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VESSELIN PETKOV
LATCHEZAR STOYANOV

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Sojourn times of trapping rays and the behavior of the modified resolvent of the Laplacian

by

Vesselin PETKOV

Département de Mathématiques, Université Bordeaux-I,
351, Cours de la Libération, 33405 Talence, France.

and

Latchezar STOYANOV

Department of Mathematics, University of Western Australia,
Nedlands, Perth 6009, Western Australia.

ABSTRACT. – Obstacles K in an odd-dimensional Euclidean space are considered which are finite disjoint unions of convex bodies with smooth boundaries. Assuming that there are no non-trivial open subsets of ∂K where the Gauss curvature vanishes, it is shown that there exists a sequence of scattering rays in the complement Ω of K such that the corresponding sequence of sojourn times tends to infinity and consists of singularities of the scattering kernel. Using this, certain information on the behavior of the modified resolvent of the Laplacian and the distribution of poles of the scattering matrix is obtained. For the same kind of obstacles K, without the additional assumption on the Gauss curvature, it is established that for almost all pairs (ω, θ) of unit vectors all singularities of the scattering kernel $s(t, \omega, \theta)$ are related to sojourn times of reflecting (ω, θ)-rays in Ω.

RÉSUMÉ. – On considère, dans un espace euclidien de dimension impaire, des obstacles K constitués d'unions disjointes finies de corps convexes à frontière régulière. Supposant qu'il n'existe pas d'ouvert non trivial dans ∂K sur lequel la courbure gaussienne s'annule, on montre l'existence d'une suite de rayons réfléchissants dans le complémentaire Ω de K dont les temps de séjours correspondant tendent vers l'infini et constituent des singularités du noyau de diffusion. À l'aide de ce résultat, on obtient certaines informations sur le comportement de la résolvante modifiée du Laplacien et sur la distribution des pôles de la matrice de diffusion. Pour le même type d'obstacles K, sans hypothèse supplémentaire sur la courbure gaussienne, on montre que pour presque toute paire (ω, θ) de vecteurs unitaires, toutes les singularités du noyau de diffusion $s(t, \omega, \theta)$ sont reliées aux temps de séjours des (ω, θ) rayons réfléchissants dans Ω.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, n odd, be an open and connected domain with C^∞ boundary $\partial\Omega$ and bounded complement

$$K = \mathbf{R}^n \backslash \Omega \subset \{x \in \mathbf{R}^n : |x| \leq \rho_0\}.$$

Consider the problem

$$\left.\begin{aligned} (\partial_t^2 - \Delta_x) u = 0 \quad &\text{in } \mathbf{R} \times \Omega, \\ u = 0 \quad &\text{on } \mathbf{R} \times \partial\Omega, \\ u(0, x) = f_1(x),\ \partial_t u(0, x) &= f_2(x). \end{aligned}\right\} \tag{1}$$

We can associate to (1) a *scattering operator*

$$S(\lambda) : L^2(\mathbf{R}^{n-1}) \to L^2(\mathbf{R}^{n-1}), \qquad \lambda \in \mathbf{R},$$

whose kernel $a(\lambda, \theta, \omega)$, called *scattering amplitude,* depends analytically on $\omega, \theta \in S^{n-1}$ (*see* [LP]). For fixed $(\theta, \omega) \in S^{n-1} \times S^{n-1}$, $a(\lambda, \theta, \omega)$ is a tempered distribution in λ and

$$a(\lambda, \theta, \omega) = \left(\frac{2\pi}{i\lambda}\right)^{(n-1)/2} \mathcal{F}_{t\to\lambda}\, s(t, \theta, \omega).$$

Here $\mathcal{F}_{t\to\lambda}$ denotes the *Fourier transform* and the distribution $s(t, \theta, \omega)$ is called the *scattering kernel* (*see* [Ma], [P1]).

The operator $S(\lambda)$ and the distribution $a(\lambda, \theta, \omega)$ admit meromorphic continuation in $\mathbf{C}$ with poles λ_j, $\operatorname{Im} \lambda_j < 0$, which are independent of θ and ω. Notice that in [LP] the scattering poles are related to $\overline{a(\lambda, \theta, \omega)}$.

One can characterize the poles λ_j using the modified resolvent of the Laplacian in Ω given by

$$R_{\varphi, \psi}(\lambda) = \varphi(x) R(\lambda) \psi(x).$$

Here the operator

$$R(\lambda) : C_0^\infty(\bar{\Omega}) \ni f \mapsto u(x, \lambda) \in C^\infty(\bar{\Omega}), \qquad \operatorname{Im} \lambda \geq 0,$$

is determined by the $(-i\lambda)$-outgoing solution of the Dirichlet problem for the reduced wave equation (*see* Section 2). The functions $\varphi(x)$, $\psi(x) \in C_0^\infty(\mathbf{R}^n)$ are chosen to be equal to 1 in some neighbourhood of K. Then $R_{\varphi,\psi}(\lambda)$ admits a meromorphic continuation in $\mathbf{C}$ the poles of which and their multiplicities coincide with these of the λ_j's. Moreover, the poles λ_j do not depend on the choice of φ and ψ (*see* [LP], [V]).

It is well known that if K is non-trapping, there exist $\varepsilon > 0$ and $d > 0$ such that $S(\lambda)$ has no poles in the domain

$$U_{\varepsilon,d} = \{\lambda \in \mathbf{C} : d - \varepsilon \operatorname{Log}(1 + |\lambda|) \leq \operatorname{Im}\lambda \leq 0\}. \tag{2}$$

On the other hand, Vainberg [V] proved that for non-trapping obstacles the estimate

$$\| R_{\varphi,\psi}(\lambda) f \|_{H^1(\Omega)} \leq C e^{\alpha|\operatorname{Im}\lambda|} \| f \|_{L^2(\Omega)} \tag{3}$$

holds for each $\lambda \in U_{\varepsilon,d}$. Here the constants $C > 0$, $\alpha > 0$ depend on $\operatorname{supp}\varphi \cup \operatorname{supp}\psi$.

The situation changes when the obstacle is trapping. In this case one expects to have an infinite number of scattering poles in $U_{\varepsilon,d}$ for all $\varepsilon > 0$ and $d > 0$. Such a result is proved in [BGR] provided the obstacle is an union of two disjoint strictly convex bodies. In the same case more precise results concerning the distribution of poles are obtained by Ikawa [I1] and Gérard [G].

The case $K = \bigcup_{j=1}^{N} K_j$, $K_i \cap K_j = \emptyset$, $i \neq j$, K_j strictly convex, has been examined by Ikawa [I2], [I3], [I4] under the condition

$$\text{(H)} \quad \begin{cases} K_\nu \cap \text{convex hull}(K_i \cup K_j) = \emptyset \text{ for each triple} \\ (i, j, \nu) \in \{1, \ldots, N\}^3, \quad i \neq j,\ i \neq \nu,\ j \neq \nu. \end{cases}$$

This condition implies for instance non-existence of periodic rays in Ω with segments tangent to $\partial\Omega$. The case when the obstacles K_j are in generic position has been studied in [PS1].

To describe the results in this paper we need several definitions. Given an obstacle K, fix an open ball B_0 of radius ρ_0 containing K. For each $\xi \in S^{n-1}$ denote by Z_ξ the hyperplane tangent to B_0 and orthogonal to ξ such that the halfspace H_ξ, determined by Z_ξ and having ξ as an inner normal, contains K. Let $\omega \in S^{n-1}$, $\theta \in S^{n-1}$. An (ω, θ)-*ray* in $\bar{\Omega}$ is a curve of the form $\gamma = \operatorname{Im}\Gamma$, where $\Gamma : \mathbf{R} \to \bar{\Omega}$ is the natural projection on $\bar{\Omega}$ of a generalized bicharacteristic of the wave equation in $T^*(\bar{\Omega} \times \mathbf{R})$ (*cf.* [MS] or Section 24.3 in [H2]) such that there exist constants $a < b$ with $\Gamma'(t) = \omega$ for $t \leq a$ and $\Gamma'(t) = \theta$ for $t \geq b$. Geometrically, such a curve γ is the trajectory of a point incoming from infinity with direction ω, moving with constant velocity in Ω, reflecting at $\partial\Omega$ following the usual law of geometrical optics and outgoing to infinity with direction θ (*cf.* [PS3], Chapter 2). In general an $(\omega,\ \theta)$-ray γ may have segments lying entirely on $\partial\Omega$; these segments, called *gliding segments,* are in fact geodesics with respect to the standard metric on $\partial\Omega$. If γ does not contain gliding segments on $\partial\Omega$ and has only finitely many reflection points, it is called a *reflecting* $(\omega,\ \theta)$*-ray* in $\bar{\Omega}$. If moreover γ has no segments tangent to ∂K, then it is called an *ordinary* reflecting $(\omega,\ \theta)$-ray.

The *sojourn time* T_γ of an $(\omega,\ \theta)$-ray γ, introduced by Guillemin [Gu], is defined by $T_\gamma = T'_\gamma - 2\,\rho_0$, where T'_γ is the length of this part of γ which is contained in $H_\omega \cap H_{-\theta}$.

Let γ be an ordinary reflecting $(\omega,\ \theta)$-ray in $\bar{\Omega}$ with successive reflection points $x_1, \ldots, x_k$ on ∂K. Note that in this case $T_\gamma = \langle \omega,\ x_1\rangle + \sum_{i=1}^{k-1} \| x_i - x_{i+1} \| - \langle \theta,\ x_k\rangle$, where $\langle\ ,\ \rangle$ denotes the standard inner product in $\mathbf{R}^n$. (*See* [Gu] or Section 2.4 in [PS3].) Denote by u_γ the orthogonal projection of x_1 on $Z = Z_\omega$. Then there exists a neighbourhood $W = W_\gamma$ of u_γ in Z such that for every $u \in W$ there are unique $\theta\,(u) \in S^{n-1}$ and points $x_1\,(u), \ldots, x_k\,(u) \in \partial K$ which are the successive reflection points of a reflecting $(\omega, \theta\,(u))$-ray in $\bar{\Omega}$ passing through u. Setting $J_\gamma\,(u) = \theta\,(u)$, one obtains a smooth map $J_\gamma : W_\gamma \longrightarrow S^{n-1}$. The ray γ is called *non-degenerate* if $\det dJ_\gamma\,(u_\gamma) \neq 0$.

The main purpose of this paper is to examine when a trapping obstacle has the following property:

(S) *There exists a sequence of directions* $(\omega_m,\ \theta_m)$ *and a sequence of ordinary reflecting* (ω_m, θ_m)*-rays* γ_m *with sojourn times* $T_m \longrightarrow \infty$ *such that*

$$-T_m \in \operatorname{sing\,supp} s\,(t,\ \theta_m,\ \omega_m). \tag{4}$$

In the case when (S) holds for a trapping obstacle K we prove that either there exist poles in $U_{\varepsilon,\,d}$ for all $\varepsilon > 0$, $d > 0$, or there exist $\varepsilon > 0$, $d > 0$ for which the domain $U_{\varepsilon,\,d}$ is free of poles but the estimate

$$\| R_{\varphi,\,\psi}\,(\lambda) f \|_{H^1\,(\Omega)} \leq C\, e^{\alpha\,|\operatorname{Im}\lambda\,|}\,(1+|\,\lambda\,|)^p\, \|\, f\, \|_{H^k\,(\Omega)}, \qquad \lambda \in U_{\varepsilon,\,d} \tag{5}$$

fails for every choice of $C > 0$, $\alpha \geq 0$, $p \in \mathbf{N}$ and $k \in \mathbf{N}$. From physical point of view it is quite natural to conjecture that (S) holds for every trapping obstacle. However, from mathematical point of view the analysis of (S) leads to many difficulties. First, for trapping obstacles the existence of a sequence $\{\gamma_m\}$ of reflecting $(\omega_m,\ \theta_m)$-rays with $T_{\gamma_m} \longrightarrow \infty$ follows from the continuity of the generalized Hamiltonian flow related to the wave operator $\Box = \partial_t^2 - \Delta_x$ (*see* Section 5). The crucial point is to prove that these rays produce singularities leading to (4). According to the results in Chapters 8 and 9 in [PS3] and [CPS], to obtain (4) it is sufficient that for all m the pair (ω_m, θ_m) has the following three properties:

(*a*) if δ and γ are different ordinary reflecting (ω_m, θ_m)-rays, then $T_\delta \neq T_\gamma$;

(*b*) for every m, γ_m is non-degenerate;

(*c*) there are no (ω_m, θ_m)-rays in $\bar{\Omega}$ containing gliding segments on $\partial\Omega$ or tangent segments to $\partial\Omega$.

