to (5.3), it follows that

$$I(f, g) = 2(2\lambda)^{1/2} \int_0^T \int (\partial_y \chi_R)(y - \sqrt{2\lambda} t, z) \\ \times (\partial_y \chi_R)(y, z) f(z) g(z) \, dx \, dt + O(h).$$

As is easily seen,

$$(2\lambda)^{1/2}\int_0^T\int(\partial_y\chi_R)(y-\sqrt{2\lambda}\,t,\,z)\,(\partial_y\chi_R)(y,\,z)\,dy\,dt=0$$

and hence we have $I(f,\,g)=O(h)$. The same bound remains true in the case that $g$ satisfies (5.2). Thus we can conclude that $I(f,\,g)=O(h)$, if either of $f$ and $g$ satisfies (5.2).

(4) Next we discuss the case in which $f$ is strictly supported inside $\Sigma_\omega$. Let $\eta_{+R}(y)$ be a non-negative smooth function such that $\eta_{+R}=0$ on $(-\infty,\,R/4]$ and $\eta_{+R}=1$ on $[R/2,\,\infty)$, and define $\eta_{-R}$ as $\eta_{-R}(y)=\eta_{+R}(-y)$. Then we may write

$$\begin{aligned} f\partial_y\chi_R &= f\eta_{+R}\,\partial_y\chi_R+f\eta_{-R}\,\partial_y\chi_R\\ &= f_+(y,\,z)\,\partial_y\chi_R+f_-(y,\,z)\,\partial_y\chi_R. \end{aligned}\tag{5.6}$$

By the result obtained in step (3), we may assume that $g$ is supported in $(c_1,\,d_n)$ and hence $g$ also admits a similar decomposition. According to this decomposition, $I(f,\,g)$ is represented as the sum of four terms. We first consider the term

$$I(f_+,\,g_+)=\alpha\,\mathrm{Im}\,(R(\lambda+i0;\,H(h))\,f_+V(h)\,\varphi,\,g_+V(h)\,\varphi).$$

As in step (3), we construct an approximate solution to

$$v(t)=\exp(-ih^{-1}\,t\,(H(h)-\lambda))\exp(i\zeta(x)/h)\,(\partial_y\chi_R)\,f_+\,\varphi.$$

Since $f_+$ has support in $[R/4,\,\infty)$ as a function of $y$, the approximate solution is obtained as

$$\exp(i\zeta(x)/h)\,(\partial_y\chi_R)\,(y-\sqrt{2\lambda}\,t,\,z)\,\eta_{+R}(y-\sqrt{2\lambda}\,t,\,z)\,f(z)\,\varphi$$

by use of the same argument as in step (3). If $x=(y,\,z)\in\operatorname{supp}\partial_y\chi_R$, $y\in\operatorname{supp}\eta_{+R}$ and $z\in\Sigma_\omega$, then it follows by partial integration that

$$\begin{aligned} &(2\lambda)^{1/2}\int_0^T(\partial_y\chi_R)(y-\sqrt{2\lambda}\,t,\,z)\,\eta_{+R}(y-\sqrt{2\lambda}\,t)\,dt\\ &\quad=(\chi_R(y,\,z)-1)\,\eta_{+R}(y). \end{aligned}$$

Since $(\partial_y \chi_R)(y, z)\, \eta_{+R}(y) = (\partial_y \chi_R)(y, z)$ for $y > 0$ and $z \in \Sigma_\omega$, we have

$$2 \int (\chi_R(y, z) - 1)\, (\partial_y \chi_R)(y, z)\, \eta_{+R}(y)^2\, dy = 1, \qquad z \in \Sigma_\omega.$$

This proves that

$$I(f_+, g_+) = \int f(z)\, g(z)\, dz + O(h).$$

A similar argument applies to

$$I(f_-, g_-) = \alpha \operatorname{Im}(f_- V(h)\, \varphi,\ R(\lambda - i0;\ H(h))\, g_- V(h)\, \varphi)$$

and we obtain

$$I(f_-, g_-) = \int f(z)\, g(z)\, dz + O(h).$$

The above approximate solution does not have support in $(-\infty, 0)$ as a function of $y$ for $t > 0$ and hence $I(f_+, g_-) = O(h)$. We consider the remaining term

$$I(f_-, g_+) = \alpha \operatorname{Im}(R(\lambda + i0;\ H(h))\, f_- V(h)\, \varphi,\ g_+ V(h)\, \varphi).$$

