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N. ORTNER
P. WAGNER

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On the evaluation of one-loop Feynman amplitudes in Euclidean quantum field theory

by

N. ORTNER and P. WAGNER

Institut für Mathematik und Geometrie, Universität Innsbruck,
Technikerstr. 13, A-6020 Innsbruck, Austria

ABSTRACT. – This paper is concerned with the explicit evaluation of Feynman integrals of the form $\int_{\mathbb{R}^n} \prod_{k=1}^{m} (q_1^2 + \cdots + q_n^2 + b_{1k}\, q_1 + \cdots + b_{nk} q_n + d_k)^{-1} \mathrm{d}q_1 \ldots \mathrm{d}q_n$ for space-time dimensions $n \leq 4$. In the physically relevant case of $n = 4$, these integrals are expressed by dilogarithms which contain as arguments inverse trigonometric functions of the parameters b_{ik}, d_k.

RÉSUMÉ. – Cet article traite du calcul explicite des intégrales de Feynman $\int_{\mathbb{R}^n} \prod_{k=1}^{m} (q_1^2 + \cdots + q_n^2 + b_{1k}\, q_1 + \cdots + b_{nk}\, q_n + d_k)^{-1}\, \mathrm{d}q_1 \ldots \mathrm{d}q_n$ pour les dimensions spatio-temporelles $n \leq 4$. Dans le cas $n = 4$, qui correspond à notre univers, ces intégrales sont représentées par des fonctions dilogarithmiques contenant comme arguments des fonctions trigonométriques inverses des paramètres b_{ik} et d_k.

1. INTRODUCTION AND NOTATIONS

In the wake of R. P. Feynman's famous articles [8], [9] (*see* also [10]), there has evolved a whole theory to express scattering cross-sections by so-called "Feynman amplitudes", which basically reduce to (often divergent) definite integrals of rational functions f (*cf.* [1], [17] and the literature cited therein). Typically, such a function f depends on some external momenta $p^{(1)}, \ldots, p^{(r)}$ and on some internal momenta $p^{(r+1)}, \ldots, p^{(r+s)}$, and it is integrated with respect to the last ones. (Of course, f can also depend on discrete spin variables; but here we shall not deal with this dependence.) Each line in a Feynman graph contributes to f a factor of the type $g/(p^2 - m^2)$, where p is a linear combination of some of the momenta $p^{(1)}, \ldots, p^{(r+s)}$, m is a mass term, and g is either a constant or a linear function of $p^{(1)}, \ldots, p^{(r+s)}$. The symbol p^2 above denotes the square of p in Minkowski's space-time, *i.e.*, $p^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2$, and hence the denominator of $g/(p^2 - m^2)$ vanishes on the "mass shell" $p^2 = m^2$. This is usually remedied by replacing $p^2 - m^2$ by $p^2 - m^2 + \mathrm{i}\,\varepsilon$ and by considering the limit, for $\varepsilon \searrow 0$, of $\int f \, \mathrm{d}p^{(r+1)} \ldots \mathrm{d}p^{(r+s)}$ viewed as a distribution in $p^{(1)}, \ldots, p^{(r)}$ (*cf.* [20], Ch. 4). Another method of regularization, fundamental for Euclidean quantum field theory, consists in the transition from the Minkowski momentum space $\mathbb{R}_p^4$ to the Euclidean space $\mathbb{R}_q^4$, by setting $p_0 = \mathrm{i}\, q_4$, $p_k = q_k$, $k = 1, 2, 3$. Eventually, the Feynman amplitudes are recovered by analytic continuation.

In this paper, we shall avoid all subtleties of renormalization theory and intend to explicitly evaluate integrals of the type

$$I_n(B, d) := \int_{\mathbb{R}^n} \prod_{k=1}^{m} (\,|\,q\,|^2 + 2\,\langle B_k^{\downarrow}, q\rangle + d_k)^{-1} \, \mathrm{d}q, \tag{1}$$

where we have: $n = 1, 2, 3$, or 4; q, $B_k^{\downarrow}$ are vectors in $\mathbb{R}^n$; $|\,q\,|^2 = q_1^2 + \cdots + q_n^2$; $d_k \in \mathbb{R}$ with $d_k > |\,B_k^{\downarrow}\,|^2$, $k = 1, \ldots, m$; $\mathrm{d}q := \mathrm{d}q_1 \ldots \mathrm{d}q_n$. (Physically, the integral in (1) corresponds to a single-loop integral in a Feynman graph involving scalar propagators only.) Furthermore, we shall suppose that $m > \frac{n}{2}$, which means that the integral in (1) is absolutely convergent (or equivalently, in the language of Feynman integrals, that no "ultraviolet divergence" occurs).

Though, in principle, our results could be obtained by combining the work from different physical references (*see* [21], [23] App. B, p. 208-216, [24], [25], [29], [34]), we aim at giving a new, coherent, and mathematically rigorous treatment, which also connects the value of the integral in (1) with the volume of simplexes on the unit sphere of dimension $n - 1$.

In Section 2, we start with a combinatorial reduction formula for integrals over simplexes, and we use it, in Section 3, Proposition 3, to reduce the integral in (1) to analogous ones over products with $n + 1$ factors only (comparable with the reduction formulae in [21], [24], [29]). Then we represent the latter integrals by sums of "Gaussian integrals" of the type $\int_{\mathbb{R}_+^n} e^{-\langle C x, x\rangle} \, dx$. Proposition 4 gives information on the signature of the occurring matrices C. In Section 4, we express such a Gaussian integral by the volume of a simplex on the unit sphere of dimension $n-1$ (Lemma 2), and we present explicit formulae in terms of logarithmic and inverse trigonometric functions in the cases of $n = 2$ and of $n = 3$ (Propositions 5 and 6). To finish up, Section 5 treats the case $n = 4$, where the result is expressed by a sum of 42 Clausen's functions (Proposition 8). (This can be compared with the evaluation of the 4-point function in perturbation theory by means of 192 dilogarithms carried out in [34], App. A, p. 72. We also refer to the procedure presented in [25] and to the literature cited there.) Note that formula (19) in Proposition 8 is not quite satisfactory since it is unsymmetric in the elements of the matrix C. The authors were not able to derive a symmetric result involving not more than 42 Clausen's functions (*cf.* also Rem. 2 to Prop. 8). We mention that representation formulae for the volume of a spherical simplex of dimension l by the polygarithm functions $\mathrm{Li}_k \left(k = 1, \ldots, \left[\frac{l}{2}\right]\right)$ are derived in [4], Ch. 5.

Let us establish some notations. We consider $\mathbb{R}^n$ as a Euclidean space with the inner product $\langle x, y\rangle := x_1 y_1 + \cdots + x_n y_n$ and write $|x| := \sqrt{\langle x, x\rangle}$. The unit sphere $\{x \in \mathbb{R}^n : |x| = 1\}$ in $\mathbb{R}^n$ is denoted by $\mathbb{S}_{n-1}$, the subscript indicating its dimension as a manifold. Similarly, we denote by Σ_{n-1} the $(n-1)$-dimensional standard simplex in $\mathbb{R}^n$, *i.e.*,

$$\Sigma_{n-1} = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0, \ldots, x_n \geq 0, \sum_{k=1}^{n} x_k = 1\right\}$$

and by $d\omega(x)$ the measure $dx_1 \ldots dx_{n-1}$ on Σ_{n-1}. The set of non-negative real numbers $\{x \in \mathbb{R} : x \geq 0\}$ is abbreviated by $\mathbb{R}_+$; we write Y for its characteristic function, *i.e.*, the Heaviside function, and furthermore, $\mathbb{R}_+^n := \underbrace{\mathbb{R}_+ \times \cdots \times \mathbb{R}_+}_{n}$. arccot stands for the principal value of this function, *i.e.*, that which has its range in the interval $(0, \pi)$. On some occasions, we shall make use of the theory of distributions, and we adopt the notations from [32]. $\langle \phi, T\rangle$ stands for the value of the distribution T on the test function ϕ. As in [15], we denote by $A_k^{\downarrow}$ and by $A_k^{\rightarrow}$ the k-th column and the k-th row of a matrix A, respectively. We abbreviate by $A_k^{\downarrow\prime}$ the k-th column vector of A where the first element is left out. $\Re w$ and $\Im w$ stand for the real and the imaginary part of a complex number w, respectively. Symbols with a hat mark on top of it (as e.g. in $dx_1 \ldots \widehat{dx_j} \ldots dx_n$)

must be understood as being omitted (*i.e.*, $\mathrm{d}x_1 \dots \mathrm{d}x_{j-1}\,\mathrm{d}x_{j+1} \dots \mathrm{d}x_n$). As differentiation symbols, we use $\partial_j = \partial_{x_j} = \dfrac{\partial}{\partial x_j}$ and Δ for the Laplacean.

2. REDUCTION OF GENERALIZED DIRICHLET AVERAGES

According to [5], Def. 5.2-1, p. 75 and Def. 4.4-1, p. 64, an integral of the type

$$F(r,\, a) := \int_{\Sigma_{m-1}} \lambda_1^{r_1-1} \cdots \lambda_m^{r_m-1} f(\langle a,\, \lambda\rangle)\, \mathrm{d}\omega(\lambda), \quad a \in \mathbb{R}^m, \ r \in \mathbb{N}^m,$$

is called a "Dirichlet average". If the numbers $a_1, \dots, a_m$ are pairwise different, then this $(m-1)$-fold integral can be represented by a finite sum over *simple* definite integrals (*cf.* [27], Lemma 1), namely

$$F(r,\, a) = \sum_{j=1}^{m} \partial_{a_j}^{r_j-1} \left[\gamma_j \int_{a_0}^{a_j} \frac{(a_j-\mu)^{|r|-2}}{(|r|-2)!} f(\mu)\, \mathrm{d}\mu\right],$$

where $\gamma_j := \prod_{k=1,\, k\neq j}^{m} (r_k-1)!/(a_j-a_k)^{r_k}$. Note that a_0 can be chosen arbitrarily, and that

$$g(t) := \int_{a_0}^{t} \frac{(t-\mu)^{|r|-2}}{(|r|-2)!} f(\mu)\, \mathrm{d}\mu$$

is an $(|r|-1)$-fold indefinite integral of f. Furthermore, the method of the proof of Lemma 1 in [27] also shows that the formula

$$F(r,\, a) = \sum_{j=1}^{m} \partial_{a_j}^{r_j-1} [\gamma_j\, g(a_j)]$$

remains valid for every $(|r|-1)$-fold indefinite integral g of f. In this way, the weighted average over the plane wave function $\lambda \mapsto f(\langle a,\, \lambda\rangle)$ is represented by a sum of evaluations, *i.e.*, 0-fold integrals, of indefinite integrals of f.

