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Quasi-classical asymptotics of local Riesz means for the Schrödinger operator in a moderate magnetic field

by

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ABSTRACT. – The object of the study is the trace of the form $\mathcal{M}(h, \mu) = \operatorname{tr}\{\psi g(H_{\mathbf{a},V})\}$ with a compactly supported function ψ, where $H_{\mathbf{a},V}$ is the Schrödinger operator with a magnetic vector-potential $\mathbf{a}$ and an electric potential V; h and μ denote the Planck constant and the intensity of the magnetic field respectively. We establish the Weyl asymptotics of $\mathcal{M}(h, \mu)$ as $h \to 0$, $\mu h \leq const$ with a remainder estimate.

RÉSUMÉ. – L'objet de l'étude est une trace de la forme $\mathcal{M}(h, \mu) = \operatorname{tr}\{\psi g(H_{\mathbf{a},V})\}$ avec une fonction à support compacte ψ, où $H_{\mathbf{a},V}$ est un opérateur de Schrödinger avec un potentiel vecteur $\mathbf{a}$ et un potentiel électrique V; h et μ signifient la constante de Planck et l'intensité du champ respectivement. On établit une asymptotique de Weyl pour $\mathcal{M}(h, \mu)$ lorsque $h \to 0$, $\mu h \leq const$ avec une estimation du reste.

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1. INTRODUCTION AND MAIN RESULT

We study in $L^2(\mathbb{R}^d)$, $d \geq 2$, the Schrödinger operator

$$\left.\begin{aligned} H_{\mathbf{a},V} &= H_{\mathbf{a},V}(h,\mu) = H_0(h,\mu) + V, \\ H_0 &= H_0(h,\mu) = \sum_{l=1}^{d} (-ih\partial_{x_l} - \mu a_l)^2. \end{aligned}\right\} \tag{1.1}$$

Here $H_0 = H_{\mathbf{a},0}$ is the free operator in the magnetic field of intensity $\mu \geq 0$ (see [2] or [10]) with a (magnetic) real-valued vector-potential $\mathbf{a} = (a_1, a_2, \ldots, a_d)$, and V is a real-valued function (electric potential). Sometimes for the sake of brevity we use the notation $\mathfrak{a} = (\mathbf{a}, V)$. We analyse the asymptotics as[2] $h \to 0$, $\mu \geq 0$, $\mu h \leq C$ of traces of the form

$$\mathcal{M}(h,\mu) = \mathcal{M}(h,\mu;\psi,g,\mathfrak{a}) = \operatorname{tr}\{\psi g(H_{\mathfrak{a}})\}. \tag{1.2}$$

Here $\psi \in C_0^\infty(\mathbb{R}^d)$ and g is a suitable function. For the main result we assume that $g(\lambda) = g_s(\lambda)$, $s \geq 0$ with

$$\left.\begin{aligned} g_s(\lambda) &= |\lambda|^s, & \lambda \leq 0, \\ g_s(\lambda) &= 0, & \lambda > 0. \end{aligned}\right\}$$

A part of intermediate results (e.g. Sect. 4), however, will be established for a more general class of functions. If $g(\lambda) = g_s(\lambda)$ one uses for $\mathcal{M}(h,\mu)$ the notation $\mathcal{M}_s(h,\mu)$ or $\mathcal{M}_s(h,\mu;\psi,\mathfrak{a})$.

In the case $d = 3$ and a homogeneous magnetic field $\mathbf{a} = (-x_2, 0, 0)$ our results can be viewed as complementary to those of [18], where the quasi-classical asymptotics of the trace (1.2) has been studied under the conditions $ch^{-\varrho} \leq \mu \leq Ch^{-\varsigma}$ for arbitrary $\varsigma \geq \varrho > 0$. In particular, the case $\mu h \to \infty$ was permitted. On the contrary, in the present paper we suppose that $\mu h \leq C$, but do not assume any lower bound on μ. One should mention that the definition of the operator H_a in [18] differs from (1.1) by $-\mu h$. This fact does not play any significant role since in the present paper this term is assumed to be bounded and therefore can be always incorporated into V. Along with [18] our local asymptotics provides a crucial step for the study of a two-term asymptotics of the global quantity $\mathcal{M}_s(h,\mu;1,a) = \operatorname{tr}\{g_s(H_a)\}$, which is finite if the potential V decreases at infinity sufficiently rapidly. For the physical motivation of $\mathcal{M}_1(h,\mu;1,a)$ we refer to papers [3], [6] and [11]-[14], where the asymptotics of $\mathcal{M}_1$ was used to study the ground state of a large electronic system. In particular, for the homogeneous magnetic field the leading term of the asymptotics

[2] Here and in what follows we denote by C and c (with or without indices) various positive constants whose precise value is of no importance.

$\mathcal{M}_1$ as $h \to 0$ uniform in μ was found in [14]. Our results on the two-term asymptotics of $\mathcal{M}_1$ in this situation are announced in [19].

Let us specify the assumptions on the Schrödinger operator for which we study the trace (1.2). The presence of the cut-off ψ in (1.2) allows one to assume that it has the form $H_{\mathfrak{a}}$ in a vicinity of supp ψ only. To distinguish it from the "true" Schrödinger operator we use the notation $\mathcal{H}$. Below $D(A)$ denotes the domain of a self-adjoint operator A.

Assumption 1.1.

(1) *The operator $\mathcal{H}$ is selfadjoint and semiboundedfrom below in $L^2(\mathbb{R}^d)$;*

(2) *There exists an open set $\mathcal{D} \subset \mathbb{R}^d$ and real-valued functions $V \in C_0^\infty(\mathbb{R}^d)$, $a_l \in C_0^\infty(\mathbb{R}^d)$, $1 \le l \le d$, such that $C_0^\infty(\mathcal{D}) \subset D(\mathcal{H})$ and*

$$\mathcal{H}u = H_{\mathfrak{a}}\, u, \qquad \forall u \in C_0^\infty(\mathcal{D}), \tag{1.3}$$

with $\mathfrak{a} = \{\mathrm{a}, V\}$.

To point out the connection of $\mathcal{H}$ with $H_{\mathfrak{a}}$ we denote it sometimes by $\mathcal{H}_{\mathfrak{a}}$. We keep notation $\mathcal{M}(h, \mu)$ for the trace $\operatorname{tr}\{\psi g(\mathcal{H}_{\mathfrak{a}})\}$. It will not cause any confusion with (1.2). We stress that no restrictions on $\mathcal{H}$ outside $\mathcal{D}$ are placed (except for the qualitative requirements of selfadjointness and semi-boundedness). In particular, $\mathcal{H}$ does not have to be local.

As a rule, we assume that $\mathcal{H}$ obeys Assumption 1.1 with $\mathcal{D} = \overset{\circ}{B}(4E)$ for some fixed $E > 0$, where $\overset{\circ}{B}(E)$ stands for the open ball $\{x : |x| < E\}$ (The corresponding closed ball is denoted by $B(E)$). The function ψ is supposed to belong to $C_0^\infty(B(E/2))$. As we shall see, the leading term of the asymptotics of $\mathcal{M}_s(h, \mu;\, \psi, \mathfrak{a})$ depends neither on μ nor a and is given by

$$\mathfrak{W}_s(h) = \mathfrak{W}_s(h;\, \psi, V) = \Xi_s\, h^{-d} \int \psi(x)\, (V_-(x))^{s+\frac{d}{2}}\, dx, \tag{1.4}$$

$$\Xi_s = \frac{|\mathbb{S}^{d-1}|}{(2\pi)^d} \int_0^1 t^{d-1}(1-t^2)^s\, dt. \tag{1.5}$$

Here $|\mathbb{S}^{d-1}|$ stands for the surface area of the $(d-1)$-dimensional unit sphere. Note that $\mathfrak{W}_s(h)$ is nothing but the classical Weyl term, which yields the leading order of the asymptotics of $\mathcal{M}_s(h, 0)$ as well (see [7]).

The error estimate in the asymptotics below will depend only on the constants in the following bounds on a_l, V and ψ:

$$\left.\begin{array}{c} |\partial_x a_l(x)| \le 1,\ |\partial_x^m a_l(x)| \le C_m,\ |m| \ge 2,\ 1 \le l \le d, \\ |\partial_x^m V(x)| \le C_m,\ |\partial_x^m \psi(x)| \le C_m,\ |m| \ge 0. \end{array}\right\} \tag{1.6}$$

In other words, the asymptotics of $\mathcal{M}_s(h, \mu; \psi, \mathfrak{a})$ is uniform in the functions a_l, V and ψ, satisfying (1.6). Note that due to the presence of the parameter μ the condition $|\partial_x a_l(x)| \leq 1$ can be always satisfied. Notice also that (1.6) contains no estimates on the function a_l itself, but only on its derivatives. This fact is quite natural, since the constant component of a_l can be chosen arbitrarily or eventually eliminated by a simple gauge transformation.

Next Theorem constitutes the main result of the paper.

THEOREM 1.2. – *Let $\mathcal{H}$ obey Assumption 1.1 with $\mathcal{D} = \overset{\circ}{B}(4E)$ for some $E > 0$, let $s \in [0, 1]$ and $0 \leq \mu \leq Ch^{-1}$, $h \in (0, h_0]$. Then for any $\psi \in C_0^\infty(B(E/2))$*

$$\left.\begin{aligned} \mathcal{M}_s(h, \mu; \psi, \mathfrak{a}) = \mathfrak{W}_s(h; \psi, V) + \langle \mu \rangle^{s+1} O(h^{s+1-d}), \\ \langle \mu \rangle = (1 + \mu^2)^{1/2}. \end{aligned}\right\} \tag{1.7}$$

The remainder estimate in (1.7) is uniform in the functions a_l, V and ψ satisfying the bounds (1.6) but may depend on s and E.

The restriction $\mathcal{D} = \overset{\circ}{B}(4E)$ is imposed for the convenience only. The result can be easily extended to arbitrary open set $\mathcal{D}$ by using an appropriate partition of unity. We emphasize again that no quantitative information on $\mathcal{H}$ for $x \notin B(4E)$ enters the answer. Note that (1.7) for $\psi = 1$ is formally consistent with [14].

The proof of Theorem 1.2 is based mainly on the auxiliary results obtained in [18]. These along with other general facts used in the proof, are collected in Sect. 2. Some preliminary estimates for the Schrödinger operator in the magnetic field are given in Sect. 3. The proof itself is divided in two steps. At first, in Sect. 4 we study the asymptotics of $\mathcal{M}(h, \mu; \psi, g)$ for a function g of a more general form that g_s under a supplementary "non-critical" condition (4.2) (see Theorem 4.1). The key fact is that the remainder estimates "do not feel" the behaviour of the operator $\mathcal{H}$ outside the ball $B(4E)$ (in the sense specified in Theorem 1.2). This is a consequence of the fact that $\mathcal{H}$ is local inside $B(4E)$. This allows one to replace $\mathcal{H}$ by an operator which has the form $H_\mathfrak{a}$ in the entire space $L^2(\mathbb{R}^d)$. Then the standard quasi-classical methods provide the asymptotics of $\mathcal{M}(h, \mu; \psi, g)$. To complete the proof of Theorem 1.2 it remains to remove the condition (4.2). This is done in Sect. 6 with the help of a method initially suggested by V.Ivrii (see [7]-[9]) which can be naturally referred to as the multiscale analysis (see also [3], [6]). The detailed description of a version of this method adjusted to our purposes, is given in Sect. 5. Schematically, it provides the asymptotics (1.7) by use of an appropriate partition of unity and Theorem 4.1 in combination with scaling-translation transformation.

Notation. $\mathcal{B}^\infty(X)$ for a domain X denotes the set of functions $f \in C^\infty(X)$ bounded along with all their derivatives. This space forms a Fréchet space for the family of natural semi-norms $|||\, f\, |||_m = \sup_x |\partial_x^m f(x)|$, $m \in \mathbb{Z}_+^d$.

A constant C is said to be uniform in $f \in \mathcal{B}^\infty(X)$ if it depends only on the constants in the estimates $|||\, f\, |||_m \le C_m$, $m \in \mathbb{Z}_+^d$.

A function g is said to belong to $\mathcal{B}^\infty(X)$ uniformly in $f \in \mathcal{B}^\infty(X)$ if the derivatives $|\partial_x^m g(x)|$, $m \in \mathbb{Z}_+^d$, are estimated by constants which are uniform in $f \in \mathcal{B}^\infty(X)$.

The basic configuration space is denoted by $\mathbb{R}^d$ or $\mathbb{R}_x^d$, $d \ge 1$. The dual space is denoted by $\mathbb{R}_\xi^d$. $B(z, E)$, $z \in \mathbb{R}^d$, $E > 0$, denotes the closed ball $\{x \in \mathbb{R}^d : |x - z| \le E\}$; $B(E) = B(0, E)$. Sometimes we use open balls $\overset{\circ}{B}(z, E) = \{x \in \mathbb{R}^d : |x - z| < E\}$ and $\overset{\circ}{B}(E)$ as well.

$B_{x,\xi}(E)$, $E > 0$, denotes the ball $\{x^2 + \xi^2 \le E^2,\ x \in \mathbb{R}_x^d,\ \xi \in \mathbb{R}_\xi^d\}$.

2. PRELIMINARIES

In this section we list some results from the theory of pseudo-differential operators (ΨDO's) to be used throughout the paper. The most of the properties below can be found in any textbook on ΨDO's. However, to be definite, we usually refer to [16]. The results on the quasi-classical asymptotics (see subsections 3, 4) are borrowed from [18], where they have been obtained in the form convenient for our needs.