To use approximation by suitable directions, we wish to prove that (*a*)-(*c*) hold for a dense set of directions $(\omega, \theta) \in S^{n-1} \times S^{n-1}$. In Section 4 we show that given an arbitrary obstacle K, for almost all $(\omega, \theta) \in S^{n-1} \times S^{n-1}$ (with respect to the Lebesgue measure in $S^{n-1} \times S^{n-1}$) every two different ordinary reflecting (ω, θ)-rays have distinct sojourn times. To obtain an analogue of this for (*c*), one must study all (ω, θ)-rays in $\bar{\Omega}$, *i.e.* all projections of generalized (ω, θ)-bicharacteristics of the wave operator. In general this is a complicated problem, especially when the set $G^\infty(K)$ of points $z \in \partial K$, where the curvature of K along some direction $\xi \in T_z(\partial K)$ vanishes of infinite order, is not empty. In Section 3 we prove that for each obstacle K for almost all $(\omega, \theta) \in S^{n-1} \times S^{n-1}$ all reflecting (ω, θ)-rays in $\bar{\Omega}$ are ordinary. This result can be applied in the case when the obstacle has the form

$$K = \bigcup_{j=1}^{N} K_j, \quad K_i \cap K_j = \emptyset \quad \text{for } i \neq j, \qquad K_j \text{ convex} \quad \text{for all } j = 1, \ldots, N, \tag{6}$$

to show that (*c*) is satisfied for almost all $(\omega, \theta) \in S^{n-1} \times S^{n-1}$ even if $G^\infty(K) \neq \emptyset$. Finally, by using Sard's theorem, one can arrange (*b*) in the same way as (*a*) and (*c*). The results in Sections 2-5 are established for obstacles with arbitrary geometry and they have an independent interest. As a consequence we get the following.

1.1. THEOREM. – *Let K have the form* (6). *Assume that there are no points $z \in \partial K$ such that the Gauss curvature $\mathcal{K}(u)$ of ∂K at u vanishes for each u in some neighbourhood U_z of z in ∂K. Then there exists a sequence of ordinary reflecting (ω_m, θ_m)-rays in $\bar{\Omega}$ with sojourn times $T_m \to \infty$ such that for t near $-T_m$ we have*

$$s(t, \theta_m, \omega_m) = A_m\, \delta^{(n-1)/2}(t + T_m) + \text{lower order singularities},$$

with $A_m \neq 0$. Moreover, either the assertion (i) *or the assertion* (ii) *in Theorem* 2.3 *in Section* 2 *holds.*

1.2. THEOREM. – *Let K have the form* (6). *Then there exists a set $\mathcal{R} \subset S^{n-1} \times S^{n-1}$, the complement of which has zero Lebesgue measure in $S^{n-1} \times S^{n-1}$, such that for all $(\omega, \theta) \in \mathcal{R}$ we have*

$$\text{sing supp } s(t, \theta, \omega) = \{-T_\gamma : \gamma \in \mathcal{L}_{\omega, \theta}\},$$

where $\mathcal{L}_{\omega, \theta}$ is the set of all (ω, θ)-rays in $\bar{\Omega}$. Moreover, for t near $-T_\gamma$ we have

$$s(t, \theta, \omega) = c_\gamma \, \delta^{(n-1)/2}(t + T_\gamma) + \text{lower order singularities}$$

with $c_\gamma \neq 0$.

For $n = 3$ the assumption of Theorem 1.1 means that there are no points $z \in \partial K$ such that the standard metric on ∂K is locally flat around z. For strictly convex obstacles the assertion of Theorem 1.2 was proved in [PS2].

Theorem 1.2 has been used by one of the authors [St] to prove an inverse scattering result related to singularities of $s(t, \theta, \omega)$. Namely, it is shown in [St] that if K and L are two obstacles, each of them being a disjoint union of finitely many compact convex bodies satisfying the condition (H), and if the equality

$$\text{sing supp } s_K(t, \theta, \omega) = \text{sing supp } s_L(t, \theta, \omega)$$

holds for almost all $(\omega, \theta) \in S^{n-1} \times S^{n-1}$, then $K = L$. Let us notice that this result does not depend on the explicite form of the singularities of the corresponding scattering kernels. For convex K and L it follows from the results of Majda [Ma].

We close this introduction by a brief discussion on the singularities produced by periodic rays. In the works of Ikawa ([I2], [I3]) and Sjöstrand and Zworski ([SjZ1], [SjZ2]) the singularities of the distribution

$$u(t) = \begin{cases} \sum_j e^{i\lambda_j t}, & t > 0, \\ \sum_j e^{-i\lambda_j t}, & t < 0, \end{cases}$$

have been exploited. Here the sumation is over all poles λ_j including their multiplicities. It is well-known that sing supp $u(t)$ is contained in the set of the periods (lengths) of all periodic rays in $\bar{\Omega}$ (*see* [PS3], Chapter 5). Thus, to prove that a period T_γ belongs to sing supp $u(t)$, it is sufficient to establish some properties (a')-(c') similar to (a)-(c). The reader may consult [PS3], Chapter 7 for results in this direction. Notice that in general, even for several strictly convex obstacles, there might be different periodic rays with rationally dependent periods or periodic reflecting rays with segments tangent to the boundary. Thus, the singularities produced by periodic rays could be canceled. One must also take care about singularities related to periodic rays having tangent or gliding segments. So, for such obstacles it seems difficult to prove that there exists a sequence $d_j \to \infty$ of singularities of $u(t)$. The condition (H) in [I3], [I4] has been introduced in order to simplify the picture of the periodic rays and, in particular, to avoid the existence of periodic rays with tangent segments to the boundary.

The novelty in this paper is that we study trapping (ω, θ)-rays with suitable directions ω, θ instead of periodic rays. This makes it possible to examine the general case of several convex obstacles and to establish the property (S). One could expect that (S) holds for every trapping obstacle K with $G^\infty(K) = \emptyset$ and the conclusions of Theorem 2.3 remain valid for such type of trapping obstacles. The results in Sections 3-5 make a progress toward the analysis of the general case.

It is an open problem if the existence of a sequence $T_m \to \infty$ of singularities always implies the assertion (i) of Theorem 2.3. From general point of view there might exist some very degenerate examples of trapping obstacles for which there are logarithmic domains free of poles and the assertion (ii) of Theorem 2.3 holds.

2. THE BEHAVIOR OF THE RESOLVENT OF THE LAPLACIAN

Consider the problem

$$\left.\begin{array}{c}(\Delta + \lambda^2)\, u(x, \lambda) = f \quad \text{in } \Omega \\ u(x, \lambda) = 0 \quad \text{on } \partial\Omega \\ u(x, \lambda) \text{ is } (-i\lambda) - \text{outgoing.}\end{array}\right\} \tag{7}$$

The last condition means that for $x = |x|\omega$, $\omega \in S^{n-1}$, and $|x| \to \infty$ we have

$$u(|x|\omega, \lambda) = \frac{e^{i\lambda|x|}}{|x|^{(n-1)/2}}\left(b(\omega, \lambda) + O\left(\frac{1}{|x|}\right)\right).$$

For $\operatorname{Im}\lambda \geq 0$ the problem (7) has an unique solution for $f \in L^2(\Omega)$ and the operator

$$R(\lambda) : C_0^\infty(\bar{\Omega}) \ni f \mapsto u(x, \lambda) \in C^\infty(\bar{\Omega}), \qquad \operatorname{Im}\lambda \geq 0,$$

admits a meromorphic extension in $\mathbf{C}$ with poles λ_j, $\operatorname{Im}\lambda_j < 0$ (*see* [LP]).

Let $\hat{s}(\lambda, \theta, \omega) = \mathcal{F}_{t\to\lambda}\, s(t, \theta, \omega)$ be the *Fourier transform* of the scattering kernel $s(t, \theta, \omega)$. It is well known ([LP], [P1], [Ma]) that

$\hat{s}(\lambda, \theta, \omega)$ has the representation

$$\hat{s}(\lambda, \theta, \omega) = c_n \lambda^{n-2} \int_{\partial\Omega} e^{-i\lambda\langle x, \theta\rangle} \left(\frac{\partial v}{\partial \nu} + i\lambda \langle \nu, \theta\rangle v\right)(x, \lambda)\, dS_x,$$

where $c_n = \text{const.}$, $\nu(x)$ is the *unit normal* to $\partial\Omega$ at $x \in \partial\Omega$ pointing into Ω, and $v(x, \lambda)$ is the solution of the problem

$$\left.\begin{array}{c}(\Delta + \lambda^2)\, u(x, \lambda) = 0 \quad \text{in}\, \Omega \\ v + e^{i\lambda\langle x, \omega\rangle} = 0 \quad \text{on}\, \partial\Omega \\ v \text{ is } (-i\lambda) - \text{outgoing.}\end{array}\right\}$$

We shall express $\hat{s}(\lambda, \theta, \omega)$ by using the operator $R(\lambda)$. Let $a > \rho_0$. Consider a function $\varphi_a \in C_0^\infty(\mathbf{R}^n)$ such that $\varphi_a(x) = 1$ for $|x| \le a$. Setting $F_a(\lambda) = [\Delta\varphi_a + 2i\lambda\langle\nabla\varphi_a, \omega\rangle]\, e^{i\lambda\langle x, \omega\rangle}$, we get

$$v(x, \lambda) + \varphi_a(x)\, e^{i\lambda\langle x, \omega\rangle} = R(\lambda)\,[(\Delta + \lambda^2)(\varphi_a\, e^{i\lambda\langle x, \omega\rangle})] = R(\lambda)\, F_a(\lambda).$$

Next, choose a function $\chi_b(x) \in C_0^\infty(\mathbf{R}^n)$ such that $\chi_b(x) = 1$ on a neighbourhood of K and $\varphi_a(x) = 1$ on $\operatorname{supp}\chi_b$. Then the normal derivative becomes

$$\left.\frac{\partial v}{\partial \nu}\right|_{\partial\Omega} = -i\lambda\, e^{i\lambda\langle x, \omega\rangle}\langle \nu, \omega\rangle_{|\partial\Omega} + \frac{\partial}{\partial\nu}(\chi_b\, R(\lambda)\, F_a(\lambda))_{|\partial\Omega}.$$

On the other hand, for $\theta \ne -\omega$ by Green's formula we have

$$\int_{\partial\Omega} e^{-i\lambda\langle x, \theta-\omega\rangle}\langle \nu, \theta + \omega\rangle\, dS_x = \int_K \frac{\partial}{\partial(\theta + \omega)}(e^{-i\lambda\langle x, \theta-\omega\rangle})\, dx = 0.$$

Thus, using Green's formula once more, we obtain

$$\begin{aligned}\hat{s}(\lambda, \theta, \omega) &= -c_n \lambda^{n-2} \int_\Omega e^{-i\lambda\langle x, \theta\rangle}(\Delta + \lambda^2)(\chi_b\, R(\lambda)\, F_a)\, dx \\ &= -c_n \lambda^{n-2} \int_\Omega e^{-i\lambda\langle x, \theta\rangle}\, [(\Delta\chi_b)\, R(\lambda)\, F_a(\lambda) \\ &\qquad + 2\langle \nabla_x \chi_b, \nabla_x (R(\lambda)\, F_a(\lambda))\rangle]\, dx. \end{aligned} \tag{8}$$

Let $\psi_c(x) \in C_0^\infty(\mathbf{R}^n)$, $c = a, b$, be cut-off functions such that $\psi_a(x) = 1$ on $\operatorname{supp}\varphi_a$, $\psi_b(x) = 1$ on $\operatorname{supp}\chi_b$. Then in the representation of $\hat{s}(\lambda, \theta, \omega)$ we can replace the resolvent $R(\lambda)$ by the *modified resolvent* $R_{a,b}(\lambda) = \psi_b(x)\, R(\lambda)\, \psi_a(x)$. Below we assume that ψ_a and ψ_b are fixed.

