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J.-P. NICOLAS

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Scattering of linear Dirac fields by a spherically symmetric Black-Hole

by

J.-P. NICOLAS

Mathematical Institute 24-29 St-Giles',
Oxford OX1 3LB, England
or
CeReMaB, Université Bordeaux-I,
Unité de Recherche Associée au CNRS n° 226,
351, cours de la Libération, 33405 Talence Cedex, France.

ABSTRACT. – We study the linear Dirac system outside a spherical Black-Hole. In the case of massless fields, we prove the existence and asymptotic completeness of classical wave operators at the horizon of the Black-Hole and at infinity.

RÉSUMÉ. – On étudie le système linéaire de Dirac à l'extérieur d'un Trou Noir sphérique. Dans le cas des champs sans masse, on montre l'existence et la complétude asymptotique des opérateurs d'onde classiques à l'horizon du Trou Noir et à l'infini.

1. INTRODUCTION

We develop a time-dependent scattering theory for the linear Dirac system on Schwarzschild-type metrics. The first time-dependent scattering results on the Schwarzschild metric were obtained by J. Dimock [8]. Using the short range at infinity of the interaction between gravity and a massless scalar field, he proved the existence and asymptotic completeness of classical wave-operators for the wave equation. The case of the Maxwell system in which the interaction is pseudo long-range has been worked out by A. Bachelot [2], and for the Regge-Wheeler equation, a complete scattering theory has been developed by A. Bachelot and A. Motet-Bachelot [3]. Our purpose in this work is to study the classical wave operators and their asymptotic completeness for the linear massless Dirac system on a general "Schwarzschild-type" metric which covers all the usual cases of spherical black-holes. The main tools are Cook's method for the existence and the results obtained in [3] for the asymptotic completeness.

Let us consider the manifold $\mathbb{R}_t \times]0, +\infty[_r \times S^2_{\theta, \phi}$ endowed with the pseudo-riemannian metric

$$g_{\mu\nu}\, dx^\mu\, dx^\nu = F(r)\, e^{2\delta(r)}\, dt^2 - [F(r)^{-1}\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\phi^2] \quad (1)$$

where $F, \delta \in C^\infty(]0, +\infty[_r)$. We assume the existence of three values r_ν of r, $0 \leq r_- < r_0 < r_+ \leq +\infty$, which are the only possible zeros of F, such that

$$F(r_\nu) = 0, \quad F'(r_\nu) = 2\kappa_\nu, \quad \kappa_\nu \neq 0, \quad \text{if } 0 < r_\nu < +\infty,$$
$$F(r) > 0 \quad \text{for } r \in]r_0, r_+[, \qquad F(r) < 0 \quad \text{for } r \in]r_-, r_0[.$$

When they are finite and non zero, r_-, r_0 and r_+ are the radii of the spheres called: horizon of the black-hole (r_0), Cauchy horizon (r_-) and cosmological horizon (r_+). κ_ν is the surface gravity at the horizon $\{r = r_\nu\}$. If r_+ is infinite, we assume moreover that

$$F(r) = 1 - \frac{r_1}{r} + O(r^{-2}), \qquad r_1 > 0,$$
$$\delta(r) = \delta(+\infty) + o(r^{-1}), \qquad r \to +\infty,$$
$$F'(r), \quad \delta'(r) = O(r^{-2}), \qquad r \to +\infty.$$

All these properties are satisfied by usual spherical black-holes (*see* [13]).

NOTATIONS. – Let (M, g) be a Riemannian manifold, $C_0^\infty(M)$ denotes the set of C^∞ functions with compact support in M, $H^k(M, g)$, $k \in \mathbb{N}$ is the Sobolev space, completion of $C_0^\infty(M)$ for the norm

$$\| f \|^2_{H^k(M)} = \sum_{j=0}^{k} \int_M \langle \nabla^j f, \nabla^j f \rangle\, d\mu,$$

where ∇^j, $d\mu$ and $\langle\, ,\, \rangle$ are respectively the covariant derivatives, the measure of volume and the hermitian product associated with the metric g. We write $L^2(M, g) = H^0(M, g)$.

If E is a distribution space on M, E_{comp} represents the subspace of elements of E with compact support in M.

The 2-dimensional euclidian sphere S^2_ω is endowed with its usual metric

$$d\omega^2 = d\theta^2 + \sin^2\theta\, d\varphi^2, \qquad 0 \le \theta \le \pi, \quad 0 \le \varphi < 2\pi.$$

2. THE COVARIANT GENERALIZATION OF THE LINEAR DIRAC SYSTEM ON SCHWARZSCHILD-TYPE METRICS

The covariant generalization of the Dirac system on the metric g has the form

$$(i\gamma^\mu \nabla_\mu - m)\Phi = 0, \qquad m \ge 0 \tag{2}$$

for a particle with mass m, where Φ is a Dirac 4-spinor, the γ^μ are the contravariant Dirac matrices on curved space-time and ∇_μ is the covariant derivation of spinor fields. We make the following choices of flat space-time Dirac matrices

$$\gamma_{\tilde{0}} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix} \quad \gamma_{\tilde{\alpha}} = \begin{pmatrix} 0 & \sigma_\alpha \\ -\sigma_\alpha & 0 \end{pmatrix} \qquad \alpha = 1, 2, 3 \tag{3}$$

where

$$\begin{aligned} \sigma_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \sigma_2 &= -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned} \tag{4}$$

are the Pauli matrices, and of local Lorentz frame

$$e^\mu_{\tilde{\alpha}} = \begin{cases} |\, g^{\mu\mu}\,|^{\frac{1}{2}} & \text{if} \quad \tilde{\alpha} = \mu, \\ 0 & \text{if} \quad \tilde{\alpha} \ne \mu. \end{cases} \tag{5}$$

We recall that flat space-time Dirac matrices are a set of 4×4 matrices $\{\gamma_{\tilde{\alpha}}\}_{0 \le \tilde{\alpha} \le 3}$ such that

$$\{\gamma_{\tilde{\alpha}}, \gamma_{\tilde{\beta}}\} = \gamma_{\tilde{\alpha}}\gamma_{\tilde{\beta}} + \gamma_{\tilde{\beta}}\gamma_{\tilde{\alpha}} = 2\,\eta_{\tilde{\alpha}\tilde{\beta}}\,\mathbf{1} \quad (\tilde{\alpha}, \tilde{\beta} = 0, 1, 2, 3) \tag{6}$$

where

$$\eta_{\tilde{\alpha}\tilde{\beta}} = \operatorname{diag}(1, -1, -1, -1) \tag{7}$$

is the Minkowski metric. The indices with a tilde refer to flat space-time and can be raised or lowered using $\eta_{\tilde{\alpha}\tilde{\beta}}$, whereas the indices without tilde refer to curved space-time and are raised or lowered using the metric g.

With these definitions, the γ^μ and ∇_μ are then defined by (*see* for example [5], [7])

$$\gamma^\mu = \gamma_{\tilde{\alpha}}\, e^{\tilde{\alpha}\mu} \tag{8}$$

and

$$\nabla_\mu = \partial_\mu + \frac{1}{2}\, G_{[\tilde\alpha\tilde\beta]}\, \omega^{\tilde\alpha\tilde\beta}{}_\mu \tag{9}$$

where

$$G_{[\tilde\alpha\tilde\beta]} = \frac{1}{4}\,[\gamma_{\tilde\alpha},\, \gamma_{\tilde\beta}] \equiv \frac{1}{4}\,(\gamma_{\tilde\alpha}\,\gamma_{\tilde\beta} - \gamma_{\tilde\beta}\,\gamma_{\tilde\alpha}) \tag{10}$$

are the generators of the spinor representation of the proper Lorentz group and

$$\begin{aligned}\omega^{\tilde\alpha\tilde\beta}{}_\mu = {} & \frac{1}{2}\, e^{\tilde\alpha\nu}\,(e^{\tilde\beta}{}_{\nu,\,\mu} - e^{\tilde\beta}{}_{\mu,\,\nu}) - \frac{1}{2}\, e^{\tilde\beta\nu}\,(e^{\tilde\alpha}{}_{\nu,\,\mu} - e^{\tilde\alpha}{}_{\mu,\,\nu}) \\ & + \frac{1}{2}\, e^{\tilde\alpha\nu}\, e^{\tilde\beta\sigma}\,(e^{\tilde\gamma}{}_{\nu,\,\sigma} - e^{\tilde\gamma}{}_{\sigma,\,\nu})\, e_{\tilde\gamma\mu} = -\omega^{\tilde\beta\tilde\alpha}{}_\mu\end{aligned} \tag{11}$$

are the coefficients of the spin connection, $_{,\,\mu}$ standing for the derivation with respect to the μ-th variable. We compute the *a priori* non zero components:

$$\begin{aligned}\omega^{\tilde t\tilde r}{}_t = {} & \frac{1}{2}\, e^{\tilde t t}\,[\partial_t\,(e^{\tilde r}{}_t) - \partial_t\,(e^{\tilde r}{}_t)] - \frac{1}{2}\, e^{\tilde r r}\,[\partial_t\,(e^{\tilde t}{}_r) - \partial_r\,(e^{\tilde t}{}_t)] \\ & + \frac{1}{2}\, e^{\tilde t t}\, e^{\tilde r r}\,[\partial_r\,(e^{\tilde t}{}_t) - \partial_t\,(e^{\tilde t}{}_r)]\, e_{\tilde t t} \\ = {} & \frac{1}{2}\, e^{\tilde r r}\,\partial_r\,(e^{\tilde t}{}_t)\,(1 + e^{\tilde t t}\, e_{\tilde t t}) = \frac{1}{2}\,(-F^{1/2})\,\partial_r\,(F^{1/2}\, e^{\delta}) \\ & \times (1 + F^{-1/2}\, e^{-\delta}\, F^{1/2}\, e^{\delta}) = -\left(\frac{F'}{2} + F\,\delta'\right) e^{\delta},\end{aligned}$$

$$\begin{aligned}\omega^{\tilde t\tilde r}{}_r = {} & \frac{1}{2}\, e^{\tilde t t}\,[\partial_r\,(e^{\tilde r}{}_t) - \partial_t\,(e^{\tilde r}{}_r)] - \frac{1}{2}\, e^{\tilde r r}\,[\partial_r\,(e^{\tilde t}{}_r) - \partial_r\,(e^{\tilde t}{}_r)] \\ & + \frac{1}{2}\, e^{\tilde t t}\, e^{\tilde r r}\,[\partial_r\,(e^{\tilde r}{}_t) - \partial_t\,(e^{\tilde r}{}_r)]\, e_{\tilde r r} = 0,\end{aligned}$$

$$\begin{aligned}\omega^{\tilde t\tilde\theta}{}_t = {} & \frac{1}{2}\, e^{\tilde t t}\,[\partial_t\,(e^{\tilde\theta}{}_t) - \partial_t\,(e^{\tilde\theta}{}_t)] - \frac{1}{2}\, e^{\tilde\theta\theta}\,[\partial_t\,(e^{\tilde t}{}_\theta) - \partial_\theta\,(e^{\tilde t}{}_t)] \\ & + \frac{1}{2}\, e^{\tilde t t}\, e^{\tilde\theta\theta}\,[\partial_\theta\,(e^{\tilde t}{}_t) - \partial_t\,(e^{\tilde t}{}_\theta)]\, e_{\tilde t t} = 0,\end{aligned}$$

$$\begin{aligned}\omega^{\tilde t\tilde\theta}{}_\theta = {} & \frac{1}{2}\, e^{\tilde t t}\,[\partial_\theta\,(e^{\tilde\theta}{}_t) - \partial_t\,(e^{\tilde\theta}{}_\theta)] - \frac{1}{2}\, e^{\tilde\theta\theta}\,[\partial_\theta\,(e^{\tilde t}{}_\theta) - \partial_\theta\,(e^{\tilde t}{}_\theta)] \\ & + \frac{1}{2}\, e^{\tilde t t}\, e^{\tilde\theta\theta}\,[\partial_\theta\,(e^{\tilde\theta}{}_t) - \partial_t\,(e^{\tilde\theta}{}_\theta)]\, e_{\tilde\theta\theta} = 0,\end{aligned}$$

$$\omega^{\tilde{t}\tilde{\varphi}}{}_t = \frac{1}{2}\, e^{\tilde{t}t}\,[\partial_t\,(e^{\tilde{\varphi}}{}_t) - \partial_t\,(e^{\tilde{\varphi}}{}_t)] - \frac{1}{2}\, e^{\tilde{\varphi}\varphi}\,[\partial_t\,(e^{\tilde{t}}{}_\varphi) - \partial_\varphi\,(e^{\tilde{t}}{}_t)] + \frac{1}{2}\, e^{\tilde{t}t}\, e^{\tilde{\varphi}\varphi}\,[\partial_\varphi\,(e^{\tilde{t}}{}_t) - \partial_t\,(e^{\tilde{t}}{}_\varphi)]\, e_{\tilde{t}t} = 0,$$

$$\omega^{\tilde{t}\tilde{\varphi}}{}_\varphi = \frac{1}{2}\, e^{\tilde{t}t}\,[\partial_\varphi\,(e^{\tilde{\varphi}}{}_t) - \partial_t\,(e^{\tilde{\varphi}}{}_\varphi)] - \frac{1}{2}\, e^{\tilde{\varphi}\varphi}\,[\partial_\varphi\,(e^{\tilde{t}}{}_\varphi) - \partial_\varphi\,(e^{\tilde{t}}{}_\varphi)] + \frac{1}{2}\, e^{\tilde{t}t}\, e^{\tilde{\varphi}\varphi}\,[\partial_\varphi\,(e^{\tilde{\varphi}}{}_t) - \partial_t\,(e^{\tilde{\varphi}}{}_\varphi)]\, e_{\tilde{\varphi}\varphi} = 0,$$