1. *Basic properties and definitions.* – By a $\beta - \Psi$DO $A = \mathrm{op}_\beta^{\mathrm{w}}\, a$ we always mean the operator with the Weyl symbol a, that is

$$(Au)(x) = (2\pi\beta)^{-d} \int\int e^{\frac{i}{\beta}(x-y)\xi}$$
$$\times\, a\left(\frac{x+y}{2}, \xi; \beta\right) u(y)\, dy d\xi,\ u \in C_0^\infty(\mathbb{R}^d),\ d \ge 1. \qquad (2.1)$$

Here $\beta \in (0, \beta_0]$, $\beta_0 > 0$, and the symbol $a \in C^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ (sometimes we drop β from the notation of a) is assumed to satisfy the condition

$$|\partial_x^{m_1} \partial_\xi^{m_2} a(x, \xi; \beta)| \le C_{m_1 m_2}\, \rho(x, \xi),\ \forall m_1, m_2, \qquad (2.2)$$

with some positive temperate weight function ρ, so that the iterated integral in the r.h.s. of (2.1) converges. In this case the symbol is said to belong to the class $\mathbf{S}(\rho)$, which is a Fréchet space with the family of natural semi-norms

$$|||a|||_{m_1, m_2, \rho} = \sup_{x, \xi} (\rho(x, \xi))^{-1} |\partial_x^{m_1} \partial_\xi^{m_2} a(x, \xi)|.$$

In the spirit of definition given at the end of Sect. 1 we say that a symbol a belongs to some class $\mathbf{S}(\rho)$ uniformly in $a' \in \mathbf{S}(\rho')$ if the semi-norms of a are bounded by constants uniform in $a' \in \mathbf{S}(\rho')$. Unless otherwise stated throughout the paper we tacitly assume that the estimates involving one or more ΨDO's are uniform in their symbols.

One can prove that the operator $A = \mathrm{op}_\beta^{\mathrm{w}} a$ with a real-valued bounded from below symbol $a(x, \xi)$, is essentially selfadjoint on $C_0^\infty(\mathbb{R}^d)$. For simplicity we denote its closure again by $\mathrm{op}_\beta^{\mathrm{w}} a$.

The following criterion will be important:

PROPOSITION 2.1.

(1) *If ρ is bounded then the operator A is bounded and*

$$\|A\| \leq C_d \sup_{|m_1| \leq k(d), |m_2| \leq k(d)} \beta^{m_2} |\partial_x^{m_1} \partial_\xi^{m_2} a(x, \xi)|,$$

where the constant C_d and the integer number $k(d)$ depend only on d.

(2) *If $\rho \in L^1(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ then $A \in \mathfrak{S}_1$ and*

$$\|A\|_1 \leq C'_d \beta^{-d} \sum_{|m_1|+|m_2| \leq 2d+2} \beta^{m_2} \int |\partial_x^{m_1} \partial_\xi^{m_2} a(x, \xi)| \, dx d\xi, \tag{2.3}$$

where C'_d depends only on the dimension d.

The first part of this Proposition is nothing but the Calderon-Vaillancourt Theorem. The second statement can be found in [16].

Let us consider now products of β – ΨDO's. Suppose first that $a_1, a_2 \in \mathcal{B}^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$. Then there exists a unique Weyl symbol $a \in \mathcal{B}^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ such that $\mathrm{op}_\beta^{\mathrm{w}} a_1 \, \mathrm{op}_\beta^{\mathrm{w}} a_2 = \mathrm{op}_\beta^{\mathrm{w}} a$. Moreover,

$$\begin{aligned} a(x, \xi) = \sum_{l=0}^{N} (-i)^l \beta^l \sum_{|m_1|+|m_2|=l} \frac{1}{m_1! \, m_2!} \left(\frac{1}{2}\right)^{|m_1|} \left(-\frac{1}{2}\right)^{|m_2|} \\ \partial_\xi^{m_1} \partial_x^{m_2} a_1(x, \xi) \, \partial_\xi^{m_2} \partial_x^{m_1} a_2(x, \xi) + \beta^{N+1} r_{N+1}(x, \xi; \beta), \\ r_{N+1}(\beta) \in \mathcal{B}^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^d), \ \forall N > 0. \end{aligned} \tag{2.4}$$

By Proposition 2.1 the operator $R_{N+1}(\beta) = \mathrm{op}_\beta^{\mathrm{w}} r_{N+1}$ is bounded uniformly in a_1, a_2. Moreover, if one of the symbols a_1 or a_2 is supported in the ball $B_{x,\xi}(E)$, $E > 0$, then

$$\| R_{N+1}(\beta) \|_1 \leq C\beta^{-d}, \qquad C = C(E).$$

The next Lemma follows directly from (2.4):

LEMMA 2.2. – *Let* $a_1, a_2 \in \mathcal{B}^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$ *and*

$$a_1(x, \xi) = 0, \qquad x, \xi \in \operatorname{supp} a_2.$$

Then

$$\| \mathrm{op}_\beta^{\mathrm{w}} a_1 \, \mathrm{op}_\beta^{\mathrm{w}} a_2 \| \leq C_N \, \beta^N, \qquad \forall N > 0.$$

If, in addition, $a_2 \in C_0^\infty(B_{x,\xi}(E))$ *with some* $E > 0$, *then*

$$\| \mathrm{op}_\beta^{\mathrm{w}} a_1 \, \mathrm{op}_\beta^{\mathrm{w}} a_2 \|_1 \leq C'_N \, \beta^N, \quad \forall N > 0. \tag{2.5}$$

The constant C'_N *in (2.5) depends on* E.

2. *Functional calculus for ΨDO's.* We give here the asymptotic expansion in powers of β for a C_0^∞-function of a β-ΨDO. To that end we assume that $A = \mathrm{op}_\beta^{\mathrm{w}} a$ with a symbol $a \in \mathbf{S}(\rho)$ independent of β such that for some $b \in \mathbb{R}$

$$a(x, \xi) + b \geq c, \qquad a \in \mathbf{S}(a + b). \tag{2.6}$$

Sometimes we refer to operators A whose symbols satisfy (2.6), as *β-admissible operators* (see [4] or [16] for more general definition).

PROPOSITION 2.3. – *Let A be β-admissible and* $g \in C_0^\infty(\mathbb{R})$. *Then for any integer* $N > 0$ *the expansion holds:*

$$g(A) = \sum_{n=0}^{N} \beta^n \, \mathrm{op}_\beta^{\mathrm{w}} a_{g,n} + \beta^{N+1} R_{g,N+1}(\beta). \tag{2.7}$$

Here $\| R_{g,N+1}(\beta) \| \leq C_N$ *and the symbols* $a_{g,n}$ *are given by the formulae*

$$\left.\begin{aligned} &a_{g,0}(x, \xi) = g(a(x, \xi)), \\ &a_{g,1}(x, \xi) = 0. \\ &a_{g,n}(x, \xi) = \sum_{k=1}^{2n-1} \frac{(-1)^k}{k!} d_{n,k} \, \partial_\lambda^k g(a(x, \xi)), \qquad n \geq 2, \end{aligned}\right\} \tag{2.8}$$

where the coefficients $d_{n,k}$ *are universal polynomials of* $\partial_x^{m_1} \partial_\xi^{m_2} a$, $m_1 + m_2 \leq n$.

This Proposition is a simplified version of a more general result established in [4] (see also [16]).

3. *Quasi-classical asymptotics*. Let A be a β-ΨDO with a real-valued symbol semi-bounded from below (and therefore essentially selfadjoint) and let $\Theta = \mathrm{op}_\beta^w\, \theta$ be a ΨDO with the real-valued symbol

$$\theta \in C_0^\infty (B_x(E) \times B_\xi(R)), \qquad E > 0, \quad R > 0. \tag{2.9}$$

We are going to discuss the asymptotics of the trace

$$\mathcal{N}_\beta = \mathcal{N}_\beta(\theta, g, A) = \mathrm{tr}\,[\Theta g(A)]$$

with a bounded function g. This quantity is finite since $\|\Theta\|_1 < \infty$ by Proposition 2.1.

First of all we study the propagator $U_\beta(t) = \exp\{-i\beta^{-1}At\}$ (see [16]). We approximate $U_\beta(t)$ by a β-Fourier integral operator (β-FIO) $G_\beta(t)$ having the kernel

$$\mathcal{G}(x, y, t) = (2\pi\beta)^{-d} \int e^{\frac{i}{\beta}(S(x,\xi,t)-y\xi)}\, v(x, \xi, t; \beta)\, d\xi, \tag{2.10}$$

with the real-valued phase $S(x, \xi, t)$ and the amplitude

$$v(x, \xi, t; \beta) = \sum_{n=0}^{N} \beta^n\, v_n(x, \xi, t),$$

where the smooth functions $v_n(., ., t)$ are compactly supported. We suppose also that

$$G_\beta(0) = \Phi, \tag{2.11}$$

where $\Phi = \mathrm{op}_\beta^w\, \varphi$ with a real-valued function φ such that

$$\left.\begin{aligned} \varphi \in C_0^\infty (B_x(2E) \times B_\xi(2R)), \\ \varphi(x, \xi) = 1, \qquad (x, \xi) \in B_x(3E/2) \times B_\xi(3R/2). \end{aligned}\right\} \tag{2.12}$$

To satisfy the initial condition (2.11) we have to assume that $S(x, \xi, 0) = x\xi$ and $v_0(x, \xi, 0) = \varphi(x, \xi)$, $v_n(x, \xi, 0) = 0$, $n \geq 1$. Note however that we do not need this information in what follows. The only fact we shall be using is given by

PROPOSITION 2.4 [18]. – *There exists a number* $T_0 > 0$ *and functions*

$$\begin{aligned} &S, v_n \in C^\infty(B_x(3E) \times B_\xi(3R) \times [-T_0, T_0]), \\ &\mathrm{supp}\, v_n(., ., t) \subset B_x(3E) \times B_\xi(3R), \qquad \forall t \in [-T_0, T_0], \end{aligned}$$

such that for any integer $N > 0$

$$\max_{|t| \leq T_0} || - i\beta\partial_t G_\beta(t) + AG_\beta(t)|| \leq C_N \beta^{N+1}.$$

To state the result on the asymptotics of $\mathcal{N}_\beta$ we first fix notation. Let $T \in (0, T_0]$ (T_0 is the number from Proposition 2.5) be some number and let $\hat{\chi} \in C_0^\infty(-T, T)$ be a real-valued function such that $\hat{\chi}(t) = \hat{\chi}(-t)$ and $\hat{\chi}(t) = 1/\sqrt{2\pi}$, $|t| \leq T/2$. Define

$$\chi_1(\tau) = (2\pi)^{-\frac{1}{2}} \int \hat{\chi}(t) e^{i\tau t} dt. \tag{2.13}$$

We assume that

$$\chi_1 \geq 0,$$

and that there exist $T_1 \in (0, T)$ and $c > 0$ such that

$$\chi_1(\tau) \geq c, \qquad |\tau| \leq T_1.$$

One can always guarantee these two properties by replacing $\hat{\chi}$ with $\hat{\chi} * \hat{\chi}$ (if necessary) and assuming $\hat{\chi} > 0$. Denote

$$\chi_\beta(\tau) = \frac{1}{\beta} \chi_1\left(\frac{\tau}{\beta}\right). \tag{2.14}$$

For a function $g \in L^1(\mathbf{R})$ we define its smoothed-out version:

$$g^{(\beta)}(\tau) = \int g(\tau - \nu) \chi_\beta(\nu) d\nu = \int g(\nu) \chi_\beta(\tau - \nu) d\nu. \tag{2.15}$$

Notice that

$$|g^{(\beta)}(\tau)| \leq \max_\nu \chi_\beta(\nu) ||g||_{L^1} \leq C\beta^{-1}. \tag{2.16}$$

To find the asymptotics of N_β we impose

Assumption 2.5. *On the set*

$$\Lambda(\lambda, \theta) = \{(x, \xi) \mid a(x, \xi) = \lambda, (x, \xi) \in \operatorname{supp} \theta\}$$

the lower bound $|\nabla a(x, \xi)| \geq \delta > 0$ *holds.*

If Assumption 2.5 is (not) satisfied, the value λ is said to be non-critical (critical) value of the operator A on supp θ. We denote $\Lambda(\lambda) = \Lambda(\lambda, 1)$.

PROPOSITION 2.6 [18]. – *Let the operator A be as specified above and let θ satisfy (2.9).*

(1) *If* $g \in C_0^\infty(\mathbf{R})$ *then*

$$\mathcal{N}_\beta(\theta, g, A) = (2\pi\beta)^{-d} \int \theta(x, \xi) g(a(x, \xi)) dx d\xi + O(\beta^{2-d}). \tag{2.17}$$

(2) *Let Assumption 2.5 be fulfilled for some* λ, $|\lambda| \leq \lambda_0$. *Then*

$$\| \Theta_{\chi_\beta} (A - \lambda) \|_1 \leq C\beta^{-d}. \tag{2.18}$$

(3) *Let* $g \in L^1(\mathbb{R})$ *be a compactly supported function and let Assumption 2.5 be fulfilled for all* $\lambda \in \operatorname{supp} g$. *Then*

$$\mathcal{N}_\beta (\theta, g^{(\beta)}, A) = (2\pi\beta)^{-d} \int \theta(x,\xi) g(a(x,\xi)) dx d\xi + O(\beta^{2-d}). \tag{2.19}$$

The constant C *in (2.18) is uniform in* λ, $|\lambda| \leq \lambda_0$.

4. *The Tauberian theorem.* The relation (2.19) provides the asymptotics of $\mathcal{N}_\beta$ for the function $g^{(\beta)}$. We are going to deduce from here the result for g itself. First we have to specify the class of functions we shall be working with.