Next, consider the domain $U_{\varepsilon, d}$ defined in the introduction and assume that $R_{a, b}(\lambda)$ satisfies the condition

$$(\mathrm{B}) \quad \begin{cases} R_{a, b}(\lambda) \text{ is analytic in } U_{\varepsilon, d} \\ \text{and there exist } C > 0,\ \alpha \geq 0,\ p \in \mathbf{N},\ k \in \mathbf{N} \\ \text{such that } \| R_{a, b}(\lambda) \varphi \|_{H^1(\Omega)} \leq C (1 + |\lambda|)^p e^{\alpha |\operatorname{Im} \lambda|} \| \varphi \|_{H^k(\Omega)} \\ \text{for each } \varphi \in C_0^\infty(\Omega) \text{ and each } \lambda \in U_{\varepsilon, d}. \end{cases}$$

Since the integration in (8) is over a compact set, we conclude that the condition (B) implies the estimate

$$|\hat{s}(\lambda, \theta, \omega)| \leq C_1 (1 + |\lambda|)^m e^{\beta |\operatorname{Im} \lambda|}, \qquad \lambda \in U_{\varepsilon, d}, \tag{9}$$

with $\beta \geq 0$, $m \in \mathbf{N}$, uniformly with respect to $(\theta, \omega) \in S^{n-1} \times S^{n-1}$.

2.1 REMARK. – The analysis in Chapter III of [LP] shows that the norm of the *scattering operator*

$$S(\lambda) : L^2(S^{n-1}) \to L^2(S^{n-1}), \qquad \lambda \in U_{\varepsilon, d},$$

is less than the norm of the modified resolvent $\varphi(x) R(\lambda) \psi(x)$, where $\varphi(x),\ \psi(x) \in C_0^\infty(\bar{\Omega})$ and $\varphi(x) = 1$, $\psi(x) = 1$ in a neighbourhood of K. The inequality (9) is sharper since it gives an estimate for the kernel of the scattering operator $S(\lambda)$.

Now we shall deduce from (9) some regularity of $s(t, \theta, \omega)$ for $t \to -\infty$. To do this we need the following lemma, which is similar to Theorem 7.3.8 in [H1].

2.2. LEMMA. – *Let $u \in \mathcal{S}'(\mathbf{R})$ be a distribution with* $\operatorname{supp} u \subset \{t : t \leq \tau\}$. *Assume that the Fourier transform $\hat{u}(\xi)$ admits an analytic continuation in the domain $U_{\varepsilon, d}$ such that for all $\zeta \in U_{\varepsilon, d}$ we have*

$$|\hat{u}(\zeta)| \leq C (1 + |\zeta|)^N e^{\alpha |Im \zeta|}, \qquad \alpha \geq 0. \tag{10}$$

Then for each $q \in \mathbf{N}$ there exist $t_q < \tau$ and $v_q \in C^q(\mathbf{R})$ such that $u = v_q$ for $t \leq t_q$.

Proof. – Choose a function $\varphi \in C_0^\infty(\mathbf{R})$ such that $\operatorname{supp} \varphi \subset (-1, 1)$ and $\int_{\mathbf{R}} \varphi(t)\, dt = 1$. Set $\varphi_\delta(t) = \frac{1}{\delta} \varphi\left(\frac{t}{\delta}\right)$, $0 < \delta \leq 1$, and consider $u \star \varphi_\delta$. Consider the path

$$\Gamma_\varepsilon : \mathbf{R} \backslash [-\gamma, \gamma] \ni \xi \mapsto \zeta = \xi + i(d - \varepsilon \operatorname{Log}(1 + |\xi|) \in U_{\varepsilon, d},$$

where $\gamma = \exp\left(\frac{d}{\varepsilon}\right) - 1$ and d is given in the definition of $U_{\varepsilon, d}$. Clearly, (10) implies

$$|\hat{u}(\zeta) \hat{\varphi}(\delta\zeta)| \leq C_M (1 + |\zeta|)^N (1 + |\delta\zeta|)^{-M} e^{(a+\delta) |\operatorname{Im} \zeta|}, \quad \zeta \in U_{\varepsilon, d}.$$

Using the analyticity of $\hat{u}(\zeta)$ in $U_{\varepsilon, d}$ and the above estimate for fixed $\delta > 0$, we can write the integral

$$(u \star \varphi_\delta)(t) = (2\pi)^{-1} \int_{\mathbf{R}} e^{it\xi} \hat{u}(\xi) \hat{\varphi}(\delta\xi) d\xi$$

as a sum of two integrals

$$(2\pi)^{-1} \int_{\Gamma_\varepsilon} e^{it\zeta} \hat{u}(\zeta) \hat{\varphi}(\delta\zeta) d\zeta + (2\pi)^{-1} \int_{|\xi| \leq \gamma} e^{it\zeta} \hat{u}(\xi) \hat{\varphi}(\delta\xi) d\xi.$$

The second integral is over a compact interval, therefore passing to a limit as $\delta \to 0$, we obtain a C^∞ function.

Next, for $\zeta \in \Gamma_\varepsilon$, $0 < \delta \leq 1$, we have the estimate

$$\begin{aligned} |\hat{u}(\zeta) \hat{\varphi}(\delta\zeta) e^{it\zeta}| &\leq C (1 + |\zeta|)^N e^{-(a+1+t) \operatorname{Im}\zeta} \\ &\leq C' (1 + |\operatorname{Re}\zeta|)^{N+\varepsilon(a+1+t)}, \end{aligned}$$

provided $a+1+t < 0$. Given $q \in \mathbf{N}$, take $t_q = \frac{1}{\varepsilon}(-N-n-1-q-\varepsilon(a+1))$. Then $t \leq t_q$ implies $\varepsilon(a+1+t) \leq -N-n-1-q$ and, since $d\zeta = F(\xi) d\xi \to d\xi$ as $|\xi| \to \infty$, for $t \leq t_q$, the integral on Γ_ε is uniformly convergent for $0 \leq \delta \leq 1$. The same is true if we take the derivatives with respect to t up to order q. Letting $\delta \to 0$ and using the fact that $\hat{\varphi}(\delta\zeta) \to 1$, by Lebesgue theorem we conclude that for $t \leq t_q$ we have $u \star \varphi_\delta \to_{\delta \to 0} f$, f being a C^q function. This completes the proof of the lemma.

Combining the estimate (9) and Lemma 2.2, we obtain the following.

2.3. THEOREM. – *Assume that there exists a sequence of ordinary reflecting (ω_m, θ_m)-rays in Ω with sojourn times T_m such that*

$$T_m \to_{m \to \infty} \infty.$$

Let $\Phi \in C_0^\infty(\mathbf{R})$ be such that $\operatorname{supp}\Phi \subset (-1, 1)$, $\Phi(t) = 1$ *for* $|t| \leq \frac{1}{2}$. *Assume that there exists a sequence $\gamma_m \to 0$ of non-zero real numbers and an integer k independent of m such that*

$$\begin{aligned} &\left| \mathcal{F}_{t \to \lambda} \left[\Phi\left(\frac{t + T_m}{\gamma_m}\right) s(t, \theta_m, \omega_m) \right] \right| \\ &\qquad \geq (c_m - o_m(1)) |\lambda|^k, \qquad |\lambda| \to \infty, \end{aligned}$$

where $c_m > 0$. Then there are two possibilities:

(i) *For each $\varepsilon > 0$ and each $d > 0$, $R_{a,b}(\lambda)$ and $S(\lambda)$ have poles in the domain $U_{\varepsilon, d}$;*

(ii) *For some* $\varepsilon > 0$, $d > 0$, $R_{a,b}(\lambda)$ *is analytic in* $U_{\varepsilon,d}$ *but for all* $\alpha \geq 0$, $p \in \mathbf{N}$, $k \in \mathbf{N}$ *we have*

$$\sup_{\lambda \in U_{\varepsilon,d},\, \|\varphi\|_{H^k(\Omega)}=1} (1 + |\lambda|)^{-p} e^{-\alpha |\operatorname{Im} \lambda|} \| R_{a,b}(\lambda) \varphi \|_{H^1(\Omega)} = +\infty.$$

2.4. REMARK. – Notice that if $R_{a,b}(\lambda)$ is analytic in $U_{\varepsilon,d}$, the same is true for $\hat{s}(\lambda, \theta, \omega)$ for all $\omega, \theta \in S^{n-1}$, and this implies the analyticity of the operator $S(\lambda)$ in $U_{\varepsilon,d}$. This shows that the cases (i) and (ii) do not depend on the choice of $\psi_a(x)$ and $\psi_b(x)$.

3. DIRECTIONS OF TANGENCY

Let X be a compact smooth $(n-1)$-dimensional submanifold of $\mathbf{R}^n$, $n \geq 2$. In this section we consider pairs $(\omega, \theta) \in S^{n-1} \times S^{n-1}$ for which there exists at least one (ω, θ)-ray for X which is not ordinary, *i.e.* it is tangent to X at some of its points. It is shown that the set of these pairs has Lebesgue measure zero in $S^{n-1} \times S^{n-1}$.

Before proceeding with the statement of the main result in this section, we introduce a notion that is slightly different from the notion of an (ω, θ)-ray but it is rather convenient for our next considerations. It was also used in our paper [CPS].

A curve γ in $\mathbf{R}^n$ is called an *(ω, θ)-trajectory for X* if it has the form $\gamma = \bigcup_{i=0}^{s} l_i$, where $l_i = [x_i, x_{i+1}]$, $i = 1, \ldots, s-1$, $x_i \in X$ for all $i = 1, \ldots, s$, while l_0 (resp. l_s) is the infinite ray starting at x_1 (resp. x_s) with direction $-\omega$ (resp. θ), and for every $i = 0, 1, \ldots, s-1$, l_i and l_{i+1} satisfy the law of reflection at x_i with respect to X.

Clearly, every reflecting (ω, θ)-ray is an (ω, θ)-trajectory, but the converse is not true in general, since an (ω, θ)-trajectory may intersect transversally X.