In the sequel, we aim at representing the "generalized" Dirichlet average

$$\int_{\Sigma_{m-1}} f(A\,\lambda)\, \mathrm{d}\omega(\lambda), \qquad A \text{ an } l \times m\text{-matrix},$$

by a sum of $(l-1)$-dimensional integrals which contain an indefinite integral of f with respect to a distinguished variable. In difference to the case of $l = 1$ explained above, we now restrict ourselves to putting $r_1 = \cdots = r_m = 1$. We mention in parentheses that a similar reduction to $(l-1)$-dimensional integrals is carried out in Proposition 4 of [27] already. (This proposition refers to the construction of the fundamental solution of an m-fold product of linear differential operators which generate an l-dimensional affine subspace).

PROPOSITION 1. – *Let $1 \leq l < m$, and A be a real-valued $l \times m$-matrix such that $A^{\downarrow}_{j_1} - A^{\downarrow}_k, \ldots, A^{\downarrow}_{j_l} - A^{\downarrow}_k$ are linearly independent in $\mathbb{R}^l$ for pairwise different indices $j_1, \ldots, j_l, k \in \{1, \ldots, m\}$. Denote by S the set*

$$A\,\Sigma_{m-1} = \left\{ \sum_{k=1}^{m} \lambda_k\, A^{\downarrow}_k : \lambda \in \Sigma_{m-1} \right\},$$

i.e., *the convex hull of $A^{\downarrow}_1, \ldots, A^{\downarrow}_m$ in $\mathbb{R}^l$, and suppose that $f \in \mathcal{C}(S)$, $g \in \mathcal{C}^{m-l}(S)$, such that $\partial_1^{m-l}\, g = f$. Then*

$$\int_{\Sigma_{m-1}} f(A\lambda)\, \mathrm{d}\omega(\lambda) = \sum_{\substack{j=(j_1,\ldots,j_l)\\ 1\leq j_1<\ldots<j_l\leq m}} \gamma_j \int_{\Sigma_{l-1}} g(A(j)\,\rho)\, \mathrm{d}\omega(\rho), \quad (2)$$

where $A(j) := (A^{\downarrow}_{j_1}, \ldots, A^{\downarrow}_{j_l})$, and

$$\gamma_j := \prod_{\substack{k=1\\ k\notin\{j_1,\ldots,j_l\}}}^{m} \frac{\det\begin{pmatrix} 1 & \cdots & 1 \\ A^{\downarrow\prime}_{j_1} & \cdots & A^{\downarrow\prime}_{j_l} \end{pmatrix}}{\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ A^{\downarrow}_k & A^{\downarrow}_{j_1} & \cdots & A^{\downarrow}_{j_l} \end{pmatrix}}.$$

Proof. – *a*) First we shall show that the set

$$M := \{(u + \langle v, \mu\rangle)^{-l} : u \in \mathbb{C}\backslash\mathbb{R},\ v \in \mathbb{R}^l\}$$

is dense in $\mathcal{E} = \mathcal{C}^{\infty}(\mathbb{R}^l_{\mu})$.

If $T \in \mathcal{E}'$ with $\langle \phi, T\rangle = 0$ for all $\phi \in M$, then we have $\phi * T = 0$ for all $\phi \in M$. Since $\phi \in \mathcal{S}'$, and $T \in \mathcal{O}'_C$, we can apply the Fourier exchange theorem (*cf.* [32], (VII,8;5), p. 268) in order to conclude that $\mathcal{F}\phi \cdot \mathcal{F}T = 0$. The Fourier transform of the special function $\phi(\mu) = (\mathrm{i} + \langle v, \mu\rangle)^{-l} \in M$ is a measure, the support of which is the half-ray in the direction v. Hence the analytic function $\mathcal{F}T$ vanishes on that half-ray, and, since v is arbitrary, $\mathcal{F}T$ must equal zero identically. Therefore, the complement $M^{\perp}$ of M in $\mathcal{E}'$ is the trivial vector space, and this implies that M is dense in $\mathcal{E}$.

b) By continuity, it is sufficient to prove the equality (2) for all functions of the type $g(\mu) = (u + \langle v, \mu\rangle)^{-l}$, $u \in \mathbb{C}\backslash\mathbb{R}$, $v \in \mathbb{R}^l$. In this case,

$$f(\mu) = \frac{(m-1)!}{(l-1)!}(-v_1)^{m-l}(u + \langle v, \mu\rangle)^{-m}.$$

We then use Feynman's first formula, *i.e.*,

$$\frac{1}{z_1 \cdots z_m} = (m-1)! \int_{\Sigma_{m-1}} \frac{d\omega(\lambda)}{(z_1\lambda_1 + \cdots + z_m\lambda_m)^m}, \tag{3}$$

valid for complex numbers $z_1, \ldots, z_m$, the convex hull in $\mathbb{C}$ of which does not contain 0 (*see* [33], p. 72, [23], Th. 7-1, p. 56). If we apply (3) on both sides of (2), then the statement of Proposition 1 is equivalent with the equation

$$(-v_1)^{m-l}\prod_{k=1}^{m}(u + \langle A_k^\downarrow, v\rangle)^{-1} = \sum_{1\le j_1<\cdots<j_l\le m}\gamma_j\prod_{r=1}^{l}(u+\langle A_{j_r}^\downarrow, v\rangle)^{-1}. \tag{4}$$

Again by a density argument, we can assume that every subset of l vectors out of $A_1^\downarrow, \ldots, A_m^\downarrow$ is linearly independent. But then (4) is just a special instance of the many-dimensional version of Lagrange's interpolation formula stated in the subsequent lemma. Indeed, we reach (4) upon setting $P = z_1^{m-l}$, $z_0 = u$, $z = -v$, and noting that

$$z(j)_1 = \det\begin{pmatrix} 1 & \cdots & 1 \\ A_{j_1}^{\downarrow\prime} & \cdots & A_{j_l}^{\downarrow\prime}\end{pmatrix} \Big/ \det(A_{j_1}^\downarrow, \ldots, A_{j_l}^\downarrow),$$

where $z(j) \in \mathbb{C}^l$ is defined below. ■

LEMMA 1. – *Let* $1 \le l < m$, $A_1^\downarrow, \ldots, A_m^\downarrow \in \mathbb{C}^l$ *such that* $A_{j_1}^\downarrow, \ldots, A_{j_l}^\downarrow$ *as well as* $A_{j_1}^\downarrow - A_k^\downarrow, \ldots, A_{j_l}^\downarrow - A_k^\downarrow$ *are linearly independent for pairwise different indices* $j_1, \ldots, j_l$, $k \in \{1, \ldots, m\}$. *For* $j = (j_1, \ldots, j_l)$, $1 \le j_1 < \cdots < j_l \le m$, *determine* $z(j) \in \mathbb{C}^l$ *by the system of linear equations* $\langle A_{j_r}^\downarrow, z(j)\rangle = 1$, $r = 1, \ldots, l$. *Then, for each complex homogeneous polynomial* $P(z_0, z)$ *of the degree* $m - l$, *we have*

$$P(z_0, z) = \sum_{\substack{j=(j_1,\ldots,j_l)\\ 1\le j_1<\cdots<j_l\le m}} P(1, z(j)) \prod_{\substack{k=1\\ k\notin\{j_1,\ldots,j_l\}}}^{m} \times \frac{(z_0 - \langle A_k^\downarrow, z\rangle)\det(A_{j_1}^\downarrow, \ldots, A_{j_l}^\downarrow)}{\det\begin{pmatrix} 1 & 1 & \cdots & 1 \\ A_k^\downarrow & A_{j_1}^\downarrow & \cdots & A_{j_l}^{\downarrow\prime}\end{pmatrix}}. \tag{5}$$

Proof. – For $j = (j_1, \ldots, j_l)$, $1 \leq j_1 < \cdots < j_l \leq m$, define the polynomial P_j by

$$P_j(z_0, z) := \prod_{\substack{k=1 \\ k \notin \{j_1, \ldots, j_l\}}}^{m} (z_0 - \langle A_k^{\downarrow}, z \rangle).$$

By the definition of $z(j)$, we have $P_j(1, z(j')) = 0$ for $j \neq j'$. The vector $(z_0, z) := (1 - \langle A_k^{\downarrow}, z(j) \rangle, z(j))$ is the solution of the system of linear equations $z_0 + \langle A_k^{\downarrow}, z \rangle = 1$, $\langle A_{j_1}^{\downarrow}, z \rangle = 1, \ldots, \langle A_{j_l}^{\downarrow}, z \rangle = 1$, and hence we deduce from Cramer's rule that

$$1 - \langle A_k^{\downarrow}, z(j) \rangle = \frac{\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ A_k^{\downarrow} & A_{j_1}^{\downarrow} & \cdots & A_{j_l}^{\downarrow} \end{pmatrix}}{\det (A_{j_1}^{\downarrow}, \ldots, A_{j_l}^{\downarrow})}.$$

This yields

$$P_j(1, z(j)) = \prod_{\substack{k=1 \\ k \notin \{j_1, \ldots, j_l\}}}^{m} (1 - \langle A_k^{\downarrow}, z(j) \rangle)$$

$$= \prod_{\substack{k=1 \\ k \notin \{j_1, \ldots, j_l\}}}^{m} \frac{\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ A_k^{\downarrow} & A_{j_1}^{\downarrow} & \cdots & A_{j_l}^{\downarrow} \end{pmatrix}}{\det (A_{j_1}^{\downarrow}, \ldots, A_{j_l}^{\downarrow})} \neq 0.$$

Therefore, the set

$$M := \{P_j : j = (j_1, \ldots, j_l), 1 \leq j_1 < \cdots < j_l \leq m\}$$

is linearly independent in the complex vector space H which consists of all homogeneous polynomials in $(z_0, z) \in \mathbb{C}^{l+1}$ of the degree $m - l$. Since H has the dimension $\binom{m}{l}$, this implies that M constitutes a basis of H. Now (5) is nothing else than the co-ordinate representation of a polynomial $P \in H$ with respect to M. ■

3. REPRESENTATION OF n-DIMENSIONAL FEYNMAN AMPLITUDES BY GAUSSIAN INTEGRALS OVER $\mathbb{R}_+^n$

As explained in the introduction, we study, in the sequel, integrals of the type

$$I_n(B, d) := \int_{\mathbb{R}^n} \prod_{k=1}^{m} (|q|^2 + 2\langle B_k^{\downarrow}, q \rangle + d_k)^{-1} \, dq, \tag{1}$$

where $m, n \in \mathbb{N}$, $m > \frac{n}{2}$, $B = (B_1^{\downarrow}, \ldots, B_m^{\downarrow})$ is a real-valued $n \times m$-matrix, $d = (d_1, \ldots, d_m) \in \mathbb{R}^m$, and $d_k > |B_k^{\downarrow}|^2$, $k = 1, \ldots, m$. To illustrate our method, let us treat first the case $B = 0$.

