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A representation independent propagator I: compact Lie groups

by

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ABSTRACT. – Conventional path integral expressions for propagators are representation dependent. Rather than having to adapt each propagator to the representation in question, it is shown that for compact Lie groups it is possible to introduce a propagator that is representation independent. For a given set of kinematical variables this propagator is a single function independent of any particular choice of fiducial vector, which monetheless, correctly propagates each element of the coherent state representation associated with these kinematical variables. Although the configuration space is in general curved, nevertheless the lattice phase-space path integral for the representation independent propagator has the form appropriate to flat space. To illustrate the general theory a representation independent propagator is explicitly constructed for the Lie group SU(2).

RÉSUMÉ. – L'expression conventionnelle des propagateurs sous forme d'intégrales de chemin dépend de la représentation choisie. Plutôt que d'adapter chaque propagateur à la représentation choisie, nous montrons que pour un groupe de Lie compact il est possible de définir un propagateur indépendant des représentations.

Pour un ensemble de variables cinématiques donné, ce propagateur est une fonction unique indépendante d'un choix de vecteur fiduciel, qui cependant, propage correctement chaque élement de la représentation d'états cohérent associée à ces variables cinématiques. Bien que l'espace des configurations soit courbe en général, l'intégrale de chemin dans l'espace des phases latticiel pour ce propagateur indépendant de la représentation, a une forme appropriée à un espace plat. Afin d'illustrer cette théorie, nous donnons une construction explicite de ce propagateur pour le groupe de Lie SU(2).

1. INTRODUCTION

In [1] a *universal propagator* has been constructed for the compact Lie group SU(2) by following the program outlined ([2], [3]). This propagator is called a universal propagator since it is independent of the chosen *fiducial vector* that fixes a coherent state representation, for example one can choose for this "vector" the ground state of the Hamilton operator of the quantum system under consideration. For the case of the affine group and the Lie group SU(2) this propagator also proved to be independent of any particular irreducible unitary representation of those Lie groups ([1], [2]).

In this paper a novel derivation of such a propagator for any compact Lie group is presented, one that clearly shows that the universal propagator introduced in [1]-[5] is indeed representation independent. Here the word representation independent is used in a dual meaning, its first meaning pertains to the fact that the universal propagator is independent of the fiducial vector and its second meaning to the fact that this propagator is also independent of the choice of the unitary irreducible representation of the Lie group G. In the case of quantum field theory these two meanings of the word representation independent are inextricably related, since the dynamics chooses a representation for the basic kinematical variables (*cf.* [6], pp. 82-83) and [7], pp. 56-57). It is therefore believed, that the concept of a representation-independent propagator holds considerable interest for quantum field theory.

Before embarking onto the construction of the representation-independent propagator for a compact Lie group G, its construction is first outlined for the case SU(2). Let S_1, S_2, and S_3 denote an irreducible representation of self-adjoint spin operators satisfying the commutation relations $[S_i,\, S_j] = i\,\varepsilon_{ijk}\, S_k$. These are the familiar commutation relations for the Lie algebra su(2); note that the physical spin operators are given by $\hat{S}_j = \hbar\, S_j$. The $(2\,s+1)$-dimensional eigenspaces of the Casimir operator $\vec{S}^2$ corresponding to the eigenvalue $s\,(s+1)$, $s = \frac{1}{2},\, 1,\, \frac{3}{2},\, 2,\, \ldots$, are denoted by $\mathbf{E}^s$. For SU(2) in the Euler angle parameterization define

$$\eta^{\theta\phi\xi} \equiv \sqrt{2\,s+1}\,\exp\,(-i\,\phi\,S_3)\,\exp\,(-i\,\theta\,S_2)\,\exp\,(-i\,\xi\,S_3)\,\eta,$$

where $\theta \in (0,\, \pi)$, $\phi \in [0,\, 2\,\pi)$, $\xi \in [-2\,\pi,\, 2\,\pi)$ (*see* [8], p. 98), and $\eta \in \mathbf{E}^s$. For all θ, ϕ, and ξ and a fixed, normalized fiducial vector η these states form a family of overcomplete states, the so called spin coherent states. These states admit the following resolution of the identity

$$I_{\mathbf{E}^s} = \int \eta^{\theta\phi\xi}\,(\eta^{\theta\phi\xi},\, \cdot\,)\, d\mu\,(\theta,\, \phi,\, \xi),$$

where $d\mu(\theta, \phi, \xi) = 1/(16\pi^2)\sin\theta\, d\theta\, d\phi\, d\xi$ is the invariant measure of SU(2) normalized to unity. The map $\mathcal{C}^s_\eta : \mathbf{E}^s \to L^2(SU(2), d\mu(\theta, \phi, \xi))$, defined for any $\psi \in \mathbf{E}^s$ by:

$$(\mathcal{C}^s_\eta \psi)(\theta, \phi, \xi) = \psi_n(\theta, \phi, \xi) \equiv (\eta^{\theta\phi\xi}, \psi),$$

yields a representation of the eigenspaces $\mathbf{E}^s$ by bounded, continuous, square integrable functions on a proper closed subspace $L^2_\eta(SU(2), d\mu(\theta, \phi, \xi))$ of $L^2(SU(2), d\mu(\theta, \phi, \xi))$. Using the resolution of the identity one finds

$$\psi_n(\theta, \phi, \xi) = \int \mathcal{K}_\eta(\theta, \phi, \xi; \theta', \phi', \xi')\, \psi_n(\theta', \phi', \xi')\, d\mu(\theta', \phi', \xi'),$$

where,

$$\mathcal{K}_\eta(\theta, \phi, \xi; \theta', \phi', \xi') = (\eta^{\theta\phi\xi}, \eta^{\theta', \phi', \xi'})$$

is the reproducing kernel, which is the kernel of a projection operator from $L^2(SU(2), d\mu(\theta, \phi, \xi))$ onto the reproducing kernel Hilbert space $L^2_\eta(SU(2), d\mu(\theta, \phi, \xi))$. An inner product in this representation is introduced as follows:

$$(\chi, \psi) = \int \chi^*_\eta(\theta, \phi, \xi)\, \psi_n(\theta, \phi, \xi)\, d\mu(\theta, \phi, \xi).$$

If $\mathcal{H}$ denotes the self-adjoint Hamilton operator for the quantum system under consideration, then the Schrödinger equation on $\mathbf{E}^s$,

$$i\hbar\, \partial_t \psi = \mathcal{H}\psi,$$

and its solution in terms of the evolution operator $U(t) = \exp(-(i/\hbar)\, t\mathcal{H})$,

$$\psi(t'') = U(t'' - t')\, \psi(t'),$$

are given in this representation by

$$i\hbar\, \partial_t \psi_n(\theta, \phi, \xi, t) = (\eta^{\theta\phi\xi}, \mathcal{H}(S_1, S_2, S_3)\, \psi(t))$$

and

$$\psi_n(\theta'', \phi'', \xi'', t'') = \int K_\eta(\theta'', \phi'', \xi'', t''; \theta', \phi', \xi', t')\, \psi_n(\theta', \phi', \xi', t')\, d\mu(\theta', \phi', \xi'),$$

where

$$K_\eta(\theta'', \phi'', \xi'', t''; \theta', \phi', \xi', t') = (\eta^{\theta''\phi''\xi''}, U(t'' - t')\, \eta^{\theta'\phi'\xi'}).$$

Clearly K_η depends on the choice of the fiducial vector η. In contrast, the *representation-independent propagator* $K(\theta'', \phi'', \xi'', t''; \theta', \phi', \xi', t')$ is a single function, *independent* of any particular fiducial vector, which nevertheless, propagates the ψ_n correctly (*cf.* Theorem 3.2), *i.e.*,

$$\psi_n(\theta'', \phi'', \xi'', t'') = \int K(\theta'', \phi'', \xi'', t''; \theta', \phi', \xi', t')\, \psi_n(\theta', \phi', \xi')\, d\mu(\theta', \phi', \xi'),$$

for any choice of fiducial vector. This function can be constructed as follows; define the differential operators

$$\left.\begin{aligned} \tilde{s}_1(p_l, l) &\equiv -\sin\phi\, p_\phi - \cot\theta\, \cos\phi\, p_\phi + \cos\phi \csc\theta\, p_\xi, \\ \tilde{s}_2(p_l, l) &\equiv \cos\phi\, p_\phi - \cot\theta\, \sin\phi\, p_\phi + \sin\phi \csc\theta\, p_\xi, \\ \tilde{s}_3(p_l, l) &\equiv p_\phi, \end{aligned}\right\} \tag{1}$$

where $l = (\theta, \phi, \xi)$ and $p_l = (-i\hbar\,\partial_\theta, -i\hbar\,\partial_\phi, -i\hbar\,\partial_\xi)$. It is then easily observed that

$$\begin{aligned} \tilde{s}_1(p_l, l)\, \psi_n(\theta, \phi, \xi) &= (\eta^{\theta\phi\xi}, S_1\psi), \\ \tilde{s}_2(p_l, l)\, \psi_n(\theta, \phi, \xi) &= (\eta^{\theta\phi\xi}, S_2\psi), \\ \tilde{s}_3(p_l, l)\, \psi_n(\theta, \phi, \xi) &= (\eta^{\theta\phi\xi}, S_3\psi), \end{aligned}$$

hold independently of η, (*cf.* Corollary 2.4). Therefore, on any one of the reproducing kernel Hilbert spaces $L^2_\eta(SU(2), d\mu(\theta, \phi, \xi))$ the Schrödinger equation takes the following form

$$i\hbar\,\partial_t\,\psi_n(\theta, \phi, \xi, t) = \mathcal{H}(\tilde{s}_1(p_l, l), \tilde{s}_2(p_l, l), \tilde{s}_3(p_l, l))\,\psi_n(\theta, \phi, \xi, t).$$

Since the representation-independent propagator is a weak solution to Schrödinger's equation, one has

$$\begin{aligned} &i\hbar\,\partial_t K(\theta, \phi, \xi, t; \theta', \phi', \xi', t') \\ &\quad = \mathcal{H}(\tilde{s}_1(p_l, l), \tilde{s}_2(p_l, l), \tilde{s}_3(p_l, l))\, K(\theta, \phi, \xi, t; \theta', \phi', \xi', t'). \end{aligned} \tag{2}$$

One now interprets (2) as a Schrödinger equation appropriate to *three separate and independent canonical* degrees of freedom. In this interpretation, $l^1 = \theta$, $l^2 = \phi$, and $l^3 = \xi$ are viewed as three "coordinates", and one is looking at the irreducible Schrödinger representation of a special class of three-variable Hamilton operators, ones where the classical Hamiltonian is restricted to have the form $\mathcal{H}(\check{s}_1(p, l), \check{s}_2(p, l), \check{s}_3(p, l))$, instead of the most general form $\mathcal{H}(p_1, p_2, p_3, l^1, l^2, l^3)$. In fact the differential operators given in (1) are elements of the left regular enveloping algebra of a three-dimensional Schrödinger representation. Based on this interpretation a formal standard phase-space path integral solution may be given for the representation-independent propagator as follows (*cf.* Proposition 3.4):

$$K(l'', t''; l', t') = \mathcal{M} \int \exp\left\{\frac{i}{\hbar} \int [p \cdot \dot{l} - \mathcal{H}(\check{s}_1(p, l), \check{s}_2(p, l), \check{s}_3(p, l)) \, dt\right\} \times \prod_{t \in [t', t'']} dp(t) \, dl(t), \tag{3}$$

where,

$$\begin{aligned} \check{s}_1(p, l) &= -\sin\phi\,\alpha - \cot\theta\,\cos\phi\,\beta + \csc\theta\,\cos\phi\,\gamma, \\ \check{s}_2(p, l) &= \cos\phi\,\alpha - \cot\theta\,\sin\phi\,\beta + \csc\theta\,\sin\phi\,\gamma, \\ \check{s}_3(p, l) &= \beta, \end{aligned}$$

$l = (\theta, \phi, \xi)$, and $p = (\alpha, \beta, \gamma)$. Despite the fact that the representation independent propagator has been constructed by interpreting the appropriate Schrödinger equation (2) as an equation for three degrees of freedom, it is nonetheless true that the classical limit corresponds to a single spin degree of freedom (*cf.* [1] and Proposition 5.1).

2. DEFINITIONS, NOTATIONS, AND PRELIMINARIES

The results of this section are derived for the case of a connected and simply connected, separable, locally compact Lie group. Denote by g a Lie algebra of symmetric operators on some Hilbert space $\mathbf{H}$ which have a common dense invariant domain $\mathbf{D}$. Let $X_1, \ldots, X_d$ be an operator basis for g, with commutation relations $[X_i, X_j] = i \sum_{k=1}^{d} c_{ij}{}^k X_k$. If $\Delta = X_1^2 + \ldots + X_d^2$ is essentially self-adjoint then there exists on $\mathbf{H}$

a unique unitary representation U of G which has g as its Lie algebra such that for all X in g, $\overline{U(X)} = \overline{X}$ (*see* [9], Theorem 5). Since the representation for g is integrable to a unique global unitary representation of the associated Lie group G, the elements of G may be parameterized by canonical coordinates of the second kind, *i.e.*

$$U_{g(l)} = \prod_{j=1}^{d} \exp(-il^j X_j) \equiv \exp(-il^1 X_1) \ldots \exp(-il^d X_d),$$

$$U^*_{g(l)} = \coprod_{j=1}^{d} \exp(il^j X_j) \equiv \exp(il^d X_d) \ldots \exp(il^1 X_1),$$

for some ordering, where l is an element of a d-dimensional parameter space $\mathcal{G}$. In general, it is not true that the common dense invariant domain $\mathbf{D}$ is also invariant under $U_{g(l)}$; this stems from the fact even if $X_j^n \psi \in \mathbf{D}$ if $\psi \in \mathbf{D}$ the series $\sum_{n=0}^{N} ((-is)^n/n!) X_j^n \psi$ does not need to converge as $N \to \infty$. However we shall need a common dense invariant domain for $X_1, \ldots, X_d$ that is also invariant under $U_{g(l)}$, therefore those vectors for which $\sum_{n=0}^{N} ((-is)^n/n!) X_j^n \psi$, $j = 1, \ldots, d$, converges absolutely are of special interest. These vectors are called analytic vectors for X_j. As shown in [10], pp. 364-365, the dense set of analytic vectors $\mathbf{A}_{U_g} \subset \mathbf{H}$ forms a common dense invariant domain for $X_1, \ldots, X_d$. We therefore, choose to work with the domain $\tilde{\mathbf{D}} = \mathbf{A}_{U_g}$ as our common dense invariant domain for the Lie algebra g.

