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# On a model for quantum friction I. Fermi's golden rule and dynamics at zero temperature

by

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ABSTRACT. – We investigate the dynamics of a quantum particle in a confining potential linearly coupled to a bosonic field at temperature zero. For a massive field we show, by employing complex deformation techniques, that Markovian semigroup which approximates the particles dynamics on the time scale $\tau = \lambda^2 t$ ($\lambda$ strength of the coupling) is determined by the resonances of the full energy operator. We also show that Markovian master equation technique leads to the right prediction for the life-time of resonances. We discuss the dissipation of the particle into its ground state both in the time mean and on the above time scale.

RÉSUMÉ. – Nous étudions la dynamique quantique d'un oscillateur anharmonique couplé à un champ de bosons massifs. À température nulle et dans la limite du couplage faible ($\lambda \to 0$), la dynamique de l'oscillateur à l'échelle de temps $\lambda^{-2}$ est décrite par un semi-groupe markovien. À l'aide de techniques de déformation spectrale nous montrons que ce semi-groupe est complètement déterminé par les résonances de l'hamiltonien total du système. Nous montrons en particulier que l'équation maîtresse markovienne prédit correctement le temps de vie des états métastables. Nous discutons également la dissipation de l'oscillateur vers son état d'équilibre à l'échelle de temps $\lambda^{-2}$ ainsi qu'en moyenne.

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# 1. INTRODUCTION

Friction, as a notion in classical physics, has been set on solid mathematical grounds by the theory of Ornstein-Uhlenbeck processes. The dynamics of the particle experiencing frictional force is governed by the Langevin equation, a second order non-linear stochastic differential equation in time variable. This equation has been widely investigated beginning with the classical paper [12]. The long-standing conjecture that the Langevin equation (an equation in macroscopic variables) can be derived from microscopics consideration, has been positively answered in 1965 by Ford, Kac, and Mazur [8]. Starting from the model of a particle linearly coupled to the chain of harmonic oscillators

$$\begin{aligned} H &= H_p + \lambda\, H_I + H_{\text{osc}} \\ &= p^2 + V\,(q) + \sum_{1}^{N} \left[ \lambda \alpha_k\, q q_k + \frac{1}{2}\,(p_k^2 + \omega_k^2\, q_k^2 - \omega_k) \right], \end{aligned} \tag{1.1}$$

($\lambda$ = friction constant) they derive the Langevin equation as follows: They first integrate the oscillator variables, under the assumption that their initial positions and momenta are given by the Gibbs distribution, and then take the limit $N \to \infty$ under the additional assumption that $\alpha_k = \alpha(\omega_k)/\sqrt{N}$, where $\alpha$ is a given function representing the ultraviolet cutoff in the interacting energies. The equation thus obtained has a memory term: It is removed by taking the ultraviolet cutoff to infinity ($\alpha$ becomes linear, $\alpha\,(\omega) = \omega$). One then recovers the Langevin equation, the contracted description of the particle interacting with the environment.

The importance of Ford-Kac-Mazur work goes beyond the classical physics: It opened the way to the quantization of notion of classical friction. One can introduce the quantum analog of Ornstein-Uhlenbeck processes and derive the quantum Langevin equation by a procedure formally analogous to the above one. Starting from the FKM seminal work, a vast number of papers has been devoted to the quantum Langevin equation (QLE). In contrast to its classical counterpart, QLE is a singular object, difficult to study [17]: to our knowledge there are no rigorous results beyond the bounded perturbations of the quadratic model.

This paper, the first in a series devoted to study of quantum (and classical [10]) notion of friction, is devoted to study of the zero temperature model (we investigate finite temperature models in [9]). The explicit Hilbert space formalism leads to the possibility of studying the model using spectral theory, and thus to relate the values obtained for a life-time of excited states (Fermi Golden Rule) with the ones obtained by the use of Davie's master equation technique. We adopt the line of thought initiated by Davies [5], namely we keep the ultraviolet cutoff in the interacting energies, and study the dynamics on a time scale $\tau = \lambda^2 t$. The loss of memory term in standard discussion of Langevin equation is obtained by removing the ultraviolet energy cutoff, and it leads to singularities in the quantum case: we obtain the loss of memory term by passing to the above time scale, on which the dynamics is Markovian (memoryless). We study the Markovian evolution and show that it is determined by the resonances of the full energy operator.

To specify the model (for the extensions *see* remarks), we take the quantum analog of (1.1), namely a particle interacting with bosonic field (usually refered to as the "bath")

$$H_\lambda = H_p + \lambda x \oplus \phi(\alpha) + H_{\text{bos}} \tag{1.2}$$

acting on $\mathcal{H} = L^2(R) \otimes \mathcal{H}_{\text{fock}}$, where $\mathcal{H}_{\text{fock}}$ is the symmetric (bosonic) Fock space, constructed from $L^2(R)$. $H_{\text{bos}}$, the free-energy operator of bosons, is the second quantization of the function $\omega(k) = |k| + m_0$ (the argument easily extends to the case when $\omega(k) = \sqrt{k^2 + m_0^2}$, *see* remark 2 below). In the sequel we will be mainly interested in the technically more accessible case of massive bosons ($m_0 > 0$). $\phi(\alpha)$, for $\alpha \in L^2(R)$ is a time-zero field,

$$\phi(\alpha) = \frac{1}{\sqrt{2}}(a(\alpha) + a^*(\overline{\alpha})),$$

where $a(\alpha)$, $a^*(\alpha)$ are the boson annihilation and creation operators. $H_p$, the Hamiltonian of a particle, is given by

$$H_p = -\Delta + V(x),$$

and we suppose that $V(x) \to \infty$. Formally, in (1.2) one should write $H_p \otimes I$ and $I \otimes H_{\text{bos}}$. However, whenever it is clear within the context, we will write $A$ for $A \otimes I$ or $I \otimes A$. $H_p$ on $L^2(R)$ has a complete set of eigenvectors $\psi_j$, and the corresponding eigenvalues (numerated in the increasing order) we denote $\{E_j\}$. The precise technical conditions on $H_p$ and $\alpha$ will be set below. For future reference, we note that for $\lambda = 0$ the system decouples, and we have that $\sigma_{\text{ac}}(H_0) = [m_0 + E_0, \infty)$, $\sigma_{\text{sc}}(H_0) = \emptyset$, $\sigma_{\text{pp}} = \{E_j\}$ with corresponding eigenvectors $\{\psi_j \otimes \Omega\}$, where $\Omega$ is the vacuum on $\mathcal{H}_{\text{fock}}$. To avoid discussion of some pathological case (*see* [3] and remark 3 below), we will assume throughout the paper that for $i \geq 0$

$$E_{i+1} - E_i > m_0. \tag{1.3}$$

Spectral properties of the Hamiltonian (1.2) have been rarely investigated, and to the best of our knowledge there are no results except on the explicitly diagonalizable quadratic model [1], [2], [3]. The technical problems are obvious: perturbation $H_I$ is not relatively compact, and even worse, in the massless model, the eigenvalues embedded in the continuum for a decoupled system ($\lambda = 0$) are thresholds. However, one still expects (at least in the generic sense) that the eigenvalues $\{E_j;\ j \geq 1\}$ of the decoupled system will dissolve into resonances after turning on the "small" perturbation. The resonances determine the life-time of the excited states, namely we expect that

