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On the second quantisation of Hilbert-Schmidt processes

by

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ABSTRACT. – Using techniques of quantum stochastic calculus, a family of representations of the extended canonical commutation relations is constructed over a space of maps from the positive half-line into the Hilbert-Schmidt operators on a complex Hilbert space.

RÉSUMÉ. – A l'aide des techniques du calcul stochastique quantique, nous construisons une famille de représentations des relations de commutation canoniques étendues, sur un espace d'applications depuis la demi-droite positive dans l'espace des opérateurs de Hilbert-Schmidt d'un espace de Hilbert complexe.

1. INTRODUCTION

Let $\Gamma_-(h)$ denote fermion Fock space over a complex Hilbert space h and let $d\Gamma(T)$ be the differential second quantisation in $\Gamma_-(h)$ of $T \in B(h)$. It is well known (*see* e.g. [Tha]) that when T is trace class, we have the following (basis independent) representation for $d\Gamma(T)$,

$$d\Gamma(T) = \sum_{i,j=1}^{\infty} \langle e_i, Te_j \rangle a^\dagger(e_i) a(e_j) \tag{1.1}$$

where $(e_i, i \in \mathbb{N})$ is a maximal orthonormal set in h, and $a(f)$ (respectively $a^\dagger(f)$) is the fermion annihilation (creation) operator with test function $f \in h$.

Now let $h = L^2(\mathbb{R})$. Physicists like to introduce the formal annihilation and creation operators $\{a(x), x \in \mathbb{R}\}$ and $\{a^\dagger(y), y \in \mathbb{R}\}$ which satisfy the improper canonical anticommutation relations (CAR's)

$$\left.\begin{array}{r}\{a(x), a(y)\} = \{a^\dagger(x), a^\dagger(y)\} = 0\\ \{a(x), a^\dagger(y)\} = \delta(x-y)\qquad\qquad\end{array}\right\} \tag{1.2}$$

where $\{A, B\} = AB + BA$ denotes the anticommutator.

Of course we cannot make rigorous sense of these relations as they stand, ($a(x)$ can be densely defined but $a^\dagger(x)$ has domain $\{0\}$) however we note that the bona fide annihilation and creation operators are formally obtained by "smearing":

$$a(f) = \int_{\mathbb{R}}\{\overline{f(x)}\, a(x)\, dx, \qquad a^\dagger(f) = \int_{\mathbb{R}} f(x)\, a^\dagger(x)\, dx \tag{1.3}$$

Now suppose that T is a Hilbert-Schmidt operator on $L^2(\mathbb{R})$ so that T is an integral operator with square-integrable kernel $k : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$.

We can then formally write

$$d\Gamma(T) = \int_{\mathbb{R}^2} k(x, y)\, a^\dagger(x)\, a(y)\, dx\, dy \tag{1.4}$$

(1.4) can be made rigorous in a weak sense, using distributions [CaRu].

In this paper we propose a generalisation of (1.1) and (1.4). We work in $h = L^2(\mathbb{R}^+, \mathfrak{H})$ where $\mathfrak{H}$ is a complex Hilbert space and aim to give meaning to "$d\Gamma(Y)$" where $Y = (Y(t), t \in \mathbb{R}^+)$ is a suitable family of linear operators in $\mathfrak{H}$.

Our main tools will be provided by quantum stochastic calculus ([HuPa 1], [Par]). This enables us to make rigorous sense of more general integrals than (1.3) as quantum stochastic integrals wherein f and $\overline{f}$ are replaced by suitable operator-valued stochastic processes and $a(x)\, dx$ (respectively $a^\dagger(x)\, dx$) are replaced by the "stochastic differentials" $dA(x)$ $(dA^\dagger(x))$, using techniques obtained by generalising the Itô stochastic calculus for functions of Brownian motion to a non-commutative framework.

In fact the theory also allows the construction of stochastic integrals with respect to the conservation differential $d\Lambda(x)$ which acts formally like $d\Gamma(\delta(x)\rangle\langle\delta(x)) = a^\dagger(x)a(x)dx$. We then find that we can make rigorous sense of $d\Gamma_t(Y)$, for each $t \in \mathbb{R}^+$ as the quantum stochastic integral

$$d\Gamma_t(Y) = \int_0^t \sum_{i,j=1}^{\infty} \langle e_i, Y(s) e_j \rangle \, d\Lambda_{i,j}(s) \tag{1.5}$$

where $(\Lambda_{i,j}, i, j \in \mathbb{N})$ is an appropriate family of conservation processes. Note that as quantum stochastic integrals are defined in an L^2-sense, we find that we must require each $Y(s)$ to be Hilbert-Schmidt rather than trace-class. We also define $d\Gamma_t(Y)$ on boson Fock space in the first instance and associate it in a natural way to a representation of the CCR's which we construct by quantum stochastic integration over a suitable space of Hilbert-Schmidt valued processes. The equivalent fermionic operators are then obtained using the unification procedure of [PaSi] and [HuPa 2].

The organisation of this paper is as follows. We introduce appropriate spaces and algebras of Hilbert-Schmidt processes in Section 2. A brief description of the sense that the term "second quantisation" will be given in this paper can be found in Section 3. In Section 4, in order to make this paper as self-contained as possible, we summarise those results about quantum stochastic integration which we will need and then use these to construct our required operator-valued processes. It is then shown in Section 5 that these yield a second quantisation in the sense of Section 3.