PROPOSITION 2. – *Let $m > \frac{n}{2}$, and $d_1, \ldots, d_m$ be pairwise different, positive numbers. Then we have:*

$$I_n(0, d) = \begin{cases} (-1)^{(n-1)/2}\, \pi \sum_{j=1}^{m} \beta_j\, d_j^{n/2-1} : n = 1, 3, 5, \ldots \\ (-1)^{n/2} \sum_{j=1}^{m} \beta_j\, d_j^{n/2-1} \ln d_j : n = 2, 4, 6, \ldots, \end{cases}$$

where $\beta_j := \frac{\pi^{n/2}}{\Gamma(n/2)} \prod_{k=1, k\neq j}^{m} (d_k - d_j)^{-1}$.

Proof. – Using Feynman's formula (3) and Fubini's Theorem we obtain

$$I_n(0, d) = \int_{\mathbb{R}^n} \left[(m-1)! \int_{\Sigma_{m-1}} \frac{d\omega(\lambda)}{(|q|^2 + \langle d, \lambda\rangle)^m} \right] dq$$
$$= (m-1)! \int_{\Sigma_{m-1}} d\omega(\lambda) \int_{\mathbb{R}^n} \frac{dq}{(|q|^2 + \langle d, \lambda\rangle)^m}.$$

By rotational invariance, the inner integral is easily computed by substituting polar co-ordinates and making use of formula 3.241,4 in [11]. This yields:

$$I_n(0, d) = \pi^{n/2}\, \Gamma(m - n/2) \int_{\Sigma_{m-1}} \langle d, \lambda\rangle^{n/2-m}\, d\omega(\lambda).$$

This Dirichlet integral can be represented by an $(m-1)$-fold indefinite integral g of the function $f(\mu) = \mu^{n/2-m}$ (*cf.* Proposition 1 or the discussion preceding it), and as such we use

$$g(\mu) = \frac{\mu^{n/2-1}(-1)^{m-1}}{\Gamma\left(\frac{n}{2}\right)\Gamma\left(m - \frac{n}{2}\right)} \times \begin{cases} (-1)^{(n-1)/2}\, \pi : n \text{ odd} \\ (-1)^{n/2} \ln \mu : n \text{ even.} \end{cases}$$

Thus Proposition 2 follows from formula (2). ■

Remarks. – 1) As pointed out by the referee, $I_n(0, d)$ could as well and more easily be obtained by decomposition into rational fractions, *i.e.*,

$$\prod_{k=1}^{m} (|q|^2 + d_k)^{-1} = \sum_{j=1}^{m} (|q|^2 + d_j)^{-1} \prod_{\substack{k=1 \\ k\neq j}}^{m} (d_k - d_j)^{-1},$$

and analytic continuation with respect to the dimension n:

$$I_n(0, d) = \lim_{R\to\infty} \sum_{j=1}^{m} \int_{|q|<R} (|q|^2 + d_j)^{-1}\,dq \prod_{\substack{k=1\\k\neq j}}^{m} (d_k - d_j)^{-1}$$
$$= \left[\sum_{j=1}^{m} \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty \frac{r^{z-1}}{r^2 + d_j}\,dr \prod_{\substack{k=1\\k\neq j}}^{m} (d_k - d_j)^{-1}\right]\Bigg|_{z=n}$$

Note that the integrals in the last line converge for $0 < \Re z < 2$ and that their meromorphic continuation with respect to z, *i.e.*, $\dfrac{\pi\, d_j^{z/2-1}}{2\sin(\pi z/2)}$, has simple poles at even dimensions n. For such values of n, one has to use instead the finite part

$$\operatorname*{Pf}_{z=n}\left(\frac{\pi\, d_j^{z/2-1}}{2\sin(\pi z/2)}\right) = (-1)^{n/2}\, d_j^{n/2-1} \ln d_j.$$

2) The value of $I_n(0, d)$ in Proposition 2 can also be interpreted as the value in 0 [multiplied by $(2\pi)^n$] of the uniquely determined temperate fundamental solution E of the product $\prod_{k=1}^{m} (d_k - \Delta)$ of Helmholtz operators, since

$$E(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \left(\prod_{k=1}^{m} (|q|^2 + d_k)^{-1}\right) e^{i\langle x, q\rangle}\,dq.$$

By application of Proposition 1 in [26], p. 82, we derive an explicit representation for E in terms of Bessel functions of the third kind:

$$E(x) = \frac{\Gamma(n/2)}{2^{n/2}\,\pi^n}\,|x|^{1-n/2} \sum_{j=1}^{m} \beta_j\, d_j^{n/4-1/2}\, K_{n/2-1}(\sqrt{d_j}\,|x|), \qquad x \neq 0.$$

To deduce the value $E(0)$ from this representation of E, the series expansion of $K_{n/2-1}$ (*cf.* [11], 8.446; 8.485; 8.445) and the identities

$$\sum_{j=1}^{m} \beta_j\, d_j^i = 0, \qquad 0 \leq i \leq m-2,$$

have to be taken into account.

In the sequel, let us suppose $m \geq n+1$ and concentrate on the case referring to a set of parameters d, B "in general position". By this, we shall understand that each set of $l \leq n+2$ columns of the matrix

$$\begin{pmatrix} 1 & \cdots & 1 \\ d_1 & \cdots & d_m \\ B_1^{\downarrow} & \cdots & B_m \end{pmatrix}$$

is linearly independent. Let us point out that the integrals (1) with parameters B, d which are not in "general position" are limiting values of the case described above, the limit computation being non-trivial, however.

PROPOSITION 3. – *Let* $1 \leq n < m$, *and* $A := \begin{pmatrix} d_1 & \cdots & d_m \\ B_1^{\downarrow} & \cdots & B_m^{\downarrow} \end{pmatrix}$ *be a real-valued* $(n+1) \times m$*-matrix such that* $d_k > |B_k^{\downarrow}|^2$, $k = 1, \ldots, m$, *and* $A_{j_1}^{\downarrow} - A_k^{\downarrow}, \ldots, A_{j_l}^{\downarrow} - A_k^{\downarrow}$ *are linearly independent for pairwise different indices* $j_1, \ldots, j_l$, $k \in \{1, \ldots, m\}$, $l \leq n+1$. *Similarly to Proposition* 1, *define for* $j = (j_1, \ldots, j_n)$, $1 \leq j_1 < \cdots < j_n \leq m$, *and* $j_{n+1} \in \{1, \ldots, m\} \backslash \{j_1, \ldots, j_n\}$:

$$\gamma_{j,\, j_{n+1}} := \prod_{\substack{k=1 \\ k \notin \{j_1, \ldots, j_{n+1}\}}}^{m} \frac{\det \begin{pmatrix} 1 & \cdots & 1 \\ B_{j_1}^{\downarrow} & \cdots & B_{j_{n+1}}^{\downarrow} \end{pmatrix}}{\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ A_k^{\downarrow} & A_{j_1}^{\downarrow} & \cdots & B_{j_{n+1}}^{\downarrow} \end{pmatrix}}.$$

(Especially, for $m = n+1$, *this means* $\gamma_{j,\, j_{n+1}} = 1$.*) Furthermore, we define the* $m \times m$*-matrix* $C = (c_{kl})$ *by*

$$c_{kl} := \frac{1}{2}(d_k + d_l) - \langle B_k^{\downarrow}, B_l^{\downarrow} \rangle, \qquad k, l = 1, \ldots, m,$$

and, for j *and* j_{n+1} *as above, the submatrices*

$$C(j) := (c_{j_r, j_s})_{r, s=1}^{n}, \qquad C(j, j_{n+1}) := (c_{j_r, j_s})_{r, s=1}^{n+1}.$$

Finally, we set $D(j, j_{n+1}) := (c_{j_r, j_s} - c_{j_{n+1}, j_s})_{r, s=1}^{n}$ *and*

$$\beta_j := \sum_{\substack{j_{n+1}=1 \\ j_{n+1} \notin \{j_1, \ldots, j_n\}}}^{m} \gamma_{j,\, j_{n+1}} \frac{\det D(j, j_{n+1})}{\det C(j, j_{n+1})}.$$

(It will be shown in Proposition 4 below that the matrices $C(j, j_{n+1})$ are non-singular). Then the following representation formula is valid:

$$\begin{aligned} I_n(B, d) :&= \int_{\mathbb{R}^n} \prod_{k=1}^{m} (|q|^2 + 2\langle B_k^{\downarrow}, q\rangle + d_k)^{-1}\, \mathrm{d}q \\ &= (-1)^{m-n-1}\, \pi^{n/2} \sum_{\substack{j=(j_1, \ldots, j_n) \\ 1 \le j_1 < \cdots < j_n \le m}} \beta_j \\ &\quad \times \int_{\mathbb{R}_+^n} \exp(-\langle C(j)\, x, x\rangle)\, \mathrm{d}x. \end{aligned} \tag{6}$$

Proof. – *a*) The application of Feynman's formula (3), the interchange of integration according to Fubini's Theorem, and the subsequent translation $u = q + B\lambda$ yield

$$\begin{aligned} I_n(B, d) &= (m-1)! \int_{\Sigma_{m-1}} \mathrm{d}\omega(\lambda) \\ &\quad \times \int_{\mathbb{R}^n} (|q|^2 + 2\langle B\lambda, q\rangle + \langle d, \lambda\rangle)^{-m}\, \mathrm{d}q \\ &= (m-1)! \int_{\Sigma_{m-1}} \mathrm{d}\omega(\lambda) \int_{\mathbb{R}^n} (|u|^2 + \langle d, \lambda\rangle - |B\lambda|^2)^{-m}\, \mathrm{d}u. \end{aligned}$$

As in the proof of Proposition 2, the inner integral, times $(m-1)!$, gives

$$\pi^{n/2}\,\Gamma\left(m - \frac{n}{2}\right) (\langle d, \lambda\rangle - |B\lambda|^2)^{n/2-m}.$$

b) The integral over Σ_{m-1} is reduced to one over Σ_n by means of Proposition 1. For this purpose, we put

$$f(\mu_1, \ldots, \mu_{n+1}) := (\mu_1 - \mu_2^2 - \cdots - \mu_{n+1}^2)^{n/2-m}.$$

(Note that $f(A\lambda) = (\langle d, \lambda\rangle - |B\lambda|^2)^{n/2-m}$ is well-defined, since the inequalities $d_k > |B_k^{\downarrow}|^2$, $k = 1, \ldots, m$, imply

$$\langle d, \lambda\rangle > \sum_{k=1}^{m} \lambda_k |B_k^{\downarrow}|^2 \ge |B\lambda|^2, \qquad \lambda \in \Sigma_{m-1},$$

by the convexity of the mapping $x \mapsto |x|^2$.) Choosing the function

$$g(\mu_1, \ldots, \mu_{n+1}) := \frac{(-1)^{m-n-1}\,\Gamma\left(1 + \frac{n}{2}\right)}{\Gamma\left(m - \frac{n}{2}\right)} (\mu_1 - \mu_2^2 - \cdots - \mu_{n+1}^2)^{-n/2-1}$$

as an $(m-n-1)$-fold indefinite integral of f with respect to μ_1, we obtain from Proposition 1

$$I_n(B, d) = (-1)^{m-n-1}\,\pi^{n/2}\,\Gamma\left(1+\frac{n}{2}\right) \sum_{\substack{i=(i_1,\dots,i_{n+1})\\ 1\le i_1<\cdots<i_{n+1}\le m}} \gamma_i \times \int_{\Sigma_n} \frac{d\omega(\rho)}{\left(\sum_{r=1}^{n+1} \rho_r\, d_{i_r} - \left|\sum_{r=1}^{n+1} \rho_r\, B^{\downarrow}_{i_r}\right|^2\right)^{n/2+1}}.$$

