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A characterization of tangent and stable tangent bundles *

by

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ABSTRACT. – We apply a result of Nagano to prove that an integrable almost tangent manifold M endowed with a vector field satisfying similar properties to those satisfied by the canonical vector field of a vector bundle admits a unique vector bundle structure such that M is isomorphic to a tangent bundle. Thus we obtain a characterization of tangent bundles. This characterization was obtained by Crampin *et al.* and Filippo *et al.* in a different way. We also extend the result to stable tangent bundles. An application to reduction of degenerate autonomous and non-autonomous Lagrangian systems is given.

Key words: Tangent bundles, stable tangent bundles, almost tangent and stable tangent structures.

RÉSUMÉ. – Nous utilisons un résultat dû à Nagano pour démontrer qu'une variété tangente presque intégrable M équipée d'un champ de vecteurs

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obéissant à des propriétés similaires à celles satisfaites par le champ de vecteurs canonique d'un fibré vectoriel admet une structure de fibré vectoriel unique telle que M soit isomorphe à un fibré tangent. Nous obtenons ainsi une caractérisation des fibrés tangents. Cette dernière a été aussi obtenue par Crampin *et al.* ainsi que par Filippo *et al.* de manière différente. Nous étendons ce résultat aux fibrés tangents stables. Nous appliquons ce résultat à la réduction des systèmes lagrangiens autonomes dégénérés ou bien non autonomes.

1. INTRODUCTION

The problem of the characterization of tangent bundles has been recently studied by several authors ([2], [5], [13]). Since the tangent bundle of an arbitrary manifold possesses a canonical almost tangent structure the starting point is to consider an almost tangent manifold M. If M is integrable and satisfies some global hypothesis, then it is possible to prove that M is an affine bundle modelled on the tangent bundle TQ of some manifold Q and hence diffeomorphic to it. Moreover, if the affine bundle admits a global section, then M is isomorphic to TQ via the isomorphism induced by the section. Similar results were obtained for cotangent bundles (*see* [12]). A different approach is due to Filippo *et al.* [5]. In fact these last authors prove that if an integrable almost tangent manifold M endowed with a vector field satisfies some global hypothesis, then there exists on M a maximal tangent bundle atlas and, hence M is a tangent bundle. However, this last fact is not sufficiently emphasized by Filippo *et al.*!

There exists an early approach in the case of the characterization of cotangent bundles due to Nagano [10]. In fact, Nagano proves that if M is a differentiable manifold endowed with a vector field satisfying the same properties of those satisfied by the canonical vector field of a vector bundle, then there exists a unique bundle structure on M over the singular submanifold S of the vector field. If, moreover, M is an exact sympletic manifold then M is isomorphic to the cotangent bundle T^*S, indeed as symplectic vector bundles.

In the present paper we use the ideas of Nagano to give a characterization of tangent and stable tangent bundles. In fact, we give a different proof of the result of Filippo *et al.* [5]: if M is an integrable almost tangent manifold endowed with a vector field C which satisfies the properties of the Liouville vector field of a tangent bundle, then there exists a unique vector bundle structure on M isomorphic to TS (S being the singular submanifold) and such that the Liouville vector field and the canonical almost tangent structure of TS are transported via the isomorphism to C and the almost tangent structure on M, respectively. Similar results are obtained for integrable almost stable tangent manifolds.

All these problems are interesting for Mechanics. In fact, tangent and stable tangent bundles are the natural framework where the Lagrangian formalism is developed in the autonomous and non-autonomous cases, respectively [9]. Also, in the reduction of degenerate Lagrangian systems one obtains local regular Lagrangians with the same dynamical information and defined on some integrable almost tangent or stable tangent manifold according to the Lagrangian be autonomous or not ([1], [7], [6]). To do this, we project the geometric structures on TQ and $\mathbb{R} \times TQ$ (the phase space of velocities and the evolution space, respectively) to the quotient spaces by the gauge distribution. Then it is important to have some criteria to decide if these manifolds are globally tangent or stable tangent bundles. It is amazing that under the hypotheses for the projectiveness we deduce that *the quotient spaces are in fact tangent and stable tangent bundles, respectively.*

The paper is structured as follows. In section 2, we recall the main results of Nagano. In section 3, we obtain the characterization of tangent bundles and in section 4 we extend the results to the case of almost stable tangent bundles. Finally, in section 5 we apply