Now introduce the following functions $\lambda_m{}^k(g(l))$, $\rho_m{}^k(g(l))$, and $U_m{}^k(l)$, which will figure in the sequel, on $\tilde{\mathbf{D}}$ one has

$$\coprod_{a=m+1}^{d} \exp(il^a X_a) X_m \prod_{b=m+1}^{d} \exp(-il^b X_b) = \sum_{k=1}^{d} \lambda_m{}^k(g(l)) X_k,$$

$$\prod_{a=1}^{m-1} \exp(-il^a X_a) X_m \coprod_{b=1}^{m-1} \exp(il^b X_b) = \sum_{k=1}^{d} \rho_m{}^k(g(l)) X_k.$$

Since the parameterization of the group G is chosen in such a way that $\det[\lambda_m{}^k(g(l))] \neq 0$ and $\det[\rho_m{}^k(g(l))] \neq 0$ the inverse matrices

$[\lambda^{-1}{}_m{}^k(g(l))]$ and $[\rho^{-1}{}_m{}^k(g(l))]$ exist. The functions $U_m{}^k(l)$ are introduced as follows, on $\tilde{\mathbf{D}}$

$$U^*_{g(l)}\, X_m\, U_{g(l)} = \sum_{k=1}^{d} U_m{}^k(l)\, X_k,$$

$$U_{g(l)}\, X_m\, U^*_{g(l)} = \sum_{k=1}^{d} U^{-1}{}_m{}^k(l)\, X_k.$$

Denote by $U(l)$ the matrix whose mk-element is $U_m{}^k$. It can be easily checked by direct calculation that $U(l)$ is given by exponentiating the adjoint representation of g,

$$U(l) = \prod_{k=1}^{d} \exp(l^k\, c_k),$$

where c_k is the matrix formed from the structure constants as follows $c_k = -c_k\,({}_i{}^j)$.

THEOREM 2.1. – *On the common dense invariant domain* $\tilde{\mathbf{D}}$ *of* $X_1, \ldots, X_d$, *the following relations hold,*

(i) *For all* $l \in \mathcal{G}$, $U^*_{g(l)}\, dU_{g(l)} = -i \sum_{k,m=1}^{d} \lambda_m{}^k(g(l))\, dl^m\, X_k$, *and* $L_{g(l_o)}\, \lambda_m{}^k(g(l)) = \lambda_m{}^k(g(l))$.

(ii) *For all* $l \in \mathcal{G}$, $dU_{g(l)}\, U^*_{g(l)} = -i \sum_{k,m=1}^{d} \rho_m{}^k(g(l))\, dl^m\, X_k$, *and* $R_{g(l_o)}\, \rho_m{}^k(g(l)) = \rho_m{}^k(g(l))$.

Proof. – (i) Since, $\tilde{\mathbf{D}}$ is a core for each of the operators $\overline{X}_k$, $k = 1, \ldots, d$ and $U_{g(l)}\, \tilde{\mathbf{D}} \subset \tilde{\mathbf{D}}$ one can define the differential of $U_{g(l)}$ for $\psi \in \tilde{\mathbf{D}}$ as follows:

$$dU_{g(l)}\, \psi \equiv \sum_{m=1}^{d} \left[\lim_{\Delta l^m \to 0} \frac{U_{g(l^1, \ldots, l^m + \Delta l^m, \ldots, l^d)}\, \psi - U_{g(l^1, \ldots, l^d)}\, \psi}{\Delta l^m} \right] dl^m.$$

Now since $U_{g(l)}$ is the product of one parameter unitary groups one finds for the differential of $U_{g(l)}$

$$dU_{g(l)}\, \psi = \sum_{m=1}^{d} \coprod_{a=1}^{m} \exp(-il^a\, X_a)\, (-i\, X_m) \prod_{b=m+1}^{d} \exp(-il^b\, X_b)\, \psi\, dl^m.$$

Therefore,

$$U^*_{g(l)}\, dU_{g(l)}\, \psi = -i \sum_{m=1}^{d} \coprod_{a=m+1}^{d} \exp(il^a X_a)\, X_m \times \prod_{b=m+1}^{d} \exp(-il^b X_b)\, \psi\, dl^m = -i \sum_{m,\,k=1}^{d} \lambda_m{}^k (g(l))\, dl^m X_k\, \psi.$$

Since $\psi \in \tilde{\mathbf{D}}$ was arbitrary, one finds that on $\tilde{\mathbf{D}} \subset \mathbf{H}$, the following relation holds

$$U^*_{g(l)}\, dU_{g(l)} = -i \sum_{k,\,m=1}^{d} \lambda_m{}^k (g(l))\, dl^m X_k, \tag{4}$$

To establish the second part of (i) let $\psi \in \tilde{\mathbf{D}}$ be arbitrary then

$$U^*_{g(l)}\, dU_{g(l)}\, \psi = U^*_{g(l)}\, U_{g(l_o)}\, dU_{g(l)}\, \psi = U^*_{g^{-1}(l_o)\, g(l)}\, dU_{g^{-1}(l_o)\, g(l)}\, \psi.$$

Therefore, using (4) and the fact that $\{X_k\}_{k=1}^d$ is an operator basis for g and that the $\{dl^m\}_{m=1}^d$ are linearly independent one finds

$$L_{g(l_o)}\, \lambda_m{}^k (g(l)) \equiv \lambda_m{}^k (g^{-1}(l_o)\, g(l)) = \lambda_m{}^k (g(l)).$$

(ii) The first part of (ii) is similar to the first part of (i). To prove the second part of (ii) one can proceed as follows, let $\psi \in \tilde{\mathbf{D}}$ be arbitrary, then

$$dU_{g(l)}\, U^*_{g(l)}\, \psi = dU_{g(l)}\, U_{g(l_o)}\, U^*_{g(l_o)}\, U^*_{g(l)}\, \psi = dU_{g(l)\, g(l_o)}\, U^*_{g(l)\, g(l_o)}\, \psi.$$

Therefore, by the same reasoning as above

$$R_{g(l_o)}\, \rho_m{}^k (g(l)) \equiv \rho_m{}^k (g(l)\, g(l_o)) = \rho_m{}^k (g(l)). \quad \square$$

Since the $\lambda_m{}^k (g(l))$ are left invariant functions on the Lie groups G, the relation (i) can be regarded as an operator version of the generalized Maurer-Cartan form on G, (*cf.* [11], p. 92).

COROLLARY 2.2. – *The functions $\lambda_m{}^k (g(l))$ and $\rho_m{}^k (g(l))$ are related as follows:*

$$\lambda_m{}^k (g(l)) = \sum_{c=1}^{d} \rho_m{}^c (g(l))\, U_c{}^k (l).$$

Proof. – Let $\psi \in \tilde{\mathbf{D}}$ be arbitrary then by Theorem 2.1 (ii),

$$dU_{g(l)}\, U^*_{g(l)}\, \psi = -i \sum_{c,\, m=1}^{d} \rho_m{}^c\,(g\,(l))\, dl^m\, X_c\, U_{g\,(l)} U^*_{g\,(l)}\, \psi.$$

Since $U_{g\,(l)}$ leaves $\tilde{\mathbf{D}}$ invariant, one can set $\phi \equiv U^*_{g\,(l)}\, \psi \in \tilde{\mathbf{D}}$ and hence obtain,

$$dU_{g\,(l)}\, \phi = -i \sum_{c,\, m=1}^{d} \rho_m{}^c\,(g\,(l))\, dl^m\, X_c\, U_{g\,(l)}\, \phi.$$

Now multiplying this relation from the left by $U^*_{g\,(l)}$ one finds

$$U^*_{g\,(l)}\, dU_{g\,(l)}\, \phi = -i \sum_{c,\, m=1}^{d} \rho_m{}^c\,(g\,(l))\, dl^m\, U^*_{g\,(l)}\, X_c\, U_{g\,(l)}\, \phi.$$

Using Theorem 2.1 (i) and the definition of the functions $U_m{}^k\,(l)$ the Corollary easily follows. $\square$

Observe that Corollary 2.2 could be proven directly from the definition of the functions $\lambda_m{}^k\,(g\,(l))$ and $\rho_m{}^k\,(g\,(l))$, however the proof given here is more general and applies to any kind of group parameterization.

COROLLARY 2.3. – *The functions $\rho_m{}^k\,(g(\,l))$ and $\lambda_m{}^k\,(g(\,l))$ satisfy the following equation*

$$\begin{aligned}
\text{(i)}\quad & \sum_{n=1}^{d} \{\partial_{l^n}\,[\rho^{-1}{}_j{}^a\,(g\,(l))]\,\rho^{-1}{}_k{}^n\,(g\,(l)) \\
& \qquad - \partial_{l^n}\,[\rho^{-1}{}_k{}^a\,(g\,(l))]\,\rho^{-1}{}_j{}^n\,(g(\,l))\} \\
& = \sum_{f=1}^{d} c_{jk}{}^f\,\rho^{-1}{}_f{}^a\,(g\,(l)),
\end{aligned}$$

$$\begin{aligned}
\text{(ii)}\quad & \sum_{n=1}^{d} \{\partial_{l^n}\,[\lambda^{-1}{}_j{}^a\,(g\,(l))]\,\lambda^{-1}{}_k{}^n\,(g\,(l)) \\
& \qquad - \partial_{l^n}\,[\lambda^{-1}{}_k{}^a\,(g\,(l))]\,\lambda^{-1}{}_j{}^n\,(g(\,l))\} \\
& = \sum_{f=1}^{d} c_{jk}{}^f\,\lambda^{-1}{}_f{}^a\,(g\,(l)),
\end{aligned}$$

where $c_{jk}{}^f$ are the structure constants for G.

Proof. – (i) Let $\psi \in \tilde{\mathbf{D}}$ be arbitrary then it follows from the Proof of Theorem 2.1 that

$$\partial_{l^m}\,\partial_{l^n}\,U_{g\,(l)}\,\psi = \partial_{l^n}\,\partial_{l^m}\,U_{g\,(l)}\,\psi,$$

holds. Now picking out the terms $\partial_{l^m}\,U_{g\,(l)}$ and $\partial_{l^n}\,U_{g\,(l)}$ in Theorem 2.1 (ii) one finds

$$\sum_{b=1}^{d} \partial_{l^m}\,(-i\,\rho_n{}^b\,(g\,(l))\,X_b\,U_{g\,(l)})\,\psi = \sum_{b=1}^{d} \partial_{l^n}\,(-i\,\rho_m{}^b\,(g\,(l))\,X_b\,U_{g\,(l)})\,\psi$$

$$\begin{aligned}&\left\{\sum_{b=1}^{d} -i\,\partial_{l^m}\,[\rho_n{}^b\,(g\,(l))]\,X_b\right.\\ &\qquad\left. - \sum_{a,\,b=1}^{d} \rho_n{}^b\,(g\,(l))\,\rho_m{}^a\,(g\,(l))\,X_a\,X_b\right\}U_{g\,(l)}\,\psi\\ &= \left\{\sum_{b=1}^{d} -i\,\partial_{l^n}\,[\rho_m{}^b\,(g\,(l))]\,X_b\right.\\ &\qquad\left. - \sum_{a,\,b=1}^{d} \rho_m{}^b\,(g\,(l))\,\rho_n{}^a\,(g\,(l))\,X_a\,X_b\right\}U_{g\,(l)}\,\psi.\end{aligned}$$

Since $U_{g\,(l)}$ leaves $\tilde{\mathbf{D}}$ invariant, one can set $\phi = U_{g\,(l)}\,\psi$ and rearranging the terms yields

$$\begin{aligned}&\sum_{f=1}^{d} \{-i\,\partial_{l^m}\,[\rho_n{}^f\,(g\,(l))] + i\,\partial_{l^n}\,[\rho_m{}^f\,(g\,(l))]\}\,X_f\,\phi\\ &\qquad = \sum_{a,\,b=1}^{d} \rho_n{}^a\,(g\,(l))\,\rho_m{}^b\,(g\,(l))\,[X_a,\,X_b]\,\phi.\end{aligned}$$