Notation. – All inner products are conjugate-linear on the left. If T is an unbounded operator in a Hilbert space $\mathfrak{H}$, its domain is denoted Dom (T). Dirac notation will be used where appropriate.

2. HILBERT-SCHMIDT PROCESSES

Let $\mathfrak{H}$ be a complex separable Hilbert space and $B(\mathfrak{H})$ be the $\star$-algebra of all bounded linear operators in $\mathfrak{H}$. $B(\mathfrak{H})$ is complete with respect to the operator norm defined by

$$\| X \| = \sup \{ \| X\psi \| ; \psi \in \mathfrak{H}, \| \psi \| = 1 \}$$

for $X \in B(\mathfrak{H})$.

An operator $T \in B(\mathfrak{H})$ is said to be *Hilbert-Schmidt* if

$$\| T \|_2^2 = \mathrm{tr}(T^\star T) = \sum_{n=1}^{\infty} \| Te_n \|^2 < \infty$$

where $\{ e_n, n \in \mathbb{N} \}$ is an arbitrary maximal orthonormal set in $\mathfrak{H}$.

Let $\mathfrak{I}_2(h)$ denote the space of all Hilbert-Schmidt operators on $\mathfrak{H}$. We collect below the following facts about such operators which can all be found in [ReSi], p. 206-210:

(i) $\mathfrak{I}_2(\mathfrak{H})$ is a $\star$-ideal in $B(\mathfrak{H})$. Moreover $\mathfrak{I}_2(\mathfrak{H})$ is a complex Hilbert space with respect to the inner product $\langle\, ,\, \rangle_2$ defined by

$$\langle A,\, B\rangle_2 = \operatorname{tr}(A^\star B) \tag{2.1}$$

(ii)

$$\|A\|_2 \leq \|A\| \tag{2.2}$$

for all $A \in \mathfrak{I}_2(\mathfrak{H})$ and

$$\left.\begin{aligned} \|AB\|_2 \leq \|A\|\ \|B\|_2 \\ \|BA\|_2 \leq \|A\|\ \|B\|_2 \end{aligned}\right\} \tag{2.3}$$

for all $A \in B(\mathfrak{H})$, $B \in \mathfrak{I}_2(\mathfrak{H})$.

(iii) Every $A \in \mathfrak{I}_2(\mathfrak{H})$ is a compact operator. Hence there exists maximal orthonormal sets $\{f_n,\, n \in \mathbb{N}\}$ and $\{g_n,\, n \in \mathbb{N}\}$ in $\mathfrak{H}$ and a sequence $\{\lambda_n,\, n \in \mathbb{N}\}$ of positive real numbers with $\lambda_n \underset{n}{\to} 0$ such that

$$A = \sum_{n=1}^{\infty} \lambda_n \langle f_n,\, .\rangle\, g_n \tag{2.4}$$

Moreover

$$\|A\|_2^2 = \sum_{n=1}^{\infty} \lambda_n^2 \tag{2.5}$$

LEMMA 2.1. – *Let $A \in \mathfrak{I}_2(\mathfrak{H})$, $\phi \in \mathfrak{H}$ with $\|\phi\| = 1$ and $\{e_n,\, n \in \mathbb{N}\}$ be a maximal orthonormal set in $\mathfrak{H}$*

(a) $$\sum_{i,\,j=1}^{\infty} |\langle e_i,\, Ae_j\rangle|^2 = \|A\|_2^2 < \infty$$

(b) $$\sum_{j=1}^{\infty} |\langle \phi,\, Ae_j\rangle|^2 \leq \|A\|_2^2 < \infty$$

Proof. – These are easily obtained by standard use of Parseval's equation and (for (b)) the Schwartz inequality. □

Now let ν be a σ-finite measure on $\mathbb{R}^+$ and $C : \mathbb{R}^+ \to B(\mathfrak{H})$ be a ν-measurable map. Let $\phi \in \mathfrak{H}$ with $\|\phi\| = 1$ be such that

(i) $\phi \notin \operatorname{Ker} C(t)$ for all $t \in \mathbb{R}^+$ except a possible set of ν-measure zero

(ii) $\displaystyle\int_0^t \|C(s)\,\phi\|^2\,\nu(ds) < \infty$ for all $t \in \mathbb{R}^+$

With the understanding that ϕ and ν are fixed once and for all, we denote by $\mathcal{C}(\mathbb{R}^+,\, \mathfrak{H})$ the linear space of all mappings which satisfy (i) and

(ii) above. $\mathcal{C}(\mathbb{R}^+, \mathfrak{H})$ is an inner product space under each of the inner products $\langle \cdot, \cdot \rangle_{\phi, \nu, t}$ defined by

$$\langle C, D \rangle_{\phi, \nu, t} = \int_0^t \langle C(s)\phi, D(s)\phi \rangle \nu(ds)$$

where $t \in \mathbb{R}^+$.