$$|(\psi_j \otimes \Omega,\ \exp(-it\, H_\lambda)\, \psi_j \otimes \Omega)| \sim \exp(-\lambda^2\, t\Gamma_j), \tag{1.4}$$

where $\Gamma_j$ is given by the Fermi Golden Rule. The physical mechanism behind (1.4) is simple: A particle interacting with the boson field at zero temperature dissipates exponentially fast (on the time scale $\tau = \lambda^2\, t$) to its ground state, radiating energy into the "bath". Our goal is to evaluate $\Gamma_j$ using complex deformation techniques, and to show that the Davies master equations approach leads to the right prediction for $\Gamma_j$. To avoid problems with thresholds, we restrict ourselves to the massive model ($m_0 > 0$). To specify the simplest set of conditions for which our results apply, we introduce some notation. For $0 < \delta < \pi/2$, let $H^2(\delta)$ be the Hardy class of the sector $\mathcal{A}(\delta) = \{z : |\arg(z)| \leq \delta\}$. $H^2(\delta)$ consists of all functions $f$ analytic in the interior of $\mathcal{A}(\delta)$ and such that

$$\|f\|^2_{H^2(\delta)} = \sup_{|\theta|<\delta} \int_{-\infty}^{\infty} |f(\exp(i\theta)\, k)|^2\, dk < \infty.$$

By $\tau(H_0)$ we denote the threshold set of the operator $H_0$,

$$\tau(H_0) = \{E_j + n m_0\ :\ j \geq 0,\ n > 0\}. \tag{1.5}$$

Let $u(\theta)\ :\ L^2(R) \to L^2(R)$ be the dilation unitary group,

$$u(\theta)\, \psi(k) = \exp(\theta/2)\, \psi(\exp(\theta)\, k), \tag{1.6}$$

and let $U(\theta) = \Gamma(u(\theta))$ be the second quantization of $u(\theta)$. We set the hypotheses:

(H1) $\alpha$ is a real, even function which can be extended to an element of $H^2(\delta)$ for some $\delta > 0$. Furthermore, the bounded operator-valued function

$$U(\theta)\, x \otimes \phi(\alpha)\, U(-\theta)\, (H_0 + i)^{-1} = x \otimes \phi(u(\theta)\, \alpha)\, (H_0 + i)^{-1}, \tag{1.7}$$

has an analytic continuation into the strip $\mathcal{S}(\delta) = \{\theta\ :\ |\mathrm{Im}(\theta)| < \delta\}$.

(H2) $x^2$ is $H_p$ bounded.

The hypotheses (H1) and (H2) ensure that for small $\lambda$, $H_\lambda$ is an essentially self-adjoint operator on $D_0 = C_0^\infty(R) \otimes F$, where $F$ is a set of finite particle vectors. The analyticity assumptions of the hypothesis (H1) are a variant of the dilation analyticity requirement for two-body potentials (*see* [15]). They are satisfied, for exemple, if $\alpha(k) = (1 + |k|^n)^{-1}$, for some integer $n > 0$. We define the function

$$\alpha_m(k) = \begin{cases} \alpha(k - m) & \text{if } k \geq m; \\ 0, & \text{otherwise.} \end{cases}$$

Our main result is:

THEOREM 1.1. – *Suppose that* (H1) *and* (H2) *are satisfied, and that for some* $j \geq 1$, $E_j \notin \tau(H_0)$. *Then for* $\lambda$ *small enough, there exists a dense set of vectors* $D$ *in* $\mathcal{H}$ *such that the matrix elements*

$$(\Phi, (H_\lambda - z)^{-1} \Psi), \quad \Phi, \Psi \in D, \tag{1.8}$$

*have a meromorphic continuation from the upper half plane onto* $O_j = \{z : |z - E_j| < C_j(\lambda)\}$. *On* $O_j$, *the functions* (1.8) *are analytic except for a simple pole (independent of* $\Phi$, $\Psi$*) at*

$$E_j(\lambda) = E_j + a_2^j \lambda^2 + \ldots \tag{1.9}$$

*Furthermore,*

$$Im(a_2^j) = -\Gamma_j = -\pi \sum_{i<j} |(x\psi_i, \psi_j)|^2 . [\alpha_{m_0}(E_j - E_i)]^2, \tag{1.10}$$

$$\mathrm{Re}(a_2^j) = \Lambda_j = -\sum_{i>0} |(x\psi_i, \psi_j)|^2 \mathrm{PV} \int_0^\infty \frac{\alpha(k)^2}{\omega(k) - (E_j - E_i)} dk. \tag{1.11}$$

PV *stands for the principal value integral.*

The following theorem is concerned with the dynamics of the system, and is a rigorous version of (1.4). We introduce the time scale $\tau = \lambda^2 t$.

THEOREM 1.2. – *Under the conditions of Theorem* 1.1, *for any two fixed constants* $b > a > 0$,

$$\lim_{\lambda \to 0} \sup_{\tau \in (a, b)} |(\psi_j \otimes \Omega, \exp(-it(H_\lambda - E_j)) \psi_j \otimes \Omega) - \exp(-\tau(\Gamma_j + i\Lambda_j))| = 0.$$

*In particular, if* $\tau = \lambda^2 t$ *is kept fixed,*

$$\lim_{\lambda \to 0} |(\psi_j \otimes \Omega, \exp(-it H_\lambda) \psi_j \otimes \Omega)| = \exp(-\tau \Gamma_j).$$

*Remark* 1. – The result can be extended to the general model where $L^2(R)$ is replaced by an arbitrary Hilbert space $\mathcal{H}_p$, $H_\lambda$ is of the form

$$H_\lambda = H_p + \lambda Q \otimes \phi(\alpha) + H_{\text{bos}}, \tag{1.12}$$

where we suppose that $H_p$ is a semi-bounded self-adjoint operator with discrete spectrum, and that $Q$ is a self-adjoint perturbation satisfying

$$\|Q^2 \psi\| \leq C(\|H_p \psi\| + \|\psi\|), \tag{1.13}$$

for some $C > 0$ and for $\psi \in D(H_p)$. If (H1) and (1.13) are satisfied, and $E_j$ is a simple eigenvalue of $H_p$ such that $E_j \notin \tau(H_0)$, then both theorems remain valid.

*Remark* 2. – One can replace condition (1.13) with

$$|(\psi, Q^2 \psi)| \leq C(\psi, H_p \psi) + (\psi, \psi), \tag{1.14}$$

for any $\psi \in \mathcal{Q}(H_p)$, the quadratic form domain of $H_p$. The passage to quadratic forms includes a number of additional technicalities: the family of operators (1.7) has to be replaced with

$$(H_0 - E_0 + 1)^{-1/2} Q \otimes \phi(u(\theta)\alpha)(H_0 - E_0 + 1)^{-1/2}. \tag{1.15}$$

In any case, if (H1) (with (1.15)) and (1.14) are satisfied, and if $E_j$ is a simple eigenvalue of $H_p$ such that $E_j \notin \tau(H_0)$, then again both theorems remain valid. In addition, the results can be easily extended to the bosonic field of an arbitrary dimension, and to a class of rotationally invariant $\omega$'s which includes $\omega(k) = \sqrt{|k|^2 + m_0^2}$. For a list of conditions which such $\omega$'s has to satisfy, we refer to [3].