3.1. THEOREM. – *There exists $\mathcal{R} \subset S^{n-1} \times S^{n-1}$ the complement of which is a countable union of compact subsets of measure zero in $S^{n-1} \times S^{n-1}$ such that for every pair $(\omega, \theta) \in \mathcal{R}$ all (ω, θ)-trajectories for X are ordinary.*

Fix a hyperplane Z in $\mathbf{R}^n$ such that X is contained in one of the open halfspaces determined by Z.

As a simple corollary of our argument in this section we also get the following theorem, which is in fact a consequence of a result of Melrose and Sjöstrand [MS], *see* also Chapter 24 in [H2].

3.2. THEOREM. – *There exists $\mathcal{T} \subset Z \times S^{n-1}$, the complement of which is a countable union of compact subsets of measure zero in $Z \times S^{n-1}$, such that for every $(x, \omega) \in \mathcal{T}$ the trajectory of the generalized geodesic flow (in the exterior domain determined by X) starting at x in direction ω has no tangencies to X.*

We now turn to the proofs of the above statements.

Fix two integers k and s with $s \geq 1$ and $0 \leq k \leq s$. Denote by $M(s, k)$ the set of those

$$\tilde{\xi} = (\omega;\, x;\, y;\, \theta) \in M_s = S^{n-1} \times X^{(s)} \times X \times S^{n-1},$$

with $x = (x_1, \ldots, x_s)$, such that there exists an (ω, θ)-trajectory for X with successive (transversal) reflection points $x_1, \ldots, x_s$, the segment $[x_k, x_{k+1}]$ of which is tangent to X at the point $y \in (x_k, x_{k+1})$. Here by x_0 (resp. x_{s+1}) we denote the orthogonal projection of x_1 on Z_ω (resp. of x_s on $Z_{-\theta}$), and by definition

$$X^{(s)} = \{(x_1, \ldots, x_s) \in X^s : x_i \neq x_j,\, i \neq j\}.$$

We are going to show that $M(s, k)$ is a smooth submanifold of M_s of dimension $2n-3$. This and Sard's theorem would then imply that for every smooth map $f : M(s, k) \to N$, N being a smooth manifold of dimension at least $2n-2$, the image $f(M(s, k))$ has measure zero in N. Setting $\lambda(\tilde{\xi}) = (y, \eta)$ for $\tilde{\xi} = (\omega;\, x;\, y;\, \theta) \in M_s$, where $\eta = \dfrac{x_{k+1} - x_k}{\| x_{k+1} - x_k \|}$, one gets a smooth map $\lambda : M_s \to X \times S^{n-1}$. Clearly, $N(s, k) = \lambda(M(s, k))$ is contained in the sphere bundle SX of X which is a smooth manifold of dimension $2n-3$. So, roughly saying, λ provides a parametrization of $M(s, k)$ by a subset of a $(2n-3)$-dimensional manifold. The restriction of λ to $M(s, k)$ defines a homeomorphism $\lambda : M(s, k) \to N(s, k)$, so one can at least conclude that the topological dimension of $M(s, k)$ is not greater than $2n-3$. However, due to the singularities of the generalized Hamiltonian flow (*see* [MS] or Section 24.3 in [H2]), the inverse map $\lambda^{-1} : N(s, k) \to M(s, k)$ has rather weak regularity properties. In general it is not locally Lipschitz and it is even not known whether it is Hölder or not. That is why the information provided by the parametrization λ is not enough for our purposes. It is necessary to consider more complicated ways to parametrize the set $M(s, k)$ and one of them is described in the proof of Lemma 3.3 below.

3.3. LEMMA. – *$M(s, k)$ is a smooth submanifold of M_s of dimension $2n-3$.*

Proof. – Clearly the sets

$$U_r(s,\,k) = \{(\omega;\, x;\, y;\, \theta) \in M_s : x_k^{(r)} \neq x_{k+1}^{(r)}\}, \qquad r = 1,\, \dots,\, n$$

are open in M_s and $M_s = \bigcup_{r=1}^{n} U_r(s,\,k)$. Thus, the lemma will be proved if we show that for each $r = 1,\, \dots,\, n$, $M_r(s,\,k) = M(s,\,k) \cap U_r(s,\,k)$ is a smooth submanifold of M_s of dimension $2n-3$. We consider only the case $r = n$, the other cases are the same.

Let us biefly explain the idea of the proof. Given an element η of $M_n(s,\,k)$, we consider a certain chart

$$\chi: \quad U \to D \subset M_s$$

with $\eta \in D$, where U is an open subset of $\mathbf{R}^{(s+3)(n-1)}$. Now it is sufficient to show that $\chi^{-1}(D \cap M_n(s,\,k))$ is a smooth submanifold of U of dimension $2n-3$. To do this we define a smooth map

$$G: \quad U \to (\mathbf{R}^{n-1})^{s-2} \times \mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \mathbf{R}^{n-1} \times \mathbf{R}$$

such that $\chi^{-1}(D \cap M_n(s,\,k)) = G^{-1}(0)$. It turns out that G is submersion at any point of $G^{-1}(0)$, and so (*cf.* [GGu]) it follows that $G^{-1}(0)$ is a smooth submanifold of U with

$$\dim G^{-1}(0) = (s+3)(n-1) - [(s+1)(n-1)+1] = 2n-3.$$

Now we pass to the detailed proof. Fix an arbitrary $\eta = (\tilde{\omega};\, \tilde{x};\, \tilde{y};\, \tilde{\theta}) \in M_n(s,\,k)$. Choose smooth charts $\varphi_i : U_i \to X$ of X around $\tilde{x}_i$ and $\psi : V \to X$ of X around $\tilde{y}$ such that $\varphi_i(U_i) \cap \varphi_{i+1}(U_{i+1}) = \emptyset$, $i = 1,\, \dots,\, s-1$, $\varphi_k(U_k) \cap \psi(V) = \emptyset$, $\varphi_{k+1}(U_{k+1}) \cap \psi(V) = \emptyset$ (for $k = 0$ or s the corresponding condition is to be deleted). Assuming $\tilde{\omega}_n > 0$, we may parametrize S^{n-1} around $\tilde{\omega}$ by

$$D_0 \ni \omega' = (\omega_1,\, \dots,\, \omega_{n-1}) \mapsto \omega = (\omega';\, \omega_n) \in S^{n-1},$$

where $\omega_n = (1 - \omega_1^2 - \dots - \omega_{n-1}^2)^{1/2}$ and D_0 is the unit open ball in $\mathbf{R}^{n-1}$. Similarly, we may assume that S^{n-1} is parametrized around $\tilde{\theta}$ by $\theta'' = (\theta_2,\, \dots,\, \theta_n) \in D_0$. In this way we get a chart

$$\chi: \quad U = D_0 \times U_1 \times \dots \times U_s \times V \times D_0 \to D \subset M_s,$$

defined by $\chi(\xi) = (\omega;\, \varphi_1(u_1),\, \dots,\, \varphi_s(u_s);\, \psi(v);\, \theta)$ for $\xi = (\omega';\, u;\, v;\, \theta'') \in U$. Here $\omega = (\omega';\, \omega_n)$, $\theta = (\theta_1;\, \theta'')$, $u_i = (u_i^{(1)},\, \dots,\, u_i^{(n-1)}) \in U_i$.

Define $F : U \to \mathbf{R}$ by $F(\xi) = \sum_{i=1}^{s-1} \| \varphi_i(u_i) - \varphi_{i+1}(u_{i+1}) \|$.

Assume that $0 < k < s$. Given $\xi = (\omega'; u; v; \theta'') \in U$ with $\chi(\xi) \in M_n(s, k)$, we have

$$\operatorname{grad}_{u_i} F(\xi) = 0, \qquad i = 2, \ldots, s-1,$$

$$\left\langle \frac{\varphi_2(u_2) - \varphi_1(u_1)}{\|\varphi_2(u_2) - \varphi_1(u_1)\|} - \omega, \frac{\partial \varphi_1}{\partial u_1^{(j)}}(u_1) \right\rangle = 0, \qquad j = 1, \ldots, n-1,$$

$$\left\langle \frac{\varphi_s(u_s) - \varphi_{s-1}(u_{s-1})}{\|\varphi_s(u_s) - \varphi_{s-1}(u_{s-1})\|} - \theta, \frac{\partial \varphi_s}{\partial u_s^{(j)}}(u_s) \right\rangle = 0, \qquad j = 1, \ldots, n-1,$$

$$\frac{\psi(v) - \varphi_k(u_k)}{\|\psi(v) - \varphi_k(u_k)\|} + \frac{\psi(v) - \varphi_{k+1}(u_{k+1})}{\|\psi(v) - \varphi_{k+1}(u_{k+1})\|} = 0,$$

$$\langle \varphi_{k+1}(u_{k+1}) - \varphi_k(u_k), N(\xi) \rangle = 0,$$

where

$$N(\xi) = \det \begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ \dfrac{\partial \psi^{(1)}}{\partial v^{(1)}}(v) & \dfrac{\partial \psi^{(2)}}{\partial v^{(1)}}(v) & \cdots & \dfrac{\partial \psi^{(n)}}{\partial v^{(1)}}(v) \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial \psi^{(1)}}{\partial v^{(n-1)}}(v) & \dfrac{\partial \psi^{(2)}}{\partial v^{(n-1)}}(v) & \cdots & \dfrac{\partial \psi^{(n)}}{\partial v^{(n-1)}} \end{pmatrix}$$

is a normal vector to X at $\psi(v)$. Here $f_1, \ldots, f_n$ are the standard basis vectors in $\mathbf{R}^n$. In correspondence with these conditions we introduce the functions

$$K_i^{(j)}(\xi) = \frac{\partial F}{\partial u_i^{(j)}}(\xi), \qquad i = 2, \ldots, s-1,\ j = 1, \ldots, n-1,$$

$$L_j(\xi) = \left\langle \frac{\varphi_2(u_2) - \varphi_1(u_1)}{\|\varphi_2(u_2) - \varphi_1(u_1)\|} - \omega, \frac{\partial \varphi_1}{\partial u_1^{(j)}}(u_1) \right\rangle, \quad j = 1, \ldots, n-1,$$

$$M_j(\xi) = \left\langle \frac{\varphi_s(u_s) - \varphi_{s-1}(u_{s-1})}{\|\varphi_s(u_s) - \varphi_{s-1}(u_{s-1})\|} - \theta, \frac{\partial \varphi_s}{\partial u_s^{(j)}}(u_s) \right\rangle, \quad j = 1, \ldots, n-1,$$

$$P_j(\xi) = \frac{\psi^{(j)}(v) - \varphi_k^{(j)}(u_k)}{\|\psi(v) - \varphi_k(u_k)\|} + \frac{\psi^{(j)}(v) - \varphi_{k+1}^{(j)}(u_{k+1})}{\|\psi(v) - \varphi_{k+1}(u_{k+1})\|}, \quad j = 1, \ldots, n-1,$$

$$Q(\xi) = \langle \varphi_{k+1}(u_{k+1}) - \varphi_k(u_k), N(\xi) \rangle.$$

Finally, define the map G by

$$G(\xi) = ((K_i^{(j)}(\xi))_{2\le i\le s-1}^{1\le j\le n-1};\ (L_j(\xi))_{1\le j\le n-1};$$
$$(M_j(\xi))_{1\le j\le n-1};\ (P_j(\xi))_{1\le j\le n-1};\ Q(\xi)).$$

Then G is smooth and, according to the above, we have

$$\chi^{-1}(D \cap M_n(s,\,k)) = G^{-1}(0).$$

Thus, to prove the lemma we have to establish that $G^{-1}(0)$ is a smooth submanifold of U of dimension $2n-3$. To do this it is sufficient to show that G is submersion at any point of $G^{-1}(0)$. Indeed, if this is true, then $G^{-1}(0)$ is a smooth submanifold with

$$\dim G^{-1}(0) = (s+3)(n-1) - [(s+1)(n-1)+1] = 2n-3.$$

Fix $\xi \in G^{-1}(0)$ and assume that

$$\sum_{i=2}^{s-1}\sum_{j=1}^{n-1} A_i^{(j)} \operatorname{grad} K_i^{(j)}(\xi) + \sum_{j=1}^{n-1} B_j \operatorname{grad} L_j(\xi) + \sum_{j=1}^{n-1} C_j \operatorname{grad} M_j(\xi)$$
$$+ \sum_{j=1}^{n-1} p_j \operatorname{grad} P_j(\xi)$$
$$+ q \operatorname{grad} Q(\xi) = 0 \tag{11}$$

for some real coefficients $A_i^{(j)}$, B_j, C_j, p_j, q. We have to show that these constants are zero.