Now making use of the commutation relations $[X_a,\,X_b] = i\sum_{f=1}^{d} c_{ab}{}^f\,X_f$ this equation becomes

$$\begin{aligned}&\sum_{f=1}^{d}\left\{\partial_{l^m}\,[\rho_n{}^f\,(g\,(l)) - \partial_{l^n}\,[\rho_m{}^f\,(g\,(l))]\right.\\ &\qquad\left. + \sum_{a,\,b=1}^{d} \rho_n{}^a\,(g\,(l))\,\rho_m{}^b\,(g\,(l))\,c_{ab}{}^f\right\}X_f\,\phi = 0.\end{aligned}$$

Finally using the fact that the operators $\{X_k\}_{k=1}^d$ form a basis for g and that $\phi \in \tilde{\mathbf{D}}$ is arbitrary one concludes

$$\{\partial_{l^m} [\rho_n{}^f (g(l)) - \partial_{l^n} [\rho_m{}^f (g(l))]\} \\ = - \sum_{s,t=1}^{d} \rho_n{}^s (g(l))\, \rho_m{}^t (g(l))\, c_{st}{}^f. \tag{5}$$

Now contracting both sides of (5) with $\rho^{-1}{}_f{}^a (g(l))$ yields

$$\sum_{f=1}^{d} \{\partial_{l^m} [\rho^{-1}{}_f{}^a (g(l))]\, \rho_n{}^f (g(l)) - \partial_{l^n} [\rho^{-1}{}_f{}^a (g(l))]\, \rho_m{}^f (g(l))\} \\ = \sum_{f=1}^{d} \sum_{s,t=1}^{d} \rho_n{}^s (g(l))\, \rho_m{}^t (g(l))\, c_{st}{}^f\, \rho^{-1}{}_f{}^a (g(l)),$$

where,

$$\sum_{k=1}^{d} \partial_{l^c} [\rho_m{}^k (g(l))]\, \rho^{-1}{}_k{}^n (g(l)) = - \sum_{k=1}^{d} \partial_{l^c} [\rho^{-1}{}_k{}^n (g(l))]\, \rho_m{}^k (g(l)),$$

has been used. Finally contract both sides with $\rho^{-1}{}_k{}^m (g(l))\, \rho^{-1}{}_j{}^n (g(l))$ to obtain the desired relation,

$$\sum_{n=1}^{d} \{\partial_{l^n} [\rho^{-1}{}_j{}^a (g(l))]\, \rho^{-1}{}_k{}^n (g(l)) - \partial_{l^n} [\rho^{-1}{}_k{}^a (g(l))]\, \rho^{-1}{}_j{}^n (g(l))\} \\ = \sum_{f=1}^{d} c_{jk}{}^f (g(l))\, \rho^{-1}{}_f{}^a (g(l)).$$

(ii) The proof of (ii) is similar to the proof of (i). □

For the remainder of this section let $U_{g(l)}$ be an irreducible unitary representation of G. If $U_{g(l)}$ is square integrable one defines the set of coherent states corresponding to the Lie group G as

$$\eta^l \equiv U_{g(l)}\, \eta; \qquad \eta \in \mathbf{H} \quad \text{and} \quad \|\eta\| = 1.$$

It can be shown that these states give rise to a resolution of the identity in the form

$$I_{\mathbf{H}} = \int_G \eta^l\, (\eta^l, \cdot)\, d\mu(l),$$

where $d\mu(l)$ denotes the suitably normalized, left invariant measure of G. For a more detailed discussion of the existence of such a resolution of the identity *see* ([12], [13]). The map $\mathcal{C}_\eta : \mathbf{H} \to L^2(G, d\mu(l))$, defined for any $\psi \in \mathbf{H}$ by:

$$(\mathcal{C}_\eta \psi)(l) = \psi_\eta(l) \equiv (\eta^l, \psi),$$

yields a representation of the Hilbert space $\mathbf{H}$ by bounded, continuous, square integrable functions on the reproducing kernel Hilbert space $L^2_\eta(G, d\mu(l)) \subset L^2(G, d\mu(l))$. Clearly the map $\mathcal{C}_\eta$ is a unitary operator from $\mathbf{H}$ onto $L^2_\eta(G, d\mu(l))$. In fact $\mathcal{C}_\eta$ intertwines the representations $U_{g(l)}$ on $\mathbf{H}$ with a subrepresentation of the left regular representation $U^L_{g(l)}$ of G on $L^2_\eta(G, d\mu(l))$.

COROLLARY 2.4. – *The unitary representation $U_{g(l)}$ intertwines the operator representation $\{X_m\}_{m=1}^d$ of g on $\mathbf{H}$, with the representation of g by right and left invariant differential operators on any one of the reproducing kernel Hilbert spaces $L^2_\eta(G, d\mu(l)) \subset L^2(G, d\mu(l))$. In fact setting $p_l = (-i\,\partial_{l^1}, \ldots, -i\,\partial_{l^d})$ the following relations hold:*

(i) *Let* $\tilde{x}_k(p_l, l) \equiv \sum_{m=1}^{d} \rho^{-1}{}_k{}^m(g(l))\,p_{l^m}$, $k = 1, \ldots, d$, *then:*

$$\tilde{x}_k(p_l, l)\,U^*_{g(l)}\psi = U^*_{g(l)}\,X_k\,\psi, \qquad \forall\,\psi \in \tilde{\mathbf{D}}.$$

(ii) *Let* $\tilde{\tilde{x}}_k(p_l, l) \equiv -\sum_{m=1}^{d} \lambda^{-1}{}_k{}^m(g(l))\,p_{l^m}$, $k = 1, \ldots, d$, *then:*

$$\tilde{\tilde{x}}_k(p_l, l)\,U_{g(l)}\psi = U_{g(l)}\,X_k\,\psi, \qquad \forall\,\psi \in \tilde{\mathbf{D}}.$$

A common dense invariant domain for these differential operators on any one of the $L^2_\eta(G, d\mu(l)) \subset L^2(G, d\mu(l))$ is given by the continuous representation of $\tilde{\mathbf{D}}$, i.e. $\tilde{\mathbf{D}}_\eta \equiv \mathcal{C}_\eta(\tilde{\mathbf{D}})$.

Proof. – (i) Let $\psi \in \tilde{\mathbf{D}}$ be arbitrary then it follows from Theorem 2.1 that

$$-i\,\partial_{l^m}\,U^*_{g(l)}\psi = \sum_{c=1}^{d} \rho_m{}^c(g(l))\,U^*_{g(l)}\,X_c\,\psi, \qquad m = 1, \ldots, d.$$

Therefore, if one contracts both sides with ${\rho^{-1}}_k{}^m (g(l))$ one finds

$$\sum_{m=1}^{d} {\rho^{-1}}_k{}^m (g(l)) (-i\, \partial_{l^m})\, U^*_{g(l)}\, \psi = U^*_{g(l)}\, X_k\, \psi, \qquad k = 1, \ldots, d,$$

i.e.

$$\tilde{x}_k (p_l, l)\, U^*_{g(l)}\, \psi = U^*_{g(l)}\, X_k\, \psi, \qquad k = 1, \ldots, d.$$

Using Corollary 2.3 (i) one obtains,

$$\begin{aligned}
&[\tilde{x}_i (p_l, l), \tilde{x}_j (p_l, l)] \\
&\quad = \sum_{m,n=1} [{\rho^{-1}}_i{}^m (g(l))\, p_{l^m}, {\rho^{-1}}_j{}^n (g(l))\, p_{l^n}] \\
&\quad = i \sum_{m=1} \sum_{n=1}^{d} \{\partial_{l^n} [{\rho^{-1}}_i{}^m (g(l))]\, {\rho^{-1}}_j{}^n (g(l)) \\
&\quad\quad - \partial_{l^n} [{\rho^{-1}}_j{}^m (g(l))]\, {\rho^{-1}}_i{}^n (g(l))\} (-i\, \partial_{l^m}) \\
&\quad = i \sum_{k=1}^{d} c_{ij}{}^k \left[\sum_{m=1}^{d} {\rho^{-1}}_k{}^m (g(l))\, p_{l^m} \right] \\
&\quad = \sum_{k=1}^{d} c_{ij}{}^k\, \tilde{x}_k (p_l, l).
\end{aligned}$$

Hence, the differential operators $\{\tilde{x}_k (p_l, l)\}_{k=1}^d$ with common dense invariant domain $\tilde{\mathbf{D}}_\eta$ form a representation of g on any one of the reproducing kernel Hilbert spaces $L^2_\eta (G, d\mu(l))$.

(ii) The proof of (ii) is similar to proof of (i). □

COROLLARY 2.5. – *The differential operators* $\{\tilde{x}_k (p_l, l)\}_{k=1}^d$ *are symmetric on any one of the reproducing kernel Hilbert spaces* $L^2_\eta (G, d\mu(l))$ *and can be identified with the generators* $\{U^L (X_k)\}_{k=1}^d$ *of a subrepresentation of the left regular representation of* G *on* $L^2_\eta (G, d\mu(l))$.

Proof. – Let $\psi \in \tilde{\mathbf{D}}$ then it follows from Corollary 2.4 that

$$\tilde{x}_k (p_l, l)\, \psi_\eta (l) = (\mathcal{C}_\eta\, X_k\, \psi)(l), \qquad k = 1, \ldots, d.$$

On the other hand let $U_{g_k(t)} = \exp(-it\,X_k)$, $k = 1, \ldots, d$, be one parameter subgroups of G. Then one can also write $(\mathcal{C}_\eta\, X_k\, \psi)(l)$ as

$$\begin{aligned}
(\mathcal{C}_\eta\, X_k\, \psi)(l) &= \left(\mathcal{C}_\eta \lim_{t\to 0} \frac{U_{g_k(t)} - I}{it}\, \psi\right)(l) \\
&\doteq \lim_{t\to 0} \frac{1}{it} \left[(U_{g_k^{-1}(t)\, g(l)}\, \eta,\, \psi) - (U_{g(l)}\, \eta,\, \psi)\right] \\
&= -i \lim_{t\to 0} \frac{\psi_n(g_k^{-1}(t)\, g(l)) - \psi_\eta(g(l))}{t} \\
&= -i \lim_{t\to 0} \frac{U^L_{g_k(t)} - I}{t}\, \psi_\eta(l) \\
&= U^L(X_k)\, \psi_\eta(l), \qquad k = 1, \ldots, d
\end{aligned}$$

where the $U^L(X_k) \equiv -i\,\text{s-}\lim\limits_{t\to 0} \dfrac{U^L_{g_k(t)} - I}{t}$, $k = 1, \ldots, d$, are the generators of a subrepresentation of the left regular representation of G on $L^2_\eta(G,\, d\mu(l))$. Hence,

$$\tilde{x}_k(p_l,\, l)\, \psi_\eta(l) = (\mathcal{C}_\eta\, X_k\, \psi)(l) = U^L(X_k)\, \psi_n(l), \qquad k = 1, \ldots, d$$

since $\psi \in \tilde{\mathbf{D}}$ was arbitrary, this is true for all $\psi_\eta(l) \in \tilde{\mathbf{D}}_\eta$. Therefore, on $\tilde{\mathbf{D}}_\eta$ one can identify $\tilde{x}_k(p_l,\, l)$ with $U^L(X_k)$, *i.e.*

$$\tilde{x}_k(p_l,\, l) = U^L(X_k), \qquad k = 1, \ldots, d.$$

Clearly the operators $\tilde{x}_k(p_l,\, l)$ are symmetric on $L^2_\eta(G,\, d\mu(l))$, since the X_k are symmetric operators on $\mathbf{H}$ and since $\mathcal{C}_\eta$ is a unitary operator from $\mathbf{H}$ onto $L^2_\eta(G,\, d\mu(l))$.

Similarly one can prove that the operators $\{\tilde{\tilde{x}}_k(p_l,\, l)\}_{k=1}^d$ are symmetric and that they can be identified with the generators of a subrepresentation of the right regular representation of G on $L^2_\eta(G,\, d\mu(l))$.