In the case where ν is Lebesgue measure, we will write

$$\langle \cdot, \cdot \rangle_{\phi, \nu, t} = \langle \cdot, \cdot \rangle_{\phi, t}$$

Let $\mathcal{C}_c(\mathbb{R}^+, \mathfrak{H})$ denote the linear space of all norm continuous maps from $\mathbb{R}^+$ to $B(\mathfrak{H})$ which satisfy (i). Note that if $C \in \mathcal{C}_c(\mathbb{R}^+, \mathfrak{H})$, the map $t \to \| C(t) \|$ is bounded on finite intervals. Clearly $\mathcal{C}_c(\mathbb{R}^+, \mathfrak{H}) \subset \mathcal{C}(\mathbb{R}^+, \mathfrak{H})$. It is not difficult to verify that $\mathcal{C}_c(\mathbb{R}^+, \mathfrak{H})$ is a $\star$-algebra where the product and involution are defined pointwise.

A ν-measurable map $C : \mathbb{R}^+ \to B(\mathfrak{H})$ is said to be a *Hilbert-Schmidt process* if

(i) each $C(t) \in \mathfrak{I}_2(\mathfrak{H})$

(ii) $\int_0^t \| C(s) \|_2^2 \nu(ds) < \infty$ for all $t \in \mathbb{R}^+$

We denote by $\mathfrak{I}_2(\mathbb{R}^+, \mathfrak{H})$ the linear space of all such maps. $\mathfrak{I}_2(\mathbb{R}^+, \mathfrak{H})$ is an inner product space with respect to each of the inner products $\langle \cdot, \cdot \rangle_{2, \nu, t}$ defined by

$$\langle C, D \rangle_{2, \nu, t} = \int_0^t \langle C(s), D(s) \rangle_2 \nu(ds)$$

for $t \in \mathbb{R}^+$.

LEMMA 2.2. – *Let* $C \in \mathfrak{I}_2(\mathbb{R}^+, \mathfrak{H})$, *then*

$$\| C \|_{\phi, \nu, t} \leq \| C \|_{2, \nu, t} < \infty$$

for all $t \in \mathbb{R}^+$.

Proof. – Use the Schwartz inequality. □

Let $\mathfrak{I}_{2, c}(\mathbb{R}^+, \mathfrak{H}) = \mathfrak{I}_2(\mathbb{R}^+, \mathfrak{H}) \cap \mathcal{C}_c(\mathbb{R}^+, \mathfrak{H})$ then by lemma 2.2 we have

$$\mathfrak{I}_{2, c}(\mathbb{R}^+, \mathfrak{H}) \subset \mathfrak{I}_2(\mathbb{R}^+, \mathfrak{H}) \subset \mathcal{C}(\mathbb{R}^+, \mathfrak{H})$$

It is not difficult to verify that $\mathfrak{I}_{2, c}(\mathbb{R}^+, \mathfrak{H})$ is a $\star$-subalgebra of $\mathcal{C}_c(\mathbb{R}^+, \mathfrak{H})$.

We consider $\mathfrak{I}_{2, c}(\mathbb{R}^+, \mathfrak{H})$ as linear operators on the space $\mathfrak{I}_2(\mathbb{R}^+, \mathfrak{H})$ where the action is defined pointwise.

Now consider the space $B(\mathfrak{I}_2(\mathbb{R}^+, \mathfrak{H}))$ of all bounded linear operators on $\mathfrak{I}_2(\mathbb{R}^+, \mathfrak{H})$. $B(\mathfrak{I}_2(\mathbb{R}^+, \mathfrak{H}))$ becomes a normed $\star$-algebra with respect to the operator norm $|||\,.\,|||_{\nu,t}$ arising from each of the norms $||\,.\,||_{2,\nu,t}$ on $\mathfrak{I}_2(\mathbb{R}^+, \mathfrak{H})$ where $t \in \mathbb{R}^+$.

LEMMA 2.3.

$$\mathfrak{I}_{2,c}(\mathbb{R}^+, \mathfrak{H}) \subset B(\mathfrak{I}_2(\mathbb{R}^+, \mathfrak{H}))$$

and if $C \in \mathfrak{I}_{2,c}(\mathbb{R}^+, \mathfrak{H})$, *then for each* $t \in \mathbb{R}^+$

$$|||\,C\,|||_{\nu,t} \leq \sup_{0 \leq s \leq t} ||\,C(s)\,||$$

Proof.

$$|||\,C\,|||_{\nu,t}^2 = \sup\{\,||\,CD\,||_{2,\nu,t}^2;\ D \in \mathfrak{I}_2(\mathbb{R}^+, \mathfrak{H}),\ ||\,D\,||_{2,\nu,t} = 1\,\}$$

However by (2.3),

$$\begin{aligned} ||\,C\,D\,||_{2,\nu,t}^2 &= \int_0^t ||\,C(s)\,D(s)\,||_2^2\,\nu(ds) \\ &\leq \sup_{0 \leq s \leq t} ||\,C(s)\,||^2\,||\,D\,||_{2,\nu,t} \end{aligned}$$

and the required result follows. □

3. SECOND QUANTISATION

Let V be a complex inner product space and $\mathcal{H}$ a complex, separable Hilbert space. We say that we have a *representation of the canonical commutation relations* or *RCCR* over V on $\mathcal{H}$ if there exists a dense linear manifold $\mathcal{E}$ in $\mathcal{H}$ and a family $\{\,a(f),\ f \in V\,\}$ of closeable linear operators in $\mathcal{H}$ such that

(i) $\mathcal{E}$ is a common domain for $\{\,a(f),\ f \in V\,\}$ and the map $f \to a(f)$ is conjugate linear,

(ii) $\mathcal{E} \subseteq \mathrm{Dom}(a(f)^\star)$ for all $f \in V$,

(iii) writing $a^\dagger(f) = a(f)^\star|_{\mathcal{E}}$, the following commutation relations hold on $\mathcal{E}$:

$$[a(f),\ a(g)] = [a^\dagger(f),\ a^\dagger(g)] = 0$$
$$[a(f),\ a^\dagger(g)] = \langle\, f,\ g\,\rangle\, I$$

for all $f,\ g \in V$.