DEFINITION 2.7. – *A function* $g \in C^\infty(\mathbb{R}\setminus\{0\})$ *is said to belong to the class* $C^{\infty, s}(\mathbb{R})$, $s \in [0, 1]$, *if (i)* $g \in C(\mathbb{R})$, $s > 0$; *(ii) for some* $r > 0$

$$g(\lambda) = 0, \qquad \lambda \geq C,$$
$$|\partial_\lambda^m g(\lambda)| \leq C_m |\lambda|^r, \qquad \forall m \geq 0, \quad \lambda \leq -C;$$

and (iii) for $|\lambda| \leq C$, $\lambda \neq 0$, *and all non-negative integer* m

$$|\partial^m g(\lambda)| \leq C_m |\lambda|^{s-m}, \qquad 0 < s < 1,$$
$$|\partial^m g(\lambda)| \leq C_m, \qquad s = 0, 1.$$

A function g *is said to belong to* $C_0^{\infty, s}(\mathbb{R})$, $s \in [0, 1]$, *if* g *is compactly supported and* $g \in C^{\infty, s}(\mathbb{R})$.

Note that $g_s \in C^{\infty, s}(\mathbb{R})$.

PROPOSITION 2.8. – *Let* A *be a selfadjoint operator and* $g \in C_0^{\infty, s}$, $s \in [0, 1]$. *Let the function* χ_1 *be as defined above. If for an operator* $B \in \mathfrak{S}_2$

$$\sup_{\lambda \in \mathcal{D}(\delta)} \| B^* \chi_\beta (A - \lambda) B \|_1 \leq Z(\beta), \tag{2.20}$$

where $\mathcal{D}(\delta) = \{\lambda : \operatorname{dist}\{\operatorname{supp} g, \lambda\} \leq \delta\}$, *with some positive function* $Z(\beta)$ *and some number* $\delta > 0$, *then*

$$\left.\begin{array}{c} \| B^* (g(A) - g^{(\beta)}(A)) B \|_1 \leq C\beta^{1+s} Z(\beta) + C'_{N_1} \beta^{N_1} \| B^* B \|_1, \\ \forall N_1 > 0. \end{array}\right\} \tag{2.21}$$

The constants C *and* C' *depend on the number* δ *and functions* g, χ_1 *only.*

Proof for $s > 0$ can be found in [18]. The case $s = 0$ can be treated analogously.

3. AUXILIARY ESTIMATES FOR THE MAGNETIC OPERATOR

1. Our objective in this section is to establish some preliminary estimates for trace norms of the form $\| \psi g(\mathcal{H}) \|_1$. Recall that we use the notation $\mathfrak{a}$ for the pair $(\mathbf{a}, V)$. Those estimates will be uniform in the Planck constant $h \in (0, h_0]$ and the field $\mathbf{a}$. Thus one can assume $\mu = 1$. The central result is Theorem 3.12 which justifies the possibility of replacing the operator $\mathcal{H}_{\mathfrak{a}}$ with $H_{\mathfrak{a}}$ in the asymptotics of $\operatorname{tr}\{\psi g(\mathcal{H}_{\mathfrak{a}})\}$ for $g \in C^{\infty}(\mathbb{R})$. For the case $d = 3$ and a homogeneous magnetic field a similar result has been obtained in [18].

We begin with the study of an operator H defined by (1.1) in the entire space $L^2(\mathbb{R}^d)$. Let $a_l \in L^2_{loc}(\mathbb{R}^d)$, $l = 1, 2, \ldots, d$. We denote by Q_l the closures of the formal differential expressions $-ih\partial_{x_l} - a_l$ on $C_0^{\infty}(\mathbb{R}^d)$. Then the operator $H_0 = H_{\mathbf{a},0} = Q_l^* Q_l$ (Here and below we assume summation over repeating indices) is selfadjoint and $C_0^{\infty}(\mathbb{R}^d)$ is its form core (see [2]). We are interested in the properties of $H = H_{\mathfrak{a}} = H_0 + V$ with a real-valued function $V \in L^{\infty}(\mathbb{R}^d)$. First we obtain estimates for $\psi(H_{\mathbf{0},V} + \lambda)^{-1}$, $\lambda > 0$ in the Neumann-Schatten classes $\mathfrak{S}_p$ of compact operators. Then by the diamagnetic inequality these estimates are extended to the case $\mathbf{a} \neq 0$ as well. Finally, to study functions of $H_{\mathfrak{a}}$ we use the formula (3.11) (see [1]), expressing a function $g \in C_0^{\infty}(\mathbb{R})$ of an arbitrary selfadjoint operator in terms of its resolvent.

Let λ_0 be a number such that $\lambda_0 \geq 1 + 2\text{v-sup}_x |V(x)|$, so that $H \geq -\lambda_0/2$. Unless otherwise stated we assume that $z \in \mathbb{C}\backslash[-\lambda_0, \infty)$ and denote $d(z) = \operatorname{dist}\{z, [-\lambda_0, \infty)\}$, $\langle z \rangle = (1 + |z|^2)^{1/2}$. Throughout the Section C denotes a constant depending only on λ_0 (and not on $\mathbf{a}$, V, h).

Recall that a compact operator A is said to belong to $\mathfrak{S}_p$, $1 \leq p < \infty$, if the value

$$\| A \|_p = \{\operatorname{tr}[A^* A]^{\frac{p}{2}}\}^{\frac{1}{p}}$$

is finite. This functional defines a norm in $\mathfrak{S}_p$. Note the inequalities

$$\left.\begin{array}{l} \| B_1 B_2 \|_p \leq \| B_1 \|_p \|B_2\|, \qquad B_1 \in \mathfrak{S}_p, \\ \| B_1 B_2 \|_p \leq \| B_1 \|_r \| B_2 \|_q, \\ B_1 \in \mathfrak{S}_r, \qquad B_2 \in \mathfrak{S}_q, \qquad r^{-1} + q^{-1} = p^{-1}. \end{array}\right\} \tag{3.1}$$

Our basic tool in the analysis of the resolvent is the so-called diamagnetic inequality [2]:

$$| ((H_{\mathbf{a},V} + \lambda)^{-\kappa} u)(x) | \leq (H_{\mathbf{0},V} + \lambda)^{-\kappa} | u(x) |,$$
$$\forall u \in L^2(\mathbb{R}^d), \qquad \lambda \geq \lambda_0, \qquad x > 0.$$

By virtue of [17, Theorem 2.13] this yields

PROPOSITION 3.1. – *Let ψ be a function such that $\psi(H_{\mathbf{0},0}+\lambda)^{-\kappa} \in \mathfrak{S}_{2n}$ with some $n \in \mathbb{N}$. Then $\psi(H_{\mathbf{a},V}+\lambda)^{-\kappa} \in \mathfrak{S}_{2n}$ as well and*

$$\|\psi(H_{\mathbf{a},V}+\lambda)^{-\kappa}\|_{2n} \le \|\psi(H_{\mathbf{0},V}+\lambda)^{-\kappa}\|_{2n}.$$

To estimate the $\mathfrak{S}_p$-norm of $\psi(H_{\mathbf{0},V}+\lambda)^{-\kappa}$ we apply the following criterion [15] which provides the bounds of $\mathfrak{S}_p$-norms for operators, having the form $f_1(-ih\partial)\, f_2(x)$:

PROPOSITION 3.2. – *Let $p \ge 2$. If $f_1, f_2 \in L^p(\mathbb{R}^d)$ then $A = f_1(-ih\partial)\, f_2(x) \in \mathfrak{S}_p$ and*

$$\|A\|_p \le C_p\, h^{-\frac{d}{p}} \|f_1\|_{L^p} \|f_2\|_{L^p}.$$

This result allows one to establish estimates for the $\mathfrak{S}_p$-norms of the operator $\chi(H_{\mathbf{0},0}+\lambda)^{-\kappa}$, $\lambda > 0$ with arbitrary $\kappa > 0$:

LEMMA 3.3. – *Let $\chi \in C_0^\infty(\mathbb{R}^d)$ and $\kappa > 0$. Then $\chi(H_{\mathbf{0},0}+\lambda)^{-\kappa} \in \mathfrak{S}_p$ for any $p \ge 2$, $p > d(2\kappa)^{-1}$ and*

$$\|\chi(H_{\mathbf{0},0}+\lambda)^{-\kappa}\|_p \le C\lambda^{-\kappa}\left(\frac{h^2}{\lambda}\right)^{-\frac{d}{2p}}.$$

Proof. – It suffices to recall that $(H_{\mathbf{0},0}+\lambda)^{-\kappa} = f(-ih\partial)$ with $f(t) = (t^2+\lambda)^{-\kappa}$ and apply Proposition 3.2. □

In combination with the diamagnetic inequality this Lemma yields

LEMMA 3.4. – *Let $\chi \in C_0^\infty(\mathbb{R}^d)$, $\kappa > 0$ and n be a natural number such that $n > d(4\kappa)^{-1}$. Then for any $\lambda \ge \lambda_0$*

$$\|\chi(H+\lambda)^{-\kappa}\|_{2n} \le C\lambda^{-\kappa+\frac{d}{4n}}\, h^{-\frac{d}{2n}}. \tag{3.2}$$

Proof. – Let $\lambda \ge \lambda_0$. It follows immediately from Proposition 3.2 that

$$\|\chi(H+\lambda)^{-\kappa}\|_{2n} \le \|\chi(H_{\mathbf{0},V}+\lambda)^{-\kappa}\|_{2n} \le 2^\kappa \|\chi(H_{\mathbf{0},0}+\lambda)^{-\kappa}\|_{2n}.$$

Now Lemma 3.3 yields (3.2). □

2. Let us study now the properties of the resolvent of H sandwiched between two functions with disjoint supports. We need the following preparatory

LEMMA 3.5. – *Let $m = 0, 1$. Then for any $\lambda \ge \lambda_0$*

$$\|Q_l^m(H+\lambda)^{-1/2}\| \le 2^{\frac{1}{2}}\lambda^{\frac{m-1}{2}}. \tag{3.3}$$

Proof. – Inequality (3.3) is obvious for $m = 0$. Further,

$$(Q_l^* Q_l u, u) \leq ((H_0 + \lambda) u, u) \leq 2((H + \lambda) u, u), \; u \in C_0^\infty(\mathbb{R}^d).$$

Here we used the fact that $\lambda \geq \lambda_0 \geq 2\text{v-sup}_x |V(x)|$. This yields (3.3) for $m = 1$. □

Note also the obvious equality

$$\begin{aligned}[H, \phi] &= (Q_l^* [Q_l, \phi] + [Q_l^*, \phi] Q_l) \\ &= -ih (Q_l^* \partial_{x_l} \phi + \partial_{x_l} \phi Q_l), \qquad \forall \phi \in \mathcal{B}^\infty(\mathbb{R}^d). \end{aligned} \tag{3.4}$$

Further, due to the resolvent identity, we have

$$\begin{aligned}(H - z)^{-1} &= (H + \lambda)^{-\frac{1}{2}} S(\lambda, z) (H + \lambda)^{-\frac{1}{2}}, \\ S(\lambda, z) &= (I + (\lambda + z)(H - z)^{-1}), \end{aligned} \tag{3.5}$$

for any $\lambda \geq \lambda_0$. Note that

$$\| S(\lambda, z) \| \leq C \frac{\lambda + \langle z \rangle}{d(z)}, \qquad \lambda \geq \lambda_0. \tag{3.6}$$

LEMMA 3.6. – *Let $\chi \in C_0^\infty(\mathbb{R}^d)$ and $\phi \in \mathcal{B}^\infty(\mathbb{R}^d)$ be such that*

$$\operatorname{dist}\{\operatorname{supp}\chi, \operatorname{supp}\phi\} \geq c, \tag{3.7}$$

and let $r, m = 0, 1$. Then for any $N > d/2$

$$\| \chi Q_l^r (H - z)^{-1} (Q_q^*)^m \phi \|_1 \leq C_N \frac{\langle z \rangle^{\frac{m+r}{2}}}{d(z)} \left[\frac{\langle z \rangle^{\frac{1}{2}}}{h} \right]^d \left[\frac{\langle z \rangle h^2}{d(z)^2} \right]^N. \tag{3.8}$$

The constant C_N depends only on the number N, the functions χ, ϕ and the constant c in (3.7).

Proof. – We start with the following simple observation. Due to (3.7), for a fixed $L \in \mathbb{N}$ one can define a family of functions $\chi^{(j)} \in C_0^\infty(\mathbb{R}^d)$, $j = 1, 2, \ldots, L$ such that

(1) Each pair $\chi^{(j)}$, ϕ satisfies (3.7);

(2) $\chi\chi^{(1)} = \chi$ and $\chi^{(j)} \chi^{(j-1)} = \chi^{(j-1)}$, $j \geq 2$.