*Remark* 3. – Although one can conjecture that under the hypotheses (H1)-(H2), for small $\lambda$ and under some reasonable genericity assumptions,

$$\sigma_{\text{ac}}(H_\lambda) = [C_1(\lambda), \infty), \qquad \sigma_{\text{sc}}(H_\lambda) = \emptyset, \; \sigma_{\text{pp}} = \{C_2(\lambda)\}, \tag{1.16}$$

for some constants $C_1(\lambda) > C_2(\lambda)$, we cannot prove this. The above result has been obtained for quadratic potentials by Arai [3] and it is certainly a very interesting problem to extend it to a more general class of potentials. In [3] Arai also discovered that, in a massive model, it can happen that all eigenvalues survive after turning on the small perturbation! He considers the case when

$$H_p = \frac{1}{2}(p^2 + \omega_0^2 q^2), \tag{1.17}$$

and $m_0 > \omega_0$, and shows (under essentially the hypothesis (H1)) that *all* embedded eigenvalues of $H_\lambda$ survive, independently of how small $\lambda$ is.

This is consistent with our result since for the model (1.17) one easily calculates that $\Gamma_j = 0$. The expression (1.10) yields that this result is certainly not "generically" true.

That on the time scale $\tau = \lambda^2 t$ the dynamics of a particle interacting with the environment is Markovian has been known since early seventies. Davies [5], [6], for a very general class of models, explicitly constructed the generator of the semigroup which approximates the dynamics of the particle subsystem on the time scale given by $\tau$. To apply his construction (*see* Section 2.5 for details), one has to regularize the Hamiltonian $H_\lambda$. Let

$$H_\lambda (L) = H_p + \lambda x_L \otimes \phi (\alpha) + H_{\text{bos}},$$

where $x_L = x$, if $|x| \leq L$, and $x_L = L\,(-L)$ if $x > L\,(< -L)$. The Davies theory gives a bounded operator $K_L$ on $\mathcal{H}_p$, such that for any $\psi \in \mathcal{H}_p$,

$$\lim_{\lambda \to 0} \sup_{\tau \in (a,\, b)} \| \exp(itH_0(L)) \exp(-itH_\lambda(L)) \psi \otimes \Omega - \exp(-\tau K_L) \psi \otimes \Omega \| = 0.$$

We will show that the operator $K_L$ is diagonal in the basis $\psi_j$ with eigenvalues which are, as $L \to \infty$, given by $\{0\} \cup \{\Gamma_j + i\,\Lambda_j;\ j \geq 1\}$. One advantage of the Davies' construction is that it applies without any changes to the case of a massless field. We refer to Section 2.6 for a comparison of the two approaches to the dynamics of non-isolated systems.

The consequence of Theorem 1.2 is that for any $\psi \in \mathcal{H}_p$ (keeping in the first limit $\tau = \lambda^2 t$ fixed)

$$\lim_{\tau \to \infty} \lim_{\lambda \to 0} |(\psi \otimes \Omega,\ \exp(-it\, H_\lambda)\, \psi \otimes \Omega)| = |(\psi,\ \psi_G)|^2, \tag{1.18}$$

where $\psi_G = \psi_0$ is the ground state of $H_p$. Physically, (1.18) stands for "on the time scale $\lambda^2 t$ the particle dissipates into ground state radiating energy into the bath". It is an interesting question, raised in [5], [7], whether or not one can improve (1.18), for example by showing that (1.18) is valid by taking first $t \to \infty$ and then taking $\lambda \to 0$. To answer that question in our model, some information on the nature of the spectrum of $H_\lambda$ is needed. For example, if (1.16) is valid, if follows from the RAGE theorem [4] that for any $\psi,\ \phi \in \mathcal{H}_p$

$$\lim_{\lambda \to 0} \lim_{T \to \infty} \frac{1}{T} \int_0^T |(\psi \otimes \Omega,\ \exp(-it\, H_\lambda)\, \phi \otimes \Omega)|^2\, dt = |(\psi,\ \psi_G)|^2 \cdot |(\phi,\ \psi_G)|^2. \tag{1.19}$$

We cannot prove (1.16), but the spectral information which we obtained as a by-product of Theorem 1.1 actually suffices for (1.19). We have

THEOREM 1.3. – *If* (H1) *and* (H2) *are satisfied and, for any* $j \geq 1$, $E_j \notin \tau(H_0)$, $\Gamma_j \neq 0$, *then* (1.19) *is valid.*

The remarks after Theorem 1.2 are also valid for Theorem 1.3.

When this work was finished, we learned from V. Bach that in reference [11] T. Okamoto and K. Yajima investigated resonances of a QED model related to (1.2). Using similar techniques, they proved a result analogous to the Theorem 1.1 above. In addition, they investigated the case when eigenvalues $\{E_j\}$ have multiplicities higher then 1. Their physical motivation was different and they did not pursue the study of the dynamical aspects of the model.

## 2. RESONANCES AND FERMI GOLDEN RULE

### 2.1. Complex deformation

If $u(\theta)$, $U(\theta)$ are given by (1.6), defining $\alpha_\theta$ as $\alpha_\theta = u(\theta)\alpha$ we get that

$$U(\theta)\, a^*(\alpha)\, U(\theta)^{-1} = a^*(\alpha_\theta), \quad U(\theta)\, a(\alpha)\, U(\theta)^{-1} = a(\alpha_\theta),$$
$$U(\theta)\, H_{\text{bos}}\, U(\theta)^{-1} = d\Gamma(\omega_\theta), \quad \omega_\theta(k) = \omega(\exp(\theta)\, k).$$

The operator

$$H_\lambda(\theta) = H_p + \lambda x \otimes \phi(\alpha_\theta) + d\Gamma(\omega_\theta)$$

is well defined on $C_0^\infty(R) \otimes F$ for $\theta \in S(\delta)$, where $F$ is the subspace of finite particle vectors. We start by showing that $H_I$ is a relatively bounded perturbation of $H_0$. Throughout this chapter we assume that the hypotheses (H1) and (H2) are satisfied.