(*Remark.* – Reexpressing the integrand in the simplex integrals on the right-hand side by integrals over $\mathbb{R}^n_q$ and applying Feynman's formula (3) results in the following reduction formula:

$$\int_{\mathbb{R}^n} \prod_{k=1}^{m} (|q|^2 + 2\langle B^{\downarrow}_k, q\rangle + d_k)^{-1}\, dq = (-1)^{m-n-1} \times \sum_{\substack{i=(i_1,\dots,i_{n+1})\\ 1\le i_1<\cdots<i_{n+1}\le m}} \gamma_i \int_{\mathbb{R}^n} \prod_{r=1}^{n+1} (|q|^2 + 2\langle B^{\downarrow}_{i_r}, q\rangle + d_{i_r})^{-1}\, dq.$$

This type of reduction is considered in [21], pp. 188, 189 and [29], pp. 1958, 1959. In these papers, the number of factors in the integral on the right-hand side is further reduced from $n+1$ to n with the help of Stokes's theorem.)

c) We insert the integral representation

$$\frac{\Gamma\left(1+\frac{n}{2}\right)}{u^{1+n/2}} = 2\int_0^\infty t^{n+1}\, e^{-ut^2}\, dt, \qquad u>0,$$

into the last integral and obtain

$$I_n(B, d) = (-1)^{m-n-1}\, 2\,\pi^{n/2} \sum_{\substack{i=(i_1,\dots,i_{n+1})\\ 1\le i_1<\cdots<i_{n+1}\le m}} \gamma_i \int_{\Sigma_n} d\omega(\rho) \times \int_0^\infty t^{n+1} \exp\left(-t^2 \left[\sum_{r=1}^{n+1} \rho_r\, d_{i_r} - \left|\sum_{r=1}^{n+1} \rho_r\, B^{\downarrow}_{i_r}\right|^2\right]\right) dt.$$

The integration over $\Sigma_n \times \mathbb{R}_+$ can be replaced with one over $\mathbb{R}_+^{n+1}$ by the substitution $x = t\,\rho$, $\mathrm{d}x = t^n\,\mathrm{d}\omega(\rho)\,\mathrm{d}t$. This yields

$$I_n(B, d) = (-1)^{m-n-1}\, 2\,\pi^{n/2} \sum_{\substack{i=(i_1,\ldots,i_{n+1})\\ 1\le i_1<\cdots<i_{n+1}\le m}} \gamma_i \times \int_{\mathbb{R}_+^{n+1}} (x_1 + \cdots + x_{n+1})\,\mathrm{e}^{-\langle C(i)\,x,\,x\rangle}\,\mathrm{d}x,$$

since

$$\langle C(i)\,x,\,x\rangle = t^2 \left[\sum_{r=1}^{n+1} \rho_r\, d_{i_r} - \left|\sum_{r=1}^{n+1} \rho_r\, B_{i_r}^{\downarrow}\right|^2\right].$$

d) Now we apply Gauβ's divergence theorem to the last integral. For a symmetric non-singular $(n+1)\times(n+1)$-matrix S, we have

$$(x_1+\cdots+x_{n+1})\,\exp(-\langle S\,x,\,x\rangle) = -\frac{1}{2}\operatorname{div}\left[S^{-1}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\exp(-\langle S\,x,\,x\rangle)\right]$$

and hence, taking into account that $C(i)$ is non-singular (*see* Proposition 4 below), we obtain:

$$I_n(B, d) = (-1)^{m-n-1}\,\pi^{n/2} \sum_{\substack{i=(i_1,\ldots,i_{n+1})\\ 1\le i_1<\cdots<i_{n+1}\le m}} \gamma_i \sum_{r=1}^{n+1}\left\langle (C(i)^{-1})_r^{\downarrow},\, \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\right\rangle \times \int_{\mathbb{R}_+^n} \exp(-\langle C(i_1,\ldots,\widehat{i_r},\ldots,i_{n+1})\,x,\,x\rangle)\,\mathrm{d}x.$$

Finally, we rearrange the summation in setting

$$j = (j_1,\ldots,j_n) := (i_1,\ldots,\widehat{i_r},\ldots,i_{n+1}), \qquad \text{and} \qquad j_{n+1} := i_r.$$

Then $\gamma_i = \gamma_{j,\,j_{n+1}}$ and

$$\left\langle (C(i)^{-1})_r^{\downarrow},\, \begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}\right\rangle = \frac{\det D(j,\, j_{n+1})}{\det C(j,\, j_{n+1})}.$$

Thus the proof is complete. ■

Remark. – The case of $m = n$ can be treated similarly by omitting the reduction step b) and the application of Gauβ's divergence theorem in d) of the proof. This gives

$$\begin{aligned} I_n(B, d) &= \pi^{n/2}\,\Gamma\left(\frac{n}{2}\right) \int_{\Sigma_{n-1}} (\langle d, \lambda\rangle - |B\lambda|^2)^{-n/2}\, d\omega(\lambda) \\ &= 2\pi^{n/2} \int_{\mathbb{R}_+^n} e^{-\langle C x, x\rangle}\, dx \qquad \text{for} \quad m = n. \end{aligned}$$

On comparison with formula (6), we conclude that the integral (1) over a product of $m > n$ factors can be expressed as a linear combination of analogous integrals with exactly n factors. In $\mathbb{R}^4$, these latter ones physically correspond to Feynman integrals attached to "box diagrams", *i.e.*, to "4-point functions".

Similarly, the case of $m = n - 1$ yields the following:

$$\begin{aligned} I_n(B, d) &= \pi^{n/2}\,\Gamma\left(\frac{n}{2} - 1\right) \int_{\Sigma_{n-2}} (\langle d, \lambda\rangle - |B\lambda|^2)^{1-n/2}\, d\omega(\lambda) \\ &= 2\pi^{n/2} \int_{\mathbb{R}_+^{n-1}} \frac{e^{-\langle C x, x\rangle}}{x_1 + \cdots + x_{n-1}}\, dx \\ &= 4\pi^{n/2} \int_{\mathbb{R}_+^n} e^{-\langle \tilde{C} x, x\rangle}\, dx \qquad \text{for} \quad m = n - 1, \end{aligned}$$

where we define

$$\tilde{c}_{kl} := \begin{cases} c_{kl} : & 1 \le k, l \le n - 1 \\ 1 : & 1 \le k \le n-1, \quad l = n \text{ or } k = n, \quad 1 \le l \le n-1 \\ 0 : & k = l = n. \end{cases}$$

Example. – Let us pause for a moment to settle the case $n = 1$, which case is both trivial and, in some sense, exceptional. From Proposition 3, we obtain, for $m \ge 2$, with $b := B = (b_1, \ldots, b_m)$, and replacing (j_1, j_2) by (i, j):

$$\begin{aligned} I_1(b, d) :&= \int_{-\infty}^{-\infty} \prod_{k=1}^{m} (q^2 + 2 b_k q + d_k)^{-1}\, dq \\ &= \pi \sum_{\substack{1 \le i, j \le m \\ i \ne j}} \left(\prod_{\substack{k=1 \\ k \notin \{i, j\}}}^{m} \frac{b_i - b_j}{\det \begin{pmatrix} d_i - d_k & d_j - d_k \\ b_i - b_k & b_j - b_k \end{pmatrix}} \right) \\ &\quad \times \frac{d_i - d_j - 2 b_i (b_i - b_j)}{\sqrt{d_i - b_i^2}\,[4(b_i - b_j)(d_i b_j - d_j b_i) - (d_i - d_j)^2]}. \end{aligned}$$

We owe to the referee the observation that $I_1(b, d)$ can be expressed in a simpler way by replacing the parameters d_k by the "internal masses" $\mu_k := \sqrt{d_k - b_k^2}$. This yields the following:

$$\int_{-\infty}^{\infty} \prod_{k=1}^{m} ((q + b_k)^2 + \mu_k^2)^{-1} \, dq$$

$$= \pi \sum_{1 \le i < j \le m} \left(\prod_{\substack{k=1 \\ k \notin \{i, j\}}}^{m} \frac{b_i - b_j}{\left\{ \begin{array}{r} (b_i - b_j)(\mu_k^2 + b_k^2) + (b_j - b_k)(\mu_i^2 + b_i^2) \\ + (b_k - b_i)(\mu_j^2 + b_j^2) \end{array} \right\}} \right)$$

$$\times \left(\frac{1}{\mu_i} + \frac{1}{\mu_j} \right) \frac{1}{(b_i - b_j)^2 + (\mu_i + \mu_j)^2}.$$

The right-hand side of the last equation can be interpreted as a sum over amplitudes corresponding to "2-tree subgraphs" of the original single loop.

We mention that $I_1(b, d)$ can also be evaluated by means of the residue theorem. This yields, though, a representation involving the complex roots of the polynomials $q^2 + 2 b_k q + d_k$, $k = 1, \ldots, m$.

The next proposition provides information on the signature of the symmetric matrix C and of its submatrices, respectively.