Therefore $\chi Q_l = \chi Q_l \chi^{(1)}$ and

$$\begin{aligned}\chi^{(j)} (H - z)^{-1} (Q_q^*)^m \phi &= -[(H - z)^{-1}, \chi^{(j)}] (Q_q^*)^m \phi \\ &= (H - z)^{-1} [H_0, \chi^{(j)}] \chi^{(j+1)} (H - z)^{-1} (Q_q^*)^m \phi. \end{aligned}$$

Thus the following representation holds:

$$\chi Q_l^r (H - z)^{-1} (Q_q^*)^m \phi$$
$$= \chi Q_l^r \prod_{j=1}^{L} \{ (H - z)^{-1} [H, \chi^{(j)}] \} (H - z)^{-1} (Q_q^*)^m \phi.$$

According to (3.5) for any $\lambda > \lambda_0$ one can write

$$(H - z)^{-1} [H, \chi^{(j)}] = (H + \lambda)^{-\frac{1}{2}} ST_j (H + \lambda)^{\frac{1}{2}}, \qquad S = S(\lambda, z),$$

where

$$T_j = (H + \lambda)^{-\frac{1}{2}} [H_0, \chi^{(j)}] (H + \lambda)^{-\frac{1}{2}}.$$

Therefore

$$\chi Q_l^r (H - z)^{-1} (Q_q^*)^m \phi = \chi Q_l^r (H + \lambda)^{-1/2}$$
$$\times \prod_{j=1}^{L} \{ ST_j \} S (H + \lambda)^{-1/2} (Q_q^*)^m \phi, \tag{3.9}$$

By (3.4)

$$T_j = -ih (Z_j + Z_j^*),$$
$$Z_j = [(H + \lambda)^{-\frac{1}{2}} Q_l^*] [\partial_{x_l} \chi^{(j)} (H + \lambda)^{-\frac{1}{2}}].$$

By (3.2) and (3.3), for any natural $N > d/2$

$$|| T_j || \leq 2h || Z_j ||_{2N} \leq C \lambda^{-\frac{1}{2} + \frac{d}{4N}} h^{1 - \frac{d}{2N}}.$$

Let $\lambda = \lambda_0 + \langle z \rangle$. Then by (3.1) and (3.6),

$$\left|\left| \prod_{j=1}^{2N} \{ ST_j \} \right|\right|_1 \leq || S ||^{2N} \prod_{j=1}^{2N} || T_j ||_{2N} \leq C_n \left[\frac{h \langle z \rangle}{d(z)} \right]^{2N} \langle z \rangle^{-N + \frac{d}{2}} h^{-d}$$
$$\leq C_N \left[\frac{\langle z \rangle^{\frac{1}{2}}}{h} \right]^d \left[\frac{h^2 \langle z \rangle}{d(z)^2} \right]^N.$$

It follows from (3.9) with $L = 2N$ that

$$||\chi Q_l^r (H - z)^{-1} (Q_q^*)^m \phi ||_1 \leq || Q_l^r (H + \lambda)^{-1/2} ||$$
$$\left\| \prod_{j=1}^{2N} \{ ST_j \} \right\|_1 || S || \, || (H + \lambda)^{-\frac{1}{2}} (Q_q^*)^m ||.$$

Using (3.3), (3.6), we obtain (3.8). □

COROLLARY 3.7. – *Let $\chi \in C_0^\infty (\mathbf{R}^d)$ and let k, n be two integers such that $n > d/4$, $k \leq 2n$. Then for $\lambda \geq \lambda_0$*

$$|| \chi (H + \lambda)^{-k} ||_{\frac{2n}{k}} \leq C \lambda^{-k} (\lambda^{\frac{1}{2}} h^{-1})^{\frac{dk}{2n}}. \tag{3.10}$$

Proof. – By (3.2) the bound (3.10) is true for $k = 1$. The further proof is by induction: assuming that (3.10) is true for some k, we shall deduce (3.10) for $k + 1$. Let $\chi_1 \in C_0^\infty (\mathbf{R}^d)$ be a function such that the pair χ, $1 - \chi_1$ satisfy (3.7). Denote $\phi = 1 - \chi_1$. Then by (3.1)

$$|| \chi (H + \lambda)^{-k-1} ||_{\frac{2n}{k+1}} \leq || \chi (H + \lambda)^{-1} \phi ||_1 \, || (H + \lambda)^{-k} ||$$
$$+ || \chi (H + \lambda)^{-1} ||_{2n} \, || \chi_1 (H + \lambda)^{-k} ||_{\frac{2n}{k}} .$$

By Lemma 3.6 the first summand is bounded by

$$C_t \, \lambda^{-k-1} \left[\frac{\lambda^{\frac{1}{2}}}{h} \right]^d \left[\frac{\lambda^{\frac{1}{2}} h}{\lambda} \right]^t \leq C_t \, \lambda^{-k-1} \left[\frac{\lambda^{\frac{1}{2}}}{h} \right]^{d-t}, \qquad \forall t \geq 0.$$

Choosing $t = d (1 - (k + 1) (2n)^{-1})$, we get (3.10). Further, by (3.2) and the inductive assumption the second term does not exceed

$$C \lambda^{-k-1} (\lambda^{\frac{1}{2}} h^{-1})^{\frac{d}{2n}} (\lambda^{\frac{1}{2}} h^{-1})^{\frac{dk}{2n}} = C \lambda^{-k-1} (\lambda^{\frac{1}{2}} h^{-1})^{\frac{d(k+1)}{2n}} .$$

Two last estimates provide (3.10) with $k+1$, which completes the proof. □

3. Let us proceed now to the study of functions of the operators with magnetic field. Our basic tool in this analysis will be the following representation established in [1]:

PROPOSITION 3.8. – *Let $g \in C_0^\infty(\mathbb{R})$. Then for any selfadjoint operator A the relation holds:*

$$g(A) = \sum_{j=0}^{n-1} \frac{1}{\pi j!} \int_{\mathbb{R}} \partial^j g(\lambda) \operatorname{Im}[i^j (A - \lambda - i)^{-1}] \, d\lambda$$
$$+ \frac{1}{\pi (n-1)!} \int_0^1 \tau^{n-1} \, d\tau$$
$$\times \int_{\mathbb{R}} \partial^n g(\lambda) \operatorname{Im}[i^n (A - \lambda - i\tau)^{-1}] \, d\lambda, \qquad \forall n \geq 2. \tag{3.11}$$

We start with

LEMMA 3.9. – *Let $g \in C_0^\infty(\mathbb{R})$ and $\chi \in C_0^\infty(\mathbb{R}^d)$. Then*

$$\| \chi g(H) \|_1 \leq C h^{-d}. \tag{3.12}$$

If the functions χ, ϕ satisfy the conditions of Lemma 3.6, then for any $N \geq 0$

$$\| \chi g(H) \phi \|_1 \leq C_N h^N. \tag{3.13}$$

Proof. – Let $\lambda \geq \lambda_0$ and $k > d/2$. The inequality

$$\| \chi g(H) \|_1 \leq \| \chi (H + \lambda)^{-k} \|_1 \, \| g(H) (H + \lambda)^k \|$$

and the bound (3.10) with $n = 2k$ lead to (3.12).

Proof of (3.13). Let $|\tau| \leq 1$. The bound (3.8) yields:

$$\left.\begin{array}{c} \| \chi (H - \lambda - i\tau)^{-1} \phi \|_1 \leq C_N \, h^{2N-d} \, \tau^{-2N-1}, \\ \forall \lambda \in \operatorname{supp} g, \qquad \forall N > d/2. \end{array}\right\} \tag{3.14}$$

Now, by (3.11)

$$g(H) = I_1^{(n)} + I_2^{(n)}, \qquad \forall n \in \mathbb{N};$$
$$I_1^{(n)} = \sum_{j=0}^{n-1} \frac{1}{\pi j!} \int_{\mathbb{R}} \partial^j g(\lambda) \operatorname{Im}\{ i^j (H - \lambda - i)^{-1} \} \, d\lambda,$$
$$I_2^{(n)} = \frac{1}{\pi (n-1)!} \int_0^1 \tau^{n-1} \, d\tau \int_{\mathbb{R}} \partial^n g(\lambda) \operatorname{Im}[i^n (H - \lambda - i\tau)^{-1}] \, d\lambda.$$

Set $n = 2N + 2$. Then in view of (3.14) $\| \chi I_1^{(n)} \|_1 \leq C_N \, h^{2N-d}$ and

$$\| \chi I_2^{(n)} \|_1 \leq C \, h^{2N-d} \int_0^1 d\tau \int_{|\lambda| \leq C} d\lambda \leq C_N \, h^{2N-d}.$$

This proves (3.13). □

So far the operator under consideration was assumed to have the form $H = H_a$ in the entire space. Now we weaken this restriction and replace H_a by an operator $\mathcal{H} = \mathcal{H}_a$ which obeys

ASSUMPTION 3.10. – *The operator $\mathcal{H}$ satisfies Assumption 1.1 with $\mathcal{D} = \overset{\circ}{B}(4E)$ for some $E > 0$.*

Below by $H = H_a$ we denote the operator from (1.3). We are going to compare $\operatorname{tr}\{\chi g(\mathcal{H})\}$ and $\operatorname{tr}\{\chi g(H)\}$ for some $\chi \in C_0^\infty(B(4E))$. At first we look at the resolvents of $\mathcal{H}$ and H. In this analysis the crucial role will be played by the following version of the resolvent identity. Let $\chi \in C_0^\infty(B(4E))$. Then for any z, $\operatorname{Im} z \neq 0$,

$$\left.\begin{aligned} \chi(\mathcal{H}-z)^{-1} &= (H-z)^{-1}\chi - (H-z)^{-1} Z (\mathcal{H}-z)^{-1}, \\ Z &= -[H, \chi] = ih(Q_l^* \partial_{x_l}\chi + \partial_{x_l}\chi Q_l). \end{aligned}\right\} \tag{3.15}$$

To prove (3.15) we use (1.3) and the fact that $C_0^\infty(B(4E)) \subset D(\mathcal{H})$. Recall that due to the boundedness of a the operator Q_l is selfadjoint and therefore one can rewrite Z as

$$Z = ih(2\, Q_l\, \partial_{x_l}\chi + ih\Delta\chi).$$

LEMMA 3.11. – *Let $\mathcal{H}$ satisfy Assumption 3.10 and $\chi \in C_0^\infty(B(3E))$ be some function. Then for any $N > d/2$*

$$\begin{aligned} &\|\chi\{(\mathcal{H}-z)^{-1} - (H-z)^{-1}\}\|_1 \\ &\quad\le C_N \left[\frac{\langle z\rangle^{\frac12}}{h}\right]^d \left[\frac{h^2\langle z\rangle}{d(z)^2}\right]^{N+\frac12} \frac{1}{|\operatorname{Im} z|}, \end{aligned} \tag{3.16}$$

The constant C_N depends only on the function χ and λ_0.

Proof. – Define $\chi_1 \in C_0^\infty(B(20E/6))$ such that $\chi_1(x) = 1$, $|x| \le 19E/6$, so that $\chi_1\chi = \chi$. Denote $\phi = 1 - \chi_1$. Due to the obvious identity

$$\begin{aligned} &\chi[(\mathcal{H}-z)^{-1} - (H-z)^{-1}] \\ &\quad= \chi[\chi_1(\mathcal{H}-z)^{-1} - (H-z)^{-1}\chi_1] - \chi(H-z)^{-1}\phi \end{aligned}$$

the problem amounts to proving the bound (3.16) for the operators

$$\begin{aligned} T_1 &= \chi[\chi_1(\mathcal{H}-z)^{-1} - (H-z)^{-1}\chi_1], \\ T_2 &= \chi(H-z)^{-1}\phi. \end{aligned}$$

By (3.8), the estimate (3.16) is obviously satisfied for T_2. Taking into account the resolvent identity (3.15), we have

$$\| T_1 \|_1 \leq h \, [2 \, \| \chi (H - z)^{-1} \, Q_l \, \partial_{x_l} \, \chi_1 \|_1 + h \, \| \chi (H - z)^{-1} \, \Delta \chi_1 \|_1] \, \| (\mathcal{H} - z)^{-1} \|.$$

By definition supp $\partial_x \chi_1$ and supp χ obey (3.7), so that the conditions of Lemma 3.6 are fulfilled. Estimating the terms in the curly brackets by means of (3.8), and the last factor by $| \operatorname{Im} z |^{-1}$, we get (3.16). □

We apply this Lemma to the study of $g(\mathcal{H}) - g(H)$:

THEOREM 3.12. – *Let $\mathcal{H}$ satisfy Assumtion 3.10. Let $g \in C^\infty(\mathbb{R}) \cap C^{\infty,1}(\mathbb{R})$. If $\chi \in C_0^\infty(B(3E))$ then for any $N \geq 0$*

$$\| \chi [g(\mathcal{H}) - g(H)] \|_1 \leq C_N \, h^N \tag{3.17}$$

and

$$\| \chi g(\mathcal{H}) \|_1 \leq C \, h^{-d}. \tag{3.18}$$

The constants C_N and C in (3.17) and (3.18) respectively depend only on λ_0, constants C_n in Definition 2.7 and the function χ.