$$\omega^{\tilde{r}\tilde{\theta}}{}_r = \frac{1}{2}\, e^{\tilde{r}r}\,[\partial_r\,(e^{\tilde{\theta}}{}_r) - \partial_r\,(e^{\tilde{\theta}}{}_r)] - \frac{1}{2}\, e^{\tilde{\theta}\theta}\,[\partial_r\,(e^{\tilde{r}}{}_\theta) - \partial_\theta\,(e^{\tilde{r}}{}_r)] + \frac{1}{2}\, e^{\tilde{r}r}\, e^{\tilde{\theta}\theta}\,[\partial_\theta\,(e^{\tilde{r}}{}_r) - \partial_r\,(e^{\tilde{r}}{}_\theta)]\, e_{\tilde{r}r} = 0,$$

$$\omega^{\tilde{r}\tilde{\theta}}{}_\theta = \frac{1}{2}\, e^{\tilde{r}r}\,[\partial_\theta\,(e^{\tilde{\theta}}{}_r) - \partial_r\,(e^{\tilde{\theta}}{}_\theta)] - \frac{1}{2}\, e^{\tilde{\theta}\theta}\,[\partial_\theta\,(e^{\tilde{r}}{}_\theta) - \partial_\theta\,(e^{\tilde{r}}{}_\theta)] + \frac{1}{2}\, e^{\tilde{r}r}\, e^{\tilde{\theta}\theta}\,[\partial_\theta\,(e^{\tilde{\theta}}{}_r) - \partial_r\,(e^{\tilde{\theta}}{}_\theta)]\, e_{\tilde{\theta}\theta} = F^{1/2},$$

$$\omega^{\tilde{r}\tilde{\varphi}}{}_r = \frac{1}{2}\, e^{\tilde{r}r}\,[\partial_r\,(e^{\tilde{\varphi}}{}_r) - \partial_r\,(e^{\tilde{\varphi}}{}_r)] - \frac{1}{2}\, e^{\tilde{\varphi}\varphi}\,[\partial r\,(e^{\tilde{r}}{}_\varphi) - \partial_\varphi\,(e^{\tilde{r}}{}_r)] + \frac{1}{2}\, e^{\tilde{r}r}\, e^{\tilde{\varphi}\varphi}\,[\partial_\varphi\,(e^{\tilde{r}}{}_r) - \partial_r\,(e^{\tilde{r}}{}_\varphi)]\, e_{\tilde{r}r} = 0,$$

$$\omega^{\tilde{r}\tilde{\varphi}}{}_\varphi = \frac{1}{2}\, e^{\tilde{r}r}\,[\partial_\varphi\,(e^{\tilde{\varphi}}{}_r) - \partial_r\,(e^{\tilde{\varphi}}{}_\varphi)] - \frac{1}{2}\, e^{\tilde{\varphi}\varphi}\,[\partial_\varphi\,(e^{\tilde{r}}{}_\varphi) - \partial_\varphi\,(e^{\tilde{r}}{}_\varphi)] + \frac{1}{2}\, e^{\tilde{r}r}\, e^{\tilde{\varphi}\varphi}\,[\partial_\varphi\,(e^{\tilde{\varphi}}{}_r) - \partial_r\,(e^{\tilde{\varphi}}{}_\varphi)]\, e_{\tilde{\varphi}\varphi} = F^{1/2}\sin\theta,$$

$$\omega^{\tilde{\theta}\tilde{\varphi}}{}_\theta = \frac{1}{2}\, e^{\tilde{\theta}\theta}\,[\partial_\theta\,(e^{\tilde{\varphi}}{}_\theta) - \partial_\theta\,(e^{\tilde{\varphi}}{}_\theta)] - \frac{1}{2}\, e^{\tilde{\varphi}\varphi}\,[\partial_\theta\,(e^{\tilde{\theta}}{}_\varphi) - \partial_\varphi\,(e^{\tilde{\theta}}{}_\theta)] + \frac{1}{2}\, e^{\tilde{\theta}\theta}\, e^{\tilde{\varphi}\varphi}\,[\partial_\varphi\,(e^{\tilde{\theta}}{}_\theta) - \partial_\theta\,(e^{\tilde{\theta}}{}_\varphi)]\, e_{\tilde{\theta}\theta} = 0,$$

$$\omega^{\tilde{\theta}\tilde{\varphi}}{}_\varphi = \frac{1}{2}\, e^{\tilde{\theta}\theta}\,[\partial_\varphi\,(e^{\tilde{\varphi}}{}_\theta) - \partial_\theta\,(e^{\tilde{\varphi}}{}_\varphi)] - \frac{1}{2}\, e^{\tilde{\varphi}\varphi}\,[\partial_\varphi\,(e^{\tilde{\theta}}{}_\varphi) - \partial_\varphi\,(e^{\tilde{\theta}}{}_\varphi)] + \frac{1}{2}\, e^{\tilde{\theta}\theta}\, e^{\tilde{\varphi}\varphi}\,[\partial_\varphi\,(e^{\tilde{\varphi}}{}_\theta) - \partial_\theta\,(e^{\tilde{\varphi}}{}_\varphi)]\, e_{\tilde{\varphi}\varphi} = \cos\theta$$

and we obtain the following expression for the linear massive Dirac equation outside a spherical black-hole:

$$\left\{\gamma^{\tilde{0}}\,\partial_t + F e^{\delta}\,\gamma^{\tilde{1}}\left(\partial_r + \frac{1}{r} + \frac{F'}{4F} + \frac{\delta'}{2}\right) + \frac{F^{1/2}\,e^{\delta}}{r}\,\gamma^{\tilde{2}}\left(\partial_\theta + \frac{1}{2}\operatorname{cotg}\theta\right)\right.$$
$$\left. + \frac{F^{1/2}\,e^{\delta}}{r\sin\theta}\,\gamma^{\tilde{3}}\,\partial_\varphi + i\,F^{1/2}\,e^{\delta}\,m\right\}\Phi = 0. \tag{12}$$

We introduce the frame with respect to which we shall express the equation, $\mathcal{R}' = \left(\frac{1}{r\sin\theta}\,\partial_\varphi,\ -\frac{1}{r}\,\partial_\theta,\ F^{1/2}\,\partial_r\right)$, image of $\mathcal{R} = \left(F^{1/2}\,\partial_r,\ \frac{1}{r}\,\partial_\theta,\ \frac{1}{r\sin\theta}\,\partial_\varphi\right)$ by the spatial rotation f with Euler angles (*see* for example [15]) $(\varphi,\ \theta,\ \psi) = (0,\ \pi/2,\ \pi)$, and the Regge-Wheeler variable r_* defined by

$$\frac{dr}{dr_*} = F\,e^{\delta}, \qquad r \in\,]\,r_0,\ r_+\,[. \tag{13}$$

The spinor

$$\Psi = T_{(f^{-1})}\,r\,F^{1/4}\,e^{\delta/2}\,\Phi, \tag{14}$$

where $T_{(f^{-1})}$ is the spin transformation associated with the rotation f^{-1}, satisfies

$$\partial_t\,\Psi = i\,H\,\Psi,$$
$$H = i\left[\gamma^{\tilde{0}}\,\gamma^{\tilde{3}}\,\partial_{r_*} - \frac{F^{1/2}\,e^{\delta}}{r}\,\gamma^{\tilde{0}}\,\gamma^{\tilde{2}}\left(\partial_\theta + \frac{1}{2}\operatorname{cotg}\theta\right)\right.$$
$$\left. + \frac{F^{1/2}\,e^{\delta}}{r\sin\theta}\,\gamma^{\tilde{0}}\,\gamma^{\tilde{1}}\,\partial_\varphi + i\,\gamma^{\tilde{0}}\,F^{1/2}\,e^{\delta}\,m\right] \tag{15}$$

on the domain $\mathbb{R}_t \times \mathbb{R}_{r_*} \times S^2_\omega$ representing the exterior of the black-hole in the variables $(t,\ r_*,\ \omega)$.

We recall (*see* [7]) that, given a spatial rotation f of angle θ around a unit vector $n = (n_1,\ n_2,\ n_3)$, its associated spin transformation T_f is

$$T_f = \operatorname{Exp}\left\{\left[n_1\,G_{[\tilde{2},\,\tilde{3}]} + n_2\,G_{[\tilde{3},\,\tilde{1}]} + n_3\,G_{[\tilde{1},\,\tilde{2}]}\right]\theta\right\} \tag{16}$$

where Exp is the exponential mapping.

3. GLOBAL CAUCHY PROBLEM

We introduce the Hilbert space

$$\mathcal{H} = \{L^2\,(\mathbb{R}_{r_*} \times S^2_\omega;\ dr_*^2 + d\omega^2)\}^4. \tag{17}$$

THEOREM 3.1. – *Given $\Psi_0 \in \mathcal{H}$, equation (15) has a unique solution Ψ such that*

$$\Psi \in \mathcal{C}(\mathbb{R}_t;\ \mathcal{H}), \quad \Psi\,|_{t=0} = \Psi_0. \tag{18}$$

Moreover, for any $t \in \mathbb{R}$

$$\|\,\Psi(t)\,\|_{\mathcal{H}} = \|\,\Psi_0\,\|_{\mathcal{H}}. \tag{19}$$

Proof. – We show that the operator

$$\tilde{H} = H + \gamma^{\tilde{0}}\, F^{1/2}\, e^{\delta}\, m \tag{20}$$

is self-adjoint with dense domain on $\mathcal{H}$. We decompose $\mathcal{H}$ using generalized spherical functions of weigths 1/2 and $-1/2$. Let

$$\mathcal{I} = \{(l,\ m,\ n);\ 2l,\ 2m,\ 2n \in \mathbb{Z};\ l - |\,m\,|,\ l - |\,n\,| \in \mathbb{N}\} \tag{21}$$

and for any half-integer m

$$\mathcal{I}_m = \{(l,\ n);\ (l,\ m,\ n) \in \mathcal{I}\}. \tag{22}$$

For $(l,\ m,\ n) \in \mathcal{I}$, we define the function T^l_{mn} of $(\varphi_1,\ \theta,\ \varphi_2)$, $\varphi_1,\ \varphi_2 \in [0,\ 2\pi\,[,\ \theta \in [0,\ \pi]$, by

$$T^l_{mn}(\varphi_1,\ \theta,\ \varphi_2) = e^{-im\,\varphi_2}\, u^l_{mn}(\theta)\, e^{-in\varphi_1} \tag{23}$$

where u^l_{mn} satisfies the following ordinary differential equations

$$\frac{d^2 u^l_{mn}}{d\theta^2} + \operatorname{cotg}\theta\,\frac{du^l_{mn}}{d\theta} + \left[l\,(l+1) - \frac{n^2 - 2\,mn\cos\theta + m^2}{\sin^2\theta}\right] u^l_{mn} = 0, \tag{24}$$

$$\frac{du^l_{mn}}{d\theta} - \frac{n - m\cos\theta}{\sin\theta}\, u^l_{mn} = -i\,[(l+m)\,(l-m+1)]^{1/2}\, u^l_{m-1,\,n}, \tag{25}$$

$$\frac{du^l_{mn}}{d\theta} + \frac{n - m\cos\theta}{\sin\theta}\, u^l_{mn} = -i\,[(l+m+1)\,(l-m)]^{1/2}\, u^l_{m+1,\,n} \tag{26}$$

and the normalization condition

$$\int_0^{\pi} |\,u^l_{mn}(\theta)\,|^2 \sin\theta\, d\theta = \frac{1}{4\pi^2}. \tag{27}$$

We know from [12], that $\{T^l_{mn}\}_{(l,\,m,\,n)\in\mathcal{I}_{\frac{1}{2}}}$ is a Hibert basis of

$$L^2\,([0,\ 2\pi\,[_{\varphi_1} \times [0,\ \pi]_\theta \times [0,\ 2\pi\,[_{\varphi_2};\ \sin^2\theta\, d\varphi_1^2 + d\theta^2 + d\varphi_2^2). \tag{28}$$

Thus, for any half-integer m,

$$\{T^l_{mn}(\varphi, \theta, 0) = e^{-in\varphi} u^l_{mn}(\theta)\}_{(l,n)\in\mathcal{I}_m}$$

is a Hilbert basis of $L^2(S^2_\omega; d\omega^2)$. In particular,

$$\mathcal{H} = \bigoplus_{(l,n)\in\mathcal{I}_{\frac{1}{2}}} \mathcal{H}_{ln} \tag{29}$$

where

$$\begin{aligned}\mathcal{H}_{ln} = \{{}^t(f_1 T^l_{-\frac{1}{2},n},\ f_2 T^l_{\frac{1}{2},n},\ f_3 T^l_{-\frac{1}{2},n},\ f_4 T^l_{\frac{1}{2},n});\\ f_i \in L^2(\mathbb{R}_{r_*}; dr_*^2),\ i = 1, 2, 3, 4\},\end{aligned} \tag{30}$$

or equivalently,

$$\mathcal{H}_{ln} = [L^2(\mathbb{R}_{r_*}; dr_*^2)]^4 \otimes F_{ln};\ F_{ln} = {}^t(T^l_{-\frac{1}{2},n},\ T^l_{\frac{1}{2},n},\ T^l_{-\frac{1}{2},n},\ T^l_{\frac{1}{2},n}) \tag{31}$$

where the $T^l_{\pm\frac{1}{2},n}$ are seen as functions of only φ, θ. Let

$$\Psi = {}^t(f_1, f_2, f_3, f_4) \otimes F_{ln} \in \mathcal{H}_{ln}.$$

Denoting $\alpha = F^{1/2} e^\delta$, the four components of $\tilde{H}\,\Psi$ are

$$i\,\partial_{r_*} f_3 T^l_{-\frac{1}{2},n} - \frac{\alpha}{r} f_4 \left(\partial_\theta + \frac{1}{2}\operatorname{cotg}\theta\right) T^l_{\frac{1}{2},n} + i\frac{\alpha}{r\sin\theta} f_4\,\partial_\varphi T^l_{\frac{1}{2},n},$$

$$-i\,\partial_{r_*} f_4 T^l_{\frac{1}{2},n} + \frac{\alpha}{r} f_3 \left(\partial_\theta + \frac{1}{2}\operatorname{cotg}\theta\right) T^l_{-\frac{1}{2},n} + i\frac{\alpha}{r\sin\theta} f_3\,\partial_\varphi T^l_{-\frac{1}{2},n},$$

$$i\,\partial_{r_*} f_1 T^l_{-\frac{1}{2},n} - \frac{\alpha}{r} f_2 \left(\partial_\theta + \frac{1}{2}\operatorname{cotg}\theta\right) T^l_{\frac{1}{2},n} + i\frac{\alpha}{r\sin\theta} f_2\,\partial_\varphi T^l_{\frac{1}{2},n},$$

$$-i\,\partial_{r_*} f_2 T^l_{\frac{1}{2},n} + \frac{\alpha}{r} f_1 \left(\partial_\theta + \frac{1}{2}\operatorname{cotg}\theta\right) T^l_{-\frac{1}{2},n} + i\frac{\alpha}{r\sin\theta} f_1\,\partial_\varphi T^l_{-\frac{1}{2},n}.$$