LEMMA 2 1. – *There exists a constant* $C > 0$ *such that for all* $\theta \in \mathcal{S}(\delta)$ *and* $\Phi \in D(H_0)$

$$\|x \otimes \phi(\alpha_\theta)\, \Phi\| \leq C\, (\|H_0\, \Phi\| + \|\Phi\|). \tag{2.1}$$

*Proof.* – Let

$$C_1 = \|\alpha/\sqrt{\omega}\|_{H^2(\delta)}, \quad C_2 = \|\alpha\|_{H^2(\delta)}$$

Then, for $\Phi \in D(H_{\text{bos}}^{1/2})$ (*see* e.g. [1])

$$\|a^*(\alpha_\theta)\Phi\| \le C_1\|H_{\text{bos}}^{1/2}\Phi\| + C_2\|\Phi\|$$
$$\|a(\alpha_\theta)\Phi\| \le C_1\|H_{\text{bos}}^{1/2}\Phi\|. \tag{2.2}$$

Consequently, there exists constant $D$ such that, in the operator sense,

$$|\phi(\alpha_\theta)|^2 \le D(H_{\text{bos}} + 1). \tag{2.3}$$

Let $C_3, C_4$ be such constants that

$$x^2 \le C_3 H_p + C_4. \tag{2.4}$$

Then, for any $\Phi \in D(H_p) \otimes D(H_{\text{bos}})$ and for any $\varepsilon > 0$

$$\begin{aligned}\|x \otimes \phi(\alpha_\theta)\Phi\|^2 &= (\Phi, x^2 \otimes |\phi^2(\alpha_\theta)|\Phi)\\ &\le D(\Phi, (C_3 H_p + C_4) \otimes (H_0 + 1)\Phi)\\ &\le D\left(\Phi, \left[\frac{\varepsilon^2}{2}(C_3 H_p + C_4)^2 + \frac{1}{2\varepsilon^2}(H_{\text{bos}} + 1)^2\right]\Phi\right)\\ &\le D\left(\Phi, \left[\frac{\varepsilon}{\sqrt{2}}(C_3 H_p + C_4) + \frac{1}{\sqrt{2}\varepsilon}(H_{\text{bos}} + 1)\right]^2\Phi\right).\end{aligned}$$

If $\varepsilon = C_3^{-1/2}$, we get that for some $C_5$,

$$\|x \otimes \phi(\alpha_\theta)\Phi\| \le C_5\sqrt{C_3}\|H_0\Phi\| + C_5\|\Phi\|. \tag{2.5}$$

■

*Remark.* – A consequence of (2.5) is that if $x^2$ is infinitesimaly small with respect to $H_p$, then so is $x \otimes \phi(\alpha_\theta)$ with respect to $H_0$.

*Remark.* – Arguing as above, one can prove the form version of the above results, namely, that if $x^2$ is relatively form bounded with respect to $H_p$ then so is $x \otimes \phi(\alpha_\theta)$ with respect to $H_0$. Also, if $x^2$ is infinitesimaly form bounded with respect to $H_p$, then so is $x \otimes \phi(\alpha_\theta)$ with respect to $H_0$.

Before stating our main technical lemma, we introduce some additional notation. Let

$$\mathcal{S}(a, b, \theta) = \{\exp(\theta)t + s \,:\, t \in \mathbf{R},\, a \le s \le b\}.$$

We numerate elements of $\tau(H_0)$ in the increasing order,

$$\tau(H_0) = \{\tau_k \,:\, \tau_k < \tau_{k+1},\, k \ge 1\},$$

and let $B(\varepsilon) = \{\lambda \in \mathbf{C} \,:\, |\lambda| \le \varepsilon\}$.

LEMMA 2.2. – *There exists* $\varepsilon > 0$ *such that*

(i) *For real* $\lambda \in B(\varepsilon)$, $H_\lambda$ *is essentially self-adjoint on* $C_0^\infty(R) \otimes F$.

(ii) $H_\lambda(\theta)$ *can be extended for* $(\lambda, \theta) \in B(\varepsilon) \times \mathcal{S}(\delta)$ *to an analytic family of type* $A$ *in each variable separately.*

(iii) *Suppose that* $\mathrm{Im}(\theta) \neq 0$. *Then for any* $j > 0$ *there exists* $\varepsilon(\lambda, j) > 0$ *such that*

$$\lim_{\lambda \to 0} \varepsilon(\lambda, j) = 0 \quad \textit{and} \quad \sigma_{\mathrm{ess}}(H_\lambda(\theta)) \cap \mathcal{S}(\tau_j + \varepsilon(\lambda, j), \tau_{j+1} - \varepsilon(\lambda, j), \theta) = \emptyset.$$

*Proof.* – (i) follows immadiately from Kato-Rellich Theorem. To prove (ii), note that the estimate (2.1) yields that $H_\lambda(\theta)$ is a closed operator with $D(H_\lambda(\theta)) = D(H_0)$. Since

$$\sigma_{\mathrm{pp}}(H_0(\theta)) = \{E_j\}, \sigma_{\mathrm{ess}}(H_0(\theta)) = \{E_j + n m_0 + t \exp(\theta) : t \geq 0, n \in \mathbf{N}\},$$

we have (*see* [14]) that for any $\theta \in \mathcal{S}(\delta)$, $H_\lambda(\theta)$ is an analytic family of type $A$ in $\lambda$ for $\lambda \in B(1/C)$, where $C$ is given by (2.1). That $H_\lambda(\theta)$ is an analytic family of type $A$ in $\theta$ follows from the hypothesis (H1). Part (iii) is a consequence of regular perturbation theory (*see*, for example, proof of Theorem XII.9 in [14]). ■

Part (i) and (iii) of the above lemma are also valid in the quadratic form model (1.14). Part (ii) is altered since $H_\lambda(\theta)$ is now an analytic family of type $B$ in each variable separately.

## 2.2. Fermi Golden Rule

The discussion of resonances in the model (1.2) follows closely that of the Auger states in helium atom [14], [15]. In the sequel

$$H_I(\theta) = x \otimes \phi(\alpha_\theta), \; H_I(0) = H_I, \; \Phi_j = \psi_j \otimes \Omega. \tag{2.6}$$

If $\mathrm{Im}(\theta) < 0$, part (iii) of Lemma 2.2 and regular perturbation theory yield that the spectrum of $H_\lambda(\theta)$ in $\mathcal{S}(\tau_j + \varepsilon(\lambda, j), \tau_{j+1} - \varepsilon(\lambda, j), \theta)$ consists of exactly one isolated and nondegenerate eigenvalue $E_j(\lambda)$, given by convergent series in $\lambda$

$$E_j(\lambda) = a_0^j + a_1^j \lambda + a_2^j \lambda^2 + \dots$$

Coefficients $a_k^j$ (and thus $E_j(\lambda)$ itself) do not depend on $\theta$ (if $\mathrm{Im}(\theta) < 0$) since they are analytic in $\theta$ for fixed $\lambda$, and since $H_\lambda(\theta)$ and $H_\lambda(\theta')$ are unitarily equivalent for $\theta - \theta'$ real. Obviously $a_0^j = E_j$, and since

$$(\Phi_j, H_I(\theta) \Phi_j) = 0, \tag{2.7}$$

we have that $a_1^j = 0$.