A representation of the anticommutation relations (RCAR) is defined similarly but with the commutators in (iii) replaced by anticommutators.

Now let $\mathcal{A} \subseteq B(V)$ be a $\star$-algebra. Given an RCCR as above we say that we have an associated *conservation map* λ if for each $T \in \mathcal{A}$, there exists a closeable operator $\lambda(T)$ in $\mathcal{H}$, such that

(i) $\mathcal{E}$ is a common domain for all $\{\lambda(T),\ T \in \mathcal{A}\}$ and the map $T \to \lambda(T)$ is linear,

(ii) $\mathcal{E} \subseteq \mathrm{Dom}(\lambda(T)^\star)$ for each $T \in \mathcal{A}$ and

$$\lambda(T)^\star = \lambda(T^\star) \quad \text{on } \mathcal{E},$$

(iii) The following commutation relations hold on $\mathcal{E}$,

$$[\lambda(S),\ \lambda(T)] = \lambda([S,\ T])$$
$$[a(f),\ \lambda(T)] = a(T^\star f)$$
$$[\lambda(T),\ a^\dagger(f)] = a^\dagger(Tf)$$

for each $S,\ T \in \mathcal{A},\ f \in V$.

Whenever, we can associate a conservation map to a RCCR we say that $(\mathcal{H},\ \mathcal{E},\ a,\ \lambda)$ is a *boson second quantisation* of the pair $(V,\ \mathcal{A})$. A fermion second quantisation is defined similarly with RCAR replacing RCCR.

One of the most familiar examples of a second quantisation is that which underpins the construction of the free quantum field. Here we take V to be a complex separable Hilbert space and $\mathcal{A} = B(V)$. $\mathcal{H}$ is the symmetric Fock space $\Gamma(V)$ defined by

$$\Gamma(V) = \mathbb{C} \oplus V \oplus (V \otimes_s V) \oplus \ldots$$

where $\otimes_s$ denotes the symmetric part of the tensor product.

$\mathcal{E}$ is the linear span of the exponential vectors $\{e(f),\ f \in V\}$ where $e(f) = \left(1,\ f,\ \frac{f \otimes f}{\sqrt{2!}},\ \ldots\right)$.

$a(f)$ is the annihilation operator with test function $f \in V$ (so that $a^\dagger(f)$ is the corresponding creation operator).

$\lambda(T)$ is the differential second quantisation of $T \in B(V)$.

The details of the above construction can be found in e.g. [Par] (p. 123-152).

In the sequel we will refer to the above boson second quantisation as the "standard model" (readers who are elementary particle physicists should not take offence!)

4. QUANTUM STOCHASTIC INTEGRALS OF HILBERT-SCHMIDT PROCESSES

In this section we take $V = L^2(\mathbb{R}^+, \mathfrak{H})$ where $\mathfrak{H}$ is a complex separable Hilbert space and work in the standard model of boson second quantisation as described above. We note that for $f \in L^2(\mathbb{R}^+)$, $\psi \in \mathfrak{H}$, the map $f(.)\,\psi \to f \otimes \psi$ extends by linearity and continuity to a canonical isomorphism between V and $L^2(\mathbb{R}^+) \otimes \mathfrak{H}$. We will use this isomorphism to identify the two spaces.

Let $\{e_n, n \in \mathbb{N}\}$ be a maximal orthonormal set in $\mathfrak{H}$. We define families of processes in $\mathcal{H}$, with common domain $\mathcal{E}$, as follows:– The *annihilation processes* $(A_j, j \in \mathbb{N})$ are $A_j = (A_j(t), t \in \mathbb{R}^+)$ where each

$$A_j(t) = a(\mathcal{X}_{[0,t)} \otimes e_j), \qquad t \in \mathbb{R}^+, \quad j \in \mathbb{N}$$

similarly the *creation processes* $(A_j^\dagger, j \in \mathbb{N})$ are given by

$$A_j^\dagger(t) = a^\dagger(\mathcal{X}_{[0,t)} \otimes e_j), \qquad t \in \mathbb{R}^+, \quad j \in \mathbb{N}$$

and the *conservation processes* $(\Lambda_{i,j}, i, j \in \mathbb{N})$ are

$$\Lambda_{i,j}(t) = \lambda(\mathcal{X}_{[0,t)} \otimes (e_i\rangle\langle e_j)), \qquad t \in \mathbb{R}^+, \quad i, j \in \mathbb{N}$$

Now for each $t \in \mathbb{R}^+$, let P_t be the orthogonal projection in V defined by continuous extension of

$$P_t(f \otimes \psi) = \mathcal{X}_{[0,t)}\, f \otimes \psi$$

where $f \in L^2(\mathbb{R}^+)$, $\psi \in \mathfrak{H}$. $\mathcal{H}$ is then canonically isomorphic to the Hilbert space tensor product $\Gamma(L^2([0, t), \mathfrak{H})) \otimes \Gamma(L^2([t, \infty), \mathfrak{H}))$ by continuous linear extension of the map which takes each $e(g)$ to $e(P_t\, g) \otimes e((I - P_t)\, g)$ for $g \in V$. We will again use this isomorphism to identify the two spaces.