Relations (25) and (26) yield

$$\left(\partial_\theta + \frac{1}{2}\operatorname{cotg}\theta\right) T^l_{\frac{1}{2},n} = \frac{n}{\sin\theta} T^l_{\frac{1}{2},n} - i\left(l + \frac{1}{2}\right) T^l_{-\frac{1}{2},n}, \tag{32}$$

$$\left(\partial_\theta + \frac{1}{2}\operatorname{cotg}\theta\right) T^l_{-\frac{1}{2},n} = \frac{-n}{\sin\theta} T^l_{-\frac{1}{2},n} - i\left(l + \frac{1}{2}\right) T^l_{\frac{1}{2},n}, \tag{33}$$

and we also have

$$\partial_\varphi T^l_{\pm\frac{1}{2},n}(\varphi,\,\theta,\,0) = -in\,T^l_{\pm\frac{1}{2},n}(\varphi,\,\theta,\,0). \tag{34}$$

Thus, the four components of $\tilde{H}\,\Psi$ are

$$\left(i\,\partial_{r_*} f_3 + i\,\frac{\alpha}{r}\left(l+\frac{1}{2}\right) f_4\right) T^l_{-\frac{1}{2},n},$$

$$\left(-i\,\partial_{r_*} f_4 - i\,\frac{\alpha}{r}\left(l+\frac{1}{2}\right) f_3\right) T^l_{\frac{1}{2},n},$$

$$\left(i\,\partial_{r_*} f_1 + i\,\frac{\alpha}{r}\left(l+\frac{1}{2}\right) f_2\right) T^l_{-\frac{1}{2},n},$$

$$\left(-i\,\partial_{r_*} f_2 - i\,\frac{\alpha}{r}\left(l+\frac{1}{2}\right) f_1\right) T^l_{\frac{1}{2},n}.$$

We see that on $\mathcal{H}_{ln}$, $\tilde{H}$ has the form

$$\tilde{H}\,|_{\mathcal{H}_{ln}} = \left(i\,\partial_{r_*} L + \frac{\alpha}{r}\left(l+\frac{1}{2}\right) M\right)_{r_*} \otimes \mathbf{1}_{\theta,\varphi} \tag{35}$$

where the matrices L et M, defined by

$$L = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \tag{36}$$

are hermitian and L is invertible. Since the function $\alpha\,r^{-1}$ belongs to $L^\infty(\mathbb{R}_{r_*})$, $\tilde{H}\,|_{\mathcal{H}_{ln}}$ is self-adjoint with domain

$$D_{ln} = [D\,(i\,\partial_{r_*})]^4 \otimes F_{ln} \simeq [H^1(\mathbb{R}_{r_*};\,dr_*^2)]^4 \otimes F_{ln} \tag{37}$$

dense in $\mathcal{H}_{ln}$. On D_{ln}, we choose the following norm

$$\Psi = {}^t(f_1,\,f_2,\,f_3,\,f_4) \otimes F_{ln} \in D_{ln},$$

$$\|\,\Psi\,\|^2_{D_{ln}} = \|\,\Psi\,\|^2_{(L^2(\mathbf{R}))^4} + \left\|\left(i\,\partial_{r_*} L + \frac{\alpha}{r}\left(l+\frac{1}{2}\right) M\right)\Psi\right\|^2_{(L^2(\mathbf{R}))^4} \tag{38}$$

and we introduce the dense subspace of $\mathcal{H}$

$$D(H) = \{\Psi = \sum_{(l,n)\in\mathcal{I}_{\frac{1}{2}}} \Psi_{ln};\ \Psi_{ln} \in D_{ln},\ \sum_{(l,n)\in\mathcal{I}_{\frac{1}{2}}} \|\,\Psi_{ln}\,\|^2_{D_{ln}} < +\infty\}. \tag{39}$$

$\tilde{H}$ is self-adjoint on $\mathcal{H}$ with domain $D(H)$, $\gamma^{\tilde{0}}\alpha m$ is self-adjoint and bounded on $\mathcal{H}$, therefore, H is self-adjoint on $\mathcal{H}$ with dense domain $D(H)$. Theorem 3.1 follows from Stone's theorem.

Q.E.D.

4. WAVE OPERATORS AT THE HORIZON

When $r \to r_0$, the operator H has the formal limit

$$H_0 = i\gamma^{\tilde{0}}\gamma^{\tilde{3}}\partial r_* \tag{40}$$

which is a self-adjoint operator on $\mathcal{H}$ with dense domain

$$D(H_0) = \{H^1[(\mathbb{R}_{r_*};\, dr_*^2);\, L^2(S_\omega^2;\, d\omega^2)]\}^4. \tag{41}$$

The spectrum of H_0 is purely absolutely continuous. We define the subspaces of incoming and outgoing waves associated with H_0 :

$$\mathcal{H}_0^\pm = \{\Psi = {}^t(u^1,\, u^2,\, u^3,\, u^4),\ u^3 = \mp u^1,\ u^4 = \pm u^2\}. \tag{42}$$

$\mathcal{H}_0^\pm$ as well as the $\mathcal{H}_{ln}$ remain stable under H_0 and we have

$$\begin{gathered}\mathcal{H} = \mathcal{H}_0^+ \oplus \mathcal{H}_0^-,\\ \forall\,\Psi_0 \in \mathcal{H}_0^\pm,\quad (e^{iH_0t}\,\Psi_0)(r_*,\,\omega) = \Psi_0(r_* \pm t,\,\omega).\end{gathered} \tag{43}$$

Since we want to compare H with H_0 in the neighbourhood of the horizon, we introduce the cut-off function

$$\begin{gathered}\chi_0 \in \mathcal{C}^\infty(\mathbb{R}_{r_*}),\quad 0 \le \chi_0 \le 1,\\ \exists\, a,\, b \in \mathbb{R},\quad a < b \text{ such that}\\ \text{for}\quad r_* < a\quad \chi_0(r_*) = 1;\qquad \text{for}\quad r_* > b\quad \chi_0(r_*) = 0\end{gathered} \tag{44}$$

together with the identifying operator

$$\mathcal{J}_0 : \begin{array}{l}\mathcal{H} \to \mathcal{H}\\ \Psi \mapsto \chi_0\,\Psi.\end{array} \tag{45}$$

We consider the classical wave operators

$$W_0^\pm\,\Psi_0 = s\text{-}\lim_{t\to\pm\infty} e^{-iHt}\,\mathcal{J}_0\,e^{iH_0t}\,\Psi_0 \quad in\ \mathcal{H}. \tag{46}$$

THEOREM 4.1. – *The operator* W_0^+ (*resp.* W_0^-) *is well-defined from* $\mathcal{H}_0^+$ (*resp.* $\mathcal{H}_0^-$) *to* $\mathcal{H}$, *is independent of the choice of* χ_0 *satisfying* (44), *moreover*

$$\forall\,\Psi_0 \in \mathcal{H}_0^\pm,\quad \|\,W_0^\pm\,\Psi_0\,\|_{\mathcal{H}} = \|\,\Psi_0\,\|_{\mathcal{H}}. \tag{47}$$

Proof. – We apply Cook's method. $\mathcal{J}_0$ being a bounded operator, it suffices to prove that for

$$\Psi_0 \in \mathcal{D}_{ln}^\pm;\quad \mathcal{D}_{ln}^\pm = \mathcal{H}_0^\pm \cap \mathcal{H}_{ln} \cap [\mathcal{C}_0^\infty(\mathbb{R}_{r_*} \times S_\omega^2)]^4,\qquad (l,\, n) \in \mathcal{I}_{\frac{1}{2}} \tag{48}$$

we have

$$\| (H \mathcal{I}_0 - \mathcal{I}_0 H_0) e^{iH_0 t} \Psi_0 \|_{\mathcal{H}} \in L^1 (\pm t > 0). \quad (49)$$

Let for $(l, n) \in \mathcal{I}_{\frac{1}{2}}$

$$\Psi_0 \in D_{ln}^{\pm}, \quad \mathrm{Supp}\, \Psi_0 \subset [-R, R]_{r_*} \times S_\omega^2, \qquad R > 0, \quad (50)$$

then

$$H e^{iH_0 t} \Psi_0 = \left(i \partial_{r_*} + \frac{\alpha}{r} \left(l + \frac{1}{2} \right) M - \alpha m \gamma^{\tilde{0}} \right) \Psi_0 (r_* + t),$$

and

$$H_0 e^{iH_0 t} \Psi_0 = i \partial_{r_*} L \Psi_0 (r_* + t).$$

Ψ_0 being compactly supported, for t large enough,

$$\begin{aligned} &\| (H \mathcal{I}_0 - \mathcal{I}_0 H_0) e^{iH_0 t} \Psi_0 \|_{\mathcal{H}} \\ &= \left\| \left(\frac{\alpha}{r} \left(l + \frac{1}{2} \right) M - \alpha m \gamma^{\tilde{0}} \right) e^{iH_0 t} \Psi_0 \right\|_{\mathcal{H}} \\ &\leq \left\| \left(l + \frac{1}{2} \right) \frac{\alpha}{r} + \alpha m \right\|_{L^\infty (-R-t,\, R-t)} \| \Psi_0 \|_{\mathcal{H}}. \end{aligned}$$

α is rapidly decreasing in r_* when $r \to r_0$, therefore

$$\| (H \mathcal{I}_0 - \mathcal{I}_0 H_0) e^{iH_0 t} \Psi_0 \|_{\mathcal{H}} \in L^1 (t > 0)$$

and W_0^+ is well-defined. The same proof can of course be applied to W_0^-. Furthermore, if $\Psi_0 \in \mathcal{H}_0^{\pm}$, we get from (43) that the energy of $e^{iH_0 t} \Psi_0$ in a domain of $\mathbb{R}_{r_*} \times S_\omega^2$ bounded to the left in r_* vanishes when t tends to infinity, which gives (47). If now we consider two different cut-off functions χ_o and χ_o', and the associated identifying operators $\mathcal{J}_0$ and $\mathcal{J}_0'$, the difference $\chi_o - \chi_o'$ is compactly supported, thus

$$\| e^{-iHt} \mathcal{J}_0 e^{iH_0 t} \Psi_0 - e^{-iHt} \mathcal{J}_0' e^{iH_0 t} \Psi_0 \|_{\mathcal{H}} \to 0, \qquad t \to \pm\infty.$$

Q.E.D.

REMARK 4.1. – *In the case where r_+ is finite, we construct in the same way classical wave operators at the cosmological horizon*

$$W_1^{\pm} \Psi_0 = s\text{-}\lim_{t \to \pm\infty} e^{-iHt} \mathcal{J}_1 e^{iH_0 t} \Psi_0 \quad in\ \mathcal{H} \quad (51)$$

where the identifying operator $\mathcal{J}_1$ *is defined by*

$$\mathcal{J}_1 : \begin{array}{l} \mathcal{H} \rightarrow \mathcal{H} \\ \Psi \rightarrow \chi_1 \Psi, \end{array} \tag{52}$$

χ_1 *being a cut-off function*

$$\begin{gathered} \chi_1 \in \mathcal{C}^\infty(\mathbb{R}_{r_*}), \quad 0 \leq \chi_1 \leq 1, \\ \exists\, a,\, b \in \mathbb{R}, \quad a < b \quad \textit{such that} \\ \textit{for} \quad r_* < a \quad \chi_1(r_*) = 0; \qquad \textit{for} \quad r_* > b \quad \chi_1(r_*) = 1. \end{gathered} \tag{53}$$

W_1^+ *(resp.* W_1^-*) is an isometry from* $\mathcal{H}_0^-$ *(resp.* $\mathcal{H}_0^+$*) to* $\mathcal{H}$ *and is independent of the choice of* χ_1 *satisfying* (53).