PROPOSITION 4. – *Let* $A := \begin{pmatrix} d_1 & \cdots & d_m \\ B_1^{\downarrow} & \cdots & B_m^{\downarrow} \end{pmatrix}$ *be a real-valued* $(n+1) \times$ m*-matrix such that* $A_{j_1}^{\downarrow} - A_k^{\downarrow}, \ldots, A_{j_l}^{\downarrow} - A_k^{\downarrow}$ *are linearly independent for pairwise different indices* $j_1, \ldots, j_l, k \in \{1, \ldots, m\}$, $l \le n+1$, *and define the* $m \times m$*-matrix* $C = (c_{kl})$ *by*

$$c_{kl} := \frac{1}{2}(d_k + d_l) - \langle B_k^{\downarrow}, B_l^{\downarrow} \rangle, \qquad k, l = 1, \ldots, m.$$

Then we have:

(1) *If* $m > n + 2$, *then* $\det C = 0$;

(2) *if* $m = n + 2$, *then* $(-1)^{m-1} \det C > 0$;

(3) *if* $(m \le n+1)$ *and* $(\exists k \in \{1, \ldots, m\} : d_k > |B_k^{\downarrow}|^2)$, *then* $(-1)^{m-1} \det C > 0$, *and, more precisely,* C *has one positive and* $m - 1$ *negative eigenvalues.*

Proof. – *a*) Since the dimension of the subspace in $\mathbb{R}^n$ spanned by $B_1^{\downarrow}, \ldots, B_m^{\downarrow}$ cannot exceed m, and since C depends on the inner products of $B_k^\downarrow$, $k = 1, \ldots, m$, only, we can suppose without restriction of generality that $m \geq n$.

b) Using the expansion theorem of Laplace (*cf.* [12], Ch. IV, § 1, (4.5), p. 104 and (4.76), p. 139) and the fact that each 3×3 subdeterminant of the matrix $(d_k + d_l)_{k,\,l=1}^m$ vanishes, we see that $\det C$ is a quadratic polynomial in $d_1, \ldots, d_m$, which is given by

$$\begin{aligned}\det C &= \det(-\langle B_k^\downarrow, B_l^\downarrow\rangle)_{k,\,l=1}^m + \frac{1}{2}\sum_{r=1}^m \sum_{s=1}^m (-1)^{r+s}(d_r + d_s) \\ &\quad \times \det(-\langle B_k^\downarrow, B_l^\downarrow\rangle)_{k,\,l=1;\,k\neq r,\,l\neq s}^m + \frac{1}{4}\sum_{1\leq r_1<r_2\leq m}\ \sum_{1\leq s_1<s_2\leq m} \\ &\quad \times (-1)^{r_1+r_2+s_1+s_2} \underbrace{\det\begin{pmatrix} d_{r_1}+d_{s_1} & d_{r_1}+d_{s_2} \\ d_{r_2}+d_{s_1} & d_{r_2}+d_{s_2}\end{pmatrix}}_{-(d_{r_1}-d_{r_2})(d_{s_1}-d_{s_2})} \\ &\quad \times \det(-\langle B_k^\downarrow, B_l^\downarrow\rangle)_{k,\,l=1;\,k\notin\{r_1,r_2\},\,l\notin\{s_1,s_2\}}^m . \end{aligned} \tag{7}$$

If $m = n$, then $\det(-\langle B_k^\downarrow, B_l^\downarrow\rangle)_{k,\,l=1}^m = (-1)^m \det^2(B)$, whereas, for $m > n$, we can identify $B_k^\downarrow$ with a vector $\tilde{B}_k^\downarrow \in \mathbb{R}^m$ by adjoining $m - n$ zero components, and this yields $\det(-\langle B_k^\downarrow, B_l^\downarrow\rangle)_{k,\,l=1}^m = (-1)^m \det^2(\tilde{B}) = 0$. If $m > n + 2$, then the remaining terms of (7) are treated similarly, and we obtain the first assertion of Proposition 4.

c) If $m = n + 2$, then the first two terms on the right hand side of (7) vanish and the third one furnishes, again by Laplace's expansion theorem:

$$\begin{aligned}\det C &= \frac{(-1)^{m-1}}{4}\Big[\sum_{1\leq r_1<r_2\leq m} (-1)^{r_1+r_2}(d_{r_1} - d_{r_2}) \\ &\qquad \times \det(B_1^\downarrow, \ldots, \widehat{B_{r_1}^\downarrow}, \ldots, \widehat{B_{r_2}^\downarrow}, \ldots, B_m^\downarrow)\Big]^2 \\ &= \frac{(-1)^{m-1}}{4}\det^2\begin{pmatrix} 1 & \cdots & 1 \\ d_1 & \cdots & d_m \\ B_1^\downarrow & \cdots & B_m^\downarrow\end{pmatrix}.\end{aligned}$$

From the last formula, the second assertion of Proposition 4 is obvious.

d) Next consider the case of $m = n + 1$. Then the first term on the right-hand side of (7) vanishes, and the remaining terms give

$$\det C = (-1)^{m-1} \det \begin{pmatrix} d \\ B_1^{\rightarrow} \\ \vdots \\ B_n^{\rightarrow} \end{pmatrix} \det \begin{pmatrix} e \\ B_1^{\rightarrow} \\ \vdots \\ B_n^{\rightarrow} \end{pmatrix} + \frac{(-1)^{m-1}}{4} \sum_{k=1}^{n} \det^2 \begin{pmatrix} e \\ d \\ B_l^{\rightarrow} \end{pmatrix}_{l=1,\dots,n;\, l\neq k}, \tag{8}$$

where $e := (1, \dots, 1) \in \mathbb{R}^m$. If $\det \begin{pmatrix} e \\ B_1^{\rightarrow} \\ \vdots \\ B_n^{\rightarrow} \end{pmatrix} = 0$, then one of the squares on the right-hand side of (8) does not vanish (since $\begin{pmatrix} e \\ d \\ B \end{pmatrix}$ has rank $n+1$ by hypothesis), and we obtain that $(-1)^{m-1} \det C > 0$. If $\det \begin{pmatrix} e \\ B_1^{\rightarrow} \\ \vdots \\ B_n^{\rightarrow} \end{pmatrix} \neq 0$, then we set $d = \alpha\, e + \gamma_1 B_1^{\rightarrow} + \cdots + \gamma_n B_n^{\rightarrow}$, and we have to show that $4\,\alpha + \sum_{k=1}^{n} \gamma_k^2 > 0$. Define $v \in \mathbb{R}^n$ by $v := \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{pmatrix}$. Then the assumption $d_k > |B_k^{\downarrow}|^2$ for $k \in \{1, \dots, m\}$ implies that

$$\begin{aligned} 4\,\alpha + \sum_{l=1}^{n} \gamma_l^2 &= 4\,\alpha + |v|^2 = 4\,\alpha + |v - 2\,B_k^{\downarrow}|^2 + 4\,\langle v,\, B_k^{\downarrow}\rangle - 4\,|B_k^{\downarrow}|^2 \\ &= |v - 2\,B_k^{\downarrow}|^2 + 4\,(d_k - |B_k^{\downarrow}|^2) > 0. \end{aligned}$$

It follows that $(-1)^{m-1} \det C > 0$ if $m = n + 1$ and $d_k > |B_k^{\downarrow}|^2$ for some $k \in \{1, \dots, m\}$.

e) In the next step we shall show that the same holds true for $m = n$. In this case, formula (7) yields:

$$\det C = (-1)^m \det^2 B + (-1)^{m-1} \sum_{k=1}^{n} \det \begin{pmatrix} d \\ B_l^{\rightarrow} \end{pmatrix}_{l=1,\,\dots,\,n;\; l\neq k}$$
$$\times \det \begin{pmatrix} e \\ B_l^{\rightarrow} \end{pmatrix}_{l=1,\,\dots,\,n;\; l\neq k} + \frac{(-1)^{m-1}}{4} \sum_{1\le k<l\le m}$$
$$\times \det^2 \begin{pmatrix} e \\ d \\ B_r^{\rightarrow} \end{pmatrix}_{r=1,\,\dots,\,n;\; r\notin\{k,\,l\}} . \tag{9}$$

If $\det \begin{pmatrix} B_1^{\rightarrow} \\ \vdots \\ B_n^{\rightarrow} \end{pmatrix} = \det B = 0$, then $B_1^{\downarrow}, \dots, B_m^{\downarrow}$ belong to a lower dimensional subspace of $\mathbb{R}^n$, and the assertion in this case follows from d) above. Hence let us suppose that $B_1^{\rightarrow}, \dots, B_n^{\rightarrow}$ is a basis of $\mathbb{R}^n$, and let $e = \alpha_1 B_1^{\rightarrow} + \dots + \alpha_n B_n^{\rightarrow}$ and $d = \beta_1 B_1^{\rightarrow} + \dots + \beta_n B_n^{\rightarrow}$ be the co-ordinate representations of e and d, respectively, corresponding to this basis. Then the assertion $(-1)^{m-1} \det C > 0$ is equivalent with the inequality

$$-4 + 4 \sum_{k=1}^{n} \alpha_k \beta_k + \sum_{1\le k<l\le n} (\alpha_k \beta_l - \alpha_l \beta_k)^2 > 0. \tag{10}$$

Define $v, w \in \mathbb{R}^n$ by $v := \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$, and $w := \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$. Then it follows that

$$\langle v, B_k^{\downarrow}\rangle = 1, \; \langle w, B_k^{\downarrow}\rangle = d_k, \qquad k = 1, \dots, m,$$

and, in particular, that $v \neq 0$, $w \neq 0$. Let us assume now that $d_k > |B_k^{\downarrow}|^2$ for at least one $k \in \{1, \dots, m\}$. This implies that the intersection of the open ball $K := \{x \in \mathbb{R}^n : |2x - w|^2 < |w|^2\}$ with the hyperplane $\varepsilon := \{x \in \mathbb{R}^n : \langle v, x\rangle = 1\}$ contains one of the vectors $B_k^{\downarrow}$, $k = 1, \dots, m$, and hence is not empty. On the other hand, the claim (10) assumes the following form:

$$\begin{aligned} 0 &< -4 + 4\langle v, w\rangle + |v \wedge w|^2 \\ &= -4 + 4\langle v, w\rangle + |v|^2 |w|^2 - \langle v, w\rangle^2 \\ &= (|v|\,|w| + \langle v, w\rangle - 2)(|v|\,|w| - \langle v, w\rangle + 2). \end{aligned}$$

Since the second factor of the last expression is always positive, this inequality is equivalent to

$$|v|\,|w| + \langle v, w\rangle > 2. \tag{11}$$

Let us show, therefore, that the condition $\varepsilon \cap K \neq \phi$ implies (11). Take $u \in \bar{K}\backslash K$ such that the tangent plane through u is parallel with ε and ε separates 0 from this tangent plane. Then it follows that $|2u - w| = |w|$ and $\langle u, v\rangle > 1$. Furthermore, $u - w/2$ is the vector joining the center of K with u, and hence it is perpendicular to ε. This yields $2u - w = \dfrac{|w|}{|v|}\,v$. Thus we conclude that

$$2 < 2\langle u, v\rangle = \langle w, v\rangle + |v|\,|w|,$$

i.e. (11). Hence we have shown that $(-1)^{m-1}\det C > 0$, provided that $m = n$ and $\exists\, k \in \{1, \ldots, m\} : d_k > |B_k^{\downarrow}|^2$.

f) Finally, let us discuss the signature of C for $m \leq n + 1$. We suppose, without restriction of generality, that $d_1 > |B_1^{\downarrow}|^2$. If C_r, $r = 1, \ldots, m$, denote the $r \times r$-submatrices $(c_{kl})_{k,l=1}^{r}$ of C, then, by what has been proven up to now, we conclude that

$$(-1)^{r-1}\det C_r > 0, \qquad r = 1, \ldots, m.$$

Since the number of sign changes in the series of numbers

$$1,\ \det C_1,\ \det C_2,\ \ldots,\ \det C_{m-1},\ \det C$$

coincides with $(n - s)/2$, s denoting the signature of C, (*cf.* [12], Ch. VII, § 2, (V), p. 229), we infer that $s = 2 - m$ and hence that C has precisely one positive eigenvalue. ■

Remarks. – 1) A rather different proof of assertion (3) in Proposition 4 for the case $m = n$ and $\det B \neq 0$ runs as follows: The matrix $M := (-\langle B_k^{\downarrow}, B_l^{\downarrow}\rangle)_{k,l=1}^{m}$ is negative definite, whereas the matrix $N := \frac{1}{2}(d_k + d_l)_{k,l=1}^{m}$ has either rank two (if d and $e = (1, \ldots, 1) \in \mathbb{R}^m$ are linearly independent), or else has rank one. In either case, N has at most one positive eigenvalue, namely $\dfrac{1}{2}\left(\sqrt{m}\,|d| + \sum_{k=1}^{m} d_k\right)$. Since the two quadratic forms corresponding to the symmetric matrices M and N can be diagonalized simultaneously (*cf.* [7], Ch. I, § 12, Th. 12.6), we conclude that $C = M + N$ has at least $m - 1$ negative eigenvalues. If $d_k > |B_k^{\downarrow}|^2$ for a $k \in \{1, \ldots, m\}$, then at least one diagonal element in C is positive, and hence C cannot be negative semi-definite, *i.e.*, it must have exactly one positive and $m-1$ negative eigenvalues. We chose to present the more geometrical proof above, since it also furnishes representations of det C in terms of the original constants d and B [comp. formulae (7), (8), (9)].