3. THE REPRESENTATION INDEPENDENT PROPAGATOR FOR A COMPACT LIE GROUP

3.1. Construction of the representation independent propagator

Let U^s_g be a d_s-dimensional irreducible unitary representation of a d-dimensional, connected and simply connected, compact Lie group G on the d_s-dimensional Hilbert space $\mathbf{H}^s$. Let $X_1, \ldots, X_d$ be the irreducible, symmetric generators of the Lie group G on $\mathbf{H}^s$. Since all operators in the family $\{X_k\}_{k=1}^d$ are finite dimensional all vectors in $\mathbf{H}^s$ are analytic vectors for this family, hence this representation of the Lie algebra is integrable to a unique unitary representation of G on $\mathbf{H}^s$ (*see* [14]). Choose the following parameterization of G,

$$U^s_{g(l)} = \prod_{k=1}^{d} \exp(-il^k X_k \equiv \exp(-il^1 X_1) \ldots \exp(-il^d X_d);$$

where $l \in \mathcal{G}$ and $\hbar = 1$. Since $U^s_{g(l)}$ is irreducible every vector is cyclic, hence, let $\eta \in \mathbf{H}^s$ be an arbitrary normalized state then the coherent states for the compact Lie group G, corresponding to the irreducible unitary representation $U^s_{g(l)}$ are defined as follows

$$\eta^l = \sqrt{d_s}\, U^s_{g(l)}\, \eta.$$

The operator $O = \int \eta^l (\eta^l, \cdot)\, d\mu(l)$ commutes with all $U^s_{g(l)}$, $l \in \mathcal{G}$, therefore, one has by Schur's Lemma that $O = \lambda\, I_{\mathbf{H}^s}$. Direct calculation shows that $\lambda = 1$, hence these states give rise to the following resolution of the identify,

$$I_{\mathbf{H}^s} = \int_G \eta^l (\eta^l, \cdot)\, d\mu(l),$$

where $d\mu(l)$ is the normalized, invariant measure of G given by

$$d\mu(l) = \gamma(l) \prod_{k=1}^{d} dl^k, \tag{6}$$

where $\gamma(l) \equiv \det[\lambda_m{}^k (g(l))]/|G|$. The map $C^s_\eta : \mathbf{H}^s \to L^2(G, d\mu(l))$, defined for any $\psi \in \mathbf{H}^s$ by:

$$(C^s_\eta \psi)(l) = \psi_\eta(l) \equiv (\eta^l, \psi) = (\eta, \sqrt{d_s}\, U^{s*}_{g(l)} \psi),$$

yields a representation of the Hilbert space $\mathbf{H}^s$ by bounded, continuous, square integrable functions on the reproducing kernel Hilbert space $L^2_\eta(G, d\mu(l))$ which is a proper subspace of $L^2(G, d\mu(l))$. Because C^s_η intertwines the irreducible representation $U^s_{g(l)}$ with a subrepresentation of the left regular representation $U^L_{g(l)}$ these representations of G are unitarily equivalent. Furthermore, it follows from Corollary 2.4 (i), since all operators in the family $\{X_k\}_{k=1}^d$ are bounded, that for any $\psi \in \mathbf{H}^s$

$$\tilde{x}_k(p_l, l)(C^s_\eta \psi)(l) = (C^s_\eta X_k \psi)(l), \qquad k = 1, \ldots, d,$$

holds independent of η. Here the right invariant differential operators $\tilde{x}_k(p_l, l)$ have been defined in Corollary 2.4 (i) as

$$\tilde{x}_k(p_l, l) = \sum_{m=1}^{d} \rho^{-1}{}_k{}^m(g(l))\, p_{l^m}, \qquad k = 1, \ldots, d,$$

where $p_l = (-i\,\partial_{l^1}, \ldots, \partial_{l^d})$. Hence, the map C^s_η intertwines the representation of the Lie algebra g on $\mathbf{H}^s$, with the representation of g by right invariant differential operators on any one of the reproducing kernel Hilbert spaces $L^2_\eta(G, d\mu(l))$. To summarize $U^s_{g(l)}$ is unitarily equivalent to a subrepresentation of the left regular representation $U^L_{g(l)}$ on $L^2_\eta(G, d\mu(l))$ and the generators of G are represented by right invariant differential operators on $L^2_\eta(G, d\mu(l))$. Since G is compact the left regular representation is completely reducible into a direct sum of all irreducible unitary representations of G, where each $U^s_{g(l)}$ occurs with multiplicity d_s (*see* [10], Theorem 7.1.4), *i.e.*

$$U^L_{g(l)} = \bigoplus_{s\in\hat{G}} d_s\, U^s_{g(l)},$$

where $\hat{G} = \{U^s\}$ is the set of all inequivalent irreducible unitary representations of G. Now consider the following. Denote by $\mathcal{H}(X_k)$ the self-adjoint hamilton operator of a quantum mechanical system on $\mathbf{H}^s$, then for $U^s_{g(l)}$ the continuous representation of the solution to Schrödinger's equation, $\psi(t) = \exp[-i(t-t')\,\mathcal{H}(X_k)]\,\psi(t')$, is given on $L^2_\eta(G, d\mu(l))$ by

$$\psi_\eta(l, t) = \int K^s_\eta(l, t; l', t')\,\psi_\eta(l', t')\,d\mu(l'),$$

where,

$$\begin{aligned} K^s_\eta(l, t; l', t') &= (\eta^l, \exp[-i(t-t')\,\mathcal{H}(X_k)]\,\eta^{l'}) \\ &= C^s_\eta(\exp[-i(t-t')\,\mathcal{H}(X_k)]\,\eta^{l'})(l) \\ &= \mathcal{U}(t-t')\,C^s_\eta(\eta^{l'})(l) \\ &= \mathcal{U}(t-t')\,d_s(\eta, U^{s*}_{g(l)}\,U^s_{g(l')}\,\eta), \end{aligned}$$

where,

$$\mathcal{U}(t-t') = \exp[-i(t-t')\,\mathcal{H}(\tilde{x}_k(p_l, l))].$$

In this construction η was arbitrary, hence it holds for any $\eta \in \mathbf{H}^s$. Therefore, one can choose any orthonormal basis (ONB) $\{\phi_j\}_{j=1}^{d}$ in $\mathbf{H}^s$ and write down the following propagator

$$
\begin{aligned}
K_{\mathbf{H}^s}(l, t; l', t') = \sum_{j=1}^{d_s} K_{\phi_j}(l, t; l', t') &= \mathcal{U}(t-t')\, d_s \operatorname{tr}(U^{s*}_{g(l)} U^s_{g(l')}) \\
&= \mathcal{U}(t-t')\, d_s\, \chi_s(g^{-1}(l)\, g(l'))
\end{aligned} \tag{7}
$$

LEMMA 3.1. – *The propagator $K_{\mathbf{H}^s}(l, t; l', t')$ given in* (7) *correctly propagates all elements of any reproducing kernel Hilbert space $L^2_\eta(G, d\mu(l))$, associated with the irreducible unitary representation $U^s_{g(l)}$.*

Proof. – Let $\eta \in \mathbf{H}^s$ be arbitrary, then for $\psi_n(l', t') \in L^2_\eta(G, d\mu(l))$ one has

$$
\begin{aligned}
&\int K_{\mathbf{H}^s}(l, t; l', t')\, \psi_n(l', t')\, d\mu(l') \\
&= \int \mathcal{U}(t-t')\, d_s\, \chi_s(g^{-1}(l)\, g(l')\, \psi_n(l', t')\, d\mu(l') \\
&= \sum_{j=1}^{d_s} d_s \mathcal{U}(t-t') \int (\phi_j, U^{s*}_{g(l)} U^s_{g(l')} \phi_j)(\eta^{l'}, \psi(t'))\, d\mu(l') \\
&= \sum_{j,n=1}^{d_s} d_s(\eta, \phi_n) \mathcal{U}(t-t') \int (\phi_j, U^{s*}_{g(l)} U^s_{g(l')} \phi_j) \\
&\quad \times (\sqrt{d_s}\, U^s_{g(l')} \phi_n, \psi(t'))\, d\mu(l') \\
&= \mathcal{U}(t-t') \left(\sqrt{d_s}\, U^s_{g(l)} \sum_{n=1}^{d_s} (\phi_n, \eta)\, \phi_n, \psi(t') \right) \\
&= (\mathcal{C}^s_\eta \exp[-i(t-t')\mathcal{H}(X_k)]\, \psi(t'))(l) \\
&= \psi_n(l, t).
\end{aligned}
$$

Therefore,

$$
\psi_\eta(l, t) = \int K_{\mathbf{H}^s}(l, t; l', t')\, \psi_\eta(l', t')\, d\mu(l'), \quad \forall\, \eta \in \mathbf{H}^s
$$

i.e. the propagator $K_{\mathbf{H}^s}(l, t; l', t')$ propagates the elements of any reproducing kernel Hilbert space $L^2_\eta(G, d\mu(l))$ correctly. □

Hence, we have succeeded in constructing for the irreducible representation $U^s_{g(l)}$ a propagator $K_{\mathbf{H}^s}$ that correctly propagates each element of an arbitrary reproducing kernel Hilbert space $L^2_\eta(G, d\mu(l))$. Using the fact that the set $\{\phi_j\}_{j=1}^{d_s}$ is an ONB one can rewrite the group character $\chi_s(g^{-1}(l)\, g(l'))$ in terms of the matrix elements $D^s_{ij}(l) \equiv (\phi_i, U^s_{g(l)}\, \phi_j)$ of U^s as follows,

$$\chi_s(g^{-1}(l)\, g(l')) = \sum_{i,j=1}^{d_s} \overline{D^s_{ij}}(l)\, D^s_{ij}(l'). \tag{8}$$

Therefore, $K_{\mathbf{H}^s}$ can be written alternatively as

$$K_{\mathbf{H}^s}(l,\, t;\, l',\, t') = \mathcal{U}(t-t') \sum_{i,j=1}^{d_s} d_s\, \overline{D^s_{ij}}(l)\, D^s_{ij}(l'). \tag{9}$$

In this construction the unitary irreducible representation $U^s_{g(l)}$ was arbitrary, hence one can introduce such a propagator for each in equivalent unitary representation of G, *i.e.* one can write down the following propagator for the left regular representation $U^L_{g(l)}$ of G on $L^2(G,\, d\mu(l))$

$$K(l,\, t;\, l',\, t') = \sum_{s\in\hat{G}} K_{\mathbf{H}^s} = \mathcal{U}(t-t') \sum_{s\in\hat{G}} \sum_{i,j=1}^{d_s} d_s\, \overline{D^s_{ij}}(l)\, D^s_{ij}(l').$$

Now it is well known from the Peter-Weyl Theorem that the functions

$$\sqrt{d_s}\, D^s_{ij}(l), \qquad s \in \hat{G}, \qquad 1 \le i,\, j \le d_s,$$

form a complete orthonormal system (ONS) in $L^2(G,\, d\mu(l))$. The completeness relation of this ONS is given by

$$\sum_{s\in\hat{G}} \sum_{i,j=1}^{d_s} d_s\, \overline{D^s_{ij}}(l)\, D^s_{ij}(l') = \delta(l;\, l'),$$

where the sum holds as a weak sum and $\delta(l;\, l')$ is defined as

$$\delta(l;\, l') \equiv \frac{1}{\gamma(l)} \prod_{k=1}^{d} \delta(l^k - l'^k).$$

Therefore, we find as our final result

$$K(l,\, t; l',\, t') = \exp\left[-i\,(t-t')\, \mathcal{H}(\tilde{x}_k(p_l,\, l))\right] \delta(l;\, l'). \tag{10}$$

This propagator, which is a tempered distribution, is clearly independent of the fiducial vector and the representation chosen for the basic kinematical variables $\{X_k\}_{k=1}^d$. The maximal set of test functions for this propagator is given by $C(G)$, the set of all continuous functions on G. Hence, we have shown the first part of the following Theorem:

THEOREM 3.2. – *The propagator $K(l, t; l', t')$ in (10) is a propagator for the left regular representation of G on $L^2(G, d\mu(l))$, that correctly propagates all elements of any reproducing kernel Hilbert space $L^2_\eta(G, d\mu(l))$, associated with an arbitrary irreducible unitary representation $U^s_{g(l)}$ of G, $s \in \hat{G}$.*

Proof. – To prove the second part of Theorem 3.2, let $U^{s'}_{g(l)}$ and $\eta \in \mathbf{H}^{s'}$ be arbitrary, then for any $\psi_\eta(l)$ in some $L^2_\eta(G, d\mu(l))$, associated with $U^{s'}_{g(l)}$ one clearly has that $\psi_\eta(l) \in C(G)$. Hence, one can write

$$\begin{aligned}\int K(l, t; l', t')\, \psi_\eta(l', t')\, d\mu(l') &= \sum_{s\in\hat{G}} \int K_{\mathbf{H}^s}(l, t; l', t') \\ &\quad \times (\sqrt{d_{s'}}\, U^{s'}_{g(l')}\eta,\ \psi(t'))\, d\mu(l') \\ &= \int K_{\mathbf{H}^{s'}}(l, t; l', t')\, \psi_\eta(l', t')\, d\mu(l') \\ &= \psi_\eta(l, t).\end{aligned}$$

The second equality holds since the elements of different representation spaces are mutually orthogonal, hence, only the s'-term remains. In the last step Lemma 3.1 has been used. □

Hence, we have constructed a propagator that is *representation independent.*

3.2. Path integral formulation of the representation independent propagator

From (10) one easily finds that the representation independent propagator is a weak solution to Schrödinger's equation, *i.e.*

$$i\,\partial_t K(l, t; l', t') = \mathcal{H}(\tilde{x}_1(p_l, l), \ldots, \tilde{x}_d(p_l, l))\, K(l, t; l', t'). \quad (11)$$

Taking in (10) the limit $t \to t'$ one finds the following initial value problem

$$\begin{aligned} &i\,\partial_t\, K(l,\, t;\, l',\, t') = \mathcal{H}\,(\tilde{x}_1\,(p_l,\, l),\, \ldots,\, \tilde{x}_d\,(p_l,\, l))\, K\,(l,\, t;\, l',\, t'),\\ &\lim_{t \to t'} K\,(l,\, t;\, l',\, t') = \delta\,(l;\, l'). \end{aligned} \tag{12}$$

One now interprets the initial value problem (12) as a Schrödinger equation appropriate to d *separate and independent canonical* degrees of freedom. Hence, $l^1, \ldots, l^d$ are viewed as d "coordinates", and one is looking at the irreducible Schrödinger representation of a special class of d-variable Hamilton operators, ones where the classical Hamiltonian is restricted to have the form $\mathcal{H}\,(\check{x}_1\,(p,\, l),\, \ldots,\, \check{x}_d\,(p,\, l))$, instead of the most general form $\mathcal{H}\,(p,\, l) = \mathcal{H}\,(p_1,\, \ldots,\, p_d,\, l^1,\, \ldots,\, l^d)$. In fact the differential operators given in Corollary 2.4 (i) are elements of the left regular enveloping algebra of the d-dimensional Schrödinger representation on $L^2\,(G,\, d\mu\,(l))$. Based on this interpretation a standard phase-space path integral solution may be given for the representation independent propagator between sharp Schrödinger states. In particular, it follows for continuous and differentiable paths that

$$\begin{aligned} K\,&(l'',\, t'';\, l',\, t')\\ &= \mathcal{M} \int \exp\left\{ i \int \left[\sum_{m=1}^{d} p_m\, \dot{l}^m - \mathcal{H}\,(\check{x}_1\,(p,\, l),\, \ldots,\, \check{x}_d\,(p,\, l)) \right] dt \right\}\\ &\quad \times \prod_{t \in [t',\, t'']} dp\,(t)\, dl\,(t), \end{aligned} \tag{13}$$

where "p_1", ..., "p_d" denote "momenta" conjugate to the "coordinates" "l^1", ..., "l^d". Note that the Hamiltonian has been used in the special form discussed above and that its arguments are given by the following functions

$$\check{x}_k\,(p,\, l) = \sum_{m=1}^{d} \rho^{-1}{}_k{}^m\,(g\,(l))\, p_m, \qquad k = 1,\, \ldots,\, d.$$

The integration over the "coordinates" is restricted to the label space $\mathcal{G}$. Since G is compact the momenta conjugate to the restricted range or periodic "coordinates" are discrete variables. Hence, the notation $\int \prod dp\,(t)$ is properly to be understood as sums rather than integrals.