Proof. – Let $\lambda \in \mathbb{R}$ and $0 < | \tau | \leq 1$. Denote

$$\delta(\lambda, \tau) = (\mathcal{H} - \lambda - i\tau)^{-1} - (H - \lambda - i\tau)^{-1}.$$

Then (3.16) yields for any $N > d/2$:

$$\| \chi \delta(\lambda, \tau) \|_1 \leq C_N \, h^{2N+1-d} \, \langle \lambda \rangle^{N + \frac{d+1}{2}} \, | \tau |^{-2N-2}, \qquad \lambda \geq -\lambda_0 \, ; \tag{3.19}$$

$$\| \chi \delta(\lambda, \tau) \|_1 \leq C_N \, h^{2N+1-d} \, \langle \lambda \rangle^{-N + \frac{d-1}{2}} \, | \tau |^{-1}, \qquad \lambda \leq -\lambda_0. \tag{3.20}$$

The representation (3.11) does not apply to the function g since it is allowed to grow as $\lambda \to -\infty$. Instead of g we use its modification. Since the operators $\mathcal{H}$, H are semi-bounded from below and g obeys (1.4), one can find a function $\tilde{g} \in C_0^\infty(\mathbb{R})$ such that $\tilde{g}(\mathcal{H}) = g(\mathcal{H})$, $\tilde{g}(H) = g(H)$ and

$$| \partial^n \, \tilde{g}(\lambda) | \leq C_n \, \langle \lambda \rangle^r \tag{3.21}$$

with constants C_n independent of $\mathcal{H}$, H. According to (3.11)

$$\tilde{g}(\mathcal{H}) - \tilde{g}(H) = I_1^{(n)} + I_2^{(n)}, \qquad \forall n \in \mathbb{N};$$

$$I_1^{(n)} = \sum_{j=0}^{n-1} \frac{1}{\pi j!} \int_{\mathbb{R}} \partial^j \, \tilde{g}(\lambda) \operatorname{Im} \{ i^j \, \delta(\lambda, i) \} \, d\lambda,$$

$$I_2^{(n)} = \frac{1}{\pi (n-1)!} \int_0^1 \tau^{n-1} \, d\tau \int_{\mathbb{R}} \partial^n \, \tilde{g}(\lambda) \operatorname{Im} \{ i^n \, \delta(\lambda, \tau) \} \, d\lambda. \tag{3.22}$$

Set $n = 2N + 3$ and choose $N > r + (d+1)/2$. In view of (3.21) and (3.19), (3.20)

$$\| \chi I_1^{(n)} \|_1 \leq C_N \, h^{2N+1-d} \left\{ \int_{\lambda \leq -\lambda_0} \langle \lambda \rangle^{-N+r+\frac{d-1}{2}} \, d\lambda \right.$$
$$\left. + \int_{-\lambda_0 \leq \lambda \leq C} \langle \lambda \rangle^{N+r+\frac{d+1}{2}} \, d\lambda \right\} \leq C h^{2N+1-d}. \tag{3.23}$$

To estimate the integral (3.22) we present it in the form

$$I_2^{(2N+3)} = I_3^{(2N+3)} + I_4^{(2N+3)},$$

$$I_3^{(2N+3)} = \frac{1}{\pi \, (n-1)\,!} \int_0^1 \tau^{2N+2} \, d\tau$$
$$\times \int_{\lambda \leq -\lambda_0} \partial^n \, \tilde{g}\,(\lambda) \, \mathrm{Im} \, \{ \, i^{2N+3} \, \delta\,(\lambda, \, \tau) \, \} \, d\lambda,$$

$$I_4^{(2N+3)} = \frac{1}{\pi \, (n-1)\,!} \int_0^1 \tau^{2N+2} \, d\tau$$
$$\times \int_{-\lambda_0 \leq \lambda \leq C} \partial^n \, \tilde{g}\,(\lambda) \, \mathrm{Im} \, \{ \, i^{2N+3} \, \delta\,(\lambda, \, \tau) \, \} \, d\lambda.$$

According to (3.21) and (3.20)

$$\| \chi I_3^{(2N+3)} \|_1 \leq C h^{2N+1-d} \int_0^1 \tau^{2N+1} \, d\tau$$
$$\times \int_{\mathbf{R}} \langle \lambda \rangle^{-N+r+\frac{d-1}{2}} \, d\lambda \leq C h^{2N+1-d}.$$

Further, by virtue of (3.19) and (3.21)

$$\| \chi I_4^{(2N+3)} \|_1 \leq C h^{2N+1-d} \int_0^1 d\tau$$
$$\times \int_{-\lambda_0 \leq \lambda \leq C} \langle \lambda \rangle^{r+N+\frac{d+1}{2}} \, d\lambda \leq C h^{2N+1-d}.$$

Combining these bounds with (3.23), we obtain from here (3.17). The estimate (3.18) follow from (3.17) and (3.12). □

4. ASYMPTOTICS IN THE NON-CRITICAL CASE

1. Here we obtain the asymptotics of the trace $\mathcal{M}\,(h, \, \mu; \, \psi, \, g, \, \mathfrak{a})$ as $h \to 0$ for an arbitrary function $g \in C^{\infty, \, s}$, assuming that $0 < \mu \leq \mu_0 < 1$.

Our result will follow from Proposition 2.6. To be able to apply the latter, we shall have to impose on the potential V a "non-critical" condition (*see* (4.2) below), so that the symbol

$$k_\mu(x, \xi; \mathfrak{a}) = (\xi - \mu \mathbf{a}(x))^2 + V(x) \tag{4.1}$$

of the operator $H_\mathfrak{a}(h, \mu)$ satisfies Assumption 2.5. All the results to be obtained will be uniform in $\mu \in (0, \mu_0]$ and in the functions $\mathbf{a}$, V, $\psi \in C_0^\infty(\mathbb{R}^d)$ satisfying the bounds (1.6). Unless otherwise stated the dependence on other parameters or functions is not controlled.

THEOREM 4.1. – *Let* $0 < \mu \leq \mu_0$, $\psi \in C_0^\infty(B(E/2))$ *and* $g \in C^{\infty, s}(\mathbb{R})$, $s \in [0, 1]$.

Suppose that the operator $\mathcal{H}$ *obeys Assumption 1.1 with* $D = \overset{\circ}{B}(4E)$ *and*

$$|V(x)| + |\partial V(x)|^2 \geq c, \qquad \forall x \in B(2E). \tag{4.2}$$

Then

$$\mathcal{M}(h, \mu) = (2\pi h)^{-d} \int \psi(x)\, g(\xi^2 + V(x))\, dx\, d\xi + O(h^{s+1-d}). \tag{4.3}$$

In particular, for $g = g_s$ *the asymptotics (1.7) holds. The remainder estimate in (4.3) is uniform in the functions* $\mathbf{a}$, V *and* ψ *satisfying the bounds (1.6), (4.2), but may depend on the function* g *and the numbers* E, μ_0.

Indeed, in the particular case $g = g_s$ the asymptotics (1.7) follows from (4.3) by integrating in ξ. In Sect. 6 we shall get rid of the condition (4.2) for this case.

Remark. – Theorem 4.1 remains true if we replace the condition (4.2) with

$$|V(x)| + |\partial V(x)|^2 + h \geq c, \qquad \forall x \in B(2E). \tag{4.2'}$$

In fact, (4.2′) implies that either $|V(x)| + |\partial V(x)|^2 \geq c/2$ for all $x \in B(2E)$ or $h \geq c/2$. In the former case the desired result follows from Theorem 4.1 with condition (4.2). In the latter case the trace $\mathcal{M}_s$ is bounded uniformly in h by Theorem 3.12. The same is true for both terms in the r.h.s. of (4.3). Therefore one can write down $\mathcal{M}_s$ in the form (4.3).

2. First of all we establish the asymptotics of $\mathcal{M}(h, \mu; \psi, g, \mathfrak{a})$ for the "true" Schrödinger operator $H = H_\mathfrak{a}$ with the same $\mathbf{a}$, $V \in C_0^\infty(\mathbb{R}^d)$ as above. Due to the semiboundedness of H from below one may think that the function g is compactly supported. Below f denotes a real-valued C_0^∞-function such that

$$\operatorname{dist}\{\operatorname{supp}(1 - f), \operatorname{supp} g\} \geq c > 0. \tag{4.4}$$

All the estimates will be uniform in the functions $\mathfrak{a}$, V and ψ satisfying the bounds (1.6) and

$$|a_l(x)| \leq C, \qquad 1 \leq l \leq d, \qquad x \in \mathbb{R}^d. \tag{4.5}$$

As was pointed out in Sect. 1 the restriction (4.5) is superfluous as far as the trace $\mathcal{M}(h, \mu; \psi, g, \mathfrak{a})$ is concerned. In the context of the ΨDO calculus, however, we need to impose this condition to be able to control the symbol $k_\mu(x, \xi; \mathfrak{a})$.

LEMMA 4.2. – *Let $\psi \in C_0^\infty(B(E/2))$ and $0 < \mu \leq \mu_0 < 1$.*

(1) *If $g \in C_0^\infty(\mathbb{R})$, then*

$$\begin{aligned}&\mathcal{M}(h, \mu; \psi, g, \mathfrak{a})\\&\quad = (2\pi h)^{-d} \int \psi(x)\, g(\xi^2 + V(x))\, dx\, d\xi + O(h^{2-d}).\end{aligned} \tag{4.6}$$

(2) *Suppose that for some λ, $|\lambda| \leq C$ the condition*

$$|V(x) - \lambda| + |\partial V(x)|^2 \geq c > 0, \qquad \forall x \in B(2E) \tag{4.7}$$

is satisfied. Then

$$\| \psi f(H)\, \chi_h(H - \lambda) \|_1 \leq C h^{-d}. \tag{4.8}$$

(3) *Let $g \in L^1(\mathbb{R})$ be a compactly supported function and let the condition (4.7) be fulfilled for all $\lambda \in \operatorname{supp} g$. Then*

$$\begin{aligned}&\mathcal{M}(h, \mu; \psi, f g^{(h)}, \mathfrak{a}) = (2\pi h)^{-d}\\&\quad \times \int \psi(x)\, g(\xi^2 + V(x))\, dx\, d\xi + O(h^{2-d}).\end{aligned} \tag{4.9}$$

For the proof we need, apart from Proposition 2.6, the following technical

LEMMA 4.3. – *Let $\psi \in C_0^\infty(B(E/2))$, $g \in C_0^\infty(\mathbb{R})$ and let the symbol θ obey (2.9) and*

$$\theta(x, \xi) = 1, \; (x, \xi) \in B_x(5E/6) \times B_\xi(R/2). \tag{4.10}$$

Then for sufficiently big $R = R(g)$

$$\| \psi g(H)(I - \Theta) \|_1 \leq C_N h^N, \; \forall N > 0. \tag{4.11}$$

The constant C_N depends on μ_0, g and the symbol θ.

Throughout this section we shall be using the following convention: for any two bounded operators $A_1 = A_1(h)$, $A_2 = A_2(h)$ we shall write $A_1 \sim A_2$ if $\| A_1 - A_2 \|_1 \leq C_N h^N$ for any $N > 0$, uniformly in the other parameters (if there are any).

Proof of Lemma 4.3. – Let $\psi_1 \in C_0^\infty(B(3E/4))$ be a function such that $\psi_1(x) = 1$, $x \in B(5E/8)$, so that the functions ψ and $\phi = 1 - \psi_1$ satisfy the condition (3.7). Let $f \in C_0^\infty(\mathbb{R})$ be a function satisfying (4.4). Then by (3.13)

$$\psi g(H) = \psi g(H) f(H) \sim \psi g(H) \psi_1 f(H).$$

In view of (3.12) it suffices to check that for large R

$$\| \psi_1 f(H)(I - \Theta) \| \leq C_N h^N, \qquad \forall N > 0. \tag{4.12}$$

It is easy to see that the symbol $k_\mu(x, \xi) = k_\mu(x, \xi; \mathbf{a}, V)$ defined by (4.1) satisfies (2.6), so that the operator H is h-admissible. Therefore one can use the representation (2.7):

$$f(H) = \sum_{n=0}^{N} h^n \operatorname{op}_h^{\mathrm{w}} a_{f,n} + h^{N+1} R_{f,N+1}(h), \ \forall N > 0, \tag{4.13}$$

where $a_{f,n}$ are given by (2.8). The bound (4.12) is obviously fulfilled for the remainder $R_{f,N+1}(h)$. Let us prove that each term in the sum satisfies (4.12) as well. According to (2.8), the estimate (4.12) amounts to proving that the product of three operators with the symbols

$$\psi_1(x), \qquad \partial_\lambda^m f(k_\mu(x, \xi)), \qquad 1 - \theta(x, \xi), \qquad m = 0, 1, \ldots,$$

obeys (4.12). Since the function f is compactly supported and the functions $\mathbf{a}$, V are bounded, we have

$$\operatorname{supp} f(k_\mu(.,.)) \subset \{(x, \xi) : |\xi| \leq C\}$$

with a suitable constant $C = C(\mathbf{a}, V, \mu_0)$. Thus

$$\operatorname{supp} \psi_1 \cap \operatorname{supp} f(k_\mu(.,.)) \subset \{(x, \xi) : |x| \leq 3E/4, \ |\xi| \leq C\}.$$

Therefore, choosing R in (4.10) large enough, one can guarantee that

$$\operatorname{supp} \psi_1 \cap \operatorname{supp} f(k_\mu(.,.)) \cap \operatorname{supp}(1 - \theta) = \emptyset.$$

Now the desired result follows from Lemma 2.2. $\square$

Let us establish a lower bound on k_μ which will guarantee the validity of Assumption 2.5:

LEMMA 4.4. – *Let* $\mathbf{a}$, V *be as in Lemma 4.2 and* $\mu \le \mu_0 < 1$. *Then for* $(x, \xi) \in \Lambda(\lambda)$, $\lambda \in \mathbb{R}$,

$$|\partial_x k_\mu(x, \xi; \mathfrak{a})|^2 + |\partial_\xi k_\mu(x, \xi; \mathfrak{a})|^2 \\ \ge (1-\mu_0)[|\partial V(x)|^2 + |\lambda - V(x)|]. \tag{4.14}$$

Proof. – Let $k_\mu(x, \xi) = k_\mu(x, \xi; \mathfrak{a})$. Let us calculate:

$$\partial_{x_j} k_\mu(x, \xi) = -2\mu(\xi_l - \mu a_l(x))\,\partial_{x_j} a_l(x) + \partial_{x_j} V(x),$$
$$\partial_{\xi_j} k_\mu(x, \xi) = 2(\xi_j - \mu a_j(x)).$$

Since $|\partial_{x_j} a_l(x)| \le 1$, we have for arbitrary $\varepsilon \in (0, 1)$

$$|\partial_x k_\mu(x, \xi)|^2 \ge (1-\varepsilon)(\partial_x V(x))^2 - 4\varepsilon^{-1}\mu^2(\xi - \mu\mathbf{a}(x))^2.$$

Thus

$$|\partial_x k_\mu(x, \xi)|^2 + |\partial_\xi k_\mu(x, \xi)|^2 \\ \ge 4(1-\varepsilon^{-1}\mu^2)(\xi - \mu\mathbf{a}(x))^2 + (1-\varepsilon)(\partial_x V(x))^2.$$