5. WAVE OPERATORS AT INFINITY (MASSLESS CASE)

In all this paragraph, we shall assume that $r_+ = +\infty$; the metric (1) is then asymptotically flat in the neighbourhood of infinity and we choose to compare H to an operator H_∞ which is equivalent to the hamiltonian operator for the Dirac equation on the Minkowski space-time. We also make the hypothesis that $m = 0$ in order to avoid long range perturbations at infinity. Let us consider on the Minkowski metric

$$ds_{\mathcal{M}}^2 = dt^2 - dx^2 - dy^2 - dz^2; \qquad x,\, y,\, z \in \mathbb{R} \tag{54}$$

the massless Dirac system

$$\{\gamma^{\tilde{0}}\, \partial_t + \gamma^{\tilde{1}}\, \partial_x + \gamma^{\tilde{2}}\, \partial_y + \gamma^{\tilde{3}}\, \partial_z\}\, \Phi = 0. \tag{55}$$

The associated hamiltonian operator, defined by

$$H_{\mathcal{M}} = i\,\gamma^{\tilde{0}}\, \{\gamma^{\tilde{1}}\, \partial_x + \gamma^{\tilde{2}}\, \partial_y + \gamma^{\tilde{3}}\, \partial_z\}, \tag{56}$$

is self-adjoint with dense domain on $[L^2(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z)]^4$ and if $\Phi \in \mathcal{C}(\mathbb{R}_t;\ [L^2(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z)]^4)$ is a solution of (55), its energy in a compact domain goes to zero when t goes to $\pm\infty$. In addition, for any $\Phi_0 \in [L^2(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z)]^4$ with a compact support contained into

$$B(0,\, R) = \{(x,\, y,\, z);\ 0 \leq \rho < R,\ \rho = (x^2 + y^2 + z^2)^{1/2}\}, \tag{57}$$

the solution Φ of (55) associated with the initial data Φ_0 satisfies

$$\Phi(t,\, x,\, y,\, z) = 0 \qquad \textit{for} \quad 0 \leq \rho \leq |\, t\,| - R. \tag{58}$$

At the point of spherical coordinates (ρ, θ, φ), we apply the spatial rotation f with Euler angles $(\pi/2, \theta, \pi - \varphi)$. The local frame $(\partial_x, \partial_y, \partial_z)$ is thus transformed by f^{-1} into

$$(\partial_{x^1}, \partial_{x^2}, \partial_{x^3}) = \left(\frac{1}{\rho \sin\theta}\, \partial_\varphi, \frac{-1}{\rho}\, \partial_\theta, \partial_\rho\right). \tag{59}$$

The spinor

$$\Psi = \rho\, T_f\, \Phi, \tag{60}$$

where T_f is the spin transformation associated with f defined in (16), satisfies

$$\partial_t \Psi = i\, H_\infty \Psi, \qquad H_\infty = i \left[\gamma^{\tilde{0}} \gamma^{\tilde{3}}\, \partial_\rho - \frac{1}{\rho}\, \gamma^{\tilde{0}} \gamma^{\tilde{2}} \left(\partial_\theta + \frac{1}{2} \operatorname{cotg}\theta\right) + \frac{1}{p \sin\theta}\, \gamma^{\tilde{0}} \gamma^{\tilde{1}}\, \partial_\varphi\right]. \tag{61}$$

The operator H_∞ on

$$\mathcal{H}_\infty = \{L^2([0, +\infty[_\rho \times S^2_\omega;\ d\rho^2 + d\omega^2)\}^4 \tag{62}$$

is unitarily equivalent to $H_{\mathcal{M}}$ on

$$\{L^2(\mathbb{R}_x \times \mathbb{R}_y \times \mathbb{R}_z;\ dx^2 + dy^2 + dz^2)\}^4.$$

Therefore, H_∞ is self-adjoint with dense domain on $\mathcal{H}_\infty$ and if $\Psi \in \mathcal{C}(\mathbb{R}_t, \mathcal{H}_\infty)$ satisfies (61), then its energy in a compact domain goes to zero when t goes to $\pm\infty$. Moreover, for

$$\Psi_0 \in \mathcal{H}_\infty; \quad \operatorname{Supp}(\Psi_0) \subset B(0, R)$$

$\Psi(t) = e^{iH_\infty t}\, \Psi_0$ satisfies

$$\Psi(t, \rho, \theta, \varphi) = 0 \qquad \textit{for} \quad 0 \leq \rho \leq |t| - R. \tag{63}$$

In order to avoid artificial long-range interactions, we choose

$$\rho = r_* \geq 0 \tag{64}$$

and we introduce the cut-off function

$$\begin{gathered} \chi_\infty \in \mathcal{C}^\infty([0, +\infty[_{r_*}), \quad 0 \leq \chi_\infty \leq 1, \\ \exists\, 0 < a < b < +\infty \quad \textit{such that} \\ \textit{for} \quad 0 \leq r_* \leq a \quad \chi_\infty(r_*) = 0, \qquad \textit{for} \quad r_* \geq b\ \chi_\infty(r_*) = 1 \end{gathered} \tag{65}$$

together with the identifying operator

$$\mathcal{J}_\infty : \quad \mathcal{H}_\infty \to \mathcal{H}; \qquad for \quad \Psi \in \mathcal{H}_\infty \begin{cases} (\mathcal{J}\,\Psi)\,|_{\{r_* \geq 0\}} = \chi_\infty\,\Psi, \\ (\mathcal{J}\,\Psi)\,|_{\{r_* \leq 0\}} = 0. \end{cases} \tag{66}$$

We define the classical wave operators

$$W_\infty^\pm\,\Psi_0 = s\text{-}\lim_{t\to\pm\infty} e^{-iHt}\,\mathcal{J}_\infty\,e^{iH_\infty t}\,\Psi_0 \quad in \;\; \mathcal{H}. \tag{67}$$

THEOREM 5.1. – *The operators $W_\infty^\pm$ are well-defined from $\mathcal{H}_\infty$ to $\mathcal{H}$, are independent of the choice of χ_∞ and*

$$\forall\,\Psi_0 \in \mathcal{H}_\infty, \quad \|\,W_\infty^\pm\,\Psi_0\,\|_{\mathcal{H}} = \|\,\Psi_0\,\|_{\mathcal{H}_\infty}. \tag{68}$$

Proof. – For $(l,\, n) \in \mathcal{I}_{\frac{1}{2}}$, we introduce the subspaces of $\mathcal{H}_\infty$

$$\mathcal{D}_{ln}^\infty = \{\Psi = {}^t(f_1,\, f_2,\, f_3,\, f_4) \otimes F_{ln} \in \mathcal{H}_\infty;\; 1 \leq i \leq 4,\; f_i \in \mathcal{C}_0^\infty\,(\mathbb{R}_{r_*}^+)\} \tag{69}$$

the direct sum of which is dense in $\mathcal{H}_\infty$. For $\Psi_0 \in \mathcal{D}_{ln}^\infty$,

$$H_\infty\,|_{\mathcal{D}_{ln}^\infty} = \left(i\,\partial_{r_*}\,L + \frac{1}{r_*}\left(l + \frac{1}{2}\right) M\right)_{r_*} \otimes \mathbf{1}_\omega \tag{70}$$

where the matrices L and M are defined by (36), and

$$\mathcal{J}_\infty\,\Psi_0 \in \mathcal{H}_{ln}. \tag{71}$$

$\mathcal{J}_\infty$ being a bounded operator, it suffices to prove that for

$$\Psi_0 \in \mathcal{D}_{ln}^\infty; \quad \operatorname{Supp}(\Psi_0) \subset B\,(0,\, R), \tag{72}$$

we have

$$\|\,(H\,\mathcal{J}_\infty - \mathcal{J}_\infty\,H_\infty)\,e^{iH_\infty t}\,\Psi_0\,\|_{\mathcal{H}} \in L^1\,(\mathbb{R}_t). \tag{73}$$

(63) yields

$$e^{iH_\infty t}\,\Psi_0 = 0 \quad in \;\; \{(t,\, r_*,\, \theta,\, \varphi);\; 0 \leq r_* \leq |\,t\,| - R\}. \tag{74}$$

Thus, for $|\,t\,|$ large enough

$$\begin{aligned} &\|\,(H\,\mathcal{J}_\infty - \mathcal{J}_\infty\,H_\infty)\,e^{iH_\infty t}\,\Psi_0\,\|_{\mathcal{H}} \\ &= \left\|\left(\frac{\alpha}{r} - \frac{1}{r_*}\right)\left(l + \frac{1}{2}\right) M\,e^{iH_\infty t}\,\Psi_0\right\|_{\mathcal{H}} \\ &\leq \left(l + \frac{1}{2}\right)\|\,\Psi_0\,\|_{\mathcal{H}_\infty}\left\|\frac{\alpha}{r} - \frac{1}{r_*}\right\|_{L^\infty\,([|t|+R,\,+\infty[_{r_*})}. \end{aligned}$$

We study the asymptotic behavior of

$$\frac{\alpha}{r} - \frac{1}{r_*} = \frac{1}{r_*}\left(F^{1/2}\, e^{\delta}\, \frac{r_*}{r} - 1\right)$$

when r_* goes to $+\infty$. The Regge-Wheeler variables r_* is defined with respect to r by

$$r_* = \frac{1}{2\,\kappa_0}\left\{\operatorname{Log}|r - r_0| - \int_{r_0}^{r}\left[\frac{1}{r - r_0} - \frac{2\,\kappa_0}{\mathrm{Fe}^{\delta}}\right] dr\right\} \tag{75}$$

where $2\,\kappa_0 = F'(r_0)$. For r larger than $r_0 + 1$, we have

$$r_* = C + \int_{r_0+1}^{r} F^{-1}\, e^{-\delta}\, dr \tag{76}$$

where

$$2\,\kappa_0\, C = -\int_{r_0}^{r_0+1}\left[\frac{1}{r - r_0} - \frac{2\,\kappa_0}{\mathrm{Fe}^{\delta}}\right] dr. \tag{77}$$

F and δ satisfy

$$\delta(r) = o(r^{-1}); \qquad F(r) = 1 - \frac{r_1}{r} + O(r^{-2}) \quad r_1 > 0; \qquad r \to +\infty$$

and therefore

$$F^{-1}(r)\, e^{-\delta(r)} = 1 + \frac{r_1}{r} + o\,(r^{-1}),$$
$$r_* = r + r_1 \operatorname{Log}(r) + o\,(\operatorname{Log}(r)),$$
$$F^{1/2}(r)\, e^{\delta(r)} = 1 - \frac{r_1}{2\,r} + o\,(r^{-1})$$

which implies

$$F^{1/2}(r)\, e^{\delta(r)}\, \frac{r_*}{r} - 1 = r_1\, \frac{\operatorname{Log}(r)}{r} + o\left(\frac{\operatorname{Log}(r)}{r}\right) = O\,(r^{-1/2}) = O\,(r_*^{-1/2}).$$

The operators $W_\infty^{\pm}$ are thus well-defined. The fact that they are isometries and do not depend upon the choice of the cut-off function can be verified using exactly the same remarks as in the case of the horizon.

Q.E.D.

6. ASYMPTOTIC COMPLETENESS OF OPERATORS $W_0^{\pm}$ AND $W_\infty^{\pm}$ (MASSLESS CASE)

We assume again that $m = 0$ and $r_+ = +\infty$. We introduce the inverse wave operators at the horizon and at infinity, defined for $\Psi_0 \in \mathcal{H}$ by

$$\tilde{W}_0^{\pm} \Psi_0 = s\text{-}\lim_{t \to \pm\infty} e^{-iH_0 t} \mathcal{J}_0^* e^{iHt} \Psi_0 \quad in \ \mathcal{H}, \tag{78}$$

$$\tilde{W}_\infty^{\pm} \Psi_0 = s\text{-}\lim_{t \to \pm\infty} e^{-iH_\infty t} \mathcal{J}_\infty^* e^{iHt} \Psi_0 \quad in \ \mathcal{H}_\infty, \tag{79}$$

where $\mathcal{J}_0^*$ and $\mathcal{J}_\infty^*$ are respectively the adjoints of $\mathcal{J}_0$ and $\mathcal{J}_\infty$. We also define the wave operators W^+ and W^- by

$$\Psi_0 \in \mathcal{H}_0^{\pm}, \quad \Psi_\infty \in \mathcal{H}_\infty, \qquad W^{\pm}(\Psi_0, \Psi_\infty) = W_0^{\pm} \Psi_0 + W_\infty^{\pm} \Psi_\infty \tag{80}$$

as well as the inverse wave operators $\tilde{W}^+$, W^-.

$$\Psi_0 \in \mathcal{H} \quad \tilde{W}^{\pm} \Psi_0 = (\tilde{W}_0^{\pm} \Psi_0, \tilde{W}_\infty^{\pm} \Psi_0). \tag{81}$$

Eventually, we define the scattering operator

$$S = \tilde{W}^+ W^-. \tag{82}$$

THEOREM 6.1. – *Operators $\tilde{W}_0^{\pm}$ (resp. $\tilde{W}_\infty^{\pm}$) are well defined from $\mathcal{H}$ into $\mathcal{H}_0^{\pm}$ (resp. from $\mathcal{H}$ into $\mathcal{H}_\infty$), are independent of the choice of χ_o (resp. χ_∞) and their norm is lower or equal to* 1. *Moreover*

$$W^{\pm} \ is \ an \ isometry \ of \ \mathcal{H}_0^{\pm} \times \mathcal{H}_\infty \ onto \ \mathcal{H}.$$

$$\tilde{W}^{\pm} \ is \ an \ isometry \ of \ \mathcal{H} \ onto \ \mathcal{H}_0^{\pm} \times \mathcal{H}_\infty.$$

$$S \ is \ an \ isometry \ of \ \mathcal{H}_0^{-} \times \mathcal{H}_\infty \ onto \ \mathcal{H}_0^{+} \times \mathcal{H}_\infty.$$

Proof. – For any solution Ψ of (15) in $\mathcal{C}(\mathbb{R}_t; \mathcal{H}_{ln})$, $(l, n) \in \mathcal{I}_{\frac{1}{2}}$, we construct asymptotic profiles at the horizon and at infinity. The idea is that each component of Ψ satisfies an equation of the form

$$(\partial_t^2 - \partial_{r_*}^2 + V(r_*)) f = 0 \tag{83}$$

where the potential V has the following properties

$$\begin{aligned} &V = V_+ - V_-; \qquad V_+, V_- \geq 0, \\ &V_+(r_*) \leq C(1 + |r_*|)^{-1-\varepsilon}, \qquad \varepsilon > 0, \\ &V_-(r_*) \leq C(1 + |r_*|)^{-2-\varepsilon}, \qquad \varepsilon > 0. \end{aligned} \tag{84}$$

We then apply the scattering results of [3]. This suffices to define $\tilde{W}_0^{\pm}$, but to prove the existence of $\tilde{W}_\infty^{\pm}$, we need to recover a solution of $(\partial_t - i H_\infty) \Psi = 0$ from the asymptotic profile at infinity.