In proving (1.10), (1.11) we follow the argument in [14]. For $\mathrm{Im}(\theta) < 0$, coefficient $a_2$ is given by

$$a_2 = \frac{1}{2\pi i} \oint_{|E_j - z| = \delta} (\Phi_j, [H_I(\theta)(H_0(\theta) - z)^{-1}]^2 \Phi_j)\, dz, \tag{2.8}$$

for $\delta$ and $\lambda$ small enough. We rewrite (2.8) as

$$a_2 = \frac{1}{2\pi i} \oint_{|E_j - z| = \delta} (\Phi_j, H_I(\theta)(H_0(\theta) - z)^{-1} H_I(\theta)\Phi_j)\, \frac{1}{E_j - z}\, dz.$$

If

$$F(\theta, z) = (\Phi_j, H_I(\theta)(H_0(\theta) - z)^{-1} H_I(\theta)\Phi_j), \tag{2.9}$$

the Cauchy integral theorem yields that

$$a_2 = -\lim_{\varepsilon \to 0} F(\theta, E_j + i\varepsilon) = -\lim_{\varepsilon \to 0} F(0, E_j + i\varepsilon).$$

As in [14], we used that $F(\theta, z)$ is defined for $\theta$ real as long as $\mathrm{Im}(z) > 0$, and is actually independent of $\theta$, as long as $\mathrm{Im}(\theta) \le 0$. The explicit calculation yields

$$F(0, E_j + i\varepsilon) = \sum_{i=0}^{\infty} |(x\psi_i, \psi_j)|^2 \int_0^{\infty} \frac{\alpha(k)^2}{\omega(k) - (E_j - E_i) - i\varepsilon}\, dk.$$

The above expression is finite, since (H2) yield that $x\psi_j \in L^2(R)$. Using the well-known formula

$$\lim_{\varepsilon \to 0} \frac{1}{x - x_0 - i\varepsilon} = i\pi\delta(x - x_0) + \mathrm{PV}\, \frac{1}{x - x_0}.$$

(or just by residue calculus), we obtain (1.10), (1.11).

In the notation of Lemma 2.2, let $\tau_k$ be chosen so that $\tau_k < E_j < \tau_{k+1}$. To finish the proof of Theorem 1.1 it suffices to show that for a dense set of vectors $D$ matrix elements

$$R(z) = (\Phi, (H_\lambda - z)^{-1}\Psi), \quad \Phi, \Psi \in D \tag{2.10}$$

have an analytic continuation from the upper half-plane onto the region

$$S(\tau_k + \varepsilon(\lambda, k), \tau_{k+1} - \varepsilon(\lambda, k), \theta_0), \quad \mathrm{Im}(\theta_0) < 0,$$

except for a simple pole at $E_j(\lambda)$. Let $D = D_\delta$ be a set of all vectors in $\mathcal{H}$ such that for $\Phi \in D_\delta$, $U(\theta)\,\Phi$ has an extension to analytic vector-valued function in strip $\mathcal{S}(\delta)$. Then, for real $\theta$ and for any $\Phi$, $\Psi \in D_\delta$

$$R(z) = (U(\theta)\,\Phi,\ (H_\lambda(\theta) - z)^{-1}\, U(\theta)\,\Psi),$$

and by analytic continuation, this formula holds (if $\mathrm{Im}(z) > 0$) for $\mathrm{Im}(\theta) < \delta$. The result follows.

From the above discussion, some spectral informations can be deduced. In the sequel we will make use of the following result which is an easy consequence of the above argument.

THEOREM 2.3. – *Suppose that for* $k > 0$ *one of the following holds:*

(i) $(\tau_k, \tau_{k+1}) \cap \{E_j : j \geq 1\} = \emptyset$.

(ii) If for some $j$, $E_j \in (\tau_k, \tau_{k+1})$, then $\Gamma_j > 0$.

Then there exists $\varepsilon(\lambda, k) > 0$ such that $\lim_{\lambda \to 0} \varepsilon(\lambda, k) = 0$ and that $H_\lambda$, for $\lambda \neq 0$, has purely absolutely continuous spectrum on $[\tau_k + \varepsilon(\lambda, k),\ \tau_{k+1} - \varepsilon(\lambda, k)]$.

The argument of this section extends without changes to the model (1.14).

## 2.3. Dynamics of the system

In this section we prove Theorem 1.2 as follows. By the Stone formula,

$$(\Phi_i,\ \exp(-it\,H_\lambda)\,\Phi_i) = \frac{1}{2\pi i}\int_\Gamma \exp(-itz)\,(\Phi_i,\ (H_\lambda - z)^{-1}\,\Phi_i)\,dz,$$

where for the contour $\Gamma$ we take the straight line $\mathrm{Im}(z) = \varepsilon > 0$. Let $\Gamma_1$ be the contour constructed as follows: $\Gamma_1$ coincides with the straight line $\mathrm{Im}(z) = c\lambda$ for a suitably chosen uniform constant $c$, and in the neigborhood of the point $E_j + i\lambda c$ goes down to enclose the resonance $E_j(\lambda)$ in such a way that for $\mathrm{Im}(z) < 0$ $\Gamma_1$ is completely contained within the set $E_j + B(C_j(\lambda))$. Since

$$(\Phi_i,\ (H_\lambda - z)^{-1}\,\Phi_i) = (\Phi_i,\ (H_\lambda(\theta) - z)^{-1}\,\Phi_i), \tag{2.11}$$

for suitable $C_j(\lambda)$ the function $f(z) = (\Phi_i,\ (H_\lambda - z)^{-1}\,\Phi_i)$ is analytic in a simple connected domain bounded by curves $\Gamma$, $\Gamma_1$, with the exception of a simple pole at resonance eigenvalue. Thus

$$\int_\Gamma \exp(-itz)\,f(z)\,dz = \exp(-it\,E_j(\lambda))\,(\Phi_i,\ P_j(\lambda)\,\Phi_i) + \int_{\Gamma_1} \exp(-itz)\,f(z)\,dz.$$

To control the integral over $\Gamma_1$ we proceed as follows. Since

$$\int_{\Gamma_1} \exp(-itz)\,(\Phi_i,\,(H_0 - z)^{-1}\,\Phi_i)\,dz = \int_{\Gamma_1} \exp(-itz)\,\frac{1}{E_j - z}\,dz = 0,$$

applying the resolvent identity we have that

$$\begin{aligned}&\int_{\Gamma_1} \exp(-itz)\,f(z)\,dz\\ &\quad = \lambda \int_{\Gamma_1} \frac{\exp(-itz)}{E_j - z}\,(\Phi_i,\,(H_\lambda - z)^{-1}\,x\psi_i \otimes a^*(\alpha)\,\Omega)\,dz.\end{aligned}$$

Since $(\psi_i \otimes \Omega,\,(H_0 - z)^{-1}\,x\psi_i \otimes a^*(\alpha)\,\Omega) = 0$, the resolvent identity yields once more

$$\begin{aligned}&\int_{\Gamma_1} \exp(-itz)\,f(z)\,dz\\ &\quad = \lambda^2 \int_{\Gamma_1} \frac{\exp(-itz)}{(E_j - z)^2}\,(x\psi_i \otimes a^*(\alpha)\,\Omega,\,(H_\lambda - z)^{-1}\,x\psi_i \otimes a^*(\alpha)\,\Omega)\,dz.\end{aligned}$$