Since $\mu \le \mu_0 < 1$, for $\varepsilon = \mu_0$ the r.h.s. has the positive lower bound

$$(1-\mu_0)[|\partial V(x)|^2 + (\xi - \mu\mathbf{a}(x))^2].$$

On the set $\Lambda(\lambda)$ we always have

$$(\xi - \mu\mathbf{a}(x))^2 = \lambda - V(x) \ge 0,$$

so that

$$|\partial_x k_\mu(x, \xi)|^2 + |\partial_\xi k_\mu(x, \xi)|^2 \ge (1-\mu_0)[|\partial V(x)|^2 + |\lambda - V(x)|],$$

which coincides with (4.14). □

Proof of Lemma 4.2. – Let us begin with proving (4.9). Without loss of generality assume that $g(\lambda)$ and $\psi(x)$ are real-valued. Let θ be the symbol from Lemma 4.3, so that by (4.11) and (2.16)

$$\psi f(H)\, g^{(h)}(H) \sim \psi f(H)\,\Theta g^{(h)}(H).$$

Using the representation (4.13) and the formula (2.4) for the product of Ψ DO's, one can show that

$$\| \psi f(H) \Theta - \mathrm{op}_h^w \tilde{\theta} - ih \, \mathrm{op}_h^w \tilde{\theta}_1 \|_1 \le C h^{2-d}.$$

Here

$$\tilde{\theta}(x, \xi) = \psi(x) f(k_\mu(x, \xi)) \theta(x, \xi)$$

and $i\tilde{\theta}_1 \in C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_\xi^d)$, $\mathrm{Im}\, \tilde{\theta}_1 = 0$, is a subprincipal symbol of the operator $\psi f(H) \Theta$, calculated by means of (2.4). Therefore

$$\begin{aligned} \mathcal{N}_h(\psi, fg^{(h)}, H) &= \mathcal{N}_h(\tilde{\theta}, g^{(h)}, H) \\ &\quad + ih\mathcal{N}_h(\tilde{\theta}_1, g^{(h)}, H) + O(h^{2-d}). \end{aligned} \tag{4.15}$$

Note that all the $\mathcal{N}_h$ here are real-valued. Hence $ih\mathcal{N}_h(\tilde{\theta}_1, g^{(h)}, H) = O(h^{2-d})$. Further, by Lemma 4.4 and (4.7) the symbol k_μ satisfies. Assumption 2.5. Thus the conditions of Proposition 2.6 are fulfilled. It follows immediately from (2.19) that the first summand in the r.h.s. of (4.15) has the asymptotics

$$\mathcal{N}_h(\tilde{\theta}, g^{(h)}, H) = (2\pi h)^{-d} \int \tilde{\theta}(x, \xi)\, g(k_\mu(x, \xi))\, dx\, d\xi + O(h^{2-d}).$$

Direct calculation shows that for sufficiently big R

$$\tilde{\theta}(x, \xi)\, g(k_\mu(x, \xi)) = \psi(x)\, g(k_\mu(x, \xi)).$$

Making the change $\xi - \mathbf{a}(x) \to \xi$, we obtain (4.9).

Analogously, (4.6) and (4.8) follow from (2.17) and (2.18) respectively. □

3. Let now the operator $\mathcal{H}$ be as in Theorem 4.1. Without loss of generality we assume that in addition to (1.6) the field $\mathbf{a}$ obeys (4.5). To pass from Lemma 4.2 to Theorem 4.1 we have to compare propagators of H and $\mathcal{H}$. We denote them by $U(t; H)$ and $U(t; \mathcal{H})$ respectively.

LEMMA 4.5. – *Let $\mathcal{H}$ be as in Theorem 4.1 and (4.5) be fulfilled. Then there exists a number $T_0 > 0$ such that for $|t| \le T_0$*

$$\| \psi f(H) [U_h(t; \mathcal{H}) - U_h(t; H)] \|_1 \le C_N h^N, \ \forall N > 0. \tag{4.16}$$

The constant C_N is uniform in ψ, $\mathbf{a}$, V satisfying the bounds (1.6), (4.5).

Proof. Step 1. – *Approximation for $U_h(t; H)$ and $U_h(t; \mathcal{H})$.* We start with an approximation for the propagator $U_h(t; H)$. Since the symbol

$k_\mu(x, \xi; \alpha)$ is smooth, we can use the approach described in Sect. 2. Let θ be the symbol from Lemma 4.3. Let $G_h(t)$ denote the h-FIO with the kernel (2.10) for some fixed $N \in \mathbb{N}$. By Proposition 2.4 one can find a number $T_0 > 0$ and the smooth functions S and v_n in (2.10) in such a way that

$$\left.\begin{array}{c} \operatorname{supp} v_n(.,.,t) \subset B_x(3E) \times B_\xi(3R), \\ \forall t \in [-T_0, T_0], \qquad \forall n \geq 0, \\ \max_{|t| \leq T_0} \| -ih\partial_t G_h(t) + HG_h(t) \| \leq C_N h^{N+1}. \end{array}\right\} \tag{4.17}$$

Since $G_h(t)$ acts into $C_0^\infty(B(3E))$ we have in view of (1.3):

$$HG_h(t) = \mathcal{H}G_h(t). \tag{4.18}$$

Using this fact and the estimate (4.17) we are going to verify that the operator $G_h(t)$ is a good approximation for both $U_h(t; \mathcal{H})$ and $U_h(t; H)$:

$$\sup_{|t| \leq T_0} \| \Theta \{ U_h(t) - G_h(t) \} \|_1 \leq C_N h^{N-d}. \tag{4.19}$$

Here $U_h(t)$ denotes any of the two propagators $U_h(t; H)$ or $U_h(t; \mathcal{H})$. For brevity we shall prove (4.19) for $U_h(t) = U_h(t; \mathcal{H})$ only. Denote

$$M_h(t) = -ih\partial_t G_h(t) + \mathcal{H}G_h(t).$$

Recall that by (2.11) $G_h(0) = \Phi$. Since the propagator $U_h(t; \mathcal{H})$ satisfies the equation

$$-ih\partial_t U_h(t) + \mathcal{H}U_h(t) = 0, \qquad U_h(0) = I,$$

the difference $U_h(t) - G_h(t) = E_h(t)$ will satisfy

$$-ih\partial_t E_h(t) + \mathcal{H}E_h(t) = -M_h(t), \qquad E_h(0) = I - \Phi.$$

Integrating this equation we get

$$E_h(t) = -\frac{i}{h} \int_0^t U_h(t-s, \mathcal{H}) M_h(s)\, ds + (I - \Phi),$$

so that

$$\max_{|t| \leq T_0} \| \Theta E_h(t) \|_1 \leq \frac{T_0}{h} \| \Theta \|_1 \max_{|t| \leq T_0} \| M_h(t) \| + \| \Theta (I - \Phi) \|_1.$$

The last term obeys (4.19) by Lemma 2.2 since in view of (2.12) the supports of θ and $1 - \varphi$ are disjoint. To estimate $M_h(t)$ we take into account the equality (4.18) and the estimate (4.17), so that

$$\max_{|t| \leq T_0} \| M_h(t) \| \leq C_N h^{N+1}.$$

In combination with the bound (2.3) for Θ this completes the proof of (4.19).

Step 2. Proof of (4.16). – We present the operator in the l.h.s. of (4.16) in the form

$$\psi f(H)(I-\Theta)[U_h(t;\mathcal{H})-U_h(t;H)]+\psi f(H)\Theta[U_h(t;\mathcal{H})-U_h(t;H)].$$

Thus its trace norm does not exceed

$$\begin{aligned}&||\psi f(H)(I-\Theta)||_1\,||U_h(t;\,\mathcal{H})-U_h(t;\,H)||\\&+||\psi f(H)||\,||\,\Theta[U_h(t;\,\mathcal{H})-U_h(t;\,H)]\,||_1\\&\leq 2\,||\psi f(H)(I-\Theta)||_1+C\,||\,\Theta[U_h(t;\,\mathcal{H})-U_h(t;\,H)]\,||_1.\end{aligned}$$

The desired bounds for the first and the second terms follow from Lemma 4.3 and the bound (4.19) respectively. □

Our next step is to prove the following analogue of Lemma 4.2 for the operator $\mathcal{H}$ satisfying Assumption 1.1.

LEMMA 4.6. – *Let $\mathcal{H}$ be as in Theorem 4.1 and let $\psi\in C_0^\infty(B(E/2))$.*
(1) *If $g\in C_0^\infty(\mathbb{R})$ then*

$$\mathcal{M}(h,\,\mu;\,\psi,\,g,\,\mathfrak{a})=(2\pi h)^{-d}\times\int\psi(x)\,g(\xi^2+V(x))\,dx\,d\xi+O(h^{2-d}).\qquad(4.20)$$

(2) *Let for some λ, $|\lambda|\leq C$ the condition (4.7) be fulfilled. Theng $\in$ $C_0^\infty(\mathbb{R})$*

$$||\psi f(\mathcal{H})\chi_h(\mathcal{H}-\lambda)||_1\leq Ch^{-d}.\qquad(4.21)$$

(3) *Let $g\in L^1(\mathbb{R})$ be a compactly supported function. Suppose that the condition (4.7) is fulfilled for all $\lambda\in\operatorname{supp}g$. Then*

$$\mathcal{M}(h,\,\mu;\,\psi,\,g^{(h)}f,\,\mathfrak{a})=(2\pi h)^{-d}\times\int\psi(x)\,g(\xi^2+V(x))\,dx\,d\xi+O(h^{2-d}).\qquad(4.22)$$

Proof. – Without loss of generality we assume that $\mathfrak{a}$ satisfies (4.5). Then the relation (4.20) follows immediately from (3.17) and (4.6).

Let us prove (4.21) and (4.22). By (3.17)

$$\begin{aligned}\psi f(\mathcal{H})\,\chi_h(\mathcal{H}-\lambda)&\sim\psi f(H)\,\chi_h(\mathcal{H}-\lambda),\\\psi f(\mathcal{H})\,g^{(h)}(\mathcal{H})&\sim\psi f(H)\,g^{(h)}(\mathcal{H}).\end{aligned}$$

Furthermore, by Lemma 4.5 and definitions (2.13), (2.14) and (2.15)

$$\psi f(H)\,\chi_h(\mathcal{H}-\lambda) \sim \psi f(H)\,\chi_h(H-\lambda),$$
$$\psi f(H)\,g^{(h)}(\mathcal{H}) \sim \psi f(H)\,g^{(h)}(H).$$

It remains to apply (4.8) and (4.9). □

4. Now we are able to prove Theorem 4.1. We shall derive it from Lemma 4.6 with the help of the Tauberian argument (Proposition 2.8).

Step 1. – Assume that g is compactly supported, *i.e.* $g \in C_0^{\infty,s}$. At first we shall prove (4.3) under the condition that (4.7) is fulfilled for all $\lambda \in \operatorname{supp} g$. Note that this automatically implies that (4.7) is fulfilled for all $\lambda \in \mathcal{D}(\delta) = \{\lambda : \operatorname{dist}\{\operatorname{supp} g, \lambda\} \le \delta\}$ with sufficiently small $\delta > 0$. Let $\psi_1 \in C_0^\infty(\mathbb{R}^d)$ be a function such that $\psi\psi_1 = \psi_1$. Set $B = \psi_1\, f(\mathcal{H})$ and $A = \mathcal{H}$. According to (3.18) $\|B^* B\|_1 \le Ch^{-d}$ and by (4.7) the bound (2.20) holds with $Z(h) = h^{-d}$. Therefore by (2.21)

$$\|\psi\psi_1\, f(\mathcal{H})\,[g(\mathcal{H}) - g^{(h)}(\mathcal{H})]\, f(\mathcal{H})\,\psi_1\|_1 \le Ch^{s+1-d}.$$

By cyclicity of trace this means that

$$\mathcal{M}(h, \mu;\, \psi, g, \mathfrak{a}) = \mathcal{M}(h, \mu;\, \psi, f^2 g^{(h)}, \mathfrak{a}) + O(h^{s+1-d}).$$

Now (4.22) provides (4.3).

Step 2. – We still assume that $g \in C_0^{\infty,s}$. However instead of (4.7) the condition (4.2) is assumed to be satisfied. Let us break up the function g into two parts: $g = g' + g''$, where $g' \in C_0^\infty(\mathbb{R})$, $g'' \in C_0^{\infty,s}(\mathbb{R})$ and $\operatorname{supp} g'' \in [-\varepsilon, \varepsilon]$. The formula (4.3) for g' follows directly from (4.20). For sufficiently small ε the lower bound (4.2) guarantees the validity of the condition (4.7) for all $\lambda \in \operatorname{supp} g''$. Consequently, according to Step 1, (4.3) holds for g''. Adding up the answers for g' and g'', we arrive at (4.3) for g.

Step 3. – Now we can prove (4.3) for any $g \in C^{\infty,s}(\mathbb{R})$. We present g as $g = g' + g''$, where $g' \in C^\infty(\mathbb{R}) \cap C^{\infty,1}(\mathbb{R})$ and $g'' \in C_0^{\infty,s}$. Referring to the semiboundedness of H, we may assume that $g'(H) = 0$. Thus according to (3.17)

$$\begin{aligned}\mathcal{M}(h, \mu;\, \psi, g', \mathfrak{a}) &= \operatorname{tr}\{\psi g'(\mathcal{H})\} = \operatorname{tr}\{\psi g'(H)\} + O(h^N)\\ &= O(h^N), \qquad \forall N > 0.\end{aligned}$$

Therefore

$$\mathcal{M}(h, \mu; \psi, g, \mathfrak{a}) = \mathcal{M}(h, \mu; \psi, g', \mathfrak{a}) + \mathcal{M}(h, \mu; \psi, g'', \mathfrak{a})$$
$$= \mathcal{M}(h, \mu; \psi, g'', \mathfrak{a}) + O(h^N).$$

Now the desired result follows from Step 2. Theorem 4.1 is proven.