Firstly, we study some spectral properties of the operator H:

PROPOSITION 6.1. – *The point spectrum of H is empty.*

A straightforward consequence of proposition 6.1 is

COROLLARY 6.1. – *For $k \in \mathbb{N}$, the direct sum of the sets*

$$\begin{aligned}\mathcal{E}_{ln}^{k} =& \{H^k \Psi;\ \Psi = {}^t(f_1, f_2, f_3, f_4) \otimes F_{ln} \in \mathcal{H}_{ln},\\ & 1 \leq i \leq 4,\ f_i \in C_0^{\infty}(\mathbb{R}_{r_*})\}; \qquad (l, n) \in \mathcal{I}_{\frac{1}{2}}\end{aligned} \tag{85}$$

is dense in $\mathcal{H}$.

Proof of proposition 6.1. – Let

$$\Psi_{ln} = \phi \otimes F_{ln} \in \mathcal{H}_{ln}; \qquad \phi = {}^t(f_1, f_2, f_3, f_4) \in [L^2(\mathbb{R}, dr_*^2)]^4 \tag{86}$$

such that

$$H\, \Psi_{ln} = \lambda\, \Psi_{ln}; \qquad \lambda \in \mathbb{R}. \tag{87}$$

Equation (87) is equivalent to

$$\begin{array}{l} f_1' = -\beta_l\, f_2 - i\,\lambda\, f_3,\\ f_2' = -\beta_l\, f_1 + i\,\lambda\, f_4,\\ f_3' = -\beta_l\, f_4 - i\,\lambda\, f_1,\\ f_4' = -\beta_l\, f_3 + i\,\lambda\, f_2, \end{array} \qquad \beta_l(r_*) = \left(l + \frac{1}{2}\right) \frac{F^{1/2}\, e^{\delta}}{r}. \tag{88}$$

We first consider the case $\lambda = 0$. Putting

$$\begin{array}{ll} g_1 = f_1 + f_2, & g_2 = f_2 - f_1,\\ g_3 = f_3 + f_4, & g_4 = f_4 - f_3, \end{array} \tag{89}$$

we see that g_1 and g_3 are solutions of

$$g' = -\beta_l . g, \tag{90}$$

while g_2 and g_4 satisfy

$$f' = \beta_l . f. \tag{91}$$

Thus $\lambda = 0$ is an eigenvalue for H if and only if there exists $l = \frac{1}{2} + k$, $k \in \mathbb{N}$, such that both equations (90) and (91) have solutions in $L^2(\mathbb{R}_{r_*};\ dr_*^2)$. β_l being smooth on $\mathbb{R}$, any solution of (90) or (91) in $L^1_{\text{loc}}(\mathbb{R})$ is necessarily smooth. Moreover, β_l decreases exponentially when r_* goes to $-\infty$, thus

$$\forall\, r_*^1 \in \mathbb{R} \quad \beta_l \in L^1(]-\infty,\ r_*^1[) \tag{92}$$

and both integral equation

$$f(r_*) = 1 + \int_{-\infty}^{r_*} \beta_l \cdot f \, dr_*, \tag{93}$$

$$g(r_*) = 1 - \int_{-\infty}^{r_*} \beta_l \cdot g \, dr_* \tag{94}$$

have a unique solution in $L^\infty(]-\infty, r_{r_*}^1[)$, which can be extended on $\mathbb{R}$ as a smooth but not square integrable function. Therefore, (90) and (91) have no non trivial solution in $L^2(\mathbb{R})$ and $\lambda = 0$ is not an eigenvalue for H.

If now we suppose $\lambda \neq 0$, the components of ϕ satisfy

$$\begin{aligned} f_1'' &= (\beta_l^2 - \lambda^2) f_1 - \beta_l' f_2, \\ f_2'' &= (\beta_l^2 - \lambda^2) f_2 - \beta_l' f_1, \\ f_3'' &= (\beta_l^2 - \lambda^2) f_3 - \beta_l' f_4, \\ f_4'' &= (\beta_l^2 - \lambda^2) f_4 - \beta_l' f_3. \end{aligned} \tag{95}$$

Functions $g_1 = f_1 + f_2$ and $g_3 = f_3 + f_4$ are eigenvectors in $L^2(\mathbb{R})$ for the operator

$$L_1 = -\partial_{r_*}^2 + \beta_l^2(r_*) - \beta_l'(r_*) \tag{96}$$

associated with the eigenvalue $\lambda^2 > 0$, whereas $g_2 = f_2 - f_1$ and $g_4 = f_4 - f_3$ are eigenvectors in $L^2(\mathbb{R})$ for the operator

$$L_2 = -\partial_{r_*}^2 + \beta_l^2(r_*) + \beta_l'(r_*) \tag{97}$$

associated with the eigenvalue $\lambda^2 > 0$. It is easily seen that potentials

$$V_1(r_*) = \beta_l^2(r_*) - \beta_l'(r_*) \tag{98}$$

and

$$V_2(r_*) = \beta_l^2(r_*) + \beta_l'(r_*) \tag{99}$$

satisfy (84). Therefore, the operators L_1 and L_2 are of the same type as the second order operators studied in [3] and have no strictly positive eigenvalue.

Q.E.D.

Proof of corollary 6.1. – For $(l, n) \in \mathcal{I}_{\frac{1}{2}}$ and $k \in \mathbb{N}$, if

$$\Psi = \phi \otimes F_{ln} \in \mathcal{H}_{ln}; \qquad \phi \in [C_0^\infty(\mathbb{R}_{r_*})]^4,$$

then Ψ belongs to $D(H^k|_{\mathcal{H}_{ln}})$. $\mathcal{E}_{ln}^k$ is well-defined and is a subset of $\mathcal{H}_{ln}$. To prove corollary 6.1 it suffices to establish that for $(l, n) \in \mathcal{I}_{\frac{1}{2}}$ and

$k \in \mathbb{N}$, $\mathcal{E}_{ln}^k$ is dense in $\mathcal{H}_{ln}$. Let

$$\Psi_0 = \phi_0 \otimes F_{ln} \in \mathcal{H}_{ln}$$

be orthogonal to $\mathcal{E}_{ln}^k$. Then, for $\phi \in [\mathcal{C}_0^\infty(\mathbb{R}_{r_*})]^4$

$$(\phi_0,\ H^k\,|_{\mathcal{H}_{ln}}\,\phi)_{L^2(\mathbb{R}_{r_*})} = 0,$$

$H^k\,|_{\mathcal{H}_{ln}}$ being here considered as an operator on $[L^2(\mathbb{R}_{r_*})]^4$. We have

$$H^k\,|_{\mathcal{H}_{ln}}\,\phi_0 = 0 \quad in\ [\mathcal{D}'(\mathbb{R}_{r_*})]^4 \tag{100}$$

where $\mathcal{D}'(\mathbb{R}_{r_*})$ is the space of distributions on $\mathbb{R}_{r_*}$. From (100), we deduce that Ψ_0 belongs to $D(H^k\,|_{\mathcal{H}_{ln}})$ and

$$H^k\,\Psi_0 = 0 \quad in\ \mathcal{H}_{ln}. \tag{101}$$

We know by proposition 6.1 that (101) has no non-trivial solution in $\mathcal{H}_{ln}$. Thus $\mathcal{E}_{ln}^k$ is dense in $\mathcal{H}_{ln}$.

Q.E.D.

We also study the spectral properties of operators L_1, L_2. We recall their definition for $l - 1/2 \in \mathbb{N}$

$$\begin{aligned} i = 1, 2, \qquad & L_i = -\partial_{r_*}^2 + V_i(r_*); \\ & V_i(r_*) = \beta_l^2(r_*) + (-1)^i\,\beta_l'(r_*). \end{aligned} \tag{102}$$

PROPOSITION 6.2. – *For $l - 1/2 \in \mathbb{N}$, the spectrum of operators L_1 and L_2 is purely absolutely continuous.*

Proof. – We already know that potentials V_1 and V_2 satisfy (84), which, from [3] implies that the singular spectrum of L_1 and L_2 is empty, that their absolutely continuous spectrum is $[0, +\infty[$ and that their point spectrum contains at the most a finite number of negative or zero eigenvalues, all of them being simple. Furthermore, V_1 and V_2 decrease exponentially when $r_* \to -\infty$ and 0 is not an eigenvalue. We show that L_1 and L_2 do not have any strictly negative eigenvalue either by a method similar to the one used in [3]. We recall that for $l - 1/2 \in \mathbb{N}$, equations

$$1 \le i \le 2 \qquad L_i\,f = 0 \tag{103}$$

both have on $\mathbb{R}_{r_*}$ a unique continuous strictly positive solution, given respectively by (93) and (94). We consider the general case of a potential

$$V \in L^\infty(\mathbb{R}_{r_*}) \cap L^2(\mathbb{R}_{r_*}) \tag{104}$$

such that there exists a function g, continuous and strictly positive on $\mathbb{R}_{r_*}$, satisfying

$$L_V\,g = 0; \qquad L_V = -\partial_{r_*}^2 + V. \tag{105}$$

Let $f \in L^2(\mathbb{R}_{r_*})$ be such that

$$L_V f = -\lambda f, \qquad \lambda > 0, \tag{106}$$

which implies

$$f \in H^2(\mathbb{R}_{r_*}). \tag{107}$$

We define the cut-off function

$$\begin{aligned} \chi \in C_0^\infty(\mathbb{R}_{r_*}), \qquad &for \quad |r_*| \leq \frac{1}{2} \\ \chi(r_*) = 1, \qquad for \quad |r_*| \geq 1 \qquad &\chi(r_*) = 0. \end{aligned} \tag{108}$$

Putting for $n \geq 1$

$$f_n(r_*) = \chi\left(\frac{r_*}{n}\right) f(r_*), \tag{109}$$

we easily see that

$$\int_{[-n,\,n]} (|f_n'|^2 + V|f_n|^2)\, dr_* = -\lambda \int_{[-\frac{n}{2},\,\frac{n}{2}]} |f|^2\, dr_* + o(1). \tag{110}$$

Thus, for n large enough

$$\int_{[-n,\,n]} [|f_n'|^2 + V|f_n|^2]\, dr_* < 0.$$

The operator $-\partial_{r_*}^2 + V$ on $L^2([-n,\, n])$ with domain $\{y \in H^2([-n,\, n]);\ y(\pm n) = 0\}$ has a strictly negative eigenvalue $-\lambda_n$ associated with an eigenvector u

$$\left.\begin{aligned} -u'' + Vu = -\lambda_n u; \qquad -n < r_* < n, \\ u(-n) = u(n) = 0. \end{aligned}\right\} \tag{111}$$

Even if it means changing u into $-u$, there exist α and β such that

$$\begin{gathered} -n \leq \alpha < \beta \leq n, \\ u(\alpha) = u(\beta) = 0, \qquad u'(\alpha) > 0, \quad u'(\beta) < 0, \\ u > 0 \qquad for \quad \alpha < r_* < \beta. \end{gathered} \tag{112}$$

We denote

$$I = \int_\alpha^\beta (u' g - ug')'\, dr_*.$$

On the one hand, we can write

$$I = u'(\beta)\, g(\beta) - u'(\alpha)\, g(\alpha),$$

g being strictly positive on $\mathbb{R}$, (112) yields

$$I < 0.$$

On the other hand

$$(u' g - ug')' = u'' g - g'' u = -\lambda_n ug,$$

thus

$$I = \lambda_n \int_\alpha^\beta ugdr_* > 0.$$

We end up with a contradiction, which means that L_V has no strictly negative eigenvalue.

Q.E.D.

We now prove the existence of the inverse wave operators $\tilde{W}_0^\pm$ and $\tilde{W}_\infty^\pm$. For $(l, n) \in \mathcal{I}_{\frac{1}{2}}$, we consider the orthogonal decomposition of $\mathcal{H}_{ln}$

$$\mathcal{H}_{ln} = \mathcal{H}_{ln}^+ \oplus \mathcal{H}_{ln}^-,$$
$$\mathcal{H}_{ln}^\pm = \{\Psi = {}^t(f_1, f_2, f_3, f_4) \otimes F_{ln} \in \mathcal{H}_{ln}; f_2 = \mp f_1, f_4 = \pm f_3\}. \quad (113)$$

Each $\mathcal{H}_{ln}^\pm$ is stable under H and by corollary 6.1, for $(l, n) \in \mathcal{I}_{\frac{1}{2}}$, $k \in \mathbb{N}$, the sets

$$\mathcal{E}_{ln}^{k\pm} = \mathcal{E}_{ln}^k \cap \mathcal{H}_{ln}^\pm = \{H^k \Psi;\ \Psi = {}^t(f_1, \mp f_1, f_3, \pm f_3) \otimes F_{ln} \in \mathcal{H}_{ln}^\pm; \\ f_1, f_3 \in C_0^\infty(\mathbb{R}_{r_*})\} \quad (114)$$

are respectively dense in $\mathcal{H}_{ln}^+$ and $\mathcal{H}_{ln}^-$. For $\Psi_0 \in \mathcal{E}_{ln}^{2\pm}$ we establish the existence of the strong limits (78) and (79) defining $\tilde{W}_0^\pm \Psi_0$ and $\tilde{W}_\infty^\pm \Psi_0$. The following lemma guarantees the existence of asymptotic profiles for Ψ_0. The details of its proof will be given after the proof of theorem 6.1.