Before we can turn to a regularized lattice prescription for the representation independent propagator, we first have to spell out what is meant by a Schrödinger representation on $L^2\,(G,\, d\mu\,(l))$. Let $\{\mathcal{L}^k,\, P_{\mathcal{L}^k}\}_{k=1}^d$

be a family of symmetric operators satisfying the following canonical commutation relations (CCR),

$$[\mathcal{L}^a, \mathcal{L}^b] = 0; \qquad [P_{\mathcal{L}^a}, P_{\mathcal{L}^b}] = 0;$$
$$[\mathcal{L}^a, P_{\mathcal{L}^b}] = i\, \delta^a{}_b\, I, \qquad a, b = 1, \ldots, d,$$

with generalized eigenkets

$$\mathcal{L}^a \,|\, l^1, \ldots, l^d\rangle = l^a \,|\, l^1, \ldots, l^d\rangle, \qquad a = 1, \ldots, d,$$
$$P_{\mathcal{L}^a} \,|\, p_1, \ldots, p_d\rangle = p_a \,|\, p_1, \ldots, p_d\rangle, \qquad a = 1, \ldots, d,$$

normalized as follows

$$\langle l''^1, \ldots, l''^d \,|\, l'^1, \ldots, l'^d\rangle = \delta\,(l''; l'),$$
$$\langle p''_1, \ldots, p''_d \,|\, p'_1, \ldots, p'_d\rangle = \prod_{k=1}^{d} \delta_{p''_k\, p'_k},$$

and giving rise to the following resolutions of the identity

$$\int |\, l\rangle \langle l \,|\, d\mu\,(l) = I,$$
$$\sum |\, p\rangle \langle p \,| = I,$$

where $|\, l\rangle \equiv |\, l^1, \ldots, l^d\rangle$ and $|\, p\rangle \equiv |\, p_1, \ldots, p_d\rangle$. Observe that on $L^2\,(G,\, d\mu\,(l))$ these operators can be represented as

$$\mathcal{L}^a \doteq l^a \qquad \text{and} \qquad P_{\mathcal{L}^a} \doteq \hat{p}_{l^a} = -i\left[\partial_{l^a} + \frac{1}{2}\,\Gamma_a(l)\right], \tag{14}$$

where $\mathbf{D}_S = C_o^\infty\,(G)$, the set of infinitely differentiable functions with compact support on G, is chosen as the common dense invariant domain of these operators. Here $\Gamma_a\,(l)$ is defined as $\Gamma_a\,(l) \equiv \partial_{l^a} \ln \gamma\,(l)$ and where $\gamma\,(l)$ is given in (6). It is straightforward to show that these operators satisfy the CCR, are symmetric with respect to the innerproduct on $L^2\,(G, d\mu\,(l))$, and that the $\delta\,(l''; l')$-normalized generalized eigenfunctions of $\hat{p}_l$ are given by

$$\langle l \,|\, p\rangle = \frac{1}{\sqrt{K\,\gamma\,(l)}}\, \exp\left(i \sum_{k=1}^{d} p_k\, l^k\right), \tag{15}$$

where K denotes the normalization constant. We call (14) a d-dimensional Schrödinger representation on $L^2\,(G,\, d\mu\,(l))$. Furthermore, the differential operators $\{\tilde{x}_k(p_l, l)\}_{k=1}^d$ become:

LEMMA 3.3. – *Using $\hat{p}_l$ given in* (14) *the right invariant differential operators $\{\tilde{x}_k(p_l, l)\}_{k=1}^d$ defined in Corollary* 2.4 *(i) can be written as:*

$$\tilde{x}_k(\hat{p}_l, l) = \sum_{m=1}^d \frac{1}{2}[\rho^{-1}{}_k{}^m(g(l))\,\hat{p}_{l^m} + \hat{p}_{l^m}\,\rho^{-1}{}_k{}^m(g(l))],$$
$$k = 1, \ldots, d. \tag{16}$$

Proof. – Since $p_{l^a} = \hat{p}_{l^a} + (i/2)\,\Gamma_a(l)$, $a = 1, \ldots, d$, the differential operators $\{\tilde{x}_k(p_{l^a}, l^a)\}_{k=1}^d$ become after substitution of this expression

$$\begin{aligned}\tilde{x}_k(\hat{p}_{l^a} + (i/2)\,\Gamma_a(l), l^a) &= \sum_{m=1}^d \rho^{-1}{}_k{}^m(g(l))\left[\hat{p}_{l^m} + \frac{i}{2}\Gamma_m(l)\right]\\ &= \sum_{m=1}^d \frac{1}{2}[\rho^{-1}{}_k{}^m(g(l))\,\hat{p}_{l^m} + \rho^{-1}{}_k{}^m(g(l))\,\hat{p}_{l^m}]\\ &\quad + \frac{i}{2}\rho^{-1}{}_k{}^m(g(l))\,\Gamma_m(l).\end{aligned}$$

Using $[\rho^{-1}{}_k{}^m(g(l)), \hat{p}_{l^m}] = i\,\partial_{l^m}\,\rho^{-1}{}_k{}^m(g(l))$ and the definition of $\Gamma_m(l)$ one finds

$$\begin{aligned}\tilde{x}_k(\hat{p}_{l^a} + (i/2)\,\Gamma_a(l), l^a) &= \sum_{m=1}^d \frac{1}{2}[\rho^{-1}{}_k{}^m(g(l))\,\hat{p}_{l^m} + \hat{p}_{l^m}\,\rho^{-1}{}_k{}^m(g(l))]\\ &\quad + \frac{i}{2\gamma(l)}\sum_{m=1}^d \partial_{l^m}[\rho^{-1}{}_k{}^m(g(l))\,\gamma(l)].\end{aligned}$$

Now since the operators $\tilde{x}_k(p_{l^a}, l^a)$ are essentially self-adjoint on any one of the reproducing kernel Hilbert-spaces $L^2_\eta(G, d\mu(l))$ (*cf.* Corollary 2.5) and $\gamma(l) \neq 0$ one finds that

$$\sum_{m=1}^d \partial_{l^m}[\rho^{-1}{}_k{}^m(g(l))\,\gamma(l)] = 0, \qquad k = 1, \ldots, d,$$

and therefore,

$$\tilde{x}_k(\hat{p}_l, l) = \sum_{m=1}^d \frac{1}{2}[\rho^{-1}{}_k{}^m(g(l))\,\hat{p}_{l^m} + \hat{p}_{l^m}\,\rho^{-1}{}_k{}^m(g(l))],$$
$$k = 1, \ldots, d. \quad \square$$

Since $\mathbf{D}_S$ is the invariant domain of the Schrödinger representation this shows that the differential operators $\{\tilde{x}_k(\hat{p}_l, l)\}_{k=1}^d$ are elements of the left regular enveloping algebra of the d-dimensional Schrödinger representation on $L^2(G, d\mu(l))$.

Now following standard procedures (e.g., *see* [15]) one can give the representations-independent propagator the following regularized lattice prescription.

PROPOSITION 3.4. – *The representation independent propagator in* (10) *can be given the following d-dimensional lattice phase-space path integral representation:*

$$K(l'', t''; l', t') = \frac{1}{\sqrt{\gamma(l'')\gamma(l')}} \lim_{\varepsilon \to 0} \int \cdots \int \sum_{\{p\}} \times \exp\left\{ i \sum_{j=0}^{N} [p_{j+1/2} \cdot (l_{j+1} - l_j) - \varepsilon \mathcal{H}(\check{x}_k(p_{j+1/2}; l_{j+1}, l_j))] \right\} \times \prod_{j=1}^{N} dl_j^1 \ldots dl_j^d, \tag{17}$$

where $l_{N+1} = l''$, $l_0 = l'$, *and* $\varepsilon = (t'' - t')/(N+1)$. *The sum* $\sum_{\{p\}}$ *appearing in* (17) *is defined as*

$$\sum_{\{p\}} \equiv \frac{1}{K} \sum_{p_{N+1/2}} \frac{1}{K} \sum_{p_{N-1/2}} \cdots \frac{1}{K} \sum_{p_{3/2}} \frac{1}{K} \sum_{p_{1/2}},$$

where K *is the normalization constant defined in* (15) *and the sums are over the spectrum of* $\hat{p}_l$. *The arguments of the Hamiltonian in* (17) *are given by the following functions:*

$$\check{x}_k(p_{j+1/2}; l_{j+1}, l_j) = \sum_{m=1}^{d} \frac{\rho^{-1}{}_k{}^m(g(l_{j+1})) + \rho^{-1}{}_k{}^m(g(l_j))}{2} p_{j+1/2^m}, \qquad k = 1, \ldots, d.$$

Proof. – To obtain the lattice phase-space path integral in (17) one can proceed as follows. Observe that

$$\begin{aligned} K(l'', t''; l', t') &= \langle l'' \,|\, \exp[-i(t''-t')\,\mathcal{H}(\tilde{x}_k(P_{\mathcal{L}^a}, \mathcal{L}^a))] \,|\, l' \rangle \\ &= \langle l'' \,|\, \mathrm{e}^{-i\varepsilon\mathcal{H}} \ldots \mathrm{e}^{-i\varepsilon\mathcal{H}} \,|\, l' \rangle \\ &= \int \ldots \int \prod_{j=0}^{N} \langle l_{j+1} \,|\, \mathrm{e}^{-i\varepsilon\mathcal{H}} \,|\, l_j \rangle \prod_{j=1}^{N} \gamma(l_j)\, dl_j^1 \ldots dl_j^d, \end{aligned}$$

where $l'' = l_{N+1}$, $l' = l_0$, and $\varepsilon \equiv (t''-t')/(N+1)$. This expression holds for any N, and therefore, it holds as well in the limit $N \to \infty$ or $\varepsilon \to 0$ *i.e.*,

$$\begin{aligned} &K(l'', t''; l', t') \\ &\quad = \lim_{\varepsilon \to 0} \int \ldots \int \prod_{j=0}^{N} \langle l_{j+1} \,|\, \mathrm{e}^{-i\varepsilon\mathcal{H}} \,|\, l_j \rangle \prod_{j=1}^{N} \gamma(l_j)\, dl_j^1 \ldots dl_j^d. \end{aligned} \tag{18}$$

Hence, one has to evaluate $\langle l_{j+1} \,|\, \mathrm{e}^{-i\varepsilon\mathcal{H}} \,|\, l_j \rangle$ for small ε. For small ε one can make the approximation

$$\begin{aligned} &\langle l_{j+1} \,|\, \mathrm{e}^{-i\varepsilon\mathcal{H}(\tilde{x}_k(P_{\mathcal{L}^a}, \mathcal{L}^a))} \,|\, l_j \rangle \\ &\quad \approx \langle l_{j+1} \,|\, \{1 - i\varepsilon\mathcal{H}(\tilde{x}_k(P_{\mathcal{L}^a};\, l_{j+1},\, l_j))\} \,|\, l_j \rangle \\ &\quad = \sum_{p_{j+1/2}} \langle l_{j+1} \,|\, p_{j+1/2} \rangle \langle p_{j+1/2} \,|\, \{1 - i\varepsilon\mathcal{H}(\tilde{x}_k(P_{\mathcal{L}^a};\, l_{j+1},\, l_j))\} \,|\, l_j \rangle \\ &\quad = \sum_{p_{j+1/2}} \langle l_{j+1} \,|\, p_{j+1/2} \rangle \langle p_{j+1/2} \,|l_j \rangle \{1 - i\varepsilon\mathcal{H}(\check{x}_k(p_{j+1/2};\, l_{j+1},\, l_j))\} \\ &\quad \approx \sum_{p_{j+1/2}} \langle l_{j+1} \,|\, p_{j+1/2} \rangle \overline{\langle l_j \,|p_{j+1/2} \rangle}\, \mathrm{e}^{-i\varepsilon\mathcal{H}(\check{x}_k(p_{j+1/2}; l_{j+1}, l_j))} \end{aligned}$$

valid to first order in ε, where

$$\check{x}_k(p_{j+1/2};\, l_{j+1},\, l_j) = \sum_{m=1}^{d} \frac{\rho^{-1}{}_k{}^m(g(l_{j+1})) + \rho^{-1}{}_k{}^m(g(l_j))}{2}\, p_{j+1/2^m}$$

$$k = 1, \ldots, d.$$

Substituting the right hand side of (15) into this expression yields

$$\langle l_{j+1} \,|\, e^{-i\varepsilon \mathcal{H}} \,|\, l_j \rangle = \gamma^{-1/2}(l_{j+1})\, \gamma^{-1/2}(l_j)\, \frac{1}{K} \times \sum_{p_{j+1/2}} \exp\{i\,[p_{j+1/2} \cdot (l_{j+1} - l_j) - \varepsilon \mathcal{H}(\check{x}_k(p_{j+1/2};\, l_{j+1},\, l_j))]\} \tag{19}$$

Now inserting (19) into (18) yields

$$K(l'',\, t'';\, l',\, t') = \frac{1}{\sqrt{\gamma(l'')\,\gamma(l')}} \lim_{\varepsilon \to 0} \int \cdots \int \sum_{\{p\}} \times \exp\left\{i \sum_{j=0}^{N} [p_{j+1/2} \cdot (l_{j+1} - l_j) - \varepsilon \mathcal{H}(\check{x}_k(p_{j+1/2};\, l_{j+1},\, l_j))]\right\} \times \prod_{j=1}^{N} dl_j^1 \ldots dl_j^d,$$

which is the desired expression. □

Observe that even though the group manifold is a *curved* space the regularized lattice expression for the representation independent propagator save for the prefactor $1/\sqrt{\gamma(l'')\,\gamma(l')}$ has the conventional from a lattice phase-space path integral on a d-dimensional *flat* space. Also note that the lattice expression for the representation independent propagator exhibits the correct time reversal symmetry.

