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SHARP BOUNDS ON THE NUMBER OF RESONANCES FOR SYMMETRIC SYSTEMS

by

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ABSTRACT. – We prove for a large class of first order systems that the number of resonances in the disk $|z| \leq r$ is $O(r^n)$, $n \geq 3$ odd being the space dimension.

RÉSUMÉ. – Nous prouvons que, pour une grande classe de systèmes du premier ordre, le nombre de résonances dans le disque $|z| \leq r$ est $O(r^n)$ pour $n \geq 3$, si n est la dimension de l'espace.

During the last ten years several approaches have been developed to prove sharp upper bounds on the number of resonances (scattering poles) for compactly supported perturbations of the Laplacian (*see* [2], [3], [5] à [8]). The simplest one (*see* [5], [6], [8]) is based on the nice properties of the kernel of the operator $R_0(z) - R_0(-z)$, $R_0(z)$ being the outgoing resolvent of the free Laplacian in $\mathbf{R}^n$. Unfortunately, these approaches do no longer work if one considers perturbations of other operators as for example first order systems of nonconstant multiplicities (*see* [4]) or the hyperbolic Laplacian (*see* [1]). The purpose of this note is to present another approach allowing to improve the bound obtained in [4] in the case of systems to the sharp one. Also, it seems to me that it could be applied successfully to the hyperbolic case in order to improve the bound obtained in [1].

Consider in $\mathbf{R}^n$, $n \geq 3$ odd, a first order matrix-valued differential operator of the form $\sum_{j=1}^{n} A_j^0 D_{x_j}$, A_j^0 being constant Hermitian $d \times d$ matrices, and denote by G_0 its selfadjoint realization on $H_0 = L^2(\mathbf{R}^n; \mathbf{C}^d)$. Suppose that the matrix $A(\xi) = \sum_{j=1}^{n} A_j^0 \xi_j$, $\xi \in \mathbf{R}^n \backslash 0$, is invertible for all ξ, *i.e.* the operator G_0 is an elliptic one. Note that in general the eigenvalues of $A(\xi)$ are continuous functions of ξ and may be of nonconstant multiplicity. As a consequence they may not be smooth functions, which makes impossible to use the methods in [3], [5], [6], [8] to obtain sharp bounds on the number of resonances associated to compactly supported perturbations of G_0.

Let $\Omega \subset \mathbf{R}^n$ be an open domain with compact complement and smooth boundary. Consider in Ω the operator $\sum_{j=1}^{n} A_j(x) D_{x_j} + B(x)$, where $A_j(x) \in C^1(\Omega; \mathbf{C}^d)$, $B(x) \in C^0(\Omega; \mathbf{C}^d)$. Suppose that $A_j(x) = A_j^0$, $B(x) = 0$ for $|x| \geq \rho$ with some $\rho > 0$ such that $\mathbf{R}^n \backslash \Omega \subset \{x \in \mathbf{R}^n : |x| \leq \rho\}$. Impose such boundary conditions that this operator admits an unique elliptic closed extension (denoted by G) on the Hilbert space $H = L^2(\Omega; \mathbf{C}^d)$ with a nonempty resolvent set. Thus, without loss of generality we can suppose that there exists z_0 with $\operatorname{Im} z_0 < 0$ such that the resolvent $R(z) = (G - z)^{-1} \in \mathcal{L}(H, H)$ is well defined in an open neighbourhood $\Lambda \subset \{\operatorname{Im} z < 0\}$ of z_0. Moreover, as we shall see later on (*see* also [6]) the cutoff resolvent $R_\chi(z) = \chi (G - z)^{-1} \chi$ admits a meromorphic continuation from Λ to the entire complex plane $\mathbf{C}$, $\chi \in C_0^\infty(\mathbf{R}^n)$ being such that $\chi = 1$ for $|x| \leq \rho + 1$. The poles of this continuation are called resonances and the multiplicity of a resonance $\lambda \in \mathbf{C}$ is defined as the rank of the residue of $R_\chi(z)$ at $z = \lambda$. As shown in [6], this definition is independent of the choice of the cutoff function χ provided that $\chi = 1$ for $|x| \leq \rho$. Denote by $N(r)$ the number of the resonances of G, counted with their multiplicities, in $\{z \in \mathbf{C} : |z| \leq r\}$. Our main result is the following theorem.