First, consider in (11) the derivatives with respect to $\omega_1, \ldots, \omega_{n-1}$. According to $\omega_n = (1-\omega_1^2 - \ldots - \omega_{n-1}^2)^{1/2}$, we get

$$\sum_{j=1}^{n-1} B_j \left(- \frac{\partial \varphi_1^{(r)}}{\partial u_1^{(j)}}(u_1) + \frac{\omega_r}{\omega_n} \cdot \frac{\partial \varphi_1^{(n)}}{\partial u_1^{(j)}}(u_1) \right) = 0 \tag{12}$$

for $r = 1, \ldots, n-1$. Note that (12) holds also for $r = n$. Setting

$$c = \frac{1}{\omega_n} \sum_{j=1}^{n-1} B_j \frac{\partial \varphi_1^{(n)}}{\partial u_1^{(j)}}(u_1),$$

(12) implies

$$c\,\omega_r = \sum_{j=1}^{n-1} B_j \frac{\partial \varphi_1^{(r)}}{\partial u_1^{(j)}}(u_1), \qquad r = 1, \ldots, n,$$

that is

$$c\,\omega = \sum_{j=1}^{n-1} B_j \frac{\partial \varphi_1}{\partial u_1^{(j)}}(u_1). \tag{13}$$

Consequently, $c\,\omega$ is a tangent vector to X at $\varphi_1(u_1)$. However, $\xi = (\omega;\ u;\ v;\ \theta) \in G^{-1}(0)$ implies $\chi(\xi) \in M_n(s,\ k) \subset M(s,\ k)$, and so $\varphi_1(u_1)$ is the first (transversal) reflection point of an $(\omega,\ \theta)$-trajectory for X. In particular, ω is not tangent to X at $\varphi_1(u_1)$. Hence $c = 0$ and now (13) yields $B_1 = \ldots = B_{n-1} = 0$.

In a similar way, considering in (11) the derivatives with respect to $\theta_2, \ldots, \theta_n$, we find $C_1 = \ldots = C_{n-1} = 0$.

Next, we are going to show that $A_m^{(1)} = \ldots = A_m^{(n-1)} = 0$ for each $m = 2, \ldots, k$. Here we assume $k \geq 2$; for $k = 1$ there is nothing to be proved in this step. Since $k \geq 2$, the functions P_j, Q, and $K_i^{(j)}$ for $i \geq 3$ do not depend on the variables $u_1^{(r)}$. On the other hand,

$$K_2^{(j)}(\xi) = \frac{\partial F}{\partial u_2^{(j)}}(\xi) = \left\langle \frac{\varphi_2(u_2) - \varphi_1(u_1)}{\|\varphi_2(u_2) - \varphi_1(u_1)\|} + \frac{\varphi_2(u_2) - \varphi_3(u_3)}{\|\varphi_2(u_2) - \varphi_3(u_3)\|},\ \frac{\partial \varphi_2}{\partial u_2^{(j)}}(u_2) \right\rangle,$$

and therefore

$$\frac{\partial K_2^{(j)}}{\partial u_1^{(r)}}(\xi) = -\,a_1 \left[\left\langle \frac{\partial \varphi_1}{\partial u_1^{(r)}}(u_1),\ \frac{\partial \varphi_2}{\partial u_2^{(j)}}(u_2) \right\rangle - \left\langle e_1,\ \frac{\partial \varphi_1}{\partial u_1^{(r)}}(u_1) \right\rangle \left\langle e_1,\ \frac{\partial \varphi_2}{\partial u_2^{(j)}}(u_2) \right\rangle \right],$$

where

$$a_i = \frac{1}{\|\varphi_i(u_i) - \varphi_{i+1}(u_{i+1})\|}, \qquad e_i = \frac{\varphi_i(u_i) - \varphi_{i+1}(u_{i+1})}{\|\varphi_i(u_i) - \varphi_{i+1}(u_{i+1})\|}.$$

Now we consider in (11) the derivatives with respect to $u_1^{(r)}$ and find

$$\sum_{j=1}^{n-1} A_2^{(j)} \frac{\partial K_2^{(j)}}{\partial u_1^{(r)}}(\xi) = 0. \tag{14}$$

Setting

$$w = \sum_{j=1}^{n-1} A_2^{(j)} \frac{\partial \varphi_2}{\partial u_2^{(j)}}(u_2) \in T_{\varphi_2(u_2)} X, \tag{15}$$

and using the expression for $\frac{\partial K_2^{(j)}}{\partial u_1^{(r)}}$ found above, we can rewrite (14) in the form $\left\langle \frac{\partial \varphi_1}{\partial u_1^{(r)}}(u_1),\, w \right\rangle - \langle e_1,\, w\rangle \left\langle e_1,\, \frac{\partial \varphi_1}{\partial u_1^{(r)}}(u_1) \right\rangle = 0.$

That is, $\left\langle \frac{\partial \varphi_1}{\partial u_1^{(r)}}(u_1),\, w - \langle e_1,\, w\rangle\, e_1 \right\rangle = 0$, and this is true for all $r = 1, \ldots, n-1$. Consequently,

$$w - \langle e_1,\, w\rangle\, e_1 = \lambda\, N_1 \tag{16}$$

for some $\lambda \in \mathbf{R}$, N_1 being an unit normal vector to X at $\varphi_1(u_1)$. Note that $\langle e_1,\, N_1\rangle \neq 0$. Taking inner product of (16) with e_1 gives $0 = \lambda\, \langle N_1,\, e_1\rangle$, and so $\lambda = 0$. Thus, by (16), $w = \langle e_1,\, w\rangle\, e_1$. If $\langle e_1,\, w\rangle \neq 0$, the latter would imply $e_1 \in T_{\varphi_2(u_2)}\, X$, which would be a contradiction with $\xi \in G^{-1}(0)$. Hence $\langle e_1,\, w\rangle = 0$ and so $w = 0$. Now (15) yields $A_2^{(1)} = \ldots = A_2^{(n-1)} = 0$.

Using the above procedure, by induction, we get $A_m^{(j)} = 0$ for all $m = 2, \ldots, k$ and $j = 1, \ldots, n-1$.

Next, if $k < s-1$, we repeat the same argument, considering the derivatives with respect to $u_s^{(r)}$, to show that $A_{s-1}^{(j)} = 0$ for $j = 1, \ldots, n-1$. In a similar way, by induction, one gets $A_m^{(j)} = 0$ for all $m = s-1$, $s-2, \ldots, k+1$ and $j = 1, \ldots, n-1$. Therefore all coefficients $A_m^{(j)}$ in (11) are zero.

Now (11) has the form

$$\sum_{j=1}^{n-1} p_j \operatorname{grad} P_j(\xi) + q \operatorname{grad} Q(\xi) = 0. \tag{17}$$

Set for convenience

$$b_1 = \frac{1}{\|\psi(v) - \varphi_k(u_k)\|}, \qquad b_2 = \frac{1}{\|\psi(v) - \varphi_{k+1}(u_{k+1})\|},$$

and note that

$$\frac{\psi(v) - \varphi_k(u_k)}{\|\psi(v) - \varphi_k(u_k)\|} = -\frac{\psi(v) - \varphi_{k+1}(u_{k+1})}{\|\psi(v) - \varphi_{k+1}(u_{k+1})\|} = e_k.$$

We have

$$\frac{\partial P_j}{\partial u_k^{(r)}}(\xi) = -b_1 \left[\frac{\partial \varphi_k^{(j)}}{\partial u_k^{(r)}}(u_k) - \left\langle e_k,\, \frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k) \right\rangle e_k^{(j)} \right]$$

and

$$\frac{\partial P_j}{\partial u_{k+1}^{(r)}}(\xi) = -b_2 \left[\frac{\partial \varphi_{k+1}^{(j)}}{\partial u_{k+1}^{(r)}}(u_{k+1}) - \left\langle e_k, \frac{\partial \varphi_{k+1}}{\partial u_{k+1}^{(r)}}(u_{k+1}) \right\rangle e_{k+1}^{(j)} \right].$$

Further, set $p_n = 0$ and $p = (p_1, \ldots, p_n) \in \mathbf{R}^n$, and consider in (17) the derivatives with respect to $u_k^{(r)}$. We get

$$-b_1 \sum_{j=1}^{n-1} p_j \left[\frac{\partial \varphi_k^{(j)}}{\partial u_k^{(r)}}(u_k) - \left\langle e_k, \frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k) \right\rangle e_k^{(j)} \right] - q \left\langle \frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k), N(\xi) \right\rangle = 0,$$

which is equivalent to

$$-b_1 \left\langle p, \frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k) \right\rangle + b_1 \langle e_k, p \rangle \left\langle e_k, \frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k) \right\rangle - q \left\langle N(\xi), \frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k) \right\rangle = 0.$$

The latter can be rewritten as

$$\left\langle b_1 p - \langle e_k, b_1 p \rangle e_k + q N(\xi), \frac{\partial \varphi_k}{\partial u_k^{(r)}}(u_k) \right\rangle = 0.$$

Since this is true for all $r = 1, \ldots, n-1$, it implies

$$b_1 p - \langle e_k, b_1 p \rangle e_k + q N(\xi) = \mu N_k \tag{18}$$

for some $\mu \in \mathbf{R}$, N_k being an unit normal vector to X at $\varphi_k(u_k)$. Taking inner product of (18) with e_k, one finds $\mu \langle N_k, e_k \rangle = q \langle N(\xi), e_k \rangle = 0$, since the segment $[\varphi_k(u_k), \varphi_{k+1}(u_{k+1})]$ is tangent to X at $\psi(v)$ and $N(\xi)$ is a normal vector to X at $\psi(v)$. On the other hand, $\langle N_k, e_k \rangle \neq 0$, and so $\mu = 0$. Now (18) implies

$$b_1 p - \langle e_k, b_1 p \rangle e_k + q N(\xi) = 0. \tag{19}$$

In a similar way, considering in (17) the derivatives with respect to $u_{k+1}^{(r)}$, one obtains

$$b_2 p - \langle e_k, b_2 p \rangle e_k - q N(\xi) = 0. \tag{20}$$

Since $b = b_1 + b_2 > 0$, combining (19) and (20) gives

$$p = \langle e_k, p \rangle e_k. \tag{21}$$

First, note that the latter and (19) imply $q = 0$. Next, $\xi \in G^{-1}(0)$ yields $\chi(\xi) \in M_n(s, k)$ and so $\varphi_k^{(n)}(u_k) \neq \varphi_{k+1}^{(n)}(u_{k+1})$. Hence $e_k^{(n)} \neq 0$. On the other hand, $p_n = 0$ by definition, therefore (21) implies $0 = \langle e_k, p\rangle e_k^{(n)}$, *i.e.* $\langle e_k, p\rangle = 0$. Again by (21) we get $p = 0$. Thus, all coefficients in (11) are zero.