Now let $F = (F(t), t \in \mathbb{R}^+)$ be a family of densely defined linear operators in $\mathcal{H}$. We say that F is an *adapted process* if

(i) $\mathcal{E} \subseteq \mathrm{Dom}\,(F(t)$ for each $t \in \mathbb{R}^+$,

(ii) $F(t) = F^\sim(t) \otimes I$ on $\mathcal{E}$ where $F^\sim(t)$ is a linear operator in $\Gamma(L^2([0, t), \mathfrak{H}))$.

An adapted process is said to be *(locally) square-integrable* if the map $t \to F(t)\, e(f)$ is measurable from $\mathbb{R}^+$ to $\mathcal{H}$ and

$$\int_0^t \|\, F(s)\, e(f)\,\|^2\, ds < \infty \quad \text{for all } f \in V,\ t \in \mathbb{R}^+$$

We denote by $\mathcal{L}^2(\mathbb{R}^+, \mathcal{H})$ the linear space of all adapted locally square integrable processes in $\mathcal{H}$. Local square integrability is sufficient to define stochastic integrals with respect to quantum noise with finitely many degrees of freedom [HuPa 1]. For the case of infinitely many degrees of freedom, as is required below, we need a stronger condition which we now describe.

For each $f \in V$, let ν_f denote the σ-finite measure on $\mathbb{R}^+$ given by

$$\nu_f([0, t)) = \int_0^t (1 + \| f(s) \|_{\mathfrak{H}}^2)\, ds \quad \text{for each } t \in \mathbb{R}^+$$

Let $\{ F_{i,j},\ 0 \leq i, j < \infty \}$ be a family of adapted processes in $\mathcal{H}$. We say that they satisfy the *Mohari-Sinha integrability condition* (to be referred to henceforth as condition (MS)) if

$$\int_0^t \sum_{i,j=0}^{\infty} \| F_{i,j}(s)\, e(f) \|^2\, \nu_f(ds) < \infty$$

for each $f \in V$, $t \in \mathbb{R}^+$ [MoSi].

Note that if condition (MS) is satisfied then each $F_{i,j} \in \mathcal{L}^2(\mathbb{R}^+, \mathcal{H})$. Furthermore, if condition (MS) is satisfied, it is shown in [Par], proposition 27.1 (p. 222-223) that the quantum stochastic integral $M = (M(t),\ t \in \mathbb{R}^+)$ exists as an adapted process in $\mathcal{H}$ where

$$\begin{aligned} dM(t) = \sum_{i=0}^{\infty} F_{i,0}(t)\, dA_i(t) + \sum_{i,j=1}^{\infty} F_{i,j}(t)\, d\Lambda_{i,j}(t) \\ + \sum_{j=0}^{\infty} F_{0,j}(t)\, dA_j^{\dagger}(t) + F_{0,0}(t)\, dt \end{aligned} \tag{4.1}$$

and we have the estimate

$$\| M(t)\, e(f) \|^2 \leq 2\, e^{\nu_f(t)} \int_0^t \sum_{i,j=0}^{\infty} \| F_{i,j}(s)\, e(f) \|^2\, \nu_f\ (ds) \tag{4.2}$$

for each $f \in V$, $t \in \mathbb{R}^+$.

Now let M^1, M^2 be stochastic integrals of the families $F_{i,j}^k\ (k = 1, 2)$, respectively such that $F_{i,j}^1\, M^2$, $M^1\, F_{i,j}$ and $\sum_{j=1}^{\infty} F_{i,j}\, F_{j,k} \in \mathcal{L}^2(\mathbb{R}^+, \mathcal{H})$ for all $0 \leq i, j, < \infty$, where each product of processes is defined pointwise and the following condition (MS′) is satisfied

$$\begin{aligned} \int_0^t \sum_{i,k=0}^{\infty} \Bigg\{ \| F_{i,k}^1(s)\, M^2(s)\, e(f) \|^2 + \| M^1(s)\, F_{i,k}(s)\, e(f) \|^2 \\ + \left\| \sum_{j=1}^{\infty} F_{i,j}(s)\, F_{j,k}(s)\, e(f) \right\|^2 \Bigg\}\, \nu_f\ (ds) < \infty \end{aligned}$$

for each $f \in V$, $t \in \mathbb{R}^+$. By combining propositions 25.26 and 27.1 in [Par], it then follows that $M^1 M^2$ exists as an adapted process in $\mathcal{H}$. Moreover, we have the *quantum Itô formula*

$$dM^1 dM^2 = M^1 dM^2 + dM^1 M^2 + dM^1 dM^2 \tag{4.3}$$

where the "Itô correction term" $dM^1 dM^2$ is evaluated by bilinear extension of the rule that all products of differentials vanish with the exception of

$$dA_i dA_j = \delta_{i,j} dt, \quad dA_i d\Lambda_{k,l} = \delta_{i,k} dA_l$$
$$d\Lambda_{i,j} dA_k^\dagger = \delta_{j,k} dA_i^\dagger, \quad d\Lambda_{i,j} d\Lambda_{k,l} = \delta_{j,k} d\Lambda_{i,l}$$