LEMMA 6.1. – *Given* $\Psi_0 \in \mathcal{E}_{ln}^{2\pm}$, $(l, n) \in \mathcal{I}_{\frac{1}{2}}$, *there exists*

$$\Psi_1 \in [C(\mathbb{R}_t;\ H^1(\mathbb{R}_{r_*})) \cap C^1(\mathbb{R}_t;\ L^2(\mathbb{R}_{r_*}))]^4 \otimes F_{ln} \quad (115)$$

such that

$$\partial_t \Psi_1 = i H_0 \Psi_1, \quad (116)$$

and

$$s\text{-}\lim_{t\to\pm\infty} \| e^{iHt} \Psi_0 - \Psi_1(t) \|_{\mathcal{H}} = 0. \quad (117)$$

Any solution of (116) in $\mathcal{C}(\mathbb{R}_t;\, \mathcal{H})$ and in particuliar Ψ_1 can be expressed in the form

$$\Psi_1(t) = e^{iH_0 t}\, \Psi_0^+ + e^{iH_0 t}\, \Psi_0^- \tag{118}$$

where

$$\Psi_0^+ \in \mathcal{H}_0^+, \qquad \Psi_0^- \in \mathcal{H}_0^-. \tag{119}$$

Thus, for a cut-off function χ_0 satisfying (44), we have

$$\lim_{t\to+\infty} \| \mathcal{J}_0\, \Psi_1(t) - e^{iH_0 t}\, \Psi_0^+ \|_{\mathcal{H}} = 0. \tag{120}$$

That is to say that for $\Psi_0 \in \mathcal{E}_{ln}^{2\varepsilon}$, $(l,\, n) \in \mathcal{I}_{\frac{1}{2}}$, $\varepsilon = +,\, -$, there exists

$$\Psi_0^+ \in \mathcal{H}_0^+ \cap \mathcal{H}_{ln}^{\varepsilon} \tag{121}$$

such that

$$\lim_{t\to+\infty} \| \mathcal{J}_0\, e^{iHt}\, \Psi_0 - e^{iH_0 t}\, \Psi_0^+ \|_{\mathcal{H}} = 0 \tag{122}$$

and of course, we can similarly prove the existence of

$$\Psi_0^- \in \mathcal{H}_0^- \cap \mathcal{H}_{ln}^{\varepsilon} \tag{123}$$

such that

$$\lim_{t\to-\infty} \| \mathcal{J}_0\, e^{iHt}\, \Psi_0 - e^{iH_0 t}\, \Psi_0^- \|_{\mathcal{H}} = 0. \tag{124}$$

From (121) to (124), we conclude that $\tilde{W}_0^{\pm}\, \Psi_0$ is well-defined for $\Psi_0 \in \mathcal{E}_{ln}^{2\varepsilon}$, $(l,\, n) \in \mathcal{I}_{\frac{1}{2}}$, $\varepsilon = +,\, -$, and

$$\tilde{W}_0^{\pm}\, \Psi_0 \in \mathcal{H}_0^{\pm}, \quad \| \tilde{W}_0^{\pm}\, \Psi_0 \|_{\mathcal{H}_0} \le \| \Psi_0 \|_{\mathcal{H}}. \tag{125}$$

Then, corollary 6.1 yields that the operator $\tilde{W}_0^+$ (resp. $\tilde{W}_0^-$) is well-defined from $\mathcal{H}$ to $\mathcal{H}_0^+$ (resp. $\mathcal{H}_0^-$) and its norm is lower or equal to 1.

In order to prove the existence of $\tilde{W}_\infty^+$, we need to compare in the neighbourhood of the future infinity the outgoing part of $\Psi_1(t)$ with a solution of

$$(\partial_t - i\, H_\infty)\, \Psi = 0. \tag{126}$$

LEMMA 6.2. – *The operator* W_0^∞

$$W_0^\infty\, \Psi_0 = s\text{-}\lim_{t\to+\infty}\, e^{-iH_\infty t}\, \mathcal{J}_\infty^*\, e^{iH_0 t}\, \Psi_0 \tag{127}$$

is well-defined from $\mathcal{H}_0^-$ *to* $\mathcal{H}_\infty$ *and is independent of the choice of* χ_∞ *satisfying (65). Of course* W_0^∞ *is defined as well from* $\mathcal{H}_0^+$ *to* $\mathcal{H}_\infty$ *and for* $\Psi_0 \in \mathcal{H}_0^+$

$$W_0^\infty\, \Psi_0 = 0.$$

Lemma 6.2, and (118), (119) yield the existence of

$$\Psi_\infty^+ \in \mathcal{H}_\infty \tag{128}$$

such that

$$\lim_{t\to+\infty} \| \mathcal{J}_\infty^* \Psi_1(t) - e^{iH_\infty t} \Psi_\infty^+ \|_{\mathcal{H}_\infty} = 0 \tag{129}$$

and therefore

$$\lim_{t\to+\infty} \| \mathcal{J}_\infty^* e^{iHt} \Psi_0 - e^{iH_\infty t} \Psi_\infty^+ \|_{\mathcal{H}_\infty} = 0. \tag{130}$$

which enables us to define $\tilde{W}_\infty^+$ on $\mathcal{E}_{ln}^{2\pm}$, $(l, n) \in \mathcal{I}_{\frac{1}{2}}$ and by density on $\mathcal{H}$. The same thing can be done for $\tilde{W}_\infty^-$. Let χ_∞ and χ'_∞ be two cut-off functions satisfying (65) and $\mathcal{J}_\infty$ and $\mathcal{J}'_\infty$ the associated identifying operators. For $t \in \mathbb{R}$, $\Psi_0 \in \mathcal{H}$

$$\begin{aligned} &\| e^{-iH_\infty t} \mathcal{J}_\infty^* e^{iHt} \Psi_0 - e^{-iH_\infty t} \mathcal{J}_\infty'^* e^{iHt} \Psi_0 \|_{\mathcal{H}_\infty} \\ &\leq \| (\chi_\infty - \chi'_\infty) e^{iHt} \Psi_0 \|_{\mathcal{H}}, \end{aligned}$$

and

$$\lim_{t\to\pm\infty} \| e^{-iH_\infty t} \mathcal{J}_\infty^* e^{iHt} \Psi_0 - e^{-iH_\infty t} \mathcal{J}_\infty'^* e^{iHt} \Psi_0 \|_{\mathcal{H}_\infty} = 0.$$

Thus, the operators $\tilde{W}_\infty^\pm$ are independent of the choice of χ_∞ and by a similar argument, $\tilde{W}_0^\pm$ are independent of the choice of χ_0.

We still have to prove that $W^\pm$ and $\tilde{W}^\pm$ are bijective isometries, which yields that S is a bijective isometry by construction. Let $\Psi \in \mathcal{H}$ and

$$\Psi_0^\pm = \tilde{W}_0^\pm \Psi, \qquad \Psi_\infty^\pm = \tilde{W}_\infty^\pm \Psi. \tag{131}$$

For χ_0 satisfying (44) and χ_∞ satisfying (65), we have

$$\lim_{t\to\pm\infty} \| \mathcal{J}_0 (e^{iHt} \Psi - e^{iH_0 t} \Psi_0^\pm) \|_{\mathcal{H}} = 0, \tag{132}$$

$$\lim_{t\to\pm\infty} \| \mathcal{J}_\infty \mathcal{J}_\infty^* e^{iHt} \Psi - \mathcal{J}_\infty e^{iH_\infty t} \Psi_\infty^\pm) \|_{\mathcal{H}} = 0, \tag{133}$$

$\mathcal{J}_\infty \mathcal{J}_\infty^*$ being simply the multiplication by χ_∞. The local energy of $e^{iHt}\Psi$ goes to 0 when t goes to $\pm\infty$, therefore

$$\lim_{t\to\pm\infty} \| (\chi_o + \chi_\infty - 1) e^{iHt} \Psi \|_{\mathcal{H}} = 0, \tag{134}$$

(132), (133) and (134) imply

$$\lim_{t\to\pm\infty} \| e^{iHt} \Psi - \mathcal{J}_0 e^{iH_0 t} \Psi_0^\pm - \mathcal{J}_\infty e^{iH_\infty t} \Psi_\infty^\pm \|_{\mathcal{H}} = 0, \tag{135}$$

which means

$$W^{\pm}\,\tilde{W}^{\pm} = \mathbf{1}_{\mathcal{H}}. \tag{136}$$

If on the other hand we consider

$$\Psi_0^{\pm} \in \mathcal{H}_0^{\pm}, \qquad \Psi_\infty^{\pm} \in \mathcal{H}_\infty \tag{137}$$

and put

$$\Psi = W^{\pm}\,(\Psi_0^{\pm},\, \Psi_\infty^{\pm}), \tag{138}$$

we have (135) from which we get

$$\lim_{t\to\pm\infty} \|\, \mathcal{J}_0^*\,(e^{iHt}\,\mathbf{\Psi} - \mathcal{J}_0\, e^{iH_0 t}\,\Psi_0^{\pm} - \mathcal{J}_\infty\, e^{iH_\infty t}\,\Psi_\infty^{\pm})\,\|_{\mathcal{H}} = 0 \tag{139}$$

$$\lim_{t\to\pm\infty} \|\, \mathcal{J}_\infty^*\,(e^{iHt}\,\mathbf{\Psi} - \mathcal{J}_0\, e^{iH_0 t}\,\Psi_0^{\pm} - \mathcal{J}_\infty\, e^{iH_\infty t}\,\Psi_\infty^{\pm})\,\|_{\mathcal{H}_\infty} = 0. \tag{140}$$

The local energy of $e^{iH_0 t}\,\Psi_0^{\pm}$ and $e^{iH_\infty t}\,\Psi_\infty^{\pm}$ goes to 0 when $|\,t\,|$ goes to $+\infty$, therefore (139) and (140) yield

$$\lim_{t\to\pm\infty} \|\, \mathcal{J}_0^*\, e^{iHt}\,\Psi - e^{iH_0 t}\,\Psi_0^{\pm}\,\|_{\mathcal{H}} = 0 \tag{141}$$

and

$$\lim_{t\to\pm\infty} \|\, \mathcal{J}_\infty^*\, e^{iHt}\,\Psi - e^{iH_\infty t}\,\Psi_\infty^{\pm}\,\|_{\mathcal{H}_\infty} = 0, \tag{142}$$

thus

$$\tilde{W}^{\pm}\,W^{\pm} = \mathbf{1}_{\mathcal{H}_0^{\pm}\times\mathcal{H}_\infty}. \tag{143}$$

(136) and (143) show that $W^{\pm}$ and $\tilde{W}^{\pm}$ are all bijections and if we choose χ_o and χ_∞ such that their supports have no intersection, we deduce from (135)

$$\|\,\Psi\,\|_{\mathcal{H}} = \|\,\Psi_0^{\pm}\,\|_{\mathcal{H}} + \|\,\Psi_\infty^{\pm}\,\|_{\mathcal{H}_\infty}. \tag{144}$$

Q.E.D.

Proof of lemma 6.1. – Let $\Psi_0 \in \mathcal{E}_{ln}^{2\varepsilon}$, $(l, n) \in \mathcal{I}_{\frac{1}{2}}$, $\varepsilon = +, -$. There exists

$$\Psi_0' = {}^t\!(f_1,\, -\varepsilon\, f_1,\, f_3,\, \varepsilon\, f_3) \otimes F_{ln} \in \mathcal{E}_{ln}^{1\varepsilon} \tag{145}$$

such that

$$\Psi_0 = i\,H\,\Psi_0' \tag{146}$$

and

$$\Psi_0'' = {}^t\!(g_1,\, -\varepsilon\, g_1,\, g_3,\, \varepsilon\, g_3) \otimes F_{ln} \in \mathcal{E}_{ln}^{0\varepsilon} \tag{147}$$

such that

$$\Psi_0' = -i\,H\,\Psi_0''. \tag{148}$$

We denote

$$\tilde{\Psi} = e^{iHt}\,\Psi_0'; \qquad \tilde{\Psi} = \tilde{\phi} \otimes F_{ln} = {}^t(\phi_1,\, -\varepsilon\phi_1,\, \phi_3,\, \varepsilon\phi_3) \otimes F_{ln} \tag{149}$$

and

$$\Psi = \partial_t\,\tilde{\Psi} = i\,H\,\tilde{\Psi}. \tag{150}$$

On the one hand, applying $\partial_t + i\,H$ to equation

$$(\partial_t - i\,H)\,\tilde{\Psi} = 0,$$

we obtain

$$(\partial_t^2 - H^2)\,\tilde{\Psi} = 0$$

which, taking into account the fact that $\tilde{\Psi}$ takes its values in $\mathcal{H}_{ln}$ can also be written

$$(\partial_t^2 - \partial_{r_*}^2 + \beta_l^2 + \varepsilon\,\beta_l')\,\phi_1 = 0, \tag{151}$$

$$(\partial_t^2 - \partial_{r_*}^2 + \beta_l^2 - \varepsilon\,\beta_l')\,\phi_3 = 0. \tag{152}$$

On the other hand

$$\phi_1\,|_{t=0} = f_1; \quad \phi_3\,|_{t=0} = f_3; \qquad f_1,\, f_3 \in C_0^\infty(\mathbb{R}_{r_*}) \tag{153}$$

and since $\Psi_0 = H^2\,\Psi_0''$

$$\partial_t\,\phi_1\,|_{t=0} = (-\partial_{r_*}^2 + \beta_l^2 + \varepsilon\beta_l')\,g_1, \qquad g_1 \in C_0^\infty(\mathbb{R}_{r_*}) \tag{154}$$

$$\partial_t\,\phi_3\,|_{t=0} = (-\partial_{r_*}^2 + \beta_l^2 - \varepsilon\beta_l')\,g_3, \qquad g_3 \in C_0^\infty(\mathbb{R}_{r_*}). \tag{155}$$