Before leaving this section it is pointed out that the path integral construction of the representation independent propagator makes no *explicit* use of the ONS $\sqrt{d_s}\, D_{ij}{}^{s}(l)$, $s \in \hat{G}$ and $i,\, j = 1,\, \ldots,\, d_s$, in $L^2(G,\, d\mu(l))$ whose existence is guaranteed by the Peter-Weyl Theorem, but merely uses the fact that it *exits* and is *complete.*

4. EXAMPLE

While the Peter-Weyl Theorem assures that the ONS $\sqrt{d_s}\, D_{ij}^{s}(l)$, $s \in \hat{G}$ and $i,\, j = 1,\, \ldots,\, d_s$ exists and is complete the construction of such a set is frequently a difficult task. The functions $\sqrt{d_s}\, D_{ij}^{s}(l)$ are known only for a limited class of groups and will now be constructed for SU(2). It turns out that this is an exercise in harmonic analysis. We will now explicitly describe the maximal set of commuting operators in $L^2(SU(2), d\mu(\theta, \phi, \xi))$ we take as their common dense domain the set $C_0^\infty(SU(2))$. Since SU(2) is a rank one group, there exists one two-sided-invariant operator C_1 in the center of the enveloping algebra $\mathcal{E}$ of SU(2). Moreover, since SU(2) is compact the maximal set of commuting right (left) invariant differential operators in the right (left) invariant enveloping algebra $\mathcal{E}^R(\mathcal{E}^L)$, can be associated with the Casimir operator of the maximal subgroup U(1) of SU(2). For SU(2) in Euler angle parameterization denote by $U^s_{g(\theta,\phi,\xi)}$ an arbitrary unitary irreducible representation of SU(2) then the operators $\{\tilde{s}_k\}_{k=1}^3$ defined in Corollary 2.4 (i) are given by:

$$\begin{aligned}
&\tilde{s}_1(p_\theta, p_\phi, p_\xi, \theta, \phi, \xi) \\
&\quad = i \sin\phi\,\partial_\theta + i\cot\theta\,\cos\phi\,\partial_\phi - i\cos\phi\csc\theta\,\partial_\xi, \\
&\tilde{s}_2(p_\theta, p_\phi, p_\xi, \theta, \phi, \xi) \\
&\quad = -i\cos\phi\,\partial_\theta + i\cot\theta\,\sin\phi\,\partial_\phi - i\sin\phi\csc\theta\,\partial_\xi, \\
&\tilde{s}_3(p_\theta, p_\phi, p_\xi, \theta, \phi, \xi) = -i\,\partial_\phi.
\end{aligned} \tag{20}$$

By Corollary 2.5 these operators can be identified with the generators of a subrepresentation of the left regular representation of SU(2), (*i.e.* belong to the right invariant Lie algebra of SU(2)). Similarly the operators $\{\tilde{\tilde{s}}_k\}_{k=1}^3$ defined in Corollary 2.4 (ii) are given by:

$$\begin{aligned}
&\tilde{\tilde{s}}_1(p_\theta, p_\phi, p_\xi, \theta, \phi, \xi) \\
&\quad = i\sin\xi\,\partial_\theta - i\csc\theta\,\cos\xi\,\partial_\phi + \cot\theta\cos\xi\,\partial_\xi, \\
&\tilde{\tilde{s}}_2(p_\theta, p_\phi, p_\xi, \theta, \phi, \xi) \\
&\quad = i\cos\xi\,\partial_\theta + i\csc\theta\,\sin\xi\,\partial_\phi - i\cot\theta\sin\xi\,\partial_\xi, \\
&\tilde{\tilde{s}}_3(p_\theta, p_\phi, p_\xi, \theta, \phi, \xi) = i\,\partial_\xi,
\end{aligned} \tag{21}$$

and can be identified with the generators of a subrepresentation of the right regular representation, (*i.e.* belong to the left invariant Lie algebra of SU(2)). From (20) and (21) we easily identify the Casimir operators of U(1) as

$$A_1 = i\,\partial_\phi \qquad B_1 = i\,\partial_\xi. \tag{22}$$

For the Casimir operator of SU(2) one finds

$$C_1 = -(1 - z^2)\,\partial_z^2 + 2\,z\,\partial_z - \frac{1}{1 - z^2}\,(\partial_\phi^2 - 2\,z\,\partial_\phi\,\partial_\xi + \partial_\xi^2), \tag{23}$$

where $z = \cos\theta$ and the identity $-\sin\theta\, \partial_{\cos\theta} = \partial_\theta$ has been used. Since, $\tilde{s}_1$, $\tilde{s}_2$, and $\tilde{s}_3$ are symmetric operators C_1 is positive definite, furthermore since C_1 is in the center of the enveloping algebra $\mathcal{E}$ and $U^s_{g(\theta,\phi,\xi)}$ is irreducible, C_1 is a multiple of the identity on any one of the reproducing kernel Hilbert spaces $L^2_\eta(SU(2))$ associated with the irreducible representation $U^s_{g(\theta,\phi,\xi)}$. This multiple of the identity is commonly denoted by $s(s+1)$, therefore,

$$C_1 = s(s+1)\, I_{L^2_\eta(SU(2))} \tag{24}$$

We shall now determine the matrix elements $D^s_{mn}(\theta, \phi, \xi)$ of the irreducible representations $U^s_{g(\theta,\phi,\xi)}$, where $s = 0, 1/2, 1, \ldots$ and $-s \leq m$, $n \leq s$, as the common eigenfunctions of the operators A_1, B_1, C_1. Equations (22) and (23) suggest that the common eigenfunctions of the operators A_1, B_1, and C_1 are of the form

$$D^s_{mn}(\theta, \phi, \xi) = e^{-i(m\phi + n\xi)}\, P^s_{mn}(\cos\theta).$$

Using this form of $D^s_{mn}(\theta, \phi, \xi)$ in (24) one finds:

$$\begin{aligned} &-(1-z^2)\frac{d^2}{d^2 z} P^s_{mn}(z) + 2z\frac{d}{dz} P^s_{mn}(z) \\ &+ \frac{1}{1-z^2}(m^2 + n^2 - 2mnz) = s(s+1)\, P^s_{mn}(z). \end{aligned}$$

It can be shown that the solution to this ordinary differential equation is uniquely determined if m and n are simultaneously integers or semi-integers (*see* [8], p. 138). In fact the functions $P^s_{mn}(z)$, which are the Wigner functions, are given by

$$\begin{aligned} P^s_{mn}(z) = &\frac{i^{m-n}}{2^m}\sqrt{\frac{(s-m)!\,(s+m)!}{(s-n)!\,(s+n)!}} \\ &\times (1-z)^{\frac{m-n}{2}}(1+z)^{\frac{m+n}{2}}\, P^{(m-n,\, m+n)}_{s-m}(z), \end{aligned}$$

where $P^{(m-n,\,m+n)}_{s-m}(z)$ are Jacobi polynomials, (*see* [8], p. 125). Also observe that $\overline{P^s_{mn}(z)} = (-1)^{n-m}\, P^s_{mn}(z)$. Therefore, one finds for the matrix elements of the irreducible representation $U^s_{g(\theta,\phi,\xi)}$ the following:

$$D^s_{mn}(\theta, \phi, \xi) = e^{-i(m\phi+n\xi)}\, P^s_{mn}(\cos\theta),$$

as pointed out above these functions form a complete ONS on $L^2(SU(2), d\mu(\theta, \phi, \xi))$. The completeness relation for this ONS takes the form

$$\sum_{s \in \hat{G}} \sum_{m, n=-s}^{s} (2s+1) \overline{D^s_{mn}(\theta'', \phi'', \xi'')} D^s_{mn}(\theta', \phi', \xi')$$
$$= \frac{16\pi^2}{\sin\theta''} \delta(\theta'' - \theta') \delta(\phi'' - \phi') \delta(\xi'' - \xi').$$

By equation (10) the representation independent propagator for SU(2) is then found to be:

$$K(\theta'', \phi'', \xi'', t''; \theta', \phi', \xi', t')$$
$$= 16\pi^2 \exp[-i(t'' - t') \mathcal{H}(\tilde{s}_1(p_l, l), \tilde{s}_2(p_l, l), \tilde{s}_3(p_l, l))]$$
$$\times \frac{1}{\sin\theta''} \delta(\theta'' - \theta') \delta(\phi'' - \phi') \delta(\xi'' - \xi'),$$

where $l = (\theta, \phi, \xi)$ and $p_l = (-i\,\partial_\theta, -i\,\partial_\phi, -i\,\partial_\xi)$. By Proposition 3.4 the regularized lattice phase-space path integral representation for the representation independent propagator for SU(2) is given by

$$K(\theta'', \phi'', \xi'', t''; \theta', \phi', \xi', t')$$
$$= \frac{16\pi^2}{\sqrt{\sin\theta'' \sin\theta'}} \lim_{\varepsilon \to 0} \int \cdots \int \sum_{\{\alpha, \beta, \gamma\}}$$
$$\times \exp\left\{ i \sum_{j=0}^{N} [\alpha_{j+1/2}(\theta_{j+1} - \theta_j) + \beta_{j+1/2}(\phi_{j+1} - \phi_j) \right.$$
$$\left. + \gamma_{j+1/2}(\xi_{j+1} - \xi_j) - \varepsilon \mathcal{H}(\check{s}_k(p_{j+1/2}; l_{j+1}, l_j))] \right\}$$
$$\times \prod_{j=1}^{N} d\theta_j\, d\phi_j\, d\xi_j,$$

where,

$$\check{s}_1(p_{j+1/2}; l_{j+1}, l_j) = -\frac{1}{2}(\sin\phi_{j+1} + \sin\phi_j)\alpha_{j+1/2}$$
$$- \frac{1}{2}(\cot\theta_{j+1}\cos\phi_{j+1} + \cot\theta_j \cos\phi_j)\beta_{j+1/2}$$
$$+ \frac{1}{2}(\cos\phi_{j+1}\csc\theta_{j+1} + \cos\phi_j \csc\theta_j)\gamma_{j+1/2},$$

$$\begin{aligned}\check{s}_2(p_{j+1/2};\, l_{j+1},\, l_j) &= \frac{1}{2}(\cos\phi_{j+1/2} + \cos\phi_j)\,\alpha_{j+1/2}\\ &\quad - \frac{1}{2}(\cot\theta_{j+1}\sin\phi_{j+1} + \cot\theta_j\sin\phi_j)\,\beta_{j+1/2}\\ &\quad + \frac{1}{2}(\csc\theta_{j+1}\sin\phi_{j+1} + \csc\theta_j\sin\phi_j)\,\gamma_{j+1/2},\\ \check{s}_3(p_{j+1/2};\, l_{j+1},\, l_j) &= \beta_{j+1/2}.\end{aligned}$$