5. MULTISCALE ANALYSIS

So far we have been interested in the asymptotics of $\mathcal{M}(h, \mu; \psi, g, \mathfrak{a})$ uniform in the functions ψ, **a**, V satisfying the bounds (1.6) and (4.2) (*see* Theorem 4.1). In this section we describe an elementary approach due to V. Ivrii (*see* [7]-[9] and also [3], [6]), which provides an explicit control of the remainder in the asymptotics in question under more general conditions on ψ, **a**, V in the case $g = g_s$, $s \in [0, 1]$.

1. We are going to study the following problem. Let $\mathcal{D} \in \mathbb{R}^d$ be an open set. Suppose that one is given two real-valued functions $f \in C(\bar{\mathcal{D}})$, $l \in C^1(\bar{\mathcal{D}})$ such that

$$\begin{aligned} &f(x) > 0, \qquad l(x) > 0, \qquad x \in \bar{\mathcal{D}};\\ &|\partial_x l(x)| \le \varrho < 1, \qquad x \in \mathcal{D}; \end{aligned} \tag{5.1}$$

$$cf(y) \le f(x) \le Cf(y), \qquad \forall x \in \mathcal{D} \cap B(y, l(y)), \quad y \in \mathcal{D}; \tag{5.2}$$

Our objective is the asymptotics of $\mathcal{M}_s$ for an operator $\mathcal{H}$ satisfying Assumption 1.1 with the domain $\mathcal{D}$ and some functions $V, a_l \in C_0^\infty(\mathbb{R}^d)$, $\psi \in C_0^\infty(\mathcal{D})$ which obey the bounds

$$\left.\begin{aligned} &|\partial_x a_l(x)| \le 1,\ |\partial_x^m a_l(x)| \le C_m l(x)^{1-|m|},\ |m| \ge 2;\\ &|\partial_x^m V(x)| \le C_m f(x)^2 l(x)^{-|m|},\ |\partial_x^m \psi(x)|\\ &\le C_m l(x)^{-|m|},\ |m| \ge 0, \end{aligned}\right\} x \in \mathcal{D}. \tag{5.3}$$

One can think of $f(x)^2$ as a measure of the size of $V(x)$, while $l(x)$ characterizes the behaviour of $V(x)$, $\mathbf{a}(x)$ and $\psi(x)$ under differentiation. Emphasize that we do not assume any uniformity of $f(x)$, $l(x)$ for $x \in \mathcal{D}$ in the parameters h, μ, so the conditions (5.3) are definitely more general than (1.6). The aim is to obtain an asymptotics of $\mathcal{M}_s$ with an explicit dependence of the remainder on the functions $f(x)$ and $l(x)$. To that end we shall use extensively the following scaling properties of the operator $\mathcal{H}_{\mathfrak{a}}$ and the trace $\mathcal{M}_s(h, \mu; \psi, \mathfrak{a})$. Let f, l be some positive numbers and let $z \in \mathbb{R}^d$. Let the unitary dilation operator $\mathcal{U}_l$ and the translation operator T_z be defined by

$$(\mathcal{U}_l u)(x) = l^{\frac{d}{2}} u(lx), \ (T_z u)(x) = u(x+z).$$

Denote

$$\hat{V}(x) = f^{-2} V(lx+z), \ \hat{\mathbf{a}}(x) = l^{-1} \mathbf{a}(lx+z), \ \hat{\psi}(x) = \psi(lx+z). \tag{5.4}$$

Define also two auxiliary parameters which will play the role of the Planck constant and the size of the magnetic field after the scaling:

$$\alpha = \frac{h}{fl}, \qquad \nu = \frac{\mu l}{f}. \tag{5.5}$$

It is clear that the operator

$$f^{-2} (\mathcal{U}_l T_z) \mathcal{H}_{\mathfrak{a}} (\mathcal{U}_l T_z)^* \tag{5.6}$$

satisfies Assumption 1.1 with the set $\hat{\mathcal{D}} = \{x \in \mathbb{R}^d : lx + z \in \mathcal{D}\}$ and the operator $H_{\mathfrak{a}}(\alpha, \nu)$, $\hat{\mathfrak{a}} = \{\hat{\mathbf{a}}, \hat{V}\}$. Therefore it is natural to denote the operator (5.6) by $\mathcal{H}_{\hat{\mathfrak{a}}}$. By the unitary equivalence of trace,

$$\mathcal{M}_s(h, \mu; \psi, \mathfrak{a}) = f^{2s} \mathcal{M}_s(\alpha, \nu; \hat{\psi}, \hat{\mathfrak{a}}). \tag{5.7}$$

Note that in the case $\mathcal{D} = \overset{\circ}{B}(z, l)$ the set $\hat{\mathcal{D}}$ is simply $\overset{\circ}{B}(1)$.

It is important that the precise form of the leading term of the asymptotics of $\mathcal{M}_s$ is irrelevant to our method. We assume instead that the leading term is given by some functional $\mathfrak{B}_s(h, \mu; \psi) = \mathfrak{B}_s(h, \mu; \psi, \mathfrak{a})$ which obeys the same properties as $\mathcal{M}_s(h, \mu; \psi, \mathfrak{a})$. First of all, it is additive:

$$\begin{aligned} \mathfrak{B}_s(h, \mu; \psi_1 + \psi_2) &= \mathfrak{B}_s(h, \mu; \psi_1) \\ &+ \mathfrak{B}_s(h, \mu; \psi_2), \ \forall \psi_1, \psi_2 \in C_0^\infty(\mathbb{R}^d). \end{aligned} \tag{5.8}$$

Secondly, as in (5.7),

$$\mathfrak{B}_s(h, \mu; \psi, \mathfrak{a}) = f^{2s} \mathfrak{B}_s(\alpha, \nu; \hat{\psi}, \hat{\mathfrak{a}}). \tag{5.9}$$

It is easy to check that these conditions are fulfilled for the Weyl coefficient defined in (1.4).

2. *Reference problem.* Our starting point is the asymptotics of $\mathcal{M}_s(h, \mu; \phi, \mathfrak{b})$ with $\mathfrak{b} = \{\mathbf{b}, W\}$ and a function ϕ satisfying the conditions (5.3) with $f = l = 1$ for $\mathcal{D} = \overset{\circ}{B}(8)$ (in other words, for $\mathbf{b}$, W, ϕ obeying (1.6) with $E = 2$). Then, by the use of an appropriate partition of unity (associated with the function $l(x)$) in combination with scaling and translation transformations we obtain an asymptotics for $\mathcal{M}_s(h, \mu; \psi, \mathfrak{a})$ under conditions (5.3). Our result will have a conditional nature: we shall deduce the asymptotics in general case, making certain assumptions on the reference problem. First of all, we assume that

$$|\mathcal{M}_s(h, \mu; \phi, \mathfrak{b}) - \mathfrak{B}_s(h, \mu; \phi, \mathfrak{b})| \leq r(h, \mu) \tag{5.10}$$

with the functional $\mathfrak{B}_s$ introduced above and a remainder $r = r(h, \mu) > 0$. The function r is supposed to be uniform in W, $\mathbf{b}$, ϕ satisfying (1.6), in the sense that it depends only on the constants in (1.6). This assumption is crucial for the approach.

We need also a sort of a non-critical condition, generalizing (4.2′). Precisely, let $F = F(t, x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, be a real-valued function such that

$$F(\tau t, \tau^{\frac{1}{2}} x) = \tau F(t, x), \qquad \forall \tau > 0. \tag{5.11}$$

Then we assume that

$$F(|W(x)| + h, \partial_x W(x)) \geq \kappa, \; x \in B(4), \tag{5.12}$$

with some $\kappa \geq 0$.

Let us sum up the hypotheses on the reference operator $\mathcal{H}_\mathfrak{b}$:

ASSUMPTION 5.1. – *If the operator $\mathcal{H}_\mathfrak{b}$ obeys Assumption 1.1 with $\mathcal{D} = \overset{\circ}{B}(8)$, and $h \in (0, h_0]$, $0 \leq \mu \leq \mu_0 h^{-\varsigma}$ for some fixed $h_0 > 0$, $\mu_0 \geq 0$, $\varsigma \geq 0$, then under the condition (5.12) the estimate (5.10) holds with a functional $\mathfrak{B}_s(h, \mu; \phi, \mathfrak{b})$, which obeys (5.8), (5.9), and some locally bounded function $r(h, \mu) > 0$, which is uniform in W, $\mathbf{b}$, ϕ satisfying (1.6).*

Note that by Theorem 4.1 this assumption is fulfilled for any $h_0 > 0$, $\mu_0 < 1$, $\kappa > 0$ and $\varsigma = 0$, $F(t, x) = t + |x|^2$, $\mathfrak{B}_s = \mathfrak{W}_s$, $r(h, \mu) = Ch^{s+1-d}$, $C = C(h_0, \mu_0, \kappa)$.

3. To apply the reference problem to that formulated in subsection 1 we have to impose the following supplementary restrictions on the functions $f(x)$, $l(x)$, $V(x)$. We suppose that

$$h_0 f(x) l(x) \geq h; \qquad \mu_0 f(x)^{\varsigma+1} l(x)^{\varsigma-1} \geq \mu h^\varsigma, \quad x \in \mathcal{D}, \tag{5.13}$$

with the parameters $h_0 > 0$, $\mu_0 \geq 0$, $\varsigma \geq 0$ introduced in Assumption 5.1. Furthermore, for some $\omega > 0$

$$F\left(|V(x)| + \frac{hf(x)}{l(x)},\ \frac{l(x)}{f(x)}\,\partial_x V(x)\right) \geq \omega\kappa f(x)^2, \qquad x \in \mathcal{D} \quad (5.14)$$

with the same number κ as in (5.12). We also need the following condition on supp ψ:

$$\bigcup B(x,\, 8l(x)) \subset \mathcal{D}, \qquad (5.15)$$

where the union is taken over those $x \in \mathcal{D}$, for which $B(x,\, l(x)) \cap$ supp $\psi \neq \emptyset$.

For any set $K \subset \bar{\mathcal{D}}$ we denote

$$R(h,\, \mu;\, K) = \int_K f(x)^{2s}\, r\left(\frac{h}{l(x)\, f(x)},\ \frac{\mu l(x)}{f(x)}\right) l(x)^{-d}\, dx, \qquad (5.16)$$

where the function r is defined in Assumption 5.1. This integral makes sense since $l(x)$ is positive in $\bar{\mathcal{D}}$.

We shall need the following notion:

DEFINITION. – *A measurable function $f : \mathbb{R}^m \to \mathbb{C}$, $m \geq 1$, is said to be of moderate variation if for* a. a. $x,\, x' \in \mathbb{R}^m$ *the condition $C_1^{-1} \leq |x|/|x'| \leq C_1$ implies $C_2^{-1} \leq |f(x)|/|f(x')| \leq C_2$ with $C_2 = C_2(C_1)$.*

Now we can state the main result of this section:

THEOREM 5.2. – *Let the functions $l(x)$, $f(x)$ satisfy (5.1), (5.2) with $\varrho < 1/8$ and Assumption 5.1 be fulfilled for any W,* $\mathbf{b}$, *ϕ satisfying (1.6) for $E = 2$, with some functions r, F of moderate variation. Let the operator $\mathcal{H}_{\mathfrak{a}}$ obey Assumption 1.1 for an open set $\mathcal{D}$ with the functions V,* $\mathfrak{a}$ *and ψ satisfying the conditions (5.3), (5.13)-(5.15) with a sufficiently big $\omega > 0$. Then*

$$|\mathcal{M}_s(h,\, \mu;\, \psi,\, \mathfrak{a}) - \mathfrak{B}_s(h,\, \mu;\, \psi,\, \mathfrak{a})| \leq CR(h,\, \mu;\, \mathcal{D}), \qquad (5.17)$$

where the constant C is uniform in the functions a_l, V, f, l, ψ satisfying (5.1)-(5.3) and (5.13)-(5.15).

4. *Particular case.* We start the proof of Theorem 5.2 with the following particular case. Let the operator $\mathcal{H}_{\mathfrak{a}}$ satisfy Assumption 1.1 for the ball $\overset{\circ}{B}(z, 8l)$ with some $z \in \mathbb{R}^d$, $l > 0$. Suppose also that the conditions (5.3), (5.13), (5.14) are fulfilled with $\psi \in C_0^\infty(B(z, l))$, $l(x) = l$, $f(x) = f > 0$, $\omega = 1$. Let the parameters α, ν and the functions $\hat{\mathfrak{a}}$, $\hat{V}$, $\hat{\psi}$ be defined by (5.5), (5.4). Obviously, these functions satisfy (1.6) for all $x \in \overset{\circ}{B}(8)$ with the constants C_m from (5.3).

LEMMA 5.3. – *Let the operator $\mathcal{H}_{\mathfrak{a}}$ be as above and Assumption 5.1 be fulfilled. Let $\psi \in C_0^\infty(B(z, l))$. Then*

$$|\mathcal{M}_s(h, \mu; \psi, \mathfrak{a}) - \mathfrak{B}_s(h, \mu; \psi, \mathfrak{a})| \leq f^{2s}\, r\left(\frac{h}{fl}, \frac{\mu l}{f}\right). \tag{5.18}$$

The function r depends only on the constants C_m in (5.3).