The scattering results obtained in [3] together with proposition 6.2 imply that for any solution

$$f \in \mathcal{C}(\mathbb{R}_t;\, H^1(\mathbb{R}_{r_*})) \cap \mathcal{C}^1(\mathbb{R}_t;\, L^2(\mathbb{R}_{r_*}))$$

of equation

$$(\partial_t^2 - \partial_{r_*}^2 + \beta_l^2 + \eta\,\beta_l')\,f = 0, \qquad \eta = +,\, -$$

with initial data

$$f\,|_{t=0} = \mu_1, \qquad \partial_t f\,|_{t=0} = (-\partial_{r_*}^2 + \beta_l^2 + \eta\beta_l')\,\mu_2$$

such that

$$i = 1,\,2 \quad \mu_i \in L^2(\mathbb{R}_{r_*}); \qquad (-\partial_{r_*}^2 + \beta_l^2 + \eta\beta_l')\,\mu_i \in L^2(\mathbb{R}_{r_*}),$$

there exists a solution

$$f_1 \in \mathcal{C}(\mathbb{R}_t;\, H^1(\mathbb{R}_{r_*})) \cap \mathcal{C}^1(\mathbb{R}_t;\, L^2(\mathbb{R}_{r_*})) \tag{156}$$

of

$$(\partial_t^2 - \partial_{r_*}^2)\, f_1 = 0 \tag{157}$$

such that

$$\lim_{t\to+\infty} \|\, f(t) - f_1(t)\,\|_{H^1(\mathbb{R}_{r_*})} + \|\,\partial_t f(t) - \partial_t f_1(t)\,\|_{L^2(\mathbb{R}_{r_*})} = 0.$$

$\tilde{\Psi}$ is the solution of (15) with initial data

$$\Psi_0' \in [\mathcal{C}_0^\infty(\mathbb{R}_{r_*})]^4 \otimes F_{ln}$$

therefore in particular,

$$\phi_1,\, \phi_2 \in \mathcal{C}(\mathbb{R}_t;\, H^1(\mathbb{R}_{r_*})) \cap \mathcal{C}^1(\mathbb{R}_t;\, L^2(\mathbb{R}_{r_*}))$$

and (151) to (155) yield the existence of

$$\tilde{\Psi}_1 \in [\mathcal{C}(\mathbb{R}_t;\, H^1(\mathbb{R}_{r_*})) \cap \mathcal{C}^1(\mathbb{R}_t;\, L^2(\mathbb{R}_{r_*}))]^4 \otimes F_{ln}$$

such that

$$(\partial_t^2 - \partial_{r_*}^2)\,\tilde{\Psi}_1 = 0$$

and

$$\lim_{t\to+\infty} \|\, e^{iHt}\,\tilde{\Psi}_0 - \tilde{\Psi}_1\,\|_{\mathcal{H}} = 0, \qquad \lim_{t\to+\infty} \|\,\partial_{r_*}(e^{iHt}\,\tilde{\Psi}_0 - \tilde{\Psi}_1)\,\|_{\mathcal{H}} = 0,$$

$$\lim_{t\to+\infty} \|\,\partial_t(e^{iHt}\,\tilde{\Psi}_0 - \tilde{\Psi}_1)\,\|_{\mathcal{H}} = 0,$$

from which we deduce

$$\lim_{t\to+\infty} \|\, e^{iHt}\,\Psi_0 - \partial_t\,\tilde{\Psi}_1\,\|_{\mathcal{H}} = 0. \tag{158}$$

Ψ_0 being an element of $\mathcal{E}_{ln}^{2\varepsilon} \subset \mathcal{E}_{ln}^{1\varepsilon}$, we can apply the previous construction to Ψ_0. We find that there exists

$$\Psi_1 \in [\mathcal{C}(\mathbb{R}_t;\, H^1(\mathbb{R}_{r_*})) \cap \mathcal{C}^1(\mathbb{R}_t;\, L^2(\mathbb{R}_{r_*}))]^4 \otimes F_{ln}$$

solution of

$$(\partial_t^2 - \partial_{r_*}^2)\,\Psi_1 = 0$$

such that

$$\lim_{t\to+\infty} \|\, e^{iHt}\,\Psi_0 - \Psi_1 \,\|_{\mathcal{H}} = 0, \qquad \lim_{t\to+\infty} \|\, \partial_{r_*}(e^{iHt}\,\Psi_0 - \Psi_1)\,\|_{\mathcal{H}} = 0, \tag{159}$$

$$\lim_{t\to+\infty} \|\, \partial_t (e^{iHt}\,\Psi_0 - \Psi_1)\,\|_{\mathcal{H}} = 0. \tag{160}$$

From (159) and (160) we deduce

$$\lim_{t\to+\infty} \|\, (\partial_t - i\,H_0)\,(e^{iHt}\,\Psi_0 - \Psi_1)\,\|_{\mathcal{H}} = 0. \tag{161}$$

$e^{iHt}\,\Psi_0$ being a solution of (15) in $\mathcal{C}(\mathbb{R}_t;\, \mathcal{H}_{ln})$, we have

$$(\partial_t - i\,H)\,e^{iHt}\,\Psi_0 = (\partial_t - i\,H_0 - i\,\beta_l\,M)\,e^{iHt}\,\Psi_0 = 0 \tag{162}$$

and by (158)

$$\lim_{t\to+\infty} \|\, i\,\beta_l\,M\,(e^{iHt}\,\Psi_0 - \partial_t\,\tilde{\Psi}_1)\,\|_{\mathcal{H}} = 0.$$

$\partial_t\,\tilde{\Psi}_1$ is identically zero in

$$\{(t,\, r_*,\, \omega);\ |\, r_*\,| \le |\, t\,| - R,\ \omega \in S^2\},$$

which is not true in general for $\tilde{\Psi}_1$, therefore

$$\lim_{t\to+\infty} \|\, i\,\beta_l\,M\,\partial_t\,\tilde{\Psi}_1\,\|_{\mathcal{H}} = 0$$

and

$$\lim_{t\to+\infty} \|\, i\,\beta_l\,M\,e^{iHt}\,\Psi_0\,\|_{\mathcal{H}} = 0. \tag{163}$$

(161), (162) and (163) give

$$\lim_{t\to+\infty} \|\, (\partial_t - i\,H_0)\,\Psi_1\,\|_{\mathcal{H}} = 0$$

and $(\partial_t - i\, H_0)\, \Psi_1$ being an element of $\mathcal{C}\,(\mathbb{R}_t;\ \mathcal{H})$ and satisfying

$$(\partial_t + i\, H_0)\, [(\partial_t - i\, H_0)\, \Psi_1] = 0$$

we must have

$$(\partial_t - i\, H_0)\, \Psi_1 = 0.$$

Q.E.D.

Proof of lemma 6.2. – Let

$$\Psi_0 \in \mathcal{H}_0^- \cap \mathcal{E}_{ln}^{0\varepsilon}, \quad (l,\ n) \in \mathcal{I}_{\frac{1}{2}}, \qquad \varepsilon = +,\ - \tag{164}$$

with

$$\mathrm{Supp}\, (\Psi_0) \subset [-R,\ R]_{r_*} \times S^2_{\theta,\, \varphi}, \qquad R > 0. \tag{165}$$

Ψ_0 can be written

$$\left.\begin{array}{c} \Psi_0 = {}^t(f_0,\ -\varepsilon\, f_0,\ f_0,\ \varepsilon\, f_0) \otimes F_{ln}, \qquad f_0 \in \mathcal{C}_0^\infty\, (\mathbb{R}_{r_*}) \\ \mathrm{Supp}\, f_0 \subset [-R,\ R] \end{array}\right. \tag{166}$$

and

$$e^{iH_0 t}\Psi_0 = {}^t(f,\ -\varepsilon\, f,\ f,\ \varepsilon\, f) \otimes F_{ln}, \qquad f\, (t,\ r_*) = f_0\, (r_* - t). \tag{167}$$

f is the solution of

$$(\partial_t^2 - \partial_{r_*}^2)\, f = 0 \tag{168}$$

associated with the initial data

$$f\, |_{t=0} = f_0, \qquad \partial_t\, f\, |_{t=0} = -\partial_{r_*}\, f_0. \tag{169}$$

Instead of applying Cook's method to operators H_∞ and H_0, which would give an apparently long-range perturbation at infinity, we work on the second order scalar equations and establish the existence of g_η solution of

$$\left.\begin{array}{c} (\partial_t^2 - \partial_{r_*}^2 + V_\eta\, (r_*))\, g_\eta = 0 \\ V_\eta\, (r_*) = \chi_\infty\, (r_*)\, \dfrac{1}{r_*^2} \left(\left(l + \dfrac{1}{2} \right)^2 + \eta \left(l + \dfrac{1}{2} \right) \right), \quad \eta = +,\ -, \end{array}\right\} \tag{170}$$

where χ_∞ is a cut-off function satisfying (65); the solution g_η being such that

$$\lim_{t\to+\infty} \| \partial_t (g_\eta - f) \|_{L^2(\mathbb{R})} = 0, \quad \lim_{t\to+\infty} \| \partial_{r_*} (g_\eta - f) \|_{L^2(\mathbb{R})} = 0, \quad (171)$$

$$\lim_{t\to+\infty} \left\| \frac{l+\frac{1}{2}}{r} (g_\eta - f) \right\|_{L^2(\mathbb{R})} = 0. \quad (172)$$

In the case where $l = 1/2$ and $\eta = -$, equations (168) and (170) are the same and it suffices to take $g_- = f$. Let us now assume

$$\left(l + \frac{1}{2}\right)^2 + \eta \left(l + \frac{1}{2}\right) > 0. \quad (173)$$

We write equations (168) and (170) in their hamiltonian form

$$\partial_t \begin{pmatrix} f \\ \partial_t f \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ -\partial_{r_*}^2 & 0 \end{pmatrix} \begin{pmatrix} f \\ \partial_t f \end{pmatrix} = -A_0 \begin{pmatrix} f \\ \partial_t f \end{pmatrix}, \quad (174)$$

$$\partial_t \begin{pmatrix} g \\ \partial_t g \end{pmatrix} = -\begin{pmatrix} 0 & -1 \\ -\partial_{r_*}^2 + V_\eta & 0 \end{pmatrix} \begin{pmatrix} g \\ \partial_t g \end{pmatrix} = -A_\eta \begin{pmatrix} g \\ \partial_t g \end{pmatrix}. \quad (175)$$

The operator $i A_0$ is skew-adjoint with dense domain on

$$\mathbb{H}_0 = BL^1(\mathbb{R}_{r_*}) \times L^2(\mathbb{R}_{r_*}) \quad (176)$$

completion of $[\mathcal{C}_0^\infty(\mathbb{R}_{r_*})]^2$ for the norm

$$\| {}^t(f_1, f_2) \|_{\mathbb{H}_0}^2 = \int_{\mathbb{R}} \{ | \partial_{r_*} f_1 |^2 + | f_2 |^2 \} \, dr_* \quad (177)$$

and $i A_\eta$ is skew-adjoint with dense domain (*cf.* [3]) on

$$\mathbb{H} = \mathbb{H}_1 \times L^2(\mathbb{R}_{r_*}) \quad (178)$$

completion of $[\mathcal{C}_0^\infty(\mathbb{R}_{r_*})]^2$ for the norm

$$\| {}^t(g_1, g_2) \|_{\mathbb{H}}^2 = \int_{\mathbb{R}} \{ | \partial_{r_*} g_1 |^2 + | g_2 |^2 + V_\eta | g_1 |^2 \} \, dr_*. \quad (179)$$

Under assumption (173), the norm (179) is equivalent to

$$||| {}^t(g_1, g_2) |||^2 = \| {}^t(g_1, g_2) \|_{\mathbb{H}_0}^2 + \left\| \frac{\left(l + \frac{1}{2}\right) \chi_\infty}{r_*} g_1 \right\|_{L^2(\mathbb{R}_{r_*})}^2. \quad (180)$$

Moreover, any solution ${}^t(g, \partial_t g) \in \mathcal{C}(\mathbb{R}_t; \mathbb{H})$ of (170) satisfies the following energy estimate: for $r_*^1 < r_*^2$ and $t \in \mathbb{R}$

$$\int_{r_*^1 < r_* < r_*^2} \{|\partial_{r_*} g(t)|^2 + |\partial_t g(t)|^2 + V_\eta(r_*)\,|g(t)|^2\}\, dr_*$$

$$\leq \int_{r_*^1 - |t| < r_* < r_*^2 + |t|} \{|\partial_{r_*} g(0)|^2 + |\partial_t g(0)|^2 + V_\eta(r_*)\,|g(0)|^2\}\, dr_* \tag{181}$$

which is very easily obtained by multiplying (170) by $\partial_t g$ and integrating by parts on the domain

$$\Omega_{t, r_*^1, r_*^2} = \{(\tau, r_*);\ \tau \in (0, t),\ r_*^1 - |t - \tau| < r_* < r_*^2 + |t - \tau|\}. \tag{182}$$

f_0 being in $\mathcal{C}_0^\infty(\mathbb{R}_{r_*})$, we can consider that

$$e^{-A_0 t}\,[{}^t(f_0,\ -\partial_{r_*} f_0)] \in \mathcal{C}(\mathbb{R}_t;\ \mathbb{H})$$

and we apply Cook's method to prove the existence in $\mathbb{H}$ of the limit

$$\begin{pmatrix} g_{0_\eta} \\ g_{1_\eta} \end{pmatrix} = s\text{-}\lim_{t \to +\infty} e^{A_\eta t}\, e^{-A_0 t} \begin{pmatrix} f_0 \\ -\partial_{r_*} f_0 \end{pmatrix}. \tag{183}$$