Now recall the discussion of Section 2 and let $X, Z \in \mathfrak{I}_2(\mathbb{R}^+, \mathfrak{H})$ and $Y \in \mathfrak{I}_{2,c}(\mathbb{R}^+, \mathfrak{H})$. We assume that

$$\max\{ \| X \|_{2,\nu_f,t}, \| Y \|_{2,\nu_f,t}, \| Z \|_{2,\nu_f,t} \} < \infty \tag{4.4}$$

for all $f \in V$, $t \in \mathbb{R}^+$.

Now let $\phi \in \mathfrak{H}$ with $\|\phi\| = 1$ and consider the process $M = (M(t), t \in \mathbb{R}^+)$ given formally as follows, for each $t \in \mathbb{R}^+$

$$M(t) = A_X(t) + \Lambda_Y(t) + A_Z^\dagger(t) \tag{4.5}$$

where

$$A_X(t) = \int_0^t \sum_{j=1}^{\infty} \langle \phi, X(s)^\star e_j \rangle dA_j(s) \tag{4.6}$$

$$\Lambda_Y(t) = \int_0^t \sum_{i,j=1}^{\infty} \langle e_i, Y(s) e_j \rangle d\Lambda_{i,j}(s) \tag{4.7}$$

$$A_Z^\dagger(t) = \int_0^t \sum_{j=1}^{\infty} \langle e_j, Z(s) \phi \rangle dA_j^\dagger(s) \tag{4.8}$$

PROPOSITION 4.1. – *M exists as an adapted process in $\mathcal{H}$.*

Proof. – The integrands in (4.6) to (4.8) are clearly adapted. Hence we have only to verify condition (MS). We obtain

$$\int_0^t \sum_{i,j=0}^{\infty} \| F_{i,j}(s)\, e(f) \|^2 \nu_f(ds)$$
$$\leq \| \psi(f) \|^2 \left\{ \int_0^t \sum_{j=1}^{\infty} | \langle \phi,\, X(s)^\star e_j \rangle |^2 \nu_f(ds) \right.$$
$$+ \int_0^t \sum_{i,j=1}^{\infty} | \langle e_i,\, Y(s)\, e_j \rangle |^2 \nu_f(ds)$$
$$\left. + \int_0^t \sum_{j=1}^{\infty} | \langle e_j,\, Z(s)\, \phi \rangle |^2 \nu_f(ds) \right\}$$
$$\leq \| \psi(f) \|^2 \{ \| X \|^2_{2,\nu_f,t} + \| Y \|^2_{2,\nu_f,t} + \| Z \|^2_{2,\nu_f,t} \}$$

$< \infty$ by (4.4), where we have used lemma 2.1 □

A standard argument using the Fourier expansion in $\mathfrak{H}$ shows that each $M(t)$ is independent of choice of orthonormal basis.

Furthermore the process M is a quantum martingale (*see* e.g. [Par], p. 180).

PROPOSITION 4.2. – $(M)^2 = (M(t)^2,\, t \in \mathbb{R}^+)$ *exists as an adapted process in* $\mathcal{H}$.

Proof. – For simplicity, we will take $X = Z = 0$. We need to show that condition (MS′) is satisfied. We have by lemma 2.1,

$$\int_0^t \sum_{i,k=0}^{\infty} \left\{ \| F^1_{i,k}(s)\, M^2(s)\, e(f) \|^2 + \| M^1(s)\, F_{i,k}(s)\, e(f) \|^2 \right.$$
$$\left. + \left\| \sum_{j=1}^{\infty} F_{i,j}(s)\, F_{j,k}(s)\, e(f) \right\|^2 \right\} \nu_f(ds)$$
$$\leq 2 \int_0^t \sum_{i,j=1}^{\infty} | \langle e_i,\, Y(s)\, e_j \rangle |^2 \| M(s)\, \psi(f) \|^2 \; \nu_f(ds)$$
$$+ \| \psi(f) \|^2 \int_0^t \left| \sum_{i,j,k=1}^{\infty} \langle e_i,\, Y(s)\, e_j \rangle \langle e_j,\, Y(s)\, e_k \rangle \right|^2 \nu_f(ds)$$

$$\leq 2 \sup_{0 \leq s \leq t} \| M(s)\, \psi(f) \|^2 \; \| Y \|^2_{2, \nu_f, t}$$

$$+ \| \psi(f) \|^2 \int_0^t \sum_{i,\, k=1}^{\infty} | \langle Y(s)^\star e_i,\, Y(s)\, e_k \rangle |^2 \, \nu_f(ds)$$