2) From Proposition 4, we conclude that the matrix $C(j, j_{n+1})$ defined in Proposition 3 is non-singular, and that $C(j)$, which appears in the Gaussian integral of formula (6), has one positive and $n-1$ negative eigenvalues. Note that $C(j)$ is a matrix with positive elements, which implies, by Brouwer's fixed point theorem, that the interior of $\mathbb{R}^n_+$ contains (a half-line of) eigenvectors of $C(j)$ corresponding to the unique positive eigenvalue of $C(j)$.

4. EVALUATION OF GAUSSIAN INTEGRALS OVER $\mathbb{R}^n_+$, $n = 2, 3$

Motivated by formula (6) in Proposition 3, we now turn towards the task of evaluating integrals of the form

$$\int_{\mathbb{R}^n_+} \mathrm{e}^{-\langle C\,x,\,x\rangle}\,\mathrm{d}x, \quad C \text{ a symmetric } n\times n\text{-matrix.} \tag{12}$$

Though we know from Proposition 4 that $\langle C(j)\,x,\,x\rangle$ in (6) defines a Minkowski metric on $\mathbb{R}^n$, we shall first assume that C in (12) is positive definite and proceed afterwards by analytic continuation.

LEMMA 2. – *Let K be an open cone in $\mathbb{R}^n$ with vertex 0, C be a symmetric positive definite real-valued $n\times n$-matrix, $\sqrt{C}$ be the unique symmetric positive definite square root of C, $f:\mathbb{R}^1_+\to\mathbb{C}$ such that $f(u)\,u^{n/2-1}\in L^1(\mathbb{R}^1_+)$. Furthermore, denote by $|A|$ the measure of a Borel set A in $\mathbb{S}_{n-1}$.*

Then $f(\langle C\,x,\,x\rangle)\in L^1(K)$ and

$$\int_K f(\langle C\,x,\,x\rangle)\,\mathrm{d}x = \frac{|\,\mathbb{S}_{n-1}\cap\sqrt{C}\,K\,|}{2\sqrt{\det C}}\int_0^\infty f(u)\,u^{n/2-1}\,\mathrm{d}u. \tag{13}$$

Proof. – If we substitute $y=\sqrt{C}\,x$ as a new variable, we obtain

$$\int_K f(\langle C\,x,\,x\rangle)\,\mathrm{d}x = \frac{1}{\sqrt{\det C}}\int_{\sqrt{C}\,K} f(|\,y\,|^2)\,\mathrm{d}y.$$

Upon introducing polar co-ordinates $y=|\,y\,|\,\omega$ and setting $u:=|\,y\,|^2$, we immediately infer formula (13). ■

PROPOSITION 5. – *Let C be a symmetric real-valued 2×2-matrix such that $\langle C x, x \rangle > 0$ for $x \in \mathbb{R}^2_+ \backslash 0$. Then*

$$\int_{\mathbb{R}^2_+} \mathrm{e}^{-\langle C x, x \rangle} \mathrm{d}x = \begin{cases} \dfrac{1}{4\sqrt{|\det C|}} \ln \left(\dfrac{c_{12} + \sqrt{|\det C|}}{c_{12} - \sqrt{|\det C|}} \right) & : \det C < 0 \\ \dfrac{1}{2\, c_{12}} : \det C = 0 \\ \dfrac{1}{2\sqrt{\det C}} \operatorname{arccot} \left(\dfrac{c_{12}}{\sqrt{\det C}} \right) & : \det C > 0. \end{cases} \tag{14}$$

Proof. – Evidently, under the assumptions made, the integral in (14) converges. If $\det C > 0$, then C is positive definite and (13) yields:

$$\int_{\mathbb{R}^2_+} \mathrm{e}^{-\langle C x, x \rangle} \, \mathrm{d}x = \frac{1}{2\sqrt{\det C}} \, | \mathbb{S}_1 \cap \sqrt{C}\, \mathbb{R}^2_+ | .$$

The length of the arc $\mathbb{S}_1 \cap \sqrt{C}\, \mathbb{R}^2_+$ equals the angle between the vectors $\sqrt{C}^{\downarrow}_1$ and $\sqrt{C}^{\downarrow}_2$. This angle is given by

$$\arccos \left(\frac{\langle \sqrt{C}^{\downarrow}_1, \sqrt{C}^{\downarrow}_2 \rangle}{|\sqrt{C}^{\downarrow}_1| \cdot |\sqrt{C}^{\downarrow}_2|} \right) = \arccos \left(\frac{c_{12}}{\sqrt{c_{11}\, c_{22}}} \right) = \operatorname{arccot} \left(\frac{c_{12}}{\sqrt{\det C}} \right),$$

since $C = \sqrt{C}^T \sqrt{C}$. Using the equation $\operatorname{arccot} z = \dfrac{1}{2\mathrm{i}} \ln \left(\dfrac{z + \mathrm{i}}{z - \mathrm{i}} \right)$, we easily arrive at formula (14) by an analytic continuation argument. ■

Remarks. – 1) Equivalently, (14) could be derived by introducing polar co-ordinates and making use of [13], 331.51a), b). We preferred to give the proof as above, in order to outline, in this easy setting already, the procedure used later on for the cases $n = 3, 4$.

2) As an example in his discussion of conditionally convergent infinite double integrals, G. H. Hardy states the formula

$$\int_0^\infty \int_0^\infty \mathrm{e}^{\left(ax^2 + 2\,hxy + by^2\right)\mathrm{i}} \, \mathrm{d}x \, \mathrm{d}y = \frac{\mathrm{i} \cos^{-1} \dfrac{h}{\sqrt{ab}}}{2\sqrt{ab - h^2}} \tag{15}$$

(*cf.* [14], p. 162). Herein, $a, b, h \in \mathbb{R}$, and the quadratic form $ax^2 + 2\,hxy + by^2$ is assumed positive definite. The left-hand side of (15) can be interpreted as the value of the "integrable distribution" (*cf.* [32], p. 256, p. 270) $T_C \in \mathcal{D}'_{L^1}(\mathbb{R}^2)$ on the testfunction 1, where

$$T_C := Y(x)\, Y(y)\, \mathrm{e}^{-\langle C \binom{x}{y}, \binom{x}{y} \rangle}, \qquad \text{and} \qquad C = -\mathrm{i} \begin{pmatrix} a & h \\ h & b \end{pmatrix}.$$

Taking into account that $T_C = \lim_{\varepsilon \searrow 0} T_{\varepsilon I + C}$ in $\mathcal{D}'_{L^1}$, we see that formula (15) is a consequence of (14).

PROPOSITION 6. – *Let C be a symmetric real-valued 3×3-matrix such that $\langle C x, x \rangle > 0$ for $x \in \mathbb{R}^3_+ \searrow 0$. Then*

$$\int_{\mathbb{R}^3_+} \mathrm{e}^{-\langle C x, x \rangle} \mathrm{d}x = \begin{cases} \dfrac{\sqrt{\pi}}{4\sqrt{|\det C|}} \ln \left(\dfrac{\eta + \sqrt{|\det C|}}{\eta - \sqrt{|\det C|}} \right) & : \det C < 0 \\ \dfrac{\sqrt{\pi}}{2\eta} : \det C = 0 \\ \dfrac{\sqrt{\pi}}{2\sqrt{\det C}} \operatorname{arccot} \left(\dfrac{\eta}{\sqrt{\det C}} \right) & : \det C > 0, \end{cases} \tag{16}$$

where $\eta := \sqrt{c_{11}\, c_{22}\, c_{33}} + \sqrt{c_{11}}\, c_{23} + \sqrt{c_{22}}\, c_{13} + \sqrt{c_{33}}\, c_{12}$.

Proof. – If C is positive definite, we use formula (13) to obtain:

$$\int_{\mathbb{R}^3_+} \mathrm{e}^{-\langle C x, x \rangle} \, \mathrm{d}x = \frac{\sqrt{\pi}}{4\sqrt{\det C}} \, | \mathbb{S}_2 \cap \sqrt{C}\, \mathbb{R}^3_+ | .$$

As is well-known, the area of a spherical triangle such as $\mathbb{S}_2 \cap \sqrt{C}\, \mathbb{R}^3_+$ equals the sum of its inner angles diminished by π. These angles can also be thought of as the angles between the planes through 0 and two of the vectors $\sqrt{C}_k^{\downarrow}$, $k = 1, 2, 3$. Hence one of these angles is given by

$$\begin{aligned} &\operatorname{angle} (\sqrt{C}_1^{\downarrow} \times \sqrt{C}_2^{\downarrow}, \sqrt{C}_1^{\downarrow} \times \sqrt{C}_3^{\downarrow}) \\ &= \arccos \left(\frac{\langle \sqrt{C}_1^{\downarrow} \times \sqrt{C}_2^{\downarrow}, \sqrt{C}_1^{\downarrow} \times \sqrt{C}_3^{\downarrow} \rangle}{|\sqrt{C}_1^{\downarrow} \times \sqrt{C}_2^{\downarrow}| \cdot |\sqrt{C}_1^{\downarrow} \times \sqrt{C}_3^{\downarrow}|} \right) \\ &= \arccos \left(\frac{c_{11}\, c_{23} - c_{12}\, c_{13}}{\sqrt{c_{11}\, c_{22} - c_{12}^2}\, \sqrt{c_{11}\, c_{33} - c_{13}^2}} \right) \\ &= \operatorname{arccot} \left(\frac{c_{11}\, c_{23} - c_{12}\, c_{13}}{\sqrt{\det C}} \right), \end{aligned}$$

and the others result from this one by cyclic permutation of the indices. Let us suppose now that C differs little from the unit matrix and apply the rule

$$\operatorname{arccot} u + \operatorname{arccot} v + \operatorname{arccot} w - \pi = \operatorname{arccot}\left(\frac{u+v+w-uvw}{1-uv-uw-vw}\right),$$

valid for $uv + uw + vw < 1$. Then an easy calculation yields:

$$\begin{aligned}\int_{\mathbb{R}^3_+} e^{-\langle C\,x,\,x\rangle}\,dx &= \frac{\sqrt{\pi}}{4\sqrt{\det C}}\operatorname{arccot}\left(\frac{\eta^2-\det C}{2\,\eta\sqrt{\det C}}\right)\\ &= \frac{\sqrt{\pi}}{2\sqrt{\det C}}\operatorname{arccot}\left(\frac{\eta}{\sqrt{\det C}}\right).\end{aligned}$$