As an example let us calculate this propagator for the following Hamilton operator

$$\mathcal{H}(\tilde{s}_1(p_l,\, l),\, \tilde{s}_2(p_l,\, l),\, \tilde{s}_3(p_l,\, l)) = \frac{1}{2I}\,C_1(\theta,\, \phi,\, \xi),$$

where $C_1(\theta,\, \phi,\, \xi)$ is the Casimir operator of SU(2) given in (23). Hence,

$$\begin{aligned}&K(\theta'',\, \phi'',\, \xi'',\, t'';\, \theta',\, \phi',\, \xi',\, t')\\ &= 16\pi^2\exp\left[-i\,\frac{(t''-t')}{2I}\,C_1(\theta'',\, \phi'',\, \xi'')\right]\\ &\quad\times\frac{1}{\sin\theta''}\,\delta(\theta''-\theta')\,\delta(\phi''-\phi')\,\delta(\xi''-\xi')\\ &= 16\pi^2\exp\left[-i\,\frac{(t''-t')}{2I}\,C_1(\theta'',\, \phi'',\, \xi'')\right]\\ &\quad\times\sum_{s\in\hat{G}}\,\sum_{m,\,n=-s}^{s}(2s+1)\,\overline{D^s_{mn}(\theta'',\, \phi'',\, \xi'')}\,D^s_{mn}(\theta',\, \phi',\, \xi')\\ &= 16\pi^2\sum_{s\in\hat{G}}(2s+1)\exp\left[-i\,\frac{(t''-t')}{2I}\,s(s+1)\right]\\ &\quad\times\chi_s(g^{-1}(\theta'',\, \phi'',\, \xi'')\,g(\theta',\, \phi',\, \xi')),\end{aligned}$$

where equations (8) and (24) have been used. Here $\chi_s(g)$ denotes the character for the representation $U^s_{g(\theta,\,\phi,\,\xi)}$, *i.e.* if one denotes the Euler angles of the element $g^{-1}(\theta'',\, \phi'',\, \xi'')\,g(\theta',\, \phi',\, \xi')$ by $(\bar{\theta},\, \bar{\phi},\, \bar{\xi})$ one finds:

$$\chi_s(g(\bar{\theta},\, \bar{\phi},\, \bar{\xi})) = \sum_{m=-s}^{s} e^{-im(\bar{\phi}+\bar{\xi})}\,P^s_{mn}(\cos\bar{\theta}).$$

Observe that, the character of the group can be expressed as a function of a single variable as follows. It is well known that the character as a function of the group is constant on conjugacy classes, *i.e.* for any two elements g and g_1 one has

$$\chi_s(g_1\, g g_1^{-1}) = \chi_s(g).$$

Therefore, to show that $\chi_s(g)$ is a function of one variable, it is sufficient to show that the conjugacy clases of SU(2) can be labeled by a single parameter. As is well known from linear algebra any unitary unimodular 2×2 matrix g can be written as $g = g_1 \gamma g_1^{-1}$, where $g_1 \in SU(2)$ and γ is of the following diagonal matrix

$$\gamma = \begin{pmatrix} e^{i(\Gamma/2)} & 0 \\ 0 & e^{-i(\Gamma/2)} \end{pmatrix}$$

Furthermore, among all matrices equivalent to g there exists only one other diagonal matrix γ' obtained from γ by complex conjugation. Therefore, each conjugacy class of elements of SU(2) is labeled by one parameter Γ, ranging from $-2\pi \leq \Gamma \leq 2\pi$ and where Γ and $-\Gamma$ give the same class. Hence, the characters $\chi_s(g)$ can be regarded as functions of one variable Γ that varies between 0 and 2π. The geometrical meaning of the parameter Γ is that it is equal to the angle of rotation corresponding to the matrix g. In terms of the Euler angles $(\theta'', \phi'', \xi'')$ and (θ', ϕ', ξ') Γ is given by

$$\begin{aligned}\Gamma = \arccos[&\cos(\theta'' - \theta')\cos(\phi'' - \phi')\cos(\xi'' - \xi') \\ &- \cos(\theta'' + \theta')\sin(\phi'' - \phi')\sin(\xi'' - \xi')].\end{aligned} \tag{25}$$

One can derive an explicit formula for $\chi_s(g)$ as a function of Γ. Note that the matrix $U^s_{\gamma(0,\Gamma,0)}$ that corresponds to $\gamma \in SU(2)$ is given by the diagonal matrix of rank $2s+1$, having diagonal elements $e^{-ia\Gamma}$, $-s \leq a \leq s$. Now let $g = g_1 \gamma g_1^{-1}$, then

$$\chi_s(g) = \operatorname{tr}(U^s_{\gamma(0,\Gamma,0)} = \sum_{a=-s}^{s} e^{-ia\Gamma} = \frac{\sin(s+1/2)\Gamma}{\sin\Gamma/2},$$

Hence, the group character can be written as

$$\chi_s(g^{-1}(\theta'', \phi'', \xi'')\, g(\theta', \phi', \xi')) = \frac{\sin(s+1/2)\Gamma}{\sin\Gamma/2},$$

where Γ is given in (25). Therefore, one finds for the representation independent propagator

$$
\begin{aligned}
K(\theta'', \phi'', \xi'', t''; \theta', \phi', \xi', t') &= \frac{16\pi^2}{\sqrt{\sin\theta''\,\sin\theta'}} \lim_{\varepsilon\to 0} \int \cdots \int \sum_{\{\alpha,\beta,\gamma\}} \\
&\times \exp\Big\{ i \sum_{j=0}^{N} [\alpha_{j+1/2}(\theta_{j+1}-\theta_j) + \beta_{j+1/2}(\phi_{j+1}-\phi_j) \\
&\qquad + \gamma_{j+1/2}(\xi_{j+1}-\xi_j) - \varepsilon\, C_1(\theta_{j+1}, \phi_{j+1}, \xi_{j+1}; \theta_j, \phi_j, \xi_j)] \Big\} \\
&\times \prod_{j=1}^{N} d\theta_j\, d\phi_j\, d\xi_j \\
&= 16\pi^2 \sum_{s\in\hat{G}} (2s+1) \exp\left[-i\,\frac{(t''-t')}{2I}\, s(s+1) \right] \frac{\sin(s+1/2)\Gamma}{\sin\Gamma/2}.
\end{aligned}
$$

This result agrres with the one found in [16] which was obtained by different methods.

5. CLASSICAL LIMIT

5.1. Classical limit

In spite of the fact that the regularized lattice phase-space path integral representation for the representation independent propagator has been constructed by interpreting the initial value problem (12) as a Schrödinger equation for d *separate and independent canonical* degrees of freedom, it should, however, be true that the classical limit for the representation independent propagator refers to the degree(s) of freedom associated with the Lie group G. In particular it is shown, that in the case of coherent sates for G, this is true, since the classical equations of motion obtained from the action functional for the representation independent propagator imply the classical equations of motion for the most general classical action functional for the coherent state path integral for G.

It is known that any compact Lie group is the direct product of its center and a finite number of simple subgroups (*cf.* [10], Theorem 3.8.2) and that all irreducible unitary representations of compact Lie groups are finite dimensional (*cf.* [17], Lemma IV.3.2). Subsequently we consider the classical limit of semisimple compact Lie groups, *i.e.* compact Lie groups which have a discrete center. This includes the physically important examples of SU(2) and SU(3). Let us denote by $\hat{X}_j = \hbar X_j$, $j = 1, \ldots, d$, the physical operators, then for finite $\hbar$ the most general action functional appropriate to the d-dimensional semisimple compact Lie group G is given by (*see* [6]):

$$\begin{aligned} I &= \int \left[i\hbar \left\langle \eta(l), \frac{d}{dt}\eta(l) \right\rangle - \left\langle \eta(l), \mathcal{H}(\hat{X}_1, \ldots, \hat{X}_d)\, \eta(l) \right\rangle \right] dt \\ &= \int \sum_{k, m=1}^{d} \lambda_m{}^k (g(l))\, \dot{l}_m \langle \eta, \hat{X}_k \eta \rangle \\ &\quad - \left\langle \eta, \mathcal{H}\left(\sum_{b=1}^{d} U_n{}^b (l)\, \hat{X}_b \right) \eta \right\rangle dt. \end{aligned} \tag{26}$$

Let us assume that the semisimple compact Lie group G we are considering has rank n, *i.e.* there exist n self-commuting operators H_r, $r = 1, \ldots, n$, that form the Cartan subalgebra H of the Lie algebra L associated with the Lie group G. Moreover, let us denote by $m = (m_1, \ldots, m_n)$ the highest weight of the finite dimensional irreducible unitary representation $(U^s_{g(l)}, \mathbf{H}_s)$ of G. Using the non-degenerate Cartan metric tensor $g_{ls} \equiv \sum_{j, k=1}^{d} c_{lk}{}^i\, c_{sj}{}^k$ we construct the Casimir operator

$$C_2 = - \sum_{l, s=1}^{d} g^{ls}\, \hat{X}_l\, \hat{X}_s$$

which satisfies

$$\begin{aligned} [C_2, \hat{X}_k] &= - \sum_{l, s=1}^{d} g^{ls} ([\hat{X}_l, \hat{X}_k]\, \hat{X}_s + \hat{X}_l\, [\hat{X}_s, \hat{X}_k]) \\ &= - \sum_{l, s=1}^{d} g^{ls} \left(\sum_{t=1}^{d} c_{lk}{}^t\, \hat{X}_t\, \hat{X}_s + \sum_{j=1}^{d} c_{sk}{}^i\, \hat{X}_l\, \hat{X}_j \right) \\ &= - \sum_{l, t=1}^{d} c_{lk}{}^t\, \hat{X}_t\, \hat{X}^l + \sum_{j, s=1}^{d} c_{sk}{}^j\, \hat{X}^s\, \hat{X}_j \\ &= - \left[\sum_{l, s=1}^{d} (c_{lks} + c_{skl}) \right] \hat{X}^s\, \hat{X}^l = 0, \end{aligned}$$

since $c_{rst} = \sum_{l=1}^{d} c_{rs}{}^{l}\, g_{lt}$ is totally antisymmetric under any interchange of its indices. Since the Cartan metric tensor is symmetric for semisimple Lie algebra it can be diagonalized, *i.e.* $g_{ls} = -\delta_{ls}$. Hence, without loss of generality we can assume that the Casimir operator is given by:

$$C_2 = \sum_{l=1}^{d} \hat{X}_l^2.$$

The operator C_2 can be written in the standard Cartan-Weyl basis of the Lie algebra L as follows:

$$C_2 = \sum_{r=1}^{n} \hbar^2\, H_r^2 + \sum_{\alpha} \hbar^2\, E_\alpha\, E_{-\alpha}, \tag{27}$$

where $\sum_{\alpha}$ denotes the sum over the nonzero roots of the Lie algebra L. When this operator acts on the highest weight vector ω_m of the irreducible unitary representation $U^s_{g\,(l)}$, one obtains, because of the condition $E_\alpha\, \omega_m = 0$ for positive roots,

$$\begin{aligned} C_2\, \omega_m &= \left\{ \sum_{r=1}^{n} \hbar^2\, m_r^2 + \sum_{\alpha>0} \hbar^2\, [E_\alpha,\, E_{-\alpha}] \right\} \omega_m \\ &= \hbar^2 \left[\sum_{r=1}^{n} m_r^2 + \sum_{\alpha>0} \sum_{j=1}^{n} \alpha^j\, H_j \right] \omega_m \\ &= \hbar^2 \left[\sum_{r=1}^{n} \left(m_r^2 + \sum_{\alpha>0} \alpha_r\, m_r \right) \right] \omega_m. \end{aligned}$$

It is well known that every irreducible unitary representation is characterized by the components of the highest weight m. By Schur's lemma every invariant operator in the carrier space of the irreducible unitary representation $U^s_{g\,(l)}$ is proportional to the identity operator, *i.e.* $C_2 = \lambda\, I_{\mathbf{H}^s}$, where λ is given in terms of the components of the highest weight m, in particular we have that

$$C_2 = \hbar^2 \sum_{r=1}^{n} (m_r^2 + 2\, g_r\, m_r)\, I_{\mathbf{H}^s}$$

where

$$g = \frac{1}{2} \sum_{\alpha>0} \alpha.$$

We now consider the classical limit of the action functional given in (26). Since we want to deal with general fiducial vectors we have to consider

$$\langle \eta,\, \hat{X}_k\, \eta \rangle = \hbar\, \chi_{k_\eta}, \qquad k = 1,\, \ldots,\, d,$$

where the χ_{k_η}, $k = 1, \ldots, d$, are real numbers given by $\chi_{k_\eta} \equiv \langle \eta,\, X_k\, \eta \rangle$. We insist on vanishing dispersion as $\hbar \to 0$ and $m \to \infty$, namely, that

$$\lim_{\substack{\hbar \to 0 \\ m \to \infty}} \sum_{r=1}^{d} \langle \eta,\, (\hat{X}_k - \langle \eta,\, \hat{X}_k\, \eta \rangle)^2\, \eta \rangle = 0, \tag{28}$$

where the limit $\hbar \to 0$ and $m \to \infty$ is taken in such a way that the product $\mu = \hbar\, m$ stays finite. We denote the set of fiducial vectors that satisfy (28) by $\mathcal{F}$. If we choose for the fiducial vector the highest weight vector of the irreducible representation $U^s_{g\,(l)}$ then we find

$$\begin{aligned} &\lim_{\substack{\hbar \to 0 \\ m \to \infty}} \sum_{k=1}^{d} \langle \omega_m,\, (\hat{X}_k - \langle \omega_m,\, \hat{X}_k\, \omega_m \rangle)^2\, \omega_m \rangle \\ &= \lim_{\substack{\hbar \to 0 \\ m \to \infty}} \left(\langle \omega_m,\, C_2\, \omega_m \rangle - \sum_{k=1}^{d} \langle \omega_m,\, \hat{X}_k\, \omega_m \rangle^2 \right) \\ &= \lim_{\substack{\hbar \to 0 \\ m \to \infty}} \left(\hbar^2 \sum_{r=1}^{n} (m_r^2 + 2\, g_r\, m_r) - \hbar^2 \sum_{r=1}^{n} m_r^2 \right) \\ &= \lim_{\substack{\hbar \to 0 \\ m \to \infty}} \hbar^2 \sum_{r=1}^{n} 2\, g_r\, m_r = 0. \end{aligned}$$