Let us divide the contour $\Gamma_1$ into two parts, $\Gamma_1^+ = \{z \in \Gamma_1 : \operatorname{Im}(z) > \lambda^2\}$ and $\Gamma_1^- = \Gamma_1 \backslash \Gamma_1^+$. We estimate

$$\begin{aligned}&\lambda^2 \left| \int_{\Gamma_1^-} \frac{\exp(-itz)}{(E_j - z)^2}\,(x\psi_i \otimes a^*(\alpha)\,\Omega,\,(H_\lambda - z)^{-1}\,x\psi_i \otimes a^*(\alpha)\,\Omega)\,dz \right|\\ &\quad = \lambda^2 \left| \int_{\Gamma_1^-} \frac{\exp(-itz)}{(E_j - z)^2}\,(x\psi_i \otimes \phi_\theta(\alpha)\,\Omega, \right.\\ &\qquad\qquad \left. (H_\lambda(\theta) - z)^{-1}\,x\psi_i \otimes \phi_\theta(\alpha)\,\Omega)\,dz \right|\\ &\quad = O(\lambda^2 \exp(\tau)).\end{aligned}$$

On the other hand, for suitable constants $c_1 > E_j$ and $c_2 < E_j$ we have (after interchanging the line of integration on left and right part of the countour $\Gamma_1^+$ and then integrating over parts of the line $\operatorname{Im}(z) = \lambda^2$ given by $\operatorname{Re}(z) > c_1$ and $\operatorname{Re}(z) < c_2$)

$$\begin{aligned}&\lambda^2 \left| \int_{\Gamma_1^+} \frac{\exp(-itz)}{(E_j - z)^2}\,(x\psi_i \otimes a^*(\alpha)\,\Omega,\,(H_\lambda - z)^{-1}\,x\psi_i \otimes a^*(\alpha)\,\Omega)\,dz \right|\\ &\quad \le \lambda^2\,[C_1(\lambda,\,\tau) + C_2(\lambda,\,\tau)]\end{aligned}$$

where

$$\left.\begin{aligned} C_1(\lambda, \tau) &= \exp(\tau) \times \int_{c_1}^{\infty} \frac{1}{(E_j - s)^2} \\ &\quad \times |(x\psi_i \otimes a^*(\alpha)\,\Omega, (H_\lambda - s - i\lambda^2)^{-1}\, x\psi_i \\ &\quad \otimes a^*(\alpha)\,\Omega)|ds, \\ C_2(\lambda, \tau) &= \exp(\tau) \times \int_{-\infty}^{c_2} \frac{1}{(E_j - s)^2} \\ &\quad \times |(x\psi_i \otimes a^*(\alpha)\,\Omega, (H_\lambda - s - i\lambda^2)^{-1}\, x\psi_i \\ &\quad \otimes a^*(\alpha)\,\Omega)|ds. \end{aligned}\right\} \tag{2.12}$$

The statement will follow if we show that

$$\lim_{\lambda \to 0} \lambda^2\, C_1(\lambda, \tau) = 0, \qquad \lim_{\lambda \to 0} \lambda^2\, C_2(\lambda, \tau) = 0. \tag{2.13}$$

We will treat the first limit in (2.13). One argues analogously for the second one.

Let $L > 0$ be fixed large number, and let $\varepsilon > 0$ be a fixed small number, and let

$$S(\varepsilon) = \bigcup_{\substack{E_j + nm \in (c_1,\, L) \\ j,\, n \geq 0}} (E_j + nm_0 - \varepsilon,\, E_j + nm_0 + \varepsilon).$$

Let

$$S'(\varepsilon) = (c_1, L) \backslash S(\varepsilon).$$

For any $s \in S'(\varepsilon)$, and for any $\Psi \in D_\delta$ (note that $x\psi_i \otimes a^*(\alpha)\,\Omega \in D_\delta$) we have that

$$\lim_{\lambda \to 0} \lambda^2 |(\Psi, (H_\lambda - s - i\lambda^2)^{-1}\, \Psi)| = 0.$$

Using that

$$\lambda^2 |(\Psi, (H_\lambda - s - i\lambda^2)^{-1}\, \Psi)| \leq ||\Psi||^2$$

we easily estimate (decomposing integral (2.12) into parts over $S(\varepsilon)$, $S'(\varepsilon)$ and $[L, \infty)$) that

$$\limsup_{\lambda \to 0} \lambda^2\, C_1(\lambda, \tau) \leq O(\varepsilon) + O(1/L). \tag{2.14}$$

Letting in (2.14) first $\varepsilon \to 0$ and then $L \to \infty$, we obtain (2.13).

A completely analogous argument yields that for $i \neq j$

$$\lim_{\lambda \to 0} \sup_{\tau \in (a,\, b)} |(\psi_i \otimes \Omega, \exp(-it\, H_\lambda)\, \psi_j \otimes \Omega)| = 0,$$

What is needed to establish (1.18).

The argument of this section applies without changes to the model (1.14).

### 2.4. Proof of Theorem 1.3

For $\varepsilon > 0$, let

$$T(\varepsilon) = \bigcup_{n>0,\, j\geq 0} (E_j + nm_0 - \varepsilon,\ E_j + nm_0 + \varepsilon), \tag{2.15}$$

and for fixed large $L > 0$, let

$$P_L^-(\lambda) = \chi_{(-\infty,\, L)}(H_\lambda), \quad P_L^+(\lambda) = \chi_{[L,\, \infty)}(H_\lambda),$$

where $\chi_A$ stands for a characteristic function of a set $A \subset R$. For $\lambda$ small enough and $\Phi,\ \Psi \in D(H_0)$ we have that

$$\frac{1}{T}\int_0^T |(\Phi,\ \exp(-it\, H_\lambda)\, \Psi)|^2\, dt$$

$$= \frac{1}{T}\int_0^T |(\Phi,\ P_L^-(\lambda)\exp(-it\, H_\lambda)\, \Psi)|^2\, dt + O(1/L^2). \tag{2.16}$$

(2.16) follows from observation that for $\lambda$ small enough

$$\begin{aligned} |(\Phi,\ P_L^+ \exp(-it\, H_\lambda)\, \Psi)| &\leq |((H_\lambda + 1)\, \Phi,\ (H_\lambda + 1)^{-2}\, P_L^+(\lambda) \\ &\quad \times \exp(-it\, H_\lambda)\, (H_\lambda + 1)\, \Psi)| \\ &\leq \frac{1}{(L+1)^2}\, \|(H_\lambda + 1)\, \Phi\| \cdot \|(H_\lambda + 1)\, \Psi\|. \end{aligned}$$

For $\lambda$ small, the bottom of the spectrum of $H_\lambda$ is an isolated, non-degenerated eigenvalue $E_0(\lambda)$. The projection on the corresponding eigenvector we denote $P_G(\lambda)$. Let

$$T'(\varepsilon) = T(\varepsilon) \cap (-\infty,\ L), \qquad T''(\varepsilon) = (-\infty,\ L)\backslash T'(\varepsilon),$$

and let

$$P_{L,\,1}^-(\lambda) = \chi_{T'(\varepsilon)}(H_\lambda), \qquad P_{L,\,2}^-(\lambda) = \chi_{T''(\varepsilon)}(H_\lambda). \tag{2.17}$$