By analytic continuation, this formula remains true as long as $\det C > 0$, and, by the same reason, we immediately deduce, from it, the two other expressions in (16), valid for $\det C \leq 0$. ■

Remarks. – 1) The above proof of Proposition 6 also yields the following formula for the area F of a spherical triangle as a function of its side lengths a, b, c:

$$F = 2\operatorname{arccot}\left(\frac{1+\cos a+\cos b+\cos c}{\sqrt{1+2\cos a\cos b\cos c-\cos^2 a-\cos^2 b-\cos^2 c}}\right).$$

2) Formula (16) can also be derived by successive integrations. The substitutions $x_1 = u x_3$, $x_2 = v x_3$ and subsequent integration with respect to x_3 yield:

$$\int_{\mathbb{R}^3_+} e^{-\langle C\,x,\,x\rangle}\,dx = \frac{\sqrt{\pi}}{4}\int_{\mathbb{R}^2_+} A^{-3/2}\,du\,dv,$$

where $A = c_{11}\,u^2 + 2\,(c_{12}\,v + c_{13})\,u + c_{22}\,v^2 + 2\,c_{23}\,v + c_{33}$. Making use of [13], 213.1, the u-integral leads to the following function in v, the indefinite integral of which is found with the help of Euler's substitutions (*cf.* [11], 2.251):

$$\begin{aligned}&\int_0^\infty A^{-3/2}\,du\\ &= \frac{1}{\sqrt{c_{22}\,v^2+2\,c_{23}\,v+c_{33}}\,\left(\sqrt{c_{11}}\,\sqrt{c_{22}\,v^2+2\,c_{23}\,v+c_{33}}+c_{12}\,v+c_{13}\right)}.\end{aligned}$$

3) The similarity of formulae (14) and (16) is not purely accidental. It rather results from a general reduction principle, which allows to express

the volume of a simplex on the $(2l)$-dimensional unit sphere by a sum over the volumes of simplexes on the $(2l-1)$-dimensional unit sphere (*cf.* [31], [30], [16], [28], and the comprehensive discussion in [2], pp. 29-34). Let us briefly sketch this reduction in our context using a procedure similar to the method of proof of Schläfli's differential formula in [18]. For the moment, let us suppose that all occurring integrals are convergent. For a symmetric real-valued $n \times n$-matrix C define

$$f(C) := \int_{\mathbb{R}_+^n} e^{-\langle Cx, x\rangle}\, dx,$$

and let $g_j(C)$, $j = 1, \ldots, n$, be the $n \times n$-matrices with the elements

$$g_j(C)_{kl} := c_{kl} \cdot (-1)^{\delta_{jk} + \delta_{jl}}, \qquad k, l = 1, \ldots, n.$$

Then we perform the integration with respect to x_j in the integral which corresponds to $f(C) + f(g_j(C))$ and obtain:

$$\begin{aligned} f(C) + f(g_j(C)) &= \int_{\mathbb{R}_+^{n-1}} \left(\int_{-\infty}^{\infty} e^{-\langle Cx, x\rangle}\, dx_j \right) dx_1 \ldots \widehat{dx_j} \ldots dx_n \\ &= \sqrt{\frac{\pi}{c_{jj}}}\, f(h_j(C)), \end{aligned} \tag{17}$$

where $h_j(C)$ is the $(n-1) \times (n-1)$-matrix resulting from the application of Gauß's algorithm to C with the j-th row as pivot row, and the subsequent removal of the j-th row and the j-th column. Since $g_n g_{n-1} \ldots g_1(C) = C$, we infer from (17) by iteration the following equation:

$$\begin{aligned} f(C) &= \sqrt{\frac{\pi}{c_{11}}}\, f(h_1(C)) - f(g_1(C)) = \cdots \\ &= \sqrt{\frac{\pi}{c_{11}}}\, f(h_1(C)) - \sqrt{\frac{\pi}{c_{22}}}\, f(h_2(g_1(C))) + \cdots + (-1)^n f(C). \end{aligned}$$

Therefore, if n is odd, we obtain:

$$f(C) = \frac{\sqrt{\pi}}{2} \sum_{j=1}^{n} \frac{(-1)^{j-1}}{\sqrt{c_{jj}}}\, f(h_j(g_{j-1} \ldots g_1(C))),$$

where $h_j(g_{j-1} \ldots g_1(C))$, $j = 1, \ldots, n$, are $(n-1) \times (n-1)$-matrices.

Combining the representation of $I_n(B, d)$ in Proposition 3 with the evaluations of Gaussians in the Propositions 5 and 6 (taking into account also Proposition 4) we now write down explicit formulae for $I_2(B, d)$ and $I_3(B, d)$.

PROPOSITION 7. – *Let* $n = 2$ *or* $n = 3$, $n < m$, *and* B, d, $C = (c_{kl})$, $C(j)$, β_j *be as in Proposition* 3. *Then*

$$I_n(B, d) := \int_{\mathbb{R}^n} \prod_{k=1}^{m} (|q|^2 + 2\langle B_k^{\downarrow}, q\rangle + d_k)^{-1}\, dq$$

$$= \begin{cases} (-1)^{m-1} \dfrac{\pi}{4} \displaystyle\sum_{\substack{j=(j_1, j_2)\\ 1 \le j_1 < j_2 \le m}} \frac{\beta_j}{\sqrt{|\det C(j)|}} \\ \quad \times \ln\left(\dfrac{c_{j_1 j_2} + \sqrt{|\det C(j)|}}{c_{j_1 j_2} - \sqrt{|\det C(j)|}}\right) & : n = 2 \\ (-1)^m \dfrac{\pi^2}{2} \displaystyle\sum_{\substack{j=(j_1, j_2, j_3)\\ 1 \le j_1 < j_2 < j_3 \le m}} \frac{\beta_j}{\sqrt{\det C(j)}} \\ \quad \times \operatorname{arccot}\left(\dfrac{\eta_j}{\sqrt{\det C(j)}}\right) & : n = 3, \end{cases} \tag{18}$$

where

$$\eta_j := \sqrt{c_{j_1 j_1}\, c_{j_2 j_2}\, c_{j_3 j_3}} + \sqrt{c_{j_1 j_1}}\, c_{j_2 j_3} + \sqrt{c_{j_2 j_2}}\, c_{j_1 j_3} + \sqrt{c_{j_3 j_3}}\, c_{j_1 j_2}. \quad \blacksquare$$

Remark. – The remaining cases of $(m, n) \in \{(2, 2), (2, 3), (3, 3)\}$ are contained in the remark to Prop. 3. E.g., for $m = 2$, $n = 3$ this yields

$$I_3(B, d) = \frac{2\pi^2}{|B_1^{\downarrow} - B_2^{\downarrow}|} \operatorname{arccot}\left(\frac{\mu_1 + \mu_2}{|B_1^{\downarrow} - B_2^{\downarrow}|}\right), \quad \mu_i := \sqrt{d_i - |B_i^{\downarrow}|^2}.$$

5. FEYNMAN AMPLITUDES IN FOUR SPACE-TIME DIMENSIONS

According to formula (13), for positive definite C, the evaluation of $\int_{\mathbb{R}_+^4} e^{-\langle C x, x\rangle}\, dx$ is reduced to the determination of the volume of a simplex T on the 3-dimensional sphere $\mathbb{S}_3$. This volume can be expressed by a sum of Lobačewskiĭ's functions which contain as arguments combinations of the dihedral angles of T (*see* [6], [3]). In our case, however, C is not positive definite, and hence we use analytic continuation as in Section 4. The result will be stated in terms of Clausen's integral (*cf.* [19], Ch. 4, (4.4), (4.5)), *i.e.*:

$$\mathrm{Cl}_2(z) := -\int_0^z \ln\left(2 \sin \frac{\mu}{2}\right) d\mu = \sum_{k=1}^{\infty} \frac{\sin(kz)}{k^2}.$$

PROPOSITION 8. – *Let C be a symmetric real-valued 4×4-matrix such that $\langle C\,x,\ x\rangle > 0$ for $x \in \mathbb{R}_+^4 \backslash 0$. Furthermore, suppose that C has one positive and three negative eigenvalues. Denote by $U = (u_{kl})$ the inverse matrix of C, and, for $\{r,\ s,\ t\} = \{1,\ 2,\ 3\}$, define the angles*

$$\psi_{rst}^0 := \arctan\left(\frac{c_{t4}\,u_{r4}\,\sqrt{|\det U|}}{u_{rs}\,u_{r4} - u_{rr}\,u_{s4}}\right),$$

$$\psi_{rst}^1 := \arctan\left(\frac{c_{t4}\,u_{r4}\,\sqrt{|\det U|}}{(u_{rs}\,u_{r4} - u_{rr}\,u_{s4})\,\sqrt{1 - c_{44}\,u_{44}}}\right),$$

$$\psi_{rst}^2 := \arctan\left(\frac{u_{r4}\,\sqrt{u_{rr}\,u_{ss} - u_{rs}^2}}{u_{rs}\,u_{r4} - u_{rr}\,u_{s4}}\right),$$

$$\psi_{rst}^3 := \arctan\left(\frac{u_{r4}}{\sqrt{u_{rr}\,u_{44} - u_{r4}^2}}\right).$$

Then

$$\int_{\mathbb{R}_+^4} e^{-\langle C\,x,\,x\rangle}\,dx = \frac{1}{16\,\sqrt{|\det C|}} \sum_{\{r,\,s,\,t\}=\{1,\,2,\,3\}} \times \left\{2\,\mathrm{Cl}_2\,(2\,\psi_{rst}^0) + \sum_{l=1}^{3} (-1)^l\,[\mathrm{Cl}_2\,(2\,\psi_{rst}^0 + 2\,\psi_{rst}^l) + \mathrm{Cl}_2\,(2\,\psi_{rst}^0 - 2\,\psi_{rst}^l)]\right\}. \tag{19}$$