THEOREM. – *Under the above assumptions, the counting function $N(r)$ satisfies the bound*

$$N(r) \leq C r^n + C, \tag{1}$$

with some constant $C > 0$ independent of r.

Proof. – Denote by $R_0(z)$ the outgoing free resolvent of G_0 defined for Im $z < 0$ by

$$R_0(z) = -i \int_0^\infty e^{-itz} e^{itG_0} dt.$$

If $\chi \in C_0^\infty(\mathbf{R}^n)$, by the Huygens principle we have $\chi e^{itG_0} \chi = 0$ for $t \geq T$ with some $T > 0$ depending on the support of χ, and hence

$$\chi R_0(z) \chi = -i \chi \int_0^T e^{-itz} e^{itG_0} dt \chi$$

defines an entire family satisfying the estimate

$$\|\chi R_0(z) \chi\| \leq e^{C\langle z \rangle}, \qquad \forall z \in \mathbf{C}, \tag{2}$$

where $\|\,.\,\|$ denotes the norm in $\mathcal{L}(H_0, H_0)$, $\langle z \rangle = 1 + |z|$.

Fix now the cutoff function $\chi \in C_0^\infty(\mathbf{R}^n)$ so that $\chi = 1$ for $|x| \leq \rho + 1$. Choose functions $\chi_1, \chi_2 \in C_0^\infty(\mathbf{R}^n)$, such that $\chi_1 = 1$ for $|x| \leq \rho + 1/3$, $\chi_1 = 0$ for $|x| \geq \rho + 1/2$, $\chi_2 = 1$ for $|x| \leq \rho + 2/3$, $\chi_2 = 0$ for $|x| \geq \rho + 5/6$. In precisely the same way as in [6] one obtains the representation

$$R_\chi(z)(1 - K(z)) = \tilde{K}(z) \quad \text{for } z \in \Lambda, \tag{3}$$

where $K(z) = K_1(z) + K_2(z)$,

$$K_1(z) = ([\chi_1, G_0] R_0(z) \eta - [\chi_1, G_0] R_0(z_0) \eta) K_3,$$

$$K_2(z) = (z - z_0) \chi_2 R(z_0) \chi,$$

$$\tilde{K}(z) = R_\chi(z_0) + (1 - \chi_1)(\chi R_0(z) \eta - \chi R_0(z_0) \eta) K_3,$$

$$K_3 = (1 - \chi_2) \chi + [\chi_2, G_0] R(z_0) \chi,$$

$\eta \in C_0^\infty(\mathbf{R}^n)$ being such that $\eta = 0$ for $|x| \leq \rho + 5/12$ or $|x| \geq \rho + 2$, $\eta = 1$ for $\rho + 11/24 \leq |x| \leq \rho + 3/2$. Denote by $\tilde{\chi}_1$ the characteristic function of $\{x \in \mathbf{R}^n : |x| \leq \rho + 1/3\}$ and let $\chi_1^\varepsilon \in C_0^\infty(\mathbf{R}^n)$ be a family of functions such that $\chi_1^\varepsilon = 1$ for $|x| \leq \rho + 1/3$, $\chi_1^\varepsilon = 0$ for $|x| \geq \rho + 1/2$, and $\chi_1^\varepsilon \to \tilde{\chi}_1$ as $\varepsilon \to 0$. Replacing the function χ_1 in the above representation by χ_1^ε gives a new one of the form

$$R_\chi(z)(1 - K^\varepsilon(z)) = \tilde{K}^\varepsilon(z) \quad \text{for } z \in \Lambda. \tag{4}$$

Combining (3) and (4) leads to

$$R_\chi(z)(1 - P^\varepsilon(z)) = \tilde{P}^\varepsilon(z) \quad \text{for } z \in \Lambda, \tag{5}$$

where $P^\varepsilon = K K_1^\varepsilon + K_2$, $\tilde{P}^\varepsilon = \tilde{K} K_1^\varepsilon + \tilde{K}^\varepsilon$. Observe now that $[\chi_1^\varepsilon, G_0] \to Q(x) \delta_S(x)$ as $\varepsilon \to 0$, where $S = \{x \in \mathbf{R}^n : |x| = \rho + 1/3\}$,

$Q \in C^\infty(S;\, \mathbf{C}^d)$. Hence, as $\varepsilon \to 0$, $K K_1^\varepsilon$ tends to an operator P_1 with a kernel of the form

$$P_1(x,\, y) = \int_S N(x,\, w)\, M(w,\, y)\, dw,$$

where $N(x,\, w)$ denotes the kernel of $K(z)$ and $M(w,\, y)$ denotes the kernel of the operator

$$M(z) = (Q\,\gamma\, R_0(z)\,\eta - Q\,\gamma\, R_0(z_0)\,\eta)\, K_3,$$

γ denotes the restriction on S. Hence

$$P_1(z) = N(z)\, M(z), \tag{6}$$

with $N \in \mathcal{L}(L^2(S;\, \mathbf{C}^d),\, H)$, $M \in \mathcal{L}(H,\, L^2(S;\, \mathbf{C}^d))$.