Hence, the highest weight vector satisfies (28), and therefore, the set $\mathcal{F}$ contains at least one vector. Since for fixed $l \in \mathcal{L}$, $U^s_{g\,(l)}$ is a unitary operator on $\mathbf{H}_s$ there exists a l_η such that

$$\eta = U^s_{g\,(l_\eta)}\, \omega_m.$$

Therefore, we find

$$\chi_{k_\eta} = \langle \eta,\, X_k\, \eta \rangle = \sum_{t=1}^{d} U_k{}^t\, (l_\eta)\, \langle \omega_m,\, X_k\, \omega_m \rangle.$$

Only the terms for which $\langle \omega_m,\, X_k\, \omega_m \rangle \neq 0$ contribute to this sum, hence we find

$$\chi_{k_\eta} = \sum_{r \in I} U_k{}^r\, (l_\eta)\, m_{i\,(r)},$$

where $I = \{r \in \{1, \ldots, d\} : X_r \in H\}$ and $m_{i\,(r)}$ denotes the component of the highest weight m for which $X_r = H_i$. For finite $\hbar$ the term that represents the classical Hamiltonian in the coherent state propagator for the Lie group G is given by

$$H(l) = \langle \eta(l), \mathcal{H}(\hat{X}_1, \ldots, \hat{X}_d)\,\eta\rangle = \left\langle \eta, \mathcal{H}\left(\sum_{m=1}^{d} U_k{}^m(l)\,\hat{X}_m\right)\eta\right\rangle.$$

Therefore, if we now take the limit $\hbar \to 0$ and $m \to \infty$ in the above mentioned sense, then the classical Hamiltonian is given by

$$\lim_{\substack{\hbar\to 0\\ m\to\infty}} H(l) = \mathcal{H}\left(\sum_{b=1}^{d} U_k{}^b(l)\,v_b\right),$$

where

$$v_k = \sum_{r\in I} U_k{}^r(l_\eta)\,\mu_{i\,(r)}, \qquad k = 1, \ldots, d.$$

Hence, the classical limit of the action functional given in (26) becomes

$$\begin{aligned} I_{cl} &= \lim_{\substack{\hbar\to 0\\ m\to\infty}} \int \left[i\hbar \left\langle \eta(l), \frac{d}{dt}\eta(l)\right\rangle - \langle \eta(l), \mathcal{H}(\hat{X}_1, \ldots, \hat{X}_d)\,\eta(l)\rangle\right] dt \\ &= \int \Bigg[\sum_{k,\,m=1}^{d} \lambda_m{}^k(g(l))\,\dot{l}^m\,v_k \\ &\quad - \mathcal{H}\left(\sum_{b=1}^{d} U_1{}^b(l)\,v_b, \ldots n \sum_{b=1}^{d} U_d{}^b(l)\,v_b\right)\Bigg]\,dt, \end{aligned} \tag{29}$$

Extremal variation of this action functional, with respect to the independent labels l^b, holding the end points fixed, yields the equations of motion

$$\sum_{b,\,s=1}^{d} v_s\,\{\partial_{l^c}\lambda_b{}^s(g(l)) - \partial l^b \lambda_c{}^s(g(l))\}\,\dot{l}^b = \sum_{a,\,f=1}^{d} \mathcal{H}^a\,\partial_{l^c}[U_a{}^f(l)]\,v_f, \tag{30}$$

where $\mathcal{H}^a$ denotes the partial derivative of $\mathcal{H}$ with respect to the a-th argument $a = 1, \ldots, d$.

Observe that the generally nonvanishing values of $v_1, \ldots, v_d$ are *vestiges of the coherent state representation induced by* η *that remain even after the limit* $\hbar \to 0$.

5.2. Classical limit of the representation independent propagator

For the representation independent propagator the classical action functional is identified as (*see* Proposition 3.4)

$$
\begin{aligned}
I_{cl} &= \int \left[\sum_{j=1}^{d} p_j \dot{l}^j - \mathcal{H}(\check{x}_1(p, l), \ldots, \check{x}_d(p, l))\right] dt \\
&= \int \left[\sum_{j=1}^{d} p_j \dot{l}^j - \mathcal{H}\right. \\
&\qquad \left. \times \left(\sum_{j=1}^{d} \rho^{-1}{}_1{}^j(g(l))\, p_j, \ldots, \sum_{j=1}^{d} \rho^{-1}{}_d{}^j(g(l))\, p_j\right)\right] dt. \qquad (31)
\end{aligned}
$$

Extremal variation of this action functional holding the end points fixed yields the equations of motion

$$\dot{l}^b = \sum_{a=1}^{d} \mathcal{H}^a\, \rho^{-1}{}_a{}^b(g(l)), \qquad (32)$$

$$\dot{p}_c = -\sum_{a,\, j=1}^{d} \mathcal{H}^a\, \partial_{l^c}\left[\rho^{-1}{}_a{}^j(g(l))\right] p_j. \qquad (33)$$

Now substitute $\mathcal{H}^s = \sum_{f=1}^{d} \rho_f{}^s(g(l))\, \dot{l}^f$ into (33), then contract both sides with $\lambda^{-1}{}_h{}^c(g(l))$, and find

$$
\begin{aligned}
\sum_{c=1}^{d} \lambda^{-1}{}_h{}^c(g(l))\, \dot{p}_c &= \sum_{c,\, s=1}^{d} \sum_{f,\, j=1}^{d} \dot{l}^f\, \lambda^{-1}{}_b{}^c(g(l)) \\
&\quad \times \partial_{l^c}\left[\rho_f{}^s(g(l))\right] \rho^{-1}{}_s{}^j(g(l))\, p_j, \qquad (34)
\end{aligned}
$$

where

$$\sum_{s=1}^{d} \rho_f{}^s(g(l))\, \partial_{l^c}\left[\rho^{-1}{}_s{}^j(g(l))\right] = \sum_{s=1}^{d} -\partial_{l^c}\left[\rho_f{}^s(g(l))\right] \rho^{-1}{}_s{}^j(g(l)).$$

has been used.

CLAIM. – *The following relation holds:*

$$\sum_{c,\,f,\,j,\,s=1}^{d} \dot{l}^f \, \lambda^{-1}{}_h{}^c (g(l)) \, \partial_{l^c} [\rho_f{}^s (g(l))] \, \rho^{-1}{}_s{}^j (g(l)) \, p_j$$

$$= - \sum_{j,\,m=1}^{d} \partial_{l^m} [\lambda^{-1}{}_h{}^j (g(l))] \dot{l}^m \, p_j. \tag{35}$$

Proof. – To establish equation (35) it is sufficient to show that

$$\partial_{l^m} \lambda^{-1}{}_h{}^b (g(l))$$

$$= - \sum_{f,\,s=1}^{d} \lambda^{-1}{}_h{}^f (g(l)) \, \partial_{l^f} [\rho_m{}^s (g(l))] \, \rho^{-1}{}_s{}^b (g(l)) \tag{36}$$

holds. Using Corollary 2.2 Equation(36) can be rewritten as

$$\sum_{t=1}^{d} \partial_{l^m} [U^{-1}{}_h{}^t (l) \, \rho^{-1}{}_t{}^b (g(l))]$$

$$= - \sum_{f,\,s,\,t=1}^{d} U^{-1}{}_h{}^t (l) \, \rho^{-1}{}_t{}^f (g(l)) \, \partial_{l^f} [\rho_m{}^s (g(l))] \, \rho^{-1}{}_s{}^b (g(l)).$$

This equation can be simplified as follows

$$\sum_{h,\,j,\,t=1}^{d} \rho_n{}^j (g(l)) \, U_j{}^h (l) \, \partial_{l^m} [U^{-1}{}_h{}^t (l) \, \rho^{-1}{}_t{}^b (g(l))]$$

$$= - \sum_{s=1}^{d} \partial_{l^n} [\rho_m{}^s (g(l))] \, \rho^{-1}{}_s{}^b (g(l))$$

$$\sum_{h,\,j=1}^{s} U^{-1}{}_h{}^f (l) \, \partial_{l^m} [\rho_n{}^j (g(l)) \, U_j{}^h (l)] = \partial_{l^n} [\rho_m{}^f (g(l))].$$

After carrying out the indicated partial differentiation of the product and rearranging the terms one finds

$$\partial_{l^m} [\rho_n{}^f (g(l))] - \partial_{l^n} [\rho_m{}^f (g(l))]$$

$$= - \sum_{j,\,h=1}^{d} \rho_n{}^j (g(l)) \, \partial_{l^m} [U_j{}^h (l)] \, U^{-1}{}_h{}^f (l).$$

Next using $\partial_{l^m} U_j{}^h(l) = \rho_m{}^s(g(l))\, c_{js}{}^n\, U_n{}^h(l)$, which along the same lines as Theorem 3.2 (ii), equation (36) finally becomes

$$\partial_{l^m}[\rho_n{}^f(g(l))] - \partial_{l^n}[\rho_m{}^f(g(l))] = -\sum_{j,\,s=1}^{d} \rho_n{}^j(g(l))\,\rho_m{}^s(g(l))\,c_{jd}{}^f,$$

which is equation (5) and therefore, establishes (35). □

If one inserts (35) into (34) one finds

$$\frac{d}{dt}\left[\sum_{j=1}^{d} \lambda^{-1}{}_h{}^j(g(l))\,p_j\right] = 0. \tag{37}$$

Therefore, one can introduce a set of integration constants, $c_1, \ldots, c_d$, such that

$$p_j = \sum_{m=1}^{d} \lambda_j{}^m(g(l))\,c_m. \tag{38}$$

If one substitutes this form of p_j into (32) and (33), one finds the following set of $2d$ equations

$$\dot{l}^b = \sum_{a=1}^{d} \mathcal{H}^a\left(\sum_{s=1}^{d} U_1{}^s(l)\,c_s, \ldots, \sum_{s=1}^{d} U_d{}^s(l)\,c_s\right)\rho^{-1}{}_a{}^b(g(l)),$$

$$\begin{aligned}\sum_{s=1}^{d} \partial_t[\lambda_c{}^s(g(l))]\,c_s = -\sum_{a=1}^{d}\sum_{j,m=1}^{d}\\ \times\,\mathcal{H}^a\left(\sum_{s=1}^{d} U_1{}^s(l)\,c_s, \ldots, \sum_{s=1}^{d} U_d{}^s(l)\,c_s\right)\\ \times\,\partial_{l^c}[\rho^{-1}{}_a{}^j(g(l))]\,\lambda_j{}^m(g(l))\,c_m.\end{aligned}$$

Carrying out the indicated partial differentiation with respect to time these equations become

$$\dot{l}^b = \sum_{a=1}^{d} \mathcal{H}^a\,\rho^{-1}{}_a{}^b(g(l)), \tag{39}$$

$$\begin{aligned}&\sum_{b,\,s=1}^{d} \partial_{l^b}[\lambda_c{}^s(g(l))]\,\dot{l}^b\,c_s\\ &\quad = -\sum_{a=1}^{d}\sum_{j,m=1}^{d} \mathcal{H}^a\,\partial_{l^c}[\rho^{-1}{}_a{}^j(g(l))]\,\lambda_j{}^m(g(l))\,c_m.\end{aligned} \tag{40}$$

Next contract (39) with $\sum_{s=1}^{d} \partial_{l^c} [\lambda_b{}^s (g(l))] c_s$, which yields

$$\sum_{b,s=1}^{d} \partial_{l^c} [\lambda_b{}^s (g(l))] \dot{l}^b c_s$$

$$= \sum_{a=1}^{d} \sum_{b,s=1}^{d} \mathcal{H}^a \rho^{-1}{}_a{}^b (g(l)) \partial_{l^c} [\lambda_b^s (g(l))] c_s, \tag{41}$$

$$\sum_{b,s=1}^{d} \partial_{l^b} [\lambda_c{}^s (g(l))] \dot{l}^b c_s$$

$$= -\sum_{a=1}^{d} \sum_{j,m=1}^{d} \mathcal{H}^a \partial_{l^c} [\rho^{-1}{}_a{}^j (g(l))] \lambda_j^m (g(l)) c_m. \tag{42}$$

Subtracting (42) from (41) yields the final result

$$\sum_{b,s=1}^{d} c_s \{\partial_{l^c} \lambda_b{}^s (g(l)) - \partial_{l^b} \lambda_c{}^s (g(l))\} \dot{l}^b$$

$$= \sum_{a,f=1}^{d} \mathcal{H}^a \partial_{l^c} [U_a{}^f (l)] c_f, \tag{43}$$

where Corollary 2.2 has been used. Among all possible allowed values of $c_1, \ldots, c_d$ are those that coincide with $v_1, \ldots, v_d$ for an arbitrary fiducial vector. Hence, the above equations can be identified with the equations of motion obtained from the most general action functional for the coherent state propagator for G [*see* Eq. (30)]. Therefore, the set of classical equations of motion for the representation independent propagator implies the set of classical equations of motion appropriate to the most general coherent state propagator for G. Thus one finds that the set of solutions of the representation independent classical equations of motion appropriate to the representation independent propagator for G includes *every possible solution* of the classical equations of motion appropriate to the most general coherent state propagator for G. Hence, we have shown the following Proposition:

PROPOSITION 5.1. – *Let G be a connected and simply connected, real compact Lie group. If $\eta \in \mathcal{F}$, then the equations of motion obtained from the action functional of the representation independent propagator imply the equations of motion obtained from the most general classical action functional for the coherent state propagator for G.*