The first term is finite by (4.4) and (4.2). For the second term, we observe that

$$\begin{aligned} \sum_{i,\, k=1}^{\infty} | \langle Y(s)^\star e_i,\, Y(s)\, e_k \rangle |^2 &= \sum_{i,\, k=1}^{\infty} | \langle e_i,\, Y(s)^2\, e_k \rangle |^2 \\ &= \sum_{k=1}^{\infty} \| Y(s)^2\, e_k \|^2 \\ &= \| Y(s)^2 \|_2^2 \leq \| Y(s) \|^2 \, \| Y(s) \|_2^2 \end{aligned}$$

by (2.3). Hence, since $Y \in \mathfrak{I}_2(\mathbb{R}^+,\, \mathfrak{H})$, we have

$$\int_0^t \sum_{i,\, k=1}^{\infty} | \langle Y(s)^\star e_i,\, Y(s)\, e_k \rangle |^2 \, \nu_f(ds)$$

$$\leq \sup_{0 \leq s \leq t} \| Y(s) \|^2 \; \| Y \|^2_{2, \nu_f, t} < \infty$$

The general case where $X,\, Z \neq 0$ is established similarly. □

5. CONSTRUCTION OF THE SECOND QUANTISATION

Fix $t \in \mathbb{R}^+$ for the remainder of this section.

We aim to show that we have a second quantisation, in the sense of Section 3, with the following data.

$V = \mathfrak{I}_2(\mathbb{R}^+,\, \mathfrak{H})$ equipped with the inner product $\langle .,\, . \rangle_{\phi, t}$

$\mathcal{H} = \Gamma(L^2(\mathbb{R}^+,\, \mathfrak{H}))$

$\mathcal{E}$ is the linear span of the exponential vectors in $\mathcal{H}$

$a(X) = A_X(t)$ for each $X \in V$

$\mathcal{A} = \mathfrak{I}_{2, c}(\mathbb{R}^+,\, \mathfrak{H})$

$\lambda(Y) = \Lambda_Y(t)$ for each $Y \in \mathcal{A}$.

We will assume that assumption (4.4) is satisfied for all $X,\, Z \in V$, $Y \in \mathcal{A}$.

Note that by lemma 2.2 and the fact that ν_f dominates Lebesgue measure for all $f \in L^2(\mathbb{R}^+,\, \mathfrak{H})$, we have

$$\| T \|_{\phi, t} \leq \| T \|_{2, \nu_f, t} \quad \text{for all } f \in L^2(\mathbb{R}^+,\, \mathfrak{H})$$

THEOREM 5.1. – $\{a(X), X \in V\}$ *defines a* RCCR *in* $\mathcal{H}$ *on the dense domain* $\mathcal{E}$.

Proof. – We must verify the conditions (i)-(iii) described in Section 3. (i) is immediate from proposition 3.1. (ii) and the closability of each $a(X)$ follows from the easily verified fact that

$$\langle e(f), A_X(t)\, e(g)\rangle = \langle A_X^\dagger(t)\, e(f), e(g)\rangle$$

for all $f, g \in L^2(\mathbb{R}^+, \mathfrak{H})$, so that $a^\dagger(X) = A_X^\dagger(t)$.

To establish (iii), we observe that proposition 3.2 allows us to use the quantum Itô formula to compute, for each $X, Z \in V$

$$\begin{aligned} d([A_X(t), A_Z^\dagger(t)] &= dA_X(t) . A_Z^\dagger(t) \\ &\quad + A_X(t) . dA_Z^\dagger(t) + dA_X(t) . dA_Z^\dagger(t) \\ &\quad - dA_Z^\dagger(t) . A_X(t) - A_Z^\dagger(t) . dA_X(t) \\ &\quad - dA_Z^\dagger(t) . dA_X(t) \end{aligned}$$

Now by adaptedness, we have

$$dA_X(t) . A_Z^\dagger(t) = A_Z^\dagger(t) . dA_X(t)$$

and

$$A_X(t) . dA_Z^\dagger(t) = dA_Z^\dagger(t) . A_X(t)$$

and by the quantum Itô formula, $dA_Z^\dagger(t) . dA_X(t) = 0$.

Hence we obtain, by (4.6), (4.8) and a further application of the quantum Itô formula,

$$\begin{aligned} d([A_X(t), A_Z^\dagger(t)] &= dA_X(t) . dA_Z^\dagger(t) \\ &= \sum_{j=1}^{\infty} \langle \phi, X(t)^\star e_j\rangle\, dA_j(t) \\ &\quad \times \sum_{k=1}^{\infty} \langle e_k, Z(t)\,\phi\rangle\, dA_k^\dagger(t) \\ &= \sum_{j=1}^{\infty} \langle X(t)\,\phi, e_j\rangle \langle e_j, Z(t)\,\phi\rangle\, dt \\ &= \langle X(t)\,\phi, Z(t)\,\phi\rangle\, dt \end{aligned}$$

As $A_X(0) = A_Z^\dagger(0) = 0$, we find that

$$\begin{aligned} [a(X), a^\dagger(Z)] &= [A_X(t), A_Z^\dagger(t)] \\ &= \int_0^t \langle X(s)\,\phi, Z(s)\,\phi\rangle\, ds = \langle X, Z\rangle_{\phi, t} \end{aligned}$$

as required.