Proof. – Due to the condition (5.13) we have $\alpha \leq h_0$, $\nu \leq \mu_0\, \alpha^{-\varsigma}$. As noted above, the functions $W = \hat{V}$, $\mathbf{b} = \hat{\mathfrak{a}}$, $\phi = \hat{\psi}$ obey the conditions (1.6) in the ball $\overset{\circ}{B}(8)$. Furthermore, by (5.14) and (5.11),

$$\begin{aligned} &F(|\hat{V}(x)| + \alpha, \partial_x \hat{V}(x)) \\ &= f^{-2} F\left(|V(y)| + \frac{hf}{l}, \frac{l}{f}\partial_y V(y)\right)\bigg|_{y = lx+z} \geq \kappa, \qquad x \in B(4). \end{aligned}$$

Thus, by Assumption 5.1,

$$|\mathcal{M}_s(\alpha, \nu; \hat{\psi}, \hat{\mathfrak{a}}) - \mathfrak{B}_s(\alpha, \nu; \hat{\psi}, \hat{\mathfrak{a}})| \leq r(\alpha, \nu),$$

which gives the desired result by (5.5), (5.7) and (5.9). □

5. *Proof of Theorem* 5.2. – To apply Lemma 5.3, we need to introduce a partition of unity associated with the function $l(x)$. Due to (5.1) we can look at $l = l(x)$ as a function which defines *a slowly varying metric* in $\mathcal{D}$ (*see* [5, Sect. 1.4] for definition), which gives rise first to a covering of $\mathcal{D}$ and then to a subordinate partition of unity.

LEMMA 5.4. – *Let $l(x)$ satisfy (5.1) with a constant $\varrho < 1$. Then*

(1) *There exists a sequence $x_k \in \mathcal{D}$, $k = 1, 2, \ldots$ such that the open balls $\overset{\circ}{B}(x_k, l(x_k))$ form a covering of $\mathcal{D}$,* i.e. $\mathcal{D} \subset \cup_k \overset{\circ}{B}(x_k, l(x_k))$. *There exists a number $N = N_\varrho$, depending on the constant ϱ in (5.1) only, such that intersection of more than N_ϱ balls is empty.*

(2) *One can choose a sequence* $\psi_k \in C_0^\infty(\overset{\circ}{B}(x_k, l(x_k)))$, $k = 1, 2, \ldots$ *such that*

$$|\partial_x^m \psi_k(x)| \leq C_m l(x_k)^{-|m|}, \forall k = 1, 2, \ldots, \tag{5.19}$$

and $\sum_k \psi_k(x) = 1,\ x \in \mathcal{D}$.

The constants C_m *in (5.19) depend only on* ϱ.

This Lemma can be proven analogously to [5, Theorem 1.4.10].

Proof of Theorem 5.2. – Let $\{x_k, \psi_k\}$ be a partition of unity subordinate to the function $l(x)$ constructed in Lemma 5.4. Denote $B_k = B(x_k, l(x_k))$. Then

$$\left.\begin{aligned} \mathcal{M}_s(h, \mu; \psi) &= \sum_k \mathcal{M}_s(h, \mu; \psi_k \psi), \\ \mathfrak{B}_s(h, \mu; \psi) &= \sum_k \mathfrak{B}_s(h, \mu; \psi_k \psi). \end{aligned}\right\} \tag{5.20}$$

The second equality follows from (5.8). We can think that the index k in (5.20) runs over the set $\mathcal{S} \in \mathbb{N}$ such that supp $\psi \cap B_k \neq \emptyset$. We are going to prove that

$$|\mathcal{M}_s(h, \mu; \psi_k\psi) - \mathfrak{B}_s(h, \mu; \psi\psi_k)| \leq CR(h, \mu; B_k), \qquad k \in \mathcal{S} \tag{5.21}$$

(*See* (5.16) for definition of $R(\,.\,)$). Here the constant C does not depend on μ, h, k. We claim that (5.21) leads to (5.17). Indeed, by (5.15) we have $B_k \subset \mathcal{D}$, $\forall k \in \mathcal{S}$ and, in addition, the intersection of more than N_ϱ (the number from Lemma 5.4) balls B_k is empty, so that we arrive at the bound

$$\sum_{k \in \mathcal{S}} R(h, \mu; B_k) \leq CR(h, \mu; \mathcal{D}), \qquad C = C(N_\varrho).$$

Now, adding up (5.21) for different $k \in \mathcal{S}$ and taking into account (5.20), we get (5.17).

Thus it remains to establish (5.21). Denote for brevity $l_k = l(x_k)$, $f_k = f(x_k)$. Notice first of all that by (5.1), (5.2) and (5.15)

$$\left.\begin{aligned} (1 - 8\varrho)\, l_k \leq l(x) \leq (1 + 8\varrho)\, l_k, \\ cf_k \leq f(x) \leq Cf_k, \end{aligned}\right\} \forall x \in B(x_k, 8 l_k), \tag{5.22}$$

with some constants c, C independent of k. Since $\varrho < 1/8$, the conditions (5.3), (5.19) imply that $\psi\psi_k$, $\mathbf{a}$, V obey the estimates (5.3) with $f = f_k$, $l = l_k$. Since F is a function of moderate variation, (5.14) implies in view of (5.22) that

$$F\left(|V(x)| + \frac{hf_k}{l_k},\ \frac{l_k}{f_k}\,\partial_x V(x)\right)$$
$$\geq cF\left(|V(x)| + \frac{hf(x)}{l(x)},\ \frac{l(x)}{f(x)}\,\partial_x V(x)\right)$$
$$\geq c\omega\kappa f(x)^2 \geq c'\omega\kappa f_k^2,\ \forall x \in B(x_k, 4l_k).$$

Thus for sufficiently big ω the conditions of Lemma 5.3 are fulfilled for $l = l_k$, $f = f_k$ and therefore one can apply the estimate (5.18). Hence the l.h.s. of (5.21) is bounded by

$$f_k^{2s}\, r\left(\frac{h}{l_k f_k},\ \frac{\mu l_k}{f_k}\right) \leq C \int_{B_k} f_k^{2s}\, r\left(\frac{h}{l_k f_k},\ \frac{\mu l_k}{f_k}\right) l_k^{-d}\, dx.$$

For r is of moderate variation, one can estimate the r.h.s. by

$$C \int_{B_k} f(x)^{2s}\, r\left(\frac{h}{l(x) f(x)},\ \frac{\mu l(x)}{f(x)}\right) l(x)^{-d}\, dx.$$

This provides (5.21). As was mentioned above, (5.21) leads to (5.17). □

6. PROOF OF THEOREM 1.2

In this short section we complete the proof of Theorem 1.2. Below we assume that the conditions of this Theorem are fulfilled. As we noted in Sect. 4, the asymptotics (4.3) for $g = g_s$ provides (1.7). Thus it remains to get rid of the condition (4.2). We do it in two steps, using a sort of a bootstrap argument: first, by means of Theorem 5.2 we prove Theorem 1.2 for $\mu \leq \mu_0 < 1$ and then, relying on this result, complete the proof in the general case.

Step 1. Proof of Theorem 1.2 for $\mu \leq \mu_0 < 1$. – By virtue of Theorem 4.1 Assumption 5.1 is fulfilled for any $h_0 > 0$, $\mu_0 < 1$, $\kappa > 0$ and $\varsigma = 0$,

$$\left.\begin{aligned} F(t, x) &= t + |x|^2, \\ \mathfrak{B}_s(h, \mu; \psi, \mathfrak{a}) &= \mathfrak{W}_s(h; \psi, V), \\ r(h, \mu) = Ch^{s+1-d}, &\qquad C = C(h_0, \mu_0, \kappa). \end{aligned}\right\} \tag{6.1}$$

The functions F, r are of moderate variation and F obeys (5.11). The validity of (5.8) and (5.9) follows immediately from (1.4).

We shall use Theorem 5.2 with $\mathcal{D} = \overset{\circ}{B}\,(4E)$. Define

$$f(x) = l(x) = A^{-1}\left[V(x)^2 + (\partial_x V(x))^4 + h^2\right]^{\frac{1}{4}}, \qquad A > 0.$$

Obviously, $f,\ l \in C^\infty(B(4E))$ and the conditions (5.1), (5.2) are fulfilled for $A \geq 1$ big enough. The estimates (5.3) for V, $\mathbf{a}$, ψ are trivial consequences of (1.6) and definition of l. Furthermore, the bounds (5.13) are obviously satisfied with $h_0 \geq A^2$, $\varsigma = 0$. The inclusion (5.15) holds for sufficiently big A, since $\operatorname{supp}\psi \subset B(E/2)$.

Let us check that the condition (5.14) is also fulfilled. For $f(x) = l(x)$, it takes the form

$$F(|V(x)| + h,\ \partial_x V(x)) = (|V(x)| + h) + |\partial_x V(x)|^2 \geq \omega\kappa l(x)^2.$$

By definition of l this holds for any $\omega \leq A^2\,\kappa^{-1}$.

Thus, conditions of Theorem 5.2 are satisfied. Therefore the estimate (5.17) holds. For $f(x) = l(x)$,

$$\begin{aligned} R(h,\ \mu;\ \mathcal{D}) &= \int_{\mathcal{D}} l(x)^{2s}\, r\left(\frac{h}{l(x)^2},\ \mu\right) l(x)^{-d}\, dx \\ &= Ch^{s+1-d} \int_{\mathcal{D}} l(x)^{d-2}\, dx \leq Ch^{s+1-d}. \end{aligned}$$

This yields (1.7).

Step 2. Completion of the proof. – According to Step 1 Assumption 5.1 is fulfilled for any $h_0 > 0$, $\mu_0 < 1$, the functions $\mathfrak{B}_s$ and r defined by (6.1) and $F = 0$, $\kappa = 0$, $\varsigma = 0$. Suppose that $\mu \geq \mu_0$ for some $\mu_0 < 1$. We apply Theorem 5.2 with $\mathcal{D}$ as above and $f(x) = 1$, $l(x) = 3\,\mu_0\,E\,(8\,\mu)^{-1}$. Then the conditions (5.3), (5.13), (5.15) are trivially fulfilled. By Theorem 5.2 the estimate (5.17) holds with an error bounded by

$$R(h,\ \mu;\ \mathcal{D}) \leq C \int_{\mathcal{D}} r(h\mu,\ \mu_0)\,\mu^d\, dx = C'\mu^{s+1}\,h^{s+1-d}.$$

In combination with Step 1 this yields Theorem 1.2.

REFERENCES

[1] W. O. Amrein, A.-M. Boutet de Monvel-Berthier and V. Georgescu, *Notes on The N-Body Problem, Part II*, University of Geneve, preprint, Geneve, 1991.

[2] J. Avron, I. Herbst and B. Simon, Schrödinger operators with magnetic fields, I., *Duke Math. J.*, Vol. **45 (4)**, 1978, pp. 847-883.

[3] C. L. FEFFERMAN, V. J. IVRII, L. A. SECO and I. M. SIGAL, The energy asymptotics of large Coulomb systems, *Lecture Notes in Physics*, Vol. **403** (E. Balslev, ed.), Springer, Heidelberg, pp. 79-99.
[4] B. HELFFER and D. ROBERT, Calcul fonctionnel par la transformation de Mellin et opérateurs admissibles, *J. Funct. Anal.*, Vol. **53**, 1983, pp. 246-268.
[5] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators, I,* Springer, Berlin, 1983.
[6] V. IVRII and I. M. SIGAL, Asymptotics of the ground state energies of Large Coulomb systems, *Ann. of Math.*, Vol. **138**, 1993, pp. 243-335.
[7] V. IVRII, *Semiclassical Microlocal Analysis and Precise Spectral Asymptotics*, École Polytechnique, Preprints, Palaiseau, 1991-1992.
[8] V. IVRII, Estimates for the number of negative eigenvalues of the Schrödinger operator with a strong magnetic field, *Soviet Math. Dokl.*, Vol. **36 (3)**, 1988, pp. 561-564.
[9] V. IVRII, Estimates for the number of negative eigenvalues of the Schrödinger operator with singular potentials, *Proc. Int. Congr. Math. Berkeley*, 1986, pp. 1084-1093.
[10] L. D. LANDAU and E. M. LIFSCHITZ, *Quantum Mechanics - Nonrelativistic Theory*, Pergamon Press, New York, 1965.
[11] E. H. LIEB and B. SIMON, The Thomas-Fermi theory of atoms, molecules and solids, *Adv. in Math.*, Vol. **23**, 1977, pp. 22-116.
[12] E. H. LIEB, J. P. SOLOVEJ and J. YNGVASON, Heavy atoms in the strong magnetic field of a neutron star, *Phys. Rev. Letters*, Vol. **69**, 1992, pp. 749-752.
[13] E. H. LIEB, J. P. SOLOVEJ and J. YNGVASON, Asymptotics of heavy atoms in high magnetic fields: I. Lowest Landau band regions, *Comm. Pure and Appl. Math.* (to appear).
[14] E. H. LIEB, J. P. SOLOVEJ and J. YNGVASON, Asymptotics of heavy atoms in high magnetic fields: II. Semiclassical regions, *Comm. Math. Phys.*, Vol. **161**, 1994, pp. 77-124.
[15] M. REED and B. SIMON, *Methods of Modern Mathematical Physics*, Vol. III, Academic Press, New York, 1979.
[16] D. ROBERT, *Autour de l'Approximation Semiclassique,* Birkhäuser, Boston, 1987.
[17] B. SIMON, *Lectures on Trace Ideals Methods*, Cambridge University Press, London, 1979.
[18] A. SOBOLEV, The quasi-classical asymptotics of local Riesz means for the Schrödinger operator in a strong homogeneous magnetic field, *Duke Math. J.*, Vol. **74 (2)**, 1994, pp. 319-429.
[19] A. SOBOLEV, *The sum of eigenvalues for the Schrödinger operator with Coulomb singularities in a homogeneous magnetic field*, University of Nantes Preprint, 1993.

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