The case $k = 0$ and $k = s$ can be treated in a similar way. We omit the arguments in these two cases. This concludes the proof of the lemma.

Proof of Theorem 3.1. – For given s and k consider the projection

$$\pi_s : \quad M_s = S^{n-1} \times X^{(s)} \times X \times S^{n-1} \longrightarrow S^{n-1} \times S^{n-1},$$

defined by $\pi_s(\omega;\, x;\, y;\, \theta) = (\omega,\, \theta)$. Since π_s is smooth and $M_r(s,\, k)$ is a smooth submanifold of dimension $2n - 3 < \dim(S^{n-1} \times S^{n-1})$, the set $L_r(s,\, k) = \pi_s(M_r(s,\, k)) \subset S^{n-1} \times S^{n-1}$ has measure zero. Let $M_r(s,\, k) = \bigcup_{j=1}^{\infty} K_j$ with K_j compact. Then $L_r(s,\, k) = \bigcup_{j=1}^{\infty} \pi_s(K_j)$ is a countable union of compact subsets of $S^{n-1} \times S^{n-1}$ with measure zero.

Finally, set $L = \bigcup_{0\le k\le s} \bigcup_{r=1}^{n} L_r(s,\, k)$ and $\mathcal{R} = (S^{n-1} \times S^{n-1})\backslash L$. Then $\mathcal{R}$ has the desired properties.

Proof of Theorem 3.2. – Consider the projections $p_s : M_s \longrightarrow S^{n-1} \times X$, defined by $p_s(\omega;\, x_1,\, \dots,\, x_s;\, y;\, \theta) = (\omega,\, x_1)$, and set

$$\mathcal{T} = (S^{n-1} \times X)\backslash \bigcup_{0\le k\le s} \bigcup_{r=1}^{n} p_s(M_r(s,\, k)).$$

The same argument as that in the proof of Theorem 1 shows that $\mathcal{T}$ has the desired properties.

3.3. Remark. – In the case when K has the form (6) with $G^\infty(K) = \emptyset$ it is possible to prove a slightly stronger result than Theorem 3.1. Namely, for each fixed $\omega \in S^{n-1}$ there exists a set $\mathcal{R}(\omega) \subset S^{n-1}$, the complement of which has Lebesgue measure zero, such that for each $\theta \in \mathcal{R}(\omega)$ all $(\omega,\, \theta)$-rays are ordinary reflecting ones (*see* [P2]).

4. SOJOURN TIMES

Let Ω be a domain in $\mathbf{R}^n$, $n \ge 2$, with bounded complement and smooth boundary $X = \partial\Omega$, and let ω be a fixed unit vector in $\mathbf{R}^n$. Our aim in this section is to prove the following.

4.1. Proposition. – *There exists $\mathcal{S}(\omega) \subset S^{n-1}$ the complement of which is a countable union of compact subsets of measure zero of S^{n-1} such that* *if $\theta \in \mathcal{S}(\omega)$, then any two different ordinary reflecting (ω, θ)-rays in Ω have distinct sojourn times.*

The rest of the section is devoted to the proof of this proposition.

Let $B = B_R$ be the open ball in $\mathbf{R}^n$ with center 0 and radius R and fix R so big that B contains X. Let $Z = Z_\omega$ be the hyperplane tangent to B and orthogonal to ω and such that the halfspace determined by Z and having ω as an inner normal contains X.

For a given integer $k \geq 1$ denote by U_k the set of those $u \in Z$ such that the trajectory $\gamma(u)$ of the generalized geodesic flow in Ω is an ordinary reflecting ray with exactly k reflection points. Denote by $J_k(u) \in S^{n-1}$ the direction of $\gamma(u)$ after the last reflection. Clearly, U_k is open in Z and $J_k : U_k \to S^{n-1}$ is smooth.

Next, we fix two arbitrary integers $k \geq 1$, $s \geq 1$. For $u \in U_k$ denote by $f(u)$ the sojourn time of the scattering ray $\gamma(u)$ (more precisely, the scattering ray determined by $\gamma(u)$). Thus, we get a smooth function $f : U_k \to \mathbf{R}$. For convenience the same function on U_s will be denoted by g, so $g : U_s \to \mathbf{R}$.

For $u \in U_k$ let $x_1(u) \in X, \ldots, x_k(u) \in X$ be the successive reflection points of $\gamma(u)$. Then $x_i : U_k \to X$ are smooth maps. Let for $y \in X$, $N(y)$ denotes the *unit normal to* X at y pointing into Ω. Then for $u \in U_k$ we have

$$\begin{aligned} J_k(u) = & \frac{x_k(u) - x_{k-1}(u)}{\| x_k(u) - x_{k-1}(u) \|} \\ & - 2 \left\langle \frac{x_k(u) - x_{k-1}(u)}{\| x_k(u) - x_{k-1}(u) \|}, N(x_k(u)) \right\rangle N(x_k(u)) \end{aligned}$$

and $f(u) = \sum_{i=0}^{k-1} \| x_{i+1}(u) - x_i(u) \| + t - 2R$, where $x_0(u)$ [resp. $x_{k+1}(u)$] denotes the orthogonal projection of $x_1(u)$ [resp. $x_k(u)$] on Z (resp. $Z_{-\theta}$), $\theta = J_k(u)$, and $t = \| x_k(u) - x_{k+1}(u) \|$. It is easy to see that $\langle \theta, x_k + t\theta - R\theta \rangle = 0$. Therefore $t = R - \langle \theta, x_k \rangle$, and so $f(u) = \sum_{i=0}^{k-1} \| x_{i+1}(u) - x_i(u) \| - \langle x_k(u), J_k(u) \rangle - R.$

For $v \in U_s$ the successive reflection points of $\gamma(v)$ will be denoted by $y_1(v), \ldots, y_s(v)$. We set $y_0(v) = v$ and define $y_{s+1}(v)$ in the same way as $x_{k+1}(u)$.

Further, denote by $W(k, s)$ the set of those $(u, v) \in U_k \times U_s$ such that $J_k(u) = J_s(v)$, $f(u) = g(v)$ and rank $dJ_k(u) = \text{rank}\, dJ_s(v) = n - 1$.

4.2. LEMMA. – *$W(k, s)$ is a smooth $(n-2)$-dimensional submanifold of $U_k \times U_s$.*

Proof. – Consider an arbitrary point $w_0 = (u_0, v_0)$ in $W(k, s)$. Then rank $dJ_k(u_0) = \text{rank}\, dJ_s(v_0) = n-1$. Clearly, there exists a neighbourhood U of w_0 in $U_k \times U_s$ such that for every $(u, v) \in W$ we have rank $dJ_k(u) = \text{rank}\, dJ_s(v) = n-1$. Define the map $L : U \to \mathbf{R}^n$ by $L(u, v) = (\lambda(u, v); (\chi^{(j)}(u,v))_{1 \le j \le n-1})$, where $\lambda(u, v) = f(u) - g(v)$, $\chi(u, v) = J_k(u) - J_s(v)$. Clearly, $W(k, s) \cap U \subset L^{-1}(0)$, and so it is sufficient ot show that L is submersion at any point of $L^{-1}(0)$.

We shall show that L is submersion at w_0. For the other points of $L^{-1}(0)$ the argument is the same. Set $\theta = J_k(u_0)$. Without loss of generality we may **assume that** $\theta^{(n)} \neq 0$.

Suppose that

$$\sum_{j=1}^{n-1} A_j \operatorname{grad} \chi^{(j)}(w_0) + C \operatorname{grad} \lambda(w_0) = 0 \tag{22}$$

for some constants A_j, C. Set $A_n = 0$ and $A = (A_1, \ldots, A_n) \in \mathbf{R}^n$. Before going on, we have to compute several derivatives. Setting $e_i = \dfrac{x_{i+1}(u_0) - x_i(u_0)}{\| x_{i+1}(u_0) - x_i(u_0) \|}$, for $p = 1, \ldots, n-1$ and $i = 1, \ldots, k-1$ we have

$$\begin{aligned}\frac{\partial}{\partial u_p} \| x_{i+1} - x_i \| (u_0) &= \frac{1}{\| x_{i+1} - x_i \|} \\ &\quad \times \left\langle x_{i+1} - x_i, \frac{\partial x_{i+1}}{\partial u_p}(u_0) - \frac{\partial x_i}{\partial u_p}(u_0) \right\rangle \\ &= \left\langle e_i, \frac{\partial x_{i+1}}{\partial u_p}(u_0) - \frac{\partial x_i}{\partial u_p}(u_0) \right\rangle.\end{aligned}$$

Note that $\left\langle e_{i-1}, \dfrac{\partial x_i}{\partial u_p}(u_0) \right\rangle = \left\langle e_i, \dfrac{\partial x_i}{\partial u_p}(u_0) \right\rangle$, since $\dfrac{\partial x_i}{\partial u_p}(u_0)$ is tangent to X at $x_i(u_0)$. Consequently,

$$\begin{aligned}\frac{\partial f}{\partial u_p}(u_0) &= \sum_{i=0}^{k-1} \left\langle e_i, \frac{\partial x_{i+1}}{\partial u_p}(u_0) - \frac{\partial x_i}{\partial u_p}(u_0) \right\rangle \\ &\quad - \left\langle \frac{\partial x_k}{\partial u_p}(u_0), J_k(u_0) \right\rangle - \left\langle x_k, \frac{\partial J_k}{\partial u_p}(u_0) \right\rangle,\end{aligned}$$

and using the fact that $\left\langle \frac{\partial x_k}{\partial u_p}(u_0),\, J_k(u_0) \right\rangle = \left\langle e_k,\, \frac{\partial x_k}{\partial u_p}(u_0) \right\rangle$, we find

$$\frac{\partial f}{\partial u_p}(u_0) = \sum_{i=1}^{k} \left\langle e_{i-1},\, \frac{\partial x_i}{\partial u_p}(u_0) \right\rangle - \sum_{i=0}^{k-1} \left\langle e_i,\, \frac{\partial x_i}{\partial u_p}(u_0) \right\rangle - \left\langle e_k,\, \frac{\partial x_k}{\partial u_p}(u_0) \right\rangle - \left\langle x_k(u_0),\, \frac{\partial J_k}{\partial u_p}(u_0) \right\rangle.$$

Since $e_0 = \omega$, this yields

$$\frac{\partial f}{\partial u_p}(u_0) = - \left\langle e_0,\, \frac{\partial x_0}{\partial u_p}(u_0) \right\rangle - \left\langle x_k(u_0),\, \frac{\partial J_k}{\partial u_p}(u_0) \right\rangle = -\left\langle x_k(u_0),\, \frac{\partial J_k}{\partial u_p}(u_0) \right\rangle.$$

According to $\langle x_k(u_0),\, \theta \rangle = 0$, we deduce

$$\frac{\partial \lambda}{\partial u_p}(u_0) = \frac{\partial f}{\partial u_p}(u_0) = -\left\langle x_k(u_0),\, \frac{\partial J_k}{\partial u_p}(u_0) \right\rangle = 0.$$

Moreover, we have $\frac{\partial \chi^{(j)}}{\partial u_p}(u_0) = \frac{\partial J_k^{(j)}}{\partial u_p}(u_0)$. Hence, considering in (22) the derivatives with respect to u_p, we obtain

$$0 = \sum_{j=1}^{n-1} A_j \frac{\partial J_k^{(j)}}{\partial u_p}(u_0) - C \left\langle x_k(u_0),\, \frac{\partial J_k^{(j)}}{\partial u_p}(u_0) \right\rangle = \left\langle A,\, \frac{\partial J_k}{\partial u_p}(u_0) \right\rangle - C \left\langle x_k(u_0),\, \frac{\partial J_k^{(j)}}{\partial u_p}(u_0) \right\rangle = \left\langle A - C\, x_k(u_0),\, \frac{\partial J_k^{(j)}}{\partial u_p}(u_0) \right\rangle$$

for all $p = 1, \ldots, n-1$. Since rank $dJ_k(u_0) = n - 1$, this yields

$$A - C\, x_k(u_0) = a\,\theta \tag{23}$$

for some $a \in \mathbf{R}$. In a similar way, considering in (22) the derivatives with respect to v_p, one gets

$$A - C\, y_s(v_0) = b\,\theta \tag{24}$$

for some $b \in \mathbf{R}$. Now combining (23) and (24) gives $C(x_k(u_0) - y_s(v_0)) = 0$. Since $x_k(u_0) \neq y_s(v_0)$ and the vector $x_k(u_0) - y_s(v_0)$ is orthogonal to θ, this implies $C = 0$. Using (23) again and taking into account that $A_n = 0$ by definition, we find $a\,\theta^{(n)} = 0$. Since $\theta^{(n)} \neq 0$ by assumption, it now follows that $a = 0$.