Since the operator G is elliptic, we have that $K_2(z)$ is an entire family of compact operators whose characteristic values satisfy the estimate

$$\mu_j(K_2(z)) \leq C\,\langle z\rangle\, j^{-1/n}, \qquad \forall\, z \in \mathbf{C}, \quad \forall\, j, \tag{7}$$

with a constant $C > 0$ independent of z and j. In particular, (7) shows that $K_2(z)^{n+1}$ is of trace class. To study $P_1(z)$ we need the following lemma.

LEMMA 1. – *Let* $\psi,\, \eta \in C_0^\infty(\mathbf{R}^n)$ *satisfy* $\operatorname{supp}\psi \cap \operatorname{supp}\eta = \emptyset$. *Then*

$$\|\psi\, \partial_x^\alpha\, R_0(z)\,\eta\| \leq (C\,m + C\,|z|)^{|\alpha|}\, e^{C\,|z|} \quad \text{for} \quad |\alpha| \leq m, \quad \forall\, z \in \mathbf{C}, \tag{8}$$

for any integer $m \geq 1$ *with a constant* $C > 0$ *independent of* α, m *and* z.

Proof. – Since $\operatorname{supp}\psi \cap \operatorname{supp}\eta = \emptyset$, by the finite speed of propagation there exists $T_1 > 0$ so that $\psi\, \partial_x^\alpha\, e^{it\,G_0}\,\eta = 0$ for $0 \leq t \leq T_1$. On the other hand, from the Huygens principle, there exists $T_2 > T_1$ so that $\psi\, \partial_x^\alpha\, e^{it\,G_0}\,\eta = 0$ for $t \geq T_2$. Choose now a function $\varphi(t) \in C_0^\infty(\mathbf{R}^+)$ such that $\varphi = 1$ in a neighbourhood of the interval $[T_1,\, T_2]$, $\varphi = 0$ for $t \geq T_2 + 1$, and

$$\|\partial_t^k\, \varphi\|_\infty \leq (C\,m)^k \quad \text{for } k \leq m. \tag{9}$$

Hence,

$$\begin{aligned}
\|\psi\, \partial_x^\alpha\, R_0(z)\,\eta\| &\leq C_1 \left\| \partial_x^\alpha \int_{-\infty}^{+\infty} \varphi(t)\, e^{-itz}\, e^{it\,G_0}\, dt \right\| \\
&\leq C^{|\alpha|+1} \left\| G_0^{|\alpha|} \int_{-\infty}^{+\infty} \varphi(t)\, e^{-itz}\, e^{it\,G_0}\, dt \right\| \\
&= C^{|\alpha|+1} \left\| \int_{-\infty}^{+\infty} \partial_t^{|\alpha|}\, (\varphi(t)\, e^{-itz})\, e^{it\,G_0}\, dt \right\| \\
&\leq C^{|\alpha|+1}\, (T_2 + 1)\, \|\partial_t^{|\alpha|}\, (\varphi(t)\, e^{-itz})\|_\infty.
\end{aligned} \tag{10}$$

On the other hand, we have for $k \leq m$;

$$\partial_t^k (\varphi(t)\, e^{-itz}) = \sum_{j=0}^{k} \binom{k}{j} (\partial_t^j \varphi)(\partial_t^{k-j} e^{-itz})$$
$$= \sum_{j=0}^{k} \binom{k}{j} (\partial_t^j \varphi)(-iz)^{k-j} e^{-itz},$$

and hence, in view of (9),

$$\|\partial_t^k (\varphi(t)\, e^{-itz})\|_\infty \leq \sum_{j=0}^{k} \binom{k}{j} (C\, m)^j\, |z|^{k-j}\, e^{(T_2+1)\,|z|}$$
$$= (C\, m + |z|)^k\, e^{(T_2+1)\,|z|}. \tag{11}$$

Now (8) follows from (10) and (11) at once. This completes the proof of Lemma 1.