Projections (2.17) have the following two properties: For any $\Phi,\ \Psi \in \mathcal{H}$ we have that

$$\lim_{\varepsilon\to 0}\lim_{\lambda\to 0} \|P_{L,\,1}^-(\lambda)\, \Phi\| = 0, \tag{2.18}$$

and, for $0 < \varepsilon < m_0$ there exists $\eta > 0$ such that for $|\lambda| < \eta$

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T |(\Phi, P_{L,2}^-(\lambda) \exp(-it H_\lambda) \Psi)|^2 dt$$
$$= \|P_G(\lambda) \Phi\|^2 \cdot \|P_G(\lambda) \Psi\|^2. \qquad (2.19)$$

To establish (2.18), note that

$$\lim_{\lambda\to 0} \|P_{L,1}^-(\lambda) \Phi\| = \|P_{L,1}^-(0) \Phi\|,$$

and that for $\varepsilon$ small, $T'(\varepsilon) \cap \sigma_{\rm pp}(H_0) = \emptyset$. To establish (2.19), note that Theorem 2.3 yields that for fixed $\varepsilon$ and $\lambda$ small enough,

$$\sigma_{\rm pp}(H_\lambda) \cap T''(\varepsilon) = E_0(\lambda).$$

(2.19) now follows from the RAGE Theorem (*see* e.g. [4], Theorem 5.8). Thus, for fixed large $L$ we have that

$$\limsup_{\lambda\to 0} \limsup_{T\to\infty} \left| \frac{1}{T} \int_0^T |(\Phi, \exp(-it H_\lambda) \Psi)|^2 dt \right.$$
$$\left. - \|P_G(0) \Phi\|^2 \cdot \|P_G(0) \Psi\|^2 \right| = o(\varepsilon) + O(1/L^2). \qquad (2.20)$$

We obtain (1.19) for $\Phi, \Psi \in D(H_0)$ by letting in (2.20) first $\varepsilon \to 0$ and then $L \to \infty$. By density, statement extends to the entire space $\mathcal{H}$.

Theorem 1.3 extends by continuity to the following setting:

THEOREM 2.4. – (i) *If $K$ and $K'$ are Hilbert-Schmidt operators on $\mathcal{H}$, we have that*

$$\lim_{\lambda\to 0} \lim_{T\to\infty} \frac{1}{T} \int_0^T \mathrm{Tr}\,[K' \exp(-it H_\lambda) K \exp(it H_\lambda)]\, dt$$
$$= (\Psi_G, K \Psi_G)(\Psi, K' \Psi_G), \qquad (2.20)$$

*where $\Psi_G = \psi_G \otimes \Omega$.*

(ii) *If $K$ is compact, and $\Psi \in \mathcal{H}$*

$$\lim_{\lambda\to 0} \lim_{T\to\infty} \frac{1}{T} \int_0^T \|K \exp(-it H_\lambda) \Psi\|^2 dt = |(\Psi_G, \Psi)|^2 \|K \Psi_G\|^2. \qquad (2.20)$$

## 2.5. Davies theory

In Chapter 2 we investigated the dynamics of the Hamiltonian (1.2) with help of spectral theory. There exists, however, a completely different approach to the dynamics of the open systems, based on the use of master equations. The master equations technique, often heuristically used in physics literature, has been raised to high art by Davies [5], [6]. We specify his general construction to the model at hand and show that it yields the right prediction for $a_2^j$, the second Rayleigh-Scrödinger coefficient in the expansion (1.9) for $E_j(\lambda)$. The usual technical assumption is that the interacting part of the Hamiltonian is a bounded operator, and consequently, in the literature, the fermionic field is more often considered then the bosonic one. Although technicalities imposed by the unboundedness of the interacting part of the Hamiltonian in the field variable can be resolved, we are forced to take a cutoff in $x$, namely to consider the Hamiltonian $H_\lambda(L)$ given by

$$H_\lambda(L) = H_p + \chi_{[-L,\,L]}\, x \otimes \phi(\alpha) + H_{\mathrm{bos}}. \tag{2.21}$$

Instead of the hypothesis (H1), throughout this section we will assume that $\alpha$ is even real function in $L^2(R)$, with a continuous Hilbert transform, such that for some $\varepsilon > 0$

$$(1 + |t|^\varepsilon)|\hat{\alpha}(t)| \in L^1(R),$$
$$\textit{where}\ \hat{\alpha}(t) = \int_0^\infty \alpha(k)^2 \exp(-i\omega(k)\,t)\,dk. \tag{2.22}$$

Under the above assumptions,

THEOREM 2.5. – *There exists an operator $K_L$, acting on $\mathcal{H}_p$, such that for any $b > a > 0$ and $\psi \in \mathcal{H}_p$*

$$\lim_{\lambda\to 0} \sup_{\tau\in(a,\,b)} \|\exp(it\,H_0)\exp(-it\,H_\lambda(L)\,\psi\otimes\Omega - \exp(-\tau\,K_L)\,\psi\otimes\Omega\| = 0.$$

*Furthermore, the operator $K_L$ is diagonal in basis $\psi_j$, with eigenvalues $\Sigma_j(L)$ such that*

$$\Sigma_0(L) = 0, \quad \lim_{L\to\infty} \Sigma(L) = \Gamma_j + i\Lambda_j, \quad j \geq 1$$

*where $\Gamma_j$, $\Lambda_j$ are given by* (1.10), (1.11).

We remark that the argument below is fairly general, and applies also to the model (1.3), whenever $Q$ is a bounded operator. It also accomodates without changes the *massless field* model, with $\Gamma_j$, $\Lambda_j$ given by (2.23), (2.24).

We now sketch the main steps of Davies theory applied to the model (2.21). Let $P$ be the orthogonal projection on $\mathcal{H}$ given by $P = I \otimes (\cdot, \Omega)\,\Omega$, and let $Q = I - P$. Since we are interested in the reduced dynamics, namely its restriction to the particle subspace, we integrate out field variables by setting

$$U_\lambda (t) = P \exp(-it\, H_\lambda (L))\, P.$$

Defining

$$V_\lambda (t) = \exp(-it\,(H_0 + \lambda Q H_I (Q)),$$

we derive the integrated form of NPRZ (Nakajima-Progogine-Resibois-Zwanzig) equation as follows (*see* also [5], [7]). First, since $PH_I$ is a bounded operator, we have a well-defined equation

$$U_\lambda (t) = V_\lambda (t)\, P - i\lambda \int_0^t V_\lambda (t-s)\, PH_I\, Q \exp(-is\, H_\lambda (L))\, Pds.$$

Since $HP_I$ is also bounded, we have

$$Q \exp(-it\, H_\lambda (L))\, P = -i\lambda \int_0^t V_\lambda (t-s)\, QH_I\, PU_\lambda (s)\, ds.$$