Applying again (23), we find $A = 0$, *i.e.* $A_1 = \ldots = A_{n-1} = 0$. Thus, L is submersion at w_0 which concludes the proof of the lemma.

Proof of Proposion. 4.1. – Define the map $\varphi : U_k \times U_s \to S^{n-1} \times S^{n-1}$ by $\varphi(u, v) = J_k(u)$. Then φ is smooth and since $\dim W(s, k) = n - 2$, $\varphi(W(s, k))$ is a countable union of compact subsets of S^{n-1} of measure zero. Set $F_k = \{u \in U_k : \operatorname{rank} dJ_k(u) \leq n - 2\}$. Then F_k is closed in U_k and so it can be represented as a countable union of compact subsets $F_k = \bigcup_i F_{k,i}$. Then by Sard's theorem $J_k(F_{k,i})$ has measure zero in S^{n-1} for all k and i. Therefore $F = \bigcup_k \bigcup_i J_k(F_{k,i})$ has measure zero in S^{n-1}. Finally, setting $\mathcal{S}(\omega) = S^{n-1} \backslash (F \cup \bigcup_k \bigcup_s J_k(W(k, s)))$, we get a subset of S^{n-1} the complement of which is a countable union of compact subsets of S^{n-1} with measure zero.

To show that $\mathcal{S}(\omega)$ has the desired property, consider an arbitrary $\theta \in \mathcal{S}(\omega)$. Assume that there exists $u \neq v$ in Z which determine ordinary reflecting (ω, θ)-rays in Ω with coinciding sojourn times. Then we have $u \in U_k$ and $v \in U_s$ for some k and s. If $\operatorname{rank} dJ_k(u) \leq n - 2$, then $u \in F_k$ and so $u \in F_{k,i}$ for some i. However, this implies $\theta = J_k(u) \in J_k(F_{k,i})$ which is a contradiction with $\theta \in \mathcal{S}(\omega)$. Hence $\operatorname{rank} dJ_k(u) = n - 1$. In the same way we find $\operatorname{rank} dJ_s(v) = n - 1$ and therefore $(u, v) \in W(k, s)$. This implies $\theta = J_k(u) = \varphi(u, v)$ which is again a contradiction with $\theta \in \mathcal{S}(\omega)$. In this way we have seen that any two different ordinary reflecting (ω, θ)-rays in Ω have distinct sojourn times which proves the proposition.

Define $\mathcal{S} = \{(\omega, \theta) \in S^{n-1} \times S^{n-1} : \theta \in \mathcal{S}(\omega)\}$. It follows by the properties of the sets $\mathcal{S}(\omega)$ that the complement of $\mathcal{S}$ in $S^{n-1} \times S^{n-1}$ has measure zero and for each $(\omega, \theta) \in \mathcal{S}$ any two different ordinary reflecting (ω, θ)-rays in Ω have distinct sojourn times.

5. EXISTENCE OF SCATTERING RAYS WITH SOJOURN TIMES TENDING TO INFINITY

Let Ω be a closed domain in $\mathbf{R}^n$, $n \geq 2$, with bounded complement and smooth boundary $\partial\Omega$. In this section we show that if the obstacle

$$K = \mathbf{R}^n \backslash \Omega^\circ$$

is trapping, then there exists a sequence of reflecting (ω_m, θ_m)-rays γ_m in Ω with sojourn times $T_{\gamma_m} \to \infty$.

Set $\mathbf{Q} = \mathbf{R} \times \Omega$ and denote by (τ, ξ) the variables dual to (t, x) in $T^*(\mathbf{Q})$. The *characteristic set* of the wave operator $\Box$ is defined by

$$\Sigma = \{(t, x, \tau, \xi) \in T^*(\bar{\mathbf{Q}})\backslash\{0\} : \tau^2 = |\xi|^2\}.$$

For $z = (t, x) \in \mathbf{R} \times \partial\Omega$ consider the *compressed cotangent bundle*

$$\tilde{T}_z^*(\bar{\mathbf{Q}}) = T_z^*(\bar{\mathbf{Q}})/N_z(\partial\mathbf{Q}) \cong T_z^*(\partial\mathbf{Q}),$$

$N_z(\partial\mathbf{Q})$ being the fiber of one-forms vanishing on $T_z(\partial\Omega)$. Set $\tilde{T}^*(\bar{\mathbf{Q}}) = T^*(\mathbf{Q}^o) \cup T^*(\partial\mathbf{Q})$, and consider the map $\sim: T^*(\bar{\mathbf{Q}}) \to \tilde{T}^*(\bar{\mathbf{Q}})$ which coincides with the identity on $\mathbf{Q}^o$, while for $(z, \zeta) \in T^*(\bar{\mathbf{Q}})$, $z \in \partial\mathbf{Q}$, we define

$$\sim(z, \zeta) = (z, \eta) \in T_z^*(\partial\mathbf{Q}), \qquad \eta = \zeta\,|_{T_z(\partial\mathbf{Q})}.$$

The image $\Sigma_b = \tilde{\Sigma} = \sim(\Sigma)$ is called the *compressed characteristic set,* and if γ is a generalized bicharacteristic of $\Box$, its image $\tilde{\gamma} = \sim(\gamma)$ is called a *compressed generalized bicharacteristic (ray).*

Let $\rho_0 > 0$ be fixed so that $K \subset B_0 = \{x \in \mathbf{R}^n : |x| \leq \rho_0\}$. Given a point $\nu = (0, x, 1, \xi) \in \Sigma_b$, $(x, \xi) \in T^*(\partial\Omega)$, consider the compressed generalized bicharacteristic $\gamma_\nu(t) = (t, x(t), 1, \xi(t)) \in \tilde{T}^*(\bar{\mathbf{Q}})$ of $\Box$, parametrized by the time t and passing through ν for $t = 0$. Denote by $T(\nu) \in \mathbf{R}^+ \cup \infty$ the maximal $T > 0$ such that $x(t) \in B_0$ for $0 \leq t \leq T(\nu)$. Denote by Σ_∞ the set of those $\nu = (0, x, 1, \xi) \in \Sigma_b$, $(x, \xi) \in T^*(\partial\Omega)$, such that $T(\nu) = \infty$. Using the continuity of the generalized Hamiltonian flow of $\Box$ (*cf.* [MS]), it is easy to see that Σ_∞ is closed in Σ_b. On the other hand, $\Sigma_\infty \neq \Sigma_b$. Indeed, take a hyperplane Π tangent to $\partial\Omega$ such that K is contained in a half-space determined by Π. Consider an arbitrary $\nu_0 = (0, x_0, 1, \xi_0) \in \Sigma_b$ with $(x_0, \xi_0) \in T(\Pi)$, $x_0 \in \partial\Omega \cap \Pi$. Then we have $T(\nu_0) < \infty$, since $\gamma_{\nu_0}(t)$ leaves B_0 for $t > 2\rho_0$.

By definition the obstacle K is *trapping* if $\Sigma_\infty \neq \emptyset$. Therefore the boundary $\partial\Sigma_\infty$ in Σ_b is not empty. Take an arbitrary $\hat{\nu} \in \partial\Sigma_\infty$. Since $\Sigma_b\backslash\Sigma_\infty \neq \emptyset$, there exists a sequence of elements $\nu_m = (0, x_m, 1, \xi_m)$ of Σ_b with $(x_m, \xi_m) \in T^*(\partial\mathbf{Q})$ such that $\nu_m \notin \Sigma_\infty$ for all m and $\nu_m \to \hat{\nu}$. Consider the compressed generalized bicharacteristics $\gamma_{\nu_m}(t) = (t, x_m(t), 1, \xi_m(t))$ passing through ν_m for $t = 0$ and such that $T(\nu_m) < \infty$. If the sequence $\{T(\nu_m)\}$ is bounded, this would imply $T(\hat{\nu}) < \infty$ in contradiction with $\hat{\nu} \in \Sigma_\infty$. Therefore $\{T(\nu_m)\}$ is unbounded and we may assume $T(\nu_m) \to_{m\to\infty} +\infty$. Set $y_m = x_m(T(\nu_m)) \in \partial B_0$, $\omega_m = \xi_m(T(\nu_m)) \in S^{n-1}$. Passing to a subsequence, we may assume that $y_m \to z \in \partial B_0$ and $\omega_m \to \omega \in S^{n-1}$.

Consider the generalized bicharacteristic $\gamma_\mu(t) = (t, y(t), 1, \xi(t))$ of $\Box$ issued from $\mu = (0, z, 1, \omega)$. By continuity we have $T(\mu) = \infty$ and $y(t) \in B_0$ for $t \geq 0$.

Let Z_ω be the hyperplane passing through z and orthogonal to ω. Denote by Z_∞ the set of those points $y \in Z_\omega$ such that the generalized bicharacteristic γ_{μ_y} passing through $\mu_y = (0, y, 1, \omega)$ has the property $T(\mu_y) = \infty$. A simple argument shows that Z_∞ is closed in Z_ω. Clearly $Z_\infty \neq Z_\omega$. Consequently, there exists a sequence $z_m \to y_0$ with $z_m \in Z_\omega \backslash Z_\infty$ for all m such that $T(\mu_{z_m}) < \infty$ for all m and $T(\mu_{z_m}) \to \infty$. In general the bicharacteristic $\gamma_{\mu_{z_m}}$ could contain gliding or glancing segments. However, if $G^\infty(K) = \emptyset$, then γ_{μ_z} can be approximated by multiple reflecting rays (*cf.* [MS]). Thus, taking (z'_m, ω'_m) sufficiently close to (z_m, ω), we obtain the following result.

5.1. Proposition. – *Let the obstacle K be such that $G^\infty(K) = \emptyset$ and $\Sigma_\infty \neq \emptyset$. Then there exists a sequence of ordinary reflecting (ω'_m, θ'_m)-rays γ_m such that $T_{\gamma_m} \to \infty$.*

5.2. Corollary. – *Let the obstacle K have the form* (6) *and let $\Sigma_\infty \neq \emptyset$. Then there exists a sequence of ordinary reflecting (ω'_m, θ'_m)-rays γ_m such that $T_{\gamma_m} \to \infty$.*

Proof. – In this case every generalized bicharacteristic γ of $\Box$ in Ω is uniquelly determined by each of its points. Moreover, every such bicharacteristic can be approximated by ordinary reflecting ones (*see* [MS]). Thus, the argument of the proof of Proposition 5.1 works even if $G^\infty(K) \neq \emptyset$.