The assumption (4.2) is now used to show that $I(f_-, g_+) = O(h)$. Let $\Theta_t$ be the mapping defined in Lemma 4.3. We note that $A(x) = \nabla \zeta(x)$ on the support of $f_-$ and $g_+$, which follows from Lemma 2.2. Taking account of this fact, we define the following two sets:

$$\Gamma_f = \{(x, \xi) : \xi = \sqrt{2\lambda}\, \omega + A(x), \quad x = (y, z) \in \operatorname{supp} f_- \cap \operatorname{supp} \chi_R\},$$
$$\Gamma_g = \{(x, \xi) : \xi = \sqrt{2\lambda}\, \omega + A(x), \quad x = (y, z) \in \operatorname{supp} g_+ \cap \operatorname{supp} \chi_R\},$$

By assumption (4.2), it follows from (4.4) that $\Theta_t \Gamma_f \cap \Gamma_g = \emptyset$ for $0 \leq t \leq T$. This, together with Lemma 4.3, implies that

$$(\exp(ih^{-1} tH(h))\, f_- V(h)\, \varphi,\ g_+ V(h)\, \varphi) = O(h^N), \qquad N \gg 1,$$

uniformly in $t$ as above. Hence we obtain $I(f_-, g_+) = O(h)$.

Summing up the result obtained in this step, we can conclude that

$$I(f, g) = 2 \int f(z)\, g(z)\, dz + O(h),$$

provided that $f$ is strictly supported in $\Sigma_\omega$.

(5) Finally we discuss the case in which $f$ is strictly supported inside the union of the gap intervals $\Lambda_{j\omega}$. Suppose that $f$ has support in $\Lambda_{j\omega}$ for some $j$, $1 \le j \le n-1$. We may again assume that $g$ is supported in $(c_1, d_n)$. We decompose $f$ and $g$ as in (5.6). Then we repeat the same argument as in step (4) to obtain that

$$I(f_+, g_+) = \int f(z)\, g(z)\, dz + O(h),$$

$$I(f_-, g_-) = \int f(z)\, g(z)\, dz + O(h)$$

and $I(f_+, g_-) = O(h)$. We consider the remaining term $I(f_-, g_+)$. Let $\Omega_j$ and $\psi_j(x)$ be as in Lemma 4.4. The function $\exp(i\psi_j(x)/h)$ takes the values $\exp(i\psi_j(x)/h) = 1$ on the support of $f_-$ and

$$\exp(i\psi_j(x)/h) = \exp(-i\,2\pi\nu_j/h) \quad \text{on } \Omega_j \cap \operatorname{supp} g_+, \tag{5.7}$$

because $\beta/h \in Z$. Hence we can write $f_-\, V(h)\,\varphi$ as

$$f_-\, V(h)\,\varphi = i\,(2\lambda)^{1/2}\, h\, \exp(i\,(\zeta(x) + \psi_j(x))/h)\,(\partial_y \chi_R)\, f_-\, \varphi + O(h^2).$$

By use of the gauge transformation, we construct an approximate solution to

$$w(t) = \exp(-ih^{-1}\, t\,(H(h) - \lambda))\, \exp(i\,(\zeta(x) + \psi_j(x))/h)\,(\partial_y \chi_R)\, f_-\, \varphi.$$

By Lemmas 2.2 and 4.4, $A(x) = \nabla\zeta(x) + \nabla\psi_j(x)$ on $\Omega_j$. Hence the approximate solution is obtained as

$$\exp(i\,(\zeta(x) + \psi_j(x))/h)\,(\partial_y \chi_R)\,(y - \sqrt{2\lambda}\,t,\, z)\,\eta_{-R}(y - \sqrt{2\lambda}\,t,\, z)\,\varphi.$$

If $x = (y, z) \in \operatorname{supp} \partial_y \chi_R$, $y \in \operatorname{supp} \eta_{+R}$ and $z \in \Lambda_{j\omega}$, then it follows that

$$(2\lambda)^{1/2} \int_0^T (\partial_y \chi_R)\,(y - \sqrt{2\lambda}\,t,\, z)\,\eta_{-R}(y - \sqrt{2\lambda}\,t)\, dt = 1$$

and also we have

$$2 \int (\partial_y \chi_R)\,(y, z)\,\eta_{+R}(y)\, dy = -2, \qquad z \in \Lambda_{j\omega}.$$

Thus we obtain from (5.7) that

$$I(f_-, g_+) = -2\,\cos(2\,\nu_j\,\pi/h) \int f(z)\, g(z)\, dz + O(h).$$

Summing up the results obtained in this step, we can conclude that

$$I(f,\, g) = 4\, \sin^2 (\nu_j\, \pi / h) \int f(z)\, g(z)\, dz + O(h),$$

if $f$ is strictly supported inside $\Lambda_{j\omega}$ for some $j$, $1 \le j \le n-1$.