We shall denote

$$\phi_0 = {}^t(f_0,\ -\partial_{r_*} f_0), \qquad \phi_\infty = {}^t(g_{0_\eta},\ g_{1_\eta}). \tag{184}$$

We have

$$\begin{aligned} \|\partial_t (e^{A_\eta t}\, e^{-A_0 t}\, \phi_0)\|_{\mathbb{H}} &= \|(A_\eta - A_0)\, e^{-A_0 t}\, \phi_0\|_{\mathbb{H}} \\ &= \|V_\eta(r_*)\, f_0(r_* - t)\|_{L^2(\mathbb{R}_{r_*})} \\ &\leq \|f_0\|_{L^2(\mathbb{R}_{r_*})}\, \|V_\eta\|_{L^\infty(r_* > t - R)} \end{aligned}$$

and for r_* large enough

$$V_\eta(r_*) = C\, r_*^{-2}, \qquad C > 0, \tag{185}$$

thus

$$\|\partial_t (e^{A_\eta t}\, e^{-A_0 t}\, \phi_0)\|_{\mathbb{H}} = O(t^{-2}); \qquad t \to +\infty,$$

and

$$\|\partial_t (e^{A_\eta t}\, e^{-A_0 t}\, \phi_0)\|_{\mathbb{H}} \in L^1(t > 0).$$

The limit (183) is therefore well-defined and if g_η is the solution of (170) such that

$$\begin{pmatrix} g_\eta(t) \\ \partial_t g_\eta(t) \end{pmatrix} = e^{-A_\eta t}\,\phi_\infty, \tag{186}$$

then

$$\lim_{t\to+\infty} \| {}^t(g_\eta, \partial_t g_\eta) - {}^t(f, \partial_t f) \|_{\mathbb{H}} = 0. \tag{187}$$

This last limit together with the equivalence of norms (179) and (180) gives (171) and (172). Moreover, for $r_* < t - R$

$$g_\eta(t, r_*) = 0 \qquad \textit{and} \qquad \partial_t g_\eta(t, r_*) = 0. \tag{188}$$

Indeed, for $t \in \mathbb{R}$, $\varepsilon > 0$ we choose $\tau \in \mathbb{R}$ such that

$$\| \phi_\infty - e^{iA_\eta \tau}\, e^{-iA_0 \tau}\, \phi_0 \|_{\mathbb{H}} \le \varepsilon, \qquad \tau \ge t. \tag{189}$$

For ${}^t(f_1, f_2) \in \mathbb{H}$, we denote

$$\mathcal{L}({}^t(f_1, f_2)) = |\partial_{r_*} f_1|^2 + V_\eta |f_1|^2 + |f_2|^2. \tag{190}$$

Let us consider

$$\begin{aligned} &\int_{r_*<t-R} \mathcal{L}(e^{-iA_\eta t}\,\phi_\infty)\, dr_* \\ &\le \int_{r_*<t-R} \mathcal{L}[e^{-iA_\eta t}(\phi_\infty - e^{iA_\eta \tau}\, e^{-iA_0\tau}\,\phi_0)]\, dr_* \\ &+ \int_{r_*<t-R} \mathcal{L}(e^{-iA_\eta (t-\tau)}\, e^{-iA_0 \tau}\,\phi_0)\, dr_*. \end{aligned}$$

(181) and (189) yield

$$\int_{r_*<t-R} \mathcal{L}(e^{-iA_\eta t}\,\phi_\infty)\, dr_* \le \varepsilon^2 + \int_{r_*<\tau-R} \mathcal{L}(e^{-iA_0\tau}\,\phi_0)\, dr_*$$

and this last integral is zero since

$$\mathrm{Supp}\,(e^{-iA_0\tau}\,\phi_0) \subset [\tau - R, \tau + R].$$

(188) is therefore satisfied and for t large enough g_η is a solution of

$$\left[\partial_t^2 - \partial_{r_*}^2 + \frac{1}{r_*^2}\left(\left(l + \frac{1}{2}\right)^2 + \eta\left(l + \frac{1}{2}\right)\right)\right] g_\eta = 0. \tag{191}$$

Let us now introduce

$$\tilde{\Psi}_\infty(t) = {}^t(g_{-\varepsilon}(t), -\varepsilon\, g_{-\varepsilon}(t), g_\varepsilon(t), \varepsilon\, g_\varepsilon(t)) \otimes F_{ln}. \tag{192}$$

There exists $t_0 > 0$ such that, for $t \geq t_0$, g_ε and $g_{-\varepsilon}$ satisfy

$$\left[\partial_t^2 - \partial_{r_*}^2 + \frac{1}{r_*^2}\left(\left(l + \frac{1}{2}\right)^2 + \varepsilon\left(l + \frac{1}{2}\right)\right)\right] g_\varepsilon = 0, \tag{193}$$

$$\left[\partial_t^2 - \partial_{r_*}^2 + \frac{1}{r_*^2}\left(\left(l + \frac{1}{2}\right)^2 - \varepsilon\left(l + \frac{1}{2}\right)\right)\right] g_{-\varepsilon} = 0, \tag{194}$$

with

$$\begin{gathered} g_\varepsilon,\ g_{-\varepsilon} \in \mathcal{C}\left([t_0,\ +\infty[;\ \mathbb{H}_1\right), \\ \partial_t\, g_\varepsilon,\ \partial_t\, g_{-\varepsilon} \in \mathcal{C}\left([t_0,\ +\infty[;\ L^2(\mathbb{R}_{r_*})\right). \end{gathered} \tag{195}$$

Moreover, for $t \geq t_0$

$$\begin{gathered} \operatorname{Supp}\left(g_\varepsilon(t),\ g_{-\varepsilon}(t),\ \partial_t\, g_\varepsilon(t),\ \partial_t\, g_{-\varepsilon}(t)\right) \\ \subset [t - R,\ +\infty[\ \subset [0,\ +\infty[. \end{gathered} \tag{196}$$

Thus, the quantities

$$\partial_t\, \tilde{\Psi}_\infty, \quad \partial_{r_*}\, \tilde{\Psi}_\infty, \qquad \left(l + \frac{1}{2}\right) r_*^{-1}\, \tilde{\Psi}_\infty$$

belong to $\mathcal{C}([t_0,\ +\infty[;\ \mathcal{H})$ and (171), (172) yield

$$\begin{gathered} \lim_{t \to +\infty} \|\, \partial_t\, (\tilde{\Psi}_\infty(t) - e^{iH_0 t}\, \Psi_0)\, \|_{\mathcal{H}} = 0, \\ \lim_{t \to +\infty} \|\, \partial_{r_*}\, (\tilde{\Psi}_\infty(t) - e^{iH_0 t}\, \Psi_0)\, \|_{\mathcal{H}} = 0, \end{gathered} \tag{197}$$

$$\lim_{t \to +\infty} \left\| \left(l + \frac{1}{2}\right) r_*^{-1}\, (\tilde{\Psi}_\infty(t) - e^{iH_0 t}\, \Psi_0) \right\|_{\mathcal{H}} = 0. \tag{198}$$

In particular, we have

$$\begin{aligned} \lim_{t \to +\infty} & \left\| \left(\partial_t + L\, \partial_{r_*} - i\left(l + \frac{1}{2}\right) r_*^{-1}\, M\right) \right. \\ & \left. \times\, (\tilde{\Psi}_\infty(t) - e^{iH_0 t}\, \Psi_0) \right\|_{\mathcal{H}} = 0. \end{aligned} \tag{199}$$

Since $e^{iH_0 t}\, \Psi_0$ is a solution of

$$(\partial_t + L\, \partial_{r_*})\, e^{iH_0 t}\, \Psi_0 = 0,$$

we have

$$\left\| \left(\partial_t + L\partial_{r_*} - i \left(l + \frac{1}{2} \right) r_*^{-1} M \right) e^{iH_0 t} \Psi_0 \right\|_{\mathcal{H}}$$
$$= \left(l + \frac{1}{2} \right) \| r_*^{-1} e^{iH_0 t} \Psi_0 \|_{\mathcal{H}} = O(t^{-1}), \qquad t \to \infty$$

and therefore

$$\lim_{t \to +\infty} \left\| \left(\partial_t + L\partial_{r_*} - i \left(l + \frac{1}{2} \right) r_*^{-1} M \right) \tilde{\Psi}_\infty (t) \right\|_{\mathcal{H}} = 0. \tag{200}$$

We introduce

$$\Psi_\infty = \tilde{\Psi}_\infty |_{\{r_* \geq 0\}}. \tag{201}$$

The quantities

$$\partial_t \Psi_\infty, \quad \partial_{r_*} \Psi_\infty, \qquad \left(l + \frac{1}{2} \right) r_*^{-1} \Psi_\infty$$

belong to $\mathcal{C}([t_0, +\infty[; \mathcal{H}_\infty^{\varepsilon l n})$ where, for $(l, n) \in \mathcal{I}_{\frac{1}{2}}$ and $\varepsilon = +, -$

$$\mathcal{H}_\infty^{\varepsilon l n} = \{ {}^t(f, -\varepsilon f, g, \varepsilon g) \otimes F_{ln} \in \mathcal{H}_\infty \}. \tag{202}$$

From (200), we get

$$\lim_{t \to +\infty} \left\| \left(\partial_t + L\partial_{r_*} - i \left(l + \frac{1}{2} \right) r_*^{-1} M \right) \Psi_\infty (t) \right\|_{\mathcal{H}_\infty} = 0 \tag{203}$$

and, the function

$$\left(\partial_t + L\partial_{r_*} - i \left(l + \frac{1}{2} \right) r_*^{-1} M \right) \Psi_\infty \in \mathcal{C}([t_0, +\infty[; \mathcal{H}_\infty^{\varepsilon l n})$$

satisfies

$$\left(\partial_t - L\partial_{r_*} + i \left(l + \frac{1}{2} \right) r_*^{-1} M \right)$$
$$\times \left[\left(\partial_t + L\partial_{r_*} - i \left(l + \frac{1}{2} \right) r_*^{-1} M \right) \Psi_\infty \right] = 0. \tag{204}$$

Therefore, we must have for $t \geq t_0$

$$\left(\partial_t + L\partial_{r_*} - i \left(l + \frac{1}{2} \right) r_*^{-1} M \right) \Psi_\infty (t) = 0 \quad \textit{in } \mathcal{H}_\infty.$$

$\mathbb{H}_1$ being a distribution space, we can write in the sense of distributions for $t \geq t_0$

$$\partial_t \left(\partial_t + L\partial_{r_*} - i \left(l + \frac{1}{2} \right) r_*^{-1} M \right) \Psi_\infty (t)$$
$$= \left(\partial_t + L\partial_{r_*} - i \left(l + \frac{1}{2} \right) r_*^{-1} M \right) \partial_t \Psi_\infty (t) = 0 \quad in\ \mathcal{H}_\infty,$$

which implies that $\partial_t \Psi_\infty$ is a solution in $\mathcal{C}([t_0, +\infty[; \mathcal{H}_\infty^{\varepsilon ln})$ of

$$(\partial_t - i H_\infty) \Psi = 0.$$

This solution can be extended to $\mathcal{C}(\mathbb{R}_t; \mathcal{H}_\infty^{\varepsilon ln})$ and we denote

$$\Psi_\infty^0 = e^{-iH_\infty t_0} \partial_t \Psi_\infty (t_0) \tag{205}$$

its initial data at $t = 0$. From (196), (197), we get

$$\lim_{t \to +\infty} \| e^{iH_\infty t} \Psi_\infty^0 - \mathcal{J}_\infty^* \partial_t (e^{iH_0 t} \Psi_0) \|_{\mathcal{H}_\infty} = 0. \tag{206}$$

The value of $\partial_t (e^{iH_0 t} \Psi_0)$ at $t = 0$ is $i H_0 \Psi_0$. H_0 is a self-adjoint operator with dense domain on $\mathcal{H}$, its point spectrum is empty and the spaces $\mathcal{H}_0^\pm$, $\mathcal{H}_{ln}^\pm$ are invariant under H_0. Therefore the direct sum of the sets

$$\{H_0 \Psi_0;\ \Psi_0 \in \mathcal{H}_0^- \cap \mathcal{E}_{ln}^{0\varepsilon}\}; \qquad (l, n) \in \mathcal{I}_{\frac{1}{2}}, \quad \varepsilon = +, - \tag{207}$$

is dense in $\mathcal{H}_0^-$. (206) shows that for an initial data $H_0 \Psi_0$ in a set of type (207), the limit

$$\Psi_\infty^0 = s\text{-}\lim_{t \to +\infty} e^{-iH_\infty t} \mathcal{J}_\infty^* e^{iH_0 t} H_0 \Psi_0 \tag{208}$$

exists in $\mathcal{H}_\infty$. The operator W_0^∞ is consequently well-defined from $\mathcal{H}_0$ into $\mathcal{H}_\infty$. Since the local energy of the solution $e^{iH_0 t} H_0 \Psi_0$ goes to zero when $|t|$ goes to $+\infty$, the limit Ψ_∞^0 is independent of the choice of χ_∞ satisfying (65).

Q.E.D.

7. CONCLUSION

The scattering theory developed in this paper is only valid for the linear massless Dirac system. In the case of a massive field and when space-time is asymptotically flat, the mass of the field induces long-range perturbations at infinity and classical wave operators will probably not exist. However, using the methods developed by J. Dollard and G. Velo [10] and by V. Enss and B. Thaller [11] about the relativistic Coulomb scattering of Dirac fields as well as the works of A. Bachelot [1] and J. Dimock and B. Kay [9] on the Klein-Gordon equation on the Schwarzschild metric, it must be possible to show the existence and asymptotic completeness of Dollard-modified wave operators at infinity.

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