It follows from the trace theorem and the above lemma that $P_1(z)$ is an entire family of trace class operators. Thus, $P(z)^{n+1} = (P_1(z) + K_2(z))^{n+1}$ forms an entire family of trace class operators. Moreover, in view of (5) we have that $R_\chi(z)(1 - P(z)^{n+1})$ is an entire family and hence by the appendix in [7] we conclude that the poles of $R_\chi(z)$, with the multiplicities, are among the zeros of the entire function

$$h(z) = \det(1 - P(z)^{n+1}).$$

Hence, to prove (1) it suffices to estimate $|h(z)|$ properly for large $|z|$. In view of (2), (6) and (7), using the inequalities $\mu_j(AB) \leq \|A\|\, \mu_j(B)$, $\mu_j(A+B) \leq \mu_{[/2j]}(A) + \mu_{[j/2]}(B)$, we have

$$\mu_j(P(z)^{n+1}) \leq e^{C\langle z\rangle}\, \mu_{[j/2]}(\gamma\, R_0(z)\,\eta - \gamma\, R_0(z_0)\,\eta) + C\,\langle z\rangle^{n+1}\, j^{-(n+1)/n},$$

with a constant $C > 0$ independent of z and j. Hence,

$$|h(z)| \leq \prod_{j=1}^{\infty} (1 + \mu_j(P(z)^{n+1}))$$
$$\leq \prod_{j=1}^{\infty} (1 + e^{C\langle z\rangle}\, \mu_j(\gamma\, R_0(z)\,\eta - \gamma\, R_0(z_0)\,\eta))^2$$
$$\times \prod_{j=1}^{\infty} (1 + C\,\langle z\rangle^{n+1}\, j^{-(n+1)/n})^2$$
$$= F_1(z)\, F_2(z). \tag{12}$$

For $F_2(z)$ we have

$$\begin{aligned} F_2(z) &= \exp\left(2\sum_{j=1}^{\infty} \log\left(1 + C\langle z\rangle^{n+1} j^{-(n+1)/n}\right)\right) \\ &\leq \exp\left(2\int_0^{\infty} \log\left(1 + C\langle z\rangle^{n+1} j^{-(n+1)/n}\right) dj\right) \\ &= \exp\left(2\langle z\rangle^n \int_0^{\infty} \log\left(1 + C\sigma^{-(n+1)/n}\right) d\sigma\right) \\ &= \exp\left(C'\langle z\rangle^n\right). \end{aligned} \tag{13}$$

To estimate $F_1(z)$ denote by Δ_S the Laplace-Beltrami operator on S and choose a function $\psi \in C_0^\infty(\mathbf{R}^n)$ such that $\psi = 1$ in a neighbourhood of S and supp $\psi \cap \operatorname{supp} \eta = \emptyset$. By the trace theorem and Lemma 1, for any integer m, we have

$$\begin{aligned} \mu_j(\gamma R_0(z)\eta) &\leq \mu_j((1-\Delta_S)^{-m}) \|\gamma R_0(z)\eta\|_{\mathcal{L}(H_0, H^{2m}(S;\mathbf{C}^d))} \\ &\leq C_1^{2m} j^{-2m/(n-1)} \sum_{|\alpha|\leq 2m+1} \|\psi\, \partial_x^\alpha R_0(z)\eta\| \\ &\leq C^{2m}(2m + \langle z\rangle)^{2m} e^{C\langle z\rangle} j^{-2m/(n-1)} \end{aligned}$$

with a constant $C > 0$ independent of m, j and z. Now, taking $2m \sim \langle z\rangle$, for any $q > 0$, we deduce

$$e^{q\langle z\rangle} \mu_j(\gamma R_0(z)\eta) \leq j^{-2} \quad \text{for } j \geq C(q)\langle z\rangle^{n-1},$$

with some constant $C(q) > 0$ independent of z and j. Using this and choosing q properly, we get

$$F_1(z) \leq \left(\prod_{j\leq C\langle z\rangle^{n-1}} e^{C\langle z\rangle}\right) \prod_{j\geq C\langle z\rangle^{n-1}} (1 + j^{-2}) \leq e^{C\langle z\rangle^n},$$

which together with (12) and (13) give

$$|h(z)| \leq e^{C\langle z\rangle^n}. \tag{14}$$

Now (1) follows from (14) and Jensen's inequality.

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