(6) If we combine all the results obtained above, the theorem can be verified by a simple argument using a partition of unity over $\Pi_\omega$. In fact, the theorem is obtained by choosing a smooth approximation to the characteristic functions of $\Sigma_\omega$ and $\Lambda_{j\omega}$, $1 \le j \le n-1$, as functions $f$ and $g$. Thus the proof is now complete.

## 6. FINAL COMMENT

As previously stated, the argument used in the proof of Theorem 4.1 extends to the higher dimensional case. We conclude the paper by making a brief comment on the three dimensional case.

Let $b(x)$, $x = (x_1,\, x_2,\, x_3) \in R^3$, be a magnetic field given by the two-form

$$b(x) = b_{12}(x)\, dx_1 \wedge dx_2 + b_{23}(x)\, dx_2 \wedge dx_3 + b_{31}(x)\, dx_3 \wedge dx_1.$$

We associate $b(x)$ to the antisymmetric matrix $B(x) = (b_{jk}(x))_{1 \le j,\, k \le 3}$, where the components $b_{jk}$ satisfy the antisymmetry condition $b_{jk} + b_{kj} = 0$ $(b_{jj} = 0)$ and the cocycle condition $\partial_j\, b_{kl} + \partial_k\, b_{lj} + \partial_l\, b_{jk} = 0$. We assume that $b_{jk}(x)$ is a real smooth function with compact support. The motion of classical particle with unit mass in the magnetic field $b$ is governed by equation

$$x'' = B(x)\, x'. \tag{6.1}$$

If we set $G = \operatorname{supp} b = \bigcup_{1 \le j,\, k \le 3} \operatorname{supp} b_{jk}$, then the classical total cross section is given by $\sigma_{\mathrm{cl}}(\omega) = \operatorname{meas}(\Sigma_\omega)$ for incident direction $\omega \in S^2$, where $\Sigma_\omega = \operatorname{Proj}_\omega(G)$ again denote the projection of $G$ onto the impact plane $\Pi_\omega$.

We define the magnetic potential $A(x) = (a_1(x),\, a_2(x),\, a_3(x))$ corresponding to the magnetic field $b$ as follows [1]:

$$a_j(x) = (4\pi)^{-1} \sum_{k=1}^{3} \partial_k \int |x - y|^{-1}\, b_{jk}(y)\, dy, \qquad 1 \le j \le 3.$$

Then $A(x)$ satisfies the relation $dA = b$ and the quantum particle of unit mass in the magnetic field $b$ is described by the Hamiltonian $H = (-i\nabla - A(x))^2/2$. As is easily seen, $A(x)$ behaves like $A(x) = O(|x|^{-2})$ as $|x| \to \infty$ and hence the perturbation $H - H_0$ to the free Hamiltonian $H_0$ is of short-range class. We now define $a(x)$ as

$$a(x) = -\int_1^\infty \sum_{k=1}^3 x_k\, a_k(sx)\, ds, \qquad x \neq 0.$$

Then we obtain the relation

$$A(x) = \nabla a(x) + E(x), \qquad x \neq 0, \tag{6.2}$$

in the same way as in the proof of Lemma 2.2, where $E(x) = (e_1(x), e_2(x), e_3(x))$ has compact support and is given by

$$e_j(x) = \int_1^\infty \sum_{k=1}^3 s x_k\, b_{jk}(sx)\, ds, \qquad x \neq 0.$$

By making use of relation (6.2) and by repeating the same argument as in section 3, we can represent the quantum total cross section $\sigma_b(\lambda, \omega)$ as in Theorem 3.5. We note that it can be defined without assuming the flux condition as in the two dimensional case.

We shall discuss the problem on the semi-classical asymptotic behavior of the total cross section $\sigma_b(\lambda, \omega; h)$. We assume that the same assumptions as in Theorem 4.1 are fulfilled for the classical system (6.1). For notational brevity, we fix the incident direction $\omega$ as $\omega = (0, 0, 1)$ and consider the straight line $\eta(z) = s\omega + z$, $s \in R$, with the impact parameter $z = (x_1, x_2) \in \Pi_\omega$. If $\Gamma_\omega = \Pi_\omega \backslash \Sigma_\omega$ is connected, then it follows from the Stokes formula that

$$\int_{\eta(z)} \langle E(x), dx \rangle = 0$$

for $z (\neq 0) \in \Gamma_\omega$. Hence we can prove by use of the same argument as in the proof of Theorem 4.1 that

$$\sigma_b(\lambda, \omega; h) = 2\, \text{meas}(\Sigma_\omega) + o(1), \qquad h \to 0.$$