Proof. – *a*) Let us assume first that C is symmetric and positive definite with $c_{kl} > 0,\ k,\ l = 1,\ \ldots,\ 4$. The four vectors $v_k := \sqrt{C}_k^{\downarrow} / |\sqrt{C}_k^{\downarrow}|$, $k = 1,\ \ldots,\ 4$, span a tetrahedron T on the unit sphere $\mathbb{S}_3$, and by Lemma 2, the integral in question equals $V/2\sqrt{\det C}$, V denoting the volume of T. Denote by v_0 the orthogonal projection of v_4 onto the spherical plane in $\mathbb{S}_3$ spanned by the vectors $v_1,\ v_2,\ v_3$. For $\{r,\ s,\ t\} = \{1,\ 2,\ 3\}$, we thereafter project v_0 onto the geodetic circle through v_s and v_t and denote this projection by w_r. By this procedure, the tetrahedron T is decomposed into the 6 so-called "orthoschemes" T_{rst} (*cf.* [2], Ch. I), which are the tetrahedra on $\mathbb{S}_3$ spanned by $v_t,\ w_r,\ v_0,\ v_4$, where $\{r,\ s,\ t\} = \{1,\ 2,\ 3\}$.

b) Let us consider now one of these orthoschemes T_{rst} for a fixed permutation r, s, t. If the dihedral angles α_1, α_2, α_3 in T_{rst} satisfy $0 < \alpha_l < \frac{\pi}{2}$, $l = 1, 2, 3$, and if we put

$$\bar{\alpha}_l := \begin{cases} \frac{\pi}{2} - \alpha_l : l = 1, 3 \\ \alpha_l : l = 2, \end{cases} \quad \text{and} \quad \tan^2 \alpha_0 := \frac{\cos^2 \bar{\alpha}_2 - \cos^2 \bar{\alpha}_1 \cos^2 \bar{\alpha}_3}{\sin^2 \bar{\alpha}_1 \sin^2 \bar{\alpha}_3},$$

where $0 \leq \Re\, \alpha_0 \leq \frac{\pi}{2}$, then the volume V_{rst} of T_{rst} is given by

$$V_{rst} = \frac{1}{4\mathrm{i}} \left\{ -2\, \Lambda_2 (\alpha^0) + \sum_{l=1}^{3} (-1)^{l-1} \left[\Lambda_2 (\alpha_0 + \bar{\alpha}_l) + \Lambda_2 (\alpha_0 - \bar{\alpha}_l) \right] \right\}, \quad (20)$$

where $\Lambda_2 (z) := -\int_0^z \ln \cos \mu \, d\mu$ is Lobačewskiĭ's function (*cf.* [3], (3.2), (3.3a), (3.4), (3.13), (3.16), (3.17)). The dihedral angles α_1, α_2, α_3 are defined as the angles between the faces of T_{rst} along the edges $\overline{v_0 v_4}$, $\overline{v_t v_4}$, and $\overline{v_t w_r}$, respectively. An elementary, yet lengthty calculation yields:

$$\alpha_0 = \arctan \left(\frac{\mathrm{i} \, (u_{rs}\, u_{r4} - u_{rr}\, u_{s4})}{c_{t4}\, u_{r4} \sqrt{\det U}} \right),$$

$$\bar{\alpha}_1 = \operatorname{arccot} \left(\frac{(u_{rs}\, u_{r4} - u_{rr}\, u_{s4}) \sqrt{c_{44}\, u_{44} - 1}}{-c_{t4}\, u_{r4} \sqrt{\det U}} \right),$$

$$\bar{\alpha}_2 = \arccos \left(\frac{u_{rs}\, u_{r4} - u_{rr}\, u_{s4}}{\sqrt{u_{rr}} \sqrt{u_{r4}^2\, u_{ss} + u_{s4}^2\, u_{rr} - 2\, u_{r4}\, u_{s4}\, u_{rs}}} \right),$$

$$\bar{\alpha}_3 = \arcsin \left(\frac{-u_{r4}}{\sqrt{u_{rr}\, u_{44}}} \right).$$

c) Next we express the volume V_{rst} in formula (20) by means of the dilogarithm function $\mathrm{Li}_2 (z) := -\int_0^z \ln (1 - \mu) \, d\mu / \mu$ (*cf.* [19], Ch. 1). Using the functional relation

$$\Lambda_2 (z) = \frac{\mathrm{i}}{2} \mathrm{Li}_2 (-e^{-2\mathrm{i}z}) + z \ln 2 - \frac{\mathrm{i}}{2} \left(z^2 - \frac{\pi^2}{12} \right),$$

(*cf.* [3], (3.15)), and the representation of inverse trigonometric functions by the logarithm, we obtain:

$$V_{rst} = \frac{1}{4} \Re \left\{ -\bar{\alpha}_1^2 + \bar{\alpha}_2^2 - \bar{\alpha}_3^2 - \mathrm{Li}_2 \left(-\frac{z_0^+}{z_0^-} \right) + \sum_{l=1}^{3} (-1)^{l-1} \mathrm{Li}_2 \left(-\frac{z_0^+ z_l^+}{z_0^- z_l^-} \right) \right\}, \quad (21)$$

where

$$\begin{aligned}
z_0^{\pm} &:= c_{t4}\, u_{r4} \sqrt{\det U} \pm (u_{rs}\, u_{r4} - u_{rr}\, u_{s4}),\\
z_1^{\pm} &:= (u_{rs}\, u_{r4} - u_{rr}\, u_{s4}) \sqrt{c_{44}\, u_{44} - 1} \pm \mathrm{i}\, c_{t4}\, u_{r4} \sqrt{\det U},\\
z_2^{\pm} &:= u_{rs}\, u_{r4} - u_{rr}\, u_{s4} \pm \mathrm{i}\, u_{r4} \sqrt{u_{rr}\, u_{ss} - u_{rs}^2},\\
z_3^{\pm} &:= \sqrt{u_{rr}\, u_{44} - u_{r4}^2} \pm \mathrm{i}\, u_{r4}.
\end{aligned}$$

d) Eventually we pass, by analytic continuation, to a matrix C which fulfils the conditions of Proposition 8. Then $\det U < 0$ and $\mathrm{i}\, z_0^{\pm}$ are conjugate complex numbers. Furthermore, $c_{t4}^2 > c_{tt}\, c_{44}$, and hence $u_{rr} u_{ss} - u_{rs}^2 = (c_{tt}\, c_{44} - c_{t4}^2)/\det C > 0$, which implies that both $z_2^{\pm}$ and $z_3^{\pm}$ are conjugate complex numbers, respectively. Finally, if C^{ad} denotes the adjoint matrix of C, then $u_{44} = C_{44}^{\mathrm{ad}}/\det C < 0$, and hence the same is true for $\mathrm{i}\, z_1^{\pm}$. Therefore, all the arguments of the dilogarithms in (21) have modulus 1. Writing out the real part in (21) with the rule $\Re\, w = \frac{1}{2}(w + \bar{w})$, taking into account that $\int_{\mathbb{R}_+^4} \mathrm{e}^{-\langle C\, x,\, x\rangle}\, \mathrm{d}x = \frac{1}{2\sqrt{\det C}} \sum_{\{r,\, s,\, t\} = \{1,\, 2,\, 3\}} V_{rst}$ remains real, and that $\Im\, \bar{\alpha}_l^2 = 0$, $l = 1, 2, 3$, we end up with formula (19), since $\Im\, \mathrm{Li}_2(\mathrm{e}^{\mathrm{i}\,\theta}) = \mathrm{Cl}_2(\theta)$, $\theta \in \mathbb{R}$ (*cf.* [19], (4.6), p. 102). Notice that the right-hand side of formula (19) is a real-analytic function of U, since we have $\arctan x + \arctan(x^{-1}) = \frac{\pi}{2}\, \mathrm{sign}\, x$ for $x \in \mathbb{R}\backslash 0$ and $\mathrm{Cl}_2(z + 2\pi) = \mathrm{Cl}_2(z)$ for $z \in \mathbb{C}$. ■

Remarks. – 1) In a similar way as it was done in Proposition 7, the combination of Proposition 8 with Proposition 3 and the remark following it yields an explicit representation of $I_4(B,\, d)$ by a sum over $42\binom{m}{4}$ Cl_2-functions. In the case of $m = 3$, $I_4(B,\, d)$ is by the remark to Prop. 3 given by $4\pi^2 \int_{\mathbb{R}_+^4} \mathrm{e}^{-\langle \tilde{C}\, x,\, x\rangle}\, \mathrm{d}x$, where $\tilde{c}_{44} = 0$. Then Prop. 8 yields a representation by 18 Cl_2-functions, since $\psi_{rst}^0 = \psi_{rst}^1 = \psi_{rst}^2$.

2) We mention that the representation in formula (19) is by no means unique, since there exist many functional equations connecting Clausen's functions with different arguments (*cf.* [19], Ch. 4). Therefore, we pose the problem to develop a simpler representation of $\int_{\mathbb{R}_+^4} \mathrm{e}^{-\langle C\, x,\, x\rangle}\, \mathrm{d}x$ than

the one by a linear combination of 42 Cl_2-functions stated in Proposition 8.

As an example for a more concise and more symmetric expression, let us write down a formula which corresponds to a matrix C wherein all elements along the main diagonal vanish. In this case, equation (19) simplifies very much, since $\psi^0_{rst} = \psi^1_{rst} = \psi^2_{rst} = \psi^3_{rst} = \psi^3_{rts}$. Suppose that $f(u)\, u \in L^1(\mathbb{R}^1_+)$, and let C be a symmetric real-valued 4×4-matrix with $c_{kk} = 0$, $k = 1, \ldots, 4$, $\det C < 0$, and $\langle C x, x\rangle > 0$ for every $x \in \mathbb{R}^4_+$ which does not lie on one of the four axes. Then

$$\int_{\mathbb{R}^4_+} f(\langle C x, x\rangle)\, \mathrm{d}x = \frac{1}{4\sqrt{|\det C|}} \times \sum_{i=1}^{3} \mathrm{Cl}_2\left(2 \operatorname{arccot}\left(\frac{\eta_i}{\sqrt{|\det C|}}\right)\right) \int_0^\infty f(u)\, u\, \mathrm{d}u,$$

where

$$\eta_1 = c_{12}\, c_{34} + c_{13}\, c_{24} - c_{23}\, c_{14}, \quad \eta_2 = c_{12}\, c_{34} + c_{23}\, c_{14} - c_{13}\, c_{24},$$
$$\eta_3 = c_{13}\, c_{24} + c_{23}\, c_{14} - c_{12}\, c_{34}.$$

In terms of hyperbolic geometry, this formula expresses the volume V of a simplex all of which vertices lie at infinity as a function of the three dihedral angles α, β, γ at an arbitrary vertex:

$$V = \frac{1}{2}\left(\mathrm{Cl}_2(\alpha) + \mathrm{Cl}_2(\beta) + \mathrm{Cl}_2(\gamma)\right),$$

cf. [22], Lemma 2, p. 18.

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