Next, we consider a fixed ordinary reflecting (ω'_m, θ'_m)-ray γ_m which is non-degenerate (*see* Section 1). We wish to replace (ω'_m, θ'_m) by a pair (ω''_m, θ''_m) sufficiently close to (ω'_m, θ'_m) for which there exist ordinary reflecting (ω''_m, θ''_m)-rays δ_m such that $\{T_{\delta_m}\}$ is an infinite sequence of isolated points in sing supp $s(-t, \theta''_m, \omega''_m)$.

Let $\mathcal{S}$ be the subset of $S^{n-1} \times S^{n-1}$ introduced at the end of Section 4. Assume that K satisfies the following condition:

$$(\mathbf{G}) \quad \begin{cases} \text{There exists a subset } \mathcal{G} \subset S^{n-1} \times S^{n-1} \text{ the complement of} \\ \text{which has Lebesgue measure zero such that for each } (\omega, \theta) \in \mathcal{G} \\ \text{all } (\omega, \theta)\text{-rays in } \Omega \text{ are ordinary reflecting ones.} \end{cases}$$

Setting $\Xi = \mathcal{G} \cap \mathcal{S}$, it suffices to take an approximation by $(\omega''_m, \theta''_m) \in \Xi$.

On the other hand, to guarantee the existence of such (ω''_m, θ''_m)-rays, we shall use a corollary of the inverse mapping theorem (*cf.* [H1], Theorem 1.1.7). Let U and V be open subsets of $\mathbf{R}^m$ and let $F : U \ni x \mapsto f(x) \in V$ be a C^∞ map. Suppose that $x_0 \in U$ is such that det $df(x_0) \neq 0$. Then $\alpha = 1/\| df(x_0)^{-1} \| > 0$, $\| \cdot \|$ being the standard norm in $\mathcal{L}(\mathbf{R}^m, \mathbf{R}^m)$ – the space of linear maps. Set $y_0 = f(x_0)$ and choose $\delta > 0$ so small that $U_\delta = \{x \in \mathbf{R}^m : \| x - x_0 \| < \delta\} \subset U$, $\| df(x) - df(x_0) \| \leq \frac{\alpha}{2}$ for $x \in U_\delta$, $V_\delta = \left\{ y \in \mathbf{R}^m : \| y - y_0 \| < \frac{\delta\alpha}{2} \right\} \subset V$. Then it follows by the inverse mapping theorem that the map f is injective on U_δ and surjective on V_δ.

In what follows we are going to construct suitable approximations for $x_0 = z'_m$ and $\omega_0 = \omega'_m$. Consider the hyperplane $Z = Z_{\omega_0}$. For ω sufficiently close to ω_0, the (ω, θ)-rays issued from $y \in Z_\omega$ in direction ω can be considered as suitable (ω, θ)-rays issued from a point $x \in Z$, provided y is close enough to x_0. Thus we obtain a C^∞ map

$$U = \mathcal{O} \times \Gamma \ni (x, \omega) \mapsto f(x, \omega) \in S^{n-1}.$$

Here $\mathcal{O} \subset Z$ is a small neighbourhood of x_0, $\Gamma \subset S^{n-1}$ is a small neighbourhood of ω_0, and $f(x, \omega)$ is the outgoing direction of the ray issued from x in direction ω. Since γ_m is non-degenerate by assumption, we have det $f'_x(x_0, \omega_0) \neq 0$. We may assume that U is chosen so small that det $f'_x(x, \omega) \neq 0$ holds for all $(x, \omega) \in \bar{U}$. Set $\max_{(x, \omega) \in U} \| (f'_x(x, \omega))^{-1} \| = \frac{1}{\alpha}$. Then there exists $\delta > 0$ such that for $(x, \omega) \in U$ with $\| x - x_0 \| < \delta$, $\| \omega - \omega_0 \| < \delta$ we have $\| f'_x(x, \omega) - f'_x(x_0, \omega_0) \| \leq \frac{1}{4}\alpha$. We may assume that δ is so small that

$$\mathcal{O}_\delta = \{x \in Z : \| x - x_0 \| < \delta\} \subset \mathcal{O},$$
$$\Gamma_\delta = \{\omega \in S^{n-1} : \| \omega - \omega_0 \| < \delta\} \subset \Gamma.$$

Clearly, for $\omega \in \Gamma_\delta$ fixed, the map $\mathcal{O}_\delta \ni x \mapsto f(x, \omega) \in S^{n-1}$ is injective. Denote $\theta_0 = f(x_0, \omega_0)$ and consider the set

$$W_\delta = \left\{ \theta \in S^{n-1} : \| \theta - \theta_0 \| < \frac{\delta\alpha}{4} \right\}.$$

Choose $\delta' \in (0, \delta)$ so small that $\| f(x_0, \omega) - \theta_0 \| < \frac{\delta\alpha}{4}$ for $\omega \in \Gamma_{\delta'}$. Then for $\omega \in \Gamma_{\delta'}$ and $\theta \in W_\delta$ we deduce $\| \theta - f(x_0, \omega) \| < \frac{\delta\alpha}{2}$, therefore, according to the above remark, for each fixed $\omega \in \Gamma_{\delta'}$ and each fixed $\theta \in W_\delta$ we can find $x_{(\omega, \theta)} \in \mathcal{O}_\delta$ with $f(x_{(\omega, \theta)}, \omega) = \theta$. This shows that, exploiting the density of Ξ in $S^{n-1} \times S^{n-1}$, we can approximate $(\omega_0, \theta_0) = (\omega'_m, \theta'_m)$

by pairs $(\omega_j, \theta_j) \in \Xi$ for which there exists (ω_j, θ_j)-rays ν_j with sojourn times T_j converging to T_{γ_m} as $j \to \infty$. Applying the results of [CPS], [PS3], we obtain a sequence of ordinary reflecting (ω''_m, θ''_m)-rays δ_m such that $T_{\delta_m} \to \infty$, and $-T_{\delta_m} \in \operatorname{sing\,supp} s(t, \theta''_m, \omega''_m)$ for all $m = 1, 2, \dots$ Moreover, near $-T_{\delta_m}$ the scattering kernel $s(t, \theta''_m, \omega''_m)$ has a singularity with leading term $A_m\, \delta^{(n-1)}(t + T_{\delta_m})$ with $A_m \neq 0$. Thus, if the reflecting rays γ_m in Proposition 5.1 are non-degenerate, we conclude that the assertion of Theorem 2.3 is true.

Finally, notice that by Theorem 3.1 the condition $(\mathbf{G})$ holds for obstacles K having the form (6).

6. PROOFS OF THEOREMS 1.1 AND 1.2

Throughout this section we assume that K has the form (6). For $z \in \partial K$ we denote by $\mathcal{K}(z)$ the *Gauss curvature* of ∂K at z. Following the argument at the end of the previous section, we need to construct a sequence of ordinary reflecting non-degenerate (ω_m, θ_m)-rays γ_m in Ω with $T_m = T_{y_m} \to \infty$. To do this we use the following.

6.1. Proposition. – *Let K have the form* (6) *and let γ be an ordinary reflecting (ω, θ)-ray with reflection points $x_1, \dots, x_k$. Assume that there exists j such that $\mathcal{K}(x_j) > 0$. Then γ is non-degenerate.*

Proof. – Consider the map J_γ defined in Section 1. Then the map $dJ_\gamma(u_\gamma)$ has the following representation (*see* [PS3], Proposition 2.4.2)

$$dJ_\gamma(u_\gamma)\, u$$
$$= M_k\, \sigma_k\, (I + \lambda_k\, M_{k-1})\, \sigma_{k-1}\, (I + \lambda_{k-1}\, M_{k-2}) \dots \sigma_2\, (I + \lambda_2\, M_1)\, \sigma_1\, u.$$

Here $\lambda_i = \| x_{i-1} - x_i \|$ for $i = 1, \dots, k$, $x_0 = u_\gamma$, σ_i is a linear map related to the symmetry with respect to the tangent plane to ∂K at x_i, and M_i are symmetric linear maps defined by $M_1 = \tilde{\psi}_1$, $M_i = \sigma_i\, M_{i-1}\, (I + \lambda_i\, M_{i-1})^{-1}\, \sigma_i + \tilde{\psi}_i$ for $i = 2, \dots, k$, $\tilde{\psi}_i$ being linear symmetric maps depending on the second fundamental form of ∂K at x_i. Since K_j are convex, we have $\tilde{\psi}_i \geq 0$ for all $i = 1, \dots, k$. Hence $M_i \geq 0$ for each i. By assumption, there exists j with $\mathcal{K}(x_j) > 0$. Then $\tilde{\psi}_j > 0$, and so $M_i > 0$ for $i = j, j+1, \dots, k$. Consequently, $dJ_\gamma(u_\gamma)\, u = 0$ implies $u = 0$ which proves the assertion.

Proof of Theorem 1.1. – By Corollary 5.2, there exists a sequence of ordinary reflecting (ω'_m, θ'_m)-rays δ_m with sojourn times $T_{\delta_m} \to \infty$. Each point $z \in \partial K$ can be approximated by points $z' \in \partial K$ such that $\mathcal{K}(z') > 0$. Using this and Proposition 6.1, we find a sequence of ordinary reflecting non-degenerate (ω_m, θ_m)-rays γ_m with sojourn times $T_{\gamma_m} \to \infty$. To complete the proof we use the argument at the end of Section 5.

Proof of Theorem 1.2. – Let $\mathcal{R}$ and $\mathcal{S}$ be the sets from Theorem 3.1 and the end of Section 4, respectively, and let $\Xi = \{(\omega, \theta) \in \mathcal{R} \cap \mathcal{S} : \omega \neq \theta\} \subset S^{n-1} \times S^{n-1}$. Given $(\omega, \theta) \in \Xi$, there are two possibilities:

1) there are no (ω, θ)-rays in Ω;

2) each (ω, θ)-ray is ordinary reflecting and different (ω, θ)-rays have different sojourn times.

Let $\mathcal{O}$ be a sphere in $\mathbf{R}^n$ which contains the obstacle K in its interior. Consider the set Γ of those $(x, \omega, y, \theta) \in \mathcal{O} \times S^{n-1} \times \mathcal{O} \times S^{n-1}$ such that $(\omega, \theta) \in \Xi$ and there exists an ordinary reflecting (ω, θ)-ray passing through x and y. Clearly, Γ is an open submanifold of $\mathcal{O} \times S^{n-1} \times \mathcal{O} \times S^{n-1}$. Since the projection

$$\pi : \quad \Gamma \ni (x, \omega, y, \theta) \mapsto (\omega, \theta) \in S^{n-1} \times S^{n-1}$$

is smooth, it follows by Sard's theorem that there exists a set $\Sigma(\Gamma) \subset S^{n-1} \times S^{n-1}$ of measure zero such that $d\pi(x, \omega, y, \theta) \neq 0$ whenever $(\omega, \theta) \notin \Sigma(\Gamma)$. As one can easily check, the last condition means that all (ω, θ)-rays with $(\omega, \theta) \notin \Sigma(\Gamma)$ are non-degenerate. Consequently, for $(\omega, \theta) \in \Xi \backslash \Sigma(\Gamma)$, all (ω, θ)-rays have the properties (*a*)-(*c*). Applying the results of Chapter 9 in [PS3], we complete the proof.

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