However, if $\Gamma_\omega$ is not connected, the shadow scattering is not in general expected. As a simple example, we consider the case in which $G = \text{supp}\, b$ is a torus and $\Sigma_\omega = \text{Proj}_\omega(G)$ is an annular domain

$$\Sigma_\omega = \{ z = (x_1, x_2) : r_1 \leq |z| \leq r_2 \}, \qquad r_2 > r_1 > 0.$$

Let $\Lambda_\omega = \{ z : |z| < r_1 \}$ be the interior disk of $\Sigma_\omega$ and let $S = G \cap Q$ be the section of the torus $G$ by the half-plane $Q = \{ x \in R^3 : x_1 > 0, x_2 = 0 \}$.

Then the Stokes formula yields that

$$\int_{\eta(z)} \langle E(x),\, dx \rangle = \int\int_S b_{31}(x_1,\, 0,\, x_3)\, dx_1\, dx_3 = \int_S b(x) = 2\pi\nu$$

for $z\,(\neq 0) \in \Lambda_\omega$. Here the value $2\pi\nu$ of the integral does not depend on the impact parameter $z \in \Lambda_\omega$ and on the section $S$. This follows from the Stokes formula at once. Thus we again repeat the same argument as in the proof of Theorem 4.1 to obtain that that

$$\sigma_b(\lambda, \omega; h) = 2\,\mathrm{meas}(\Sigma_\omega) + 4\sin^2(\nu\pi/h)\,\mathrm{meas}(\Lambda_\omega) + o(1), \qquad h \to 0.$$

We see that the shadow scattering is also violated in the higher dimensional case.

## REFERENCES

[1] A. Boutet de Monvel and R. Purice, Limiting absorption principle for Schrödinger Hamiltonians with magnetic fields, *Commun. Partial Differ. Eqs.*, Vol. **19**, 1994, pp. 89-117.

[2] V. Enss and B. Simon, Finite total cross sections in nonrelativistic quantum mechanics, *Commun. Math. Phys.*, Vol. **76**, 1980, pp. 177-209.

[3] C. Gérard and A. Martinez, Principe d'absorption limite pour des opérateurs de Schrödinger à longue portée, *C. R. Acad. Sci. Paris,* Vol. **306**, 1988, pp. 121-123.

[4] T. Ikebe and Y. Saitō, Limiting absorption method and absolute continuity for the Schrödinger operators, *J. Math. Kyoto Univ.*, Vol. **7**, 1972, pp. 513-542.

[5] H. Isozaki and H. Kitada, A remark on the microlocal resolvent estimates for two-body Schrödinger operators, *Publ. RIMS Kyoto Univ.*, Vol. **21**, 1985, pp. 889-910.

[6] A. Iwatsuka, Spectral representation for Schrödinger operators with magnetic vector potentials, *J. Math. Kyoto Univ.*, Vol. **22**, 1982, pp. 223-242.

[7] S. T. Kuroda, On the essential spectrum of Schrödinger operators with vector potentials, *Sci. Papers College Gen. Ed. Univ. Tokyo,* Vol. **23**, 1973, pp. 87-91.

[8] M. Loss and B. Thaller, Scattering of particles by long-range magnetic fields, *Ann. of Phys.*, Vol. **176**, 1987, pp. 159-180.

[9] P. A. Perry, *Scattering Theory by the Enss Method,* Mathematical Reports 1, Harwood Academic, 1983.

[10] D. Robert, *Autour de l'Approximation Semi-classique*, Birkhäuser, 1987.

[11] D. Robert and H. Tamura, Semi-classical estimates for resolvents and asymptotics for total scattering cross-sections, *Ann. Inst. Henri Poincaré,* Vol. **46**, 1987, pp. 415-442.

[12] D. Robert and H. Tamura, Asymptotic behavior of scattering amplitudes in semi-classical and low energy limits, *Ann. Inst. Fourier Grenoble,* Vol. **39**, 1989, pp. 155-192.

[13] S. N. M. Ruijsenaars, The Aharonov-Bohm effect and scattering theory, *Ann. of Phys.*, Vol. **146**, 1983, pp. 1-34.
[14] D. R. Yafaev, *Mathematical Scattering Theory, General Theory,* Translations of Mathematical Monographs, Vol. **105**, A.M.S., 1992.
[15] K. Yajima, The quasi-classical limit of scattering amplitude – finite range potentials –, Lect. Notes in Math., 1159, Springer, 1985.

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