The other commutation relations are obtained by a similar argument. □

THEOREM 5.2. – $\lambda : \mathcal{A} \to B(\mathcal{H})$ *is a conservation map.*

Proof. – We verify the conditions (i) to (iii) in the definition. (i) is immediate. (ii) and the closeability of each $\lambda(Y)$ follows from the easily verified fact that

$$\langle e(f), \Lambda_Y(t)\, e(g) \rangle = \langle \Lambda_{Y^*}(t)\, e(f), e(g) \rangle$$

for all $f, g \in L^2(\mathbb{R}^+, \mathfrak{H})$, $Y \in \mathcal{A}$.

To establish (iii), we again use quantum Itô's formula, adaptedness considerations and (4.7) to show that for $W, Y \in \mathcal{A}$,

$$\begin{aligned}
&d([\Lambda_W(t), \Lambda_Y(t)])\\
&\quad = d\Lambda_W(t) . \Lambda_Y(t) + \Lambda_W(t) . d\Lambda_Y(t) + d\Lambda_W(t) . \Lambda_Y(t)\\
&\qquad - d\Lambda_Y(t)\, \Lambda_W(t) - \Lambda_Y(t) . d\Lambda_W(t) - d\Lambda_Y(t) . d\Lambda_W(t)\\
&\quad = d\Lambda_W(t) . d\Lambda_Y(t) - d\Lambda_Y(t) . d\Lambda_W(t)\\
&\quad = \sum_{i,j=1}^{\infty} \langle e_i, W(t)\, e_j \rangle\, d\Lambda_{i,j}(t) . \sum_{k,l=1}^{\infty} \langle e_k, Y(t)\, e_l \rangle\, d\Lambda_{k,l}(t)\\
&\qquad - \sum_{k,l=1}^{\infty} \langle e_k, Y(t)\, e_l \rangle\, d\Lambda_{k,l}(t) . \sum_{i,j=1}^{\infty} \langle e_i, W(t)\, e_j \rangle\, d\Lambda_{i,j}(t)\\
&\quad = \sum_{i,j,l=1}^{\infty} \{ \langle W(t)^\star e_i, e_j \rangle \langle e_j, Y(t)\, e_l \rangle\\
&\qquad - \langle Y(t)^\star e_i, e_j \rangle \langle e_j, W(t)\, e_l \rangle . d\Lambda_{i,l}(t)\\
&\quad = \sum_{i,l=1}^{\infty} \langle e_i, [W(t), Y(t)]\, e_l \rangle\, d\Lambda_{i,l}(t) = d\Lambda_{[W,Y]}(t)
\end{aligned}$$

Hence since $\Lambda_W(0) = \Lambda_Y(0) = 0$, we have

$$[\lambda(W), \lambda(Y)] = [\Lambda_W(t), \Lambda_Y(t)] = \Lambda_{[W,Y]}(t) = \lambda([W, Y])$$

as required.

For $Y \in \mathcal{A}$, $X \in V$, we have by a similar argument to the above

$$\begin{aligned}
d([A_X(t), \Lambda_Y(t)]) &= dA_X(t) . d\Lambda_Y(t)\\
&= \sum_{j=1}^{\infty} \langle X(t)\, \phi, Y(s)\, e_j \rangle\, dA_j(t)\\
&= dA_{Y^* X}(t)
\end{aligned}$$

Hence $[a(X), \lambda(Y)] = a(Y^\star X)$ as required.

The other commutation relation is established similarly. □

COROLLARY 5.3. – *$(\mathcal{H}, \mathcal{E}, a, \lambda)$ is a boson second quantisation of the pair $(V, \mathcal{A})$*

We close by indicating how the results of this section can be extended to the fermion case. Following [PaSi] (*see* also [HuPa 2] and [Par]), we define an adapted process $J = (J(t), t \in \mathbb{R}^+)$ in $\mathcal{H}$ by continuous linear extension of

$$J(t)\, e(f) = e((I - 2P_t)\, f) \tag{5.1}$$

then each $J(t)$ is self-adjoint and unitary. Moreover we obtain fermion annihilation and creation processes in $\mathcal{H}$ by defining

$$F_j(t) = \int_0^t J(s)\, dA_j(s), \qquad F_j^\dagger(t) = \int_0^t J(s)\, dA_j^\dagger(s)$$

for $j \in \mathbb{N}$, $t \in \mathbb{R}^+$. To obtain a RCAR, we define for $X, Z \in \mathfrak{J}_2(\mathbb{R}^+, \mathfrak{H})$, (*cf.* (4.6) and (4.8))

$$F_X(t) = \int_0^t \sum_{j=1}^\infty \langle \phi, X(s)^\star e_j \rangle\, dF_j(s)$$

$$F_Z^\dagger(t) = \int_0^t \sum_{j=1}^\infty \langle e_j, Z(s)\, \phi \rangle\, dF_j^\dagger(s)$$

then a similar argument to that of theorem 5.1 above shows that the CAR's are satisfied *i.e.*

$$\{F_X(t), F_Z(t)\} = \{F_X^\dagger(t), F_Z^\dagger(t)\} = 0$$
$$\{F_X(t), F_Z^\dagger(t)\} = \langle X, Z \rangle_{\phi, t}$$

The same conservation map as described in theorem 5.2 (and defined by (4.7)) works in the fermion case so that we indeed have a fermion second quantisation.

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