Combining the above equations and using that $V_\lambda (t)\, P = U_0 (t)$, we obtain the NPRZ equation,

$$U_\lambda (t) = U_0 (t) - \lambda^2 \int_0^t ds \int_0^s ds_1 \times U_0 (t-s)\, PH_I\, QV_\lambda (s - s_1)\, QH_I\, PU_\lambda (s_1), \quad (2.23)$$

Introducing new variables $\tau = \lambda^2\, t$, $x = \lambda^2\, s_1$, we get

$$U_\lambda (\tau/\lambda^2) = U_0 (\tau/\lambda^2) + \int_0^\tau U_0 ((\tau - x)/\lambda^2)\, K(\lambda,\, \tau - x)\, U_\lambda (x/\lambda^2)\, dx,$$

where

$$K(\lambda,\, \tau) = \int_0^{\tau/\lambda^2} U_0 (-s)\, PH_I\, QV_\lambda (s)\, Q\, H_I\, Pds.$$

$K(\lambda,\, \tau)$ is bounded operator on $\mathcal{H}$. One can immediately extract the operator which should give the dissipative part of the dynamics in second order perturbation theory, explicitly

$$K = \int_0^\infty U_0 (-s)\, PH_I\, QU_0 (s)\, Q\, H_I\, Pds.$$

Let

$$K_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} \exp(it\, H_p)\, K\, \exp(-it\, H_p)\, P\, dt = \sum_j P_j\, K\, P_j,$$

where $P_j$ is the projection on $\psi_j \otimes \Omega$. The following two theorems are at heart of the Davies theory.

THEOREM 2.6. –

$$\lim_{\lambda \to 0} \sup_{\lambda^2\, t \in (0,\, a)} \| \exp(-(it\, H_p + \lambda^2\, tK))\, \psi \otimes \Omega - \exp(-(it\, H_p + \lambda^2\, tK_{\mathrm{av}}))\, \psi \otimes \Omega \| = 0.$$

THEOREM 2.7. – *Suppose that for every $a > 0$ there exists a positive constant $C_a$ such that*

$$\sup_{\tau \in (0,\, a)} \|K(\lambda,\, \tau)\| \le C_a.$$

*Suppose also that for any $c > b > 0$ fixed,*

$$\lim_{\lambda \to 0} \sup_{\tau\, \in (b,\, c)} \|K(\lambda,\, \tau) - K\| = 0.$$

*Then,*

$$\lim_{\lambda \to 0} \sup_{\lambda^2 t\, \in (0,\, a)} \|U_\lambda(s) - \exp(-(it\, H_p + t\lambda^2\, K))\| = 0.$$

The proofs of the above two theorems are somewhat technical [5]. It is even more involved to check that their conditions are satisfied in our model. Using condition (2.22), it can be done following line by line the Davies argument in [5].

To finish the proof of the Theorem 2.5, we need the following

LEMMA 2.8. – *Operator $K_{\mathrm{av}}$ restricted to the subspace generated by $\{\psi_j \otimes \Omega\}$ has a discrete spectrum, and its eigenvalues and eigenvectors are given by*

$$K_{av}\, \psi_j \otimes \Omega = (\Gamma_j(L) + i\Lambda_j(L))\, \psi_j \otimes \Omega,$$

$$\Gamma_0 = 0,\ \Gamma_j(L) = \pi \sum_{i=1}^{j-1} |(x_L\, \psi_i,\, \psi_j)|^2 \cdot |\alpha_m(E_j - E_i)|^2\ j \ge 1,$$

$$\Lambda_j(L) = -\sum_{i>0} |(x_L\, \psi_i,\, \psi_j)|^2\, \mathrm{PV} \int_0^\infty \frac{\alpha(k)^2}{\omega(k) - (E_j - E_i)}\, dk.$$

*Proof.* – We have

$$(\psi_i \otimes \Omega,\ K_{\text{av}}\, \psi_i \otimes \Omega) = (\psi_i \otimes \Omega,\ K\, \psi_i \otimes \Omega)$$
$$= \int_0^\infty \sum_{i>0} \exp\left(-it\,(E_i - E_j)\right)|(x_L\, \psi_i,\ \psi_j)|^2$$
$$\times \left[\int_0^\infty \exp\left(-i\omega\,(k)\,t\right)\alpha\,(k)^2\,dk\right] dt,$$

and the result follows from the well known formula

$$\int_0^\infty \exp\left(i\,(\alpha - \beta)\,t\right) dt = \pi\,\delta\,(\alpha - \beta) + \text{PV}\,\frac{i}{\alpha - \beta}.$$

■

Setting $K_{\text{av}} = K_L$, we finish the proof of Theorem 3.1.

### 2.6. Some remarks

To obtain dissipation, both in the time mean and on the time scale $\tau$, the hypothesis that $E_j \notin \tau\,(H_0)$ has been crucial. It is essentially saying that embedded eigenvalues of $H_0$ are not thresholds (whose presence invalidates regular perturbation theory). Intuitively, it is clearly "generically" satisfied. One can make that statement rigorous for a finite-dimensional system coupled to a heat bath.

When $m_0 = 0$, all embedded eigenvalues are thresholds, and our argument breaks down. Although one expects that embeded eigenvalues again dissolve into resonances after "turning on" small perturbations, the available techniques (even in much simpler models) are too crude to address such issues. One way of dealing with thresholds could be by adding small perturbations to the field, namely to consider

$$H_\lambda = H_p + \lambda H_I + H_{\text{bos}} + m_0\, R,$$

where $m_0 > 0$ is a small parameter, and $R$ acts on $\mathcal{H}_{\text{fock}}$ as $R = 1 - (\cdot, \Omega)\Omega$. One then can proceed as in previous sections. The expression for Fermi Golden Rule has a well-defined limit as $m_0 \to 0$, namely

$$\lim_{m_0 \to 0} \Gamma_j\,(m_0) = -\pi \sum_{i<j} |(x\psi_i,\ \psi_j)|^2\,[\alpha\,(E_j - E_i)]^2, \tag{2.24}$$

$$\lim_{m_0 \to 0} \Lambda_j\,(m_0) = -\sum_{i>0} |(x\psi_i,\ \psi_j)|^2\,\text{PV}\int_0^\infty \frac{\alpha\,(k)^2}{k - (E_j - E_i)}\,dk. \tag{2.25}$$

However, the analytic continuation of resolvent matrix elements takes place on the domain depending on $m_0$. Thus, in studying dynamics, one can at best take $m_0$ and $\lambda$ to zero simultaneously, and formal exchange of limits (first $m_0$ and then $\lambda$), although probably yielding the right result, is not allowed.

It is perhaps worth mentioning that master equation approach, although very powerful, also requires renormalization of the Hamiltonian (even for the positive mass model!). One cannot exchange limit $L \to \infty$ and $\lambda \to 0$ in Theorem 2.5. Again, $L$ and $\lambda$ at best can be taken simultaneously to zero (but only after considerable technical effort). Thus, for a positive mass model, our results on dymanics are stronger then the existing ones. For the massless model, all existing approaches require some regularization of the Hamiltonian. It remains as an open (and we believe hard) question to obtain result analogous to Theorems 1.1. and 1.2. for the model (1.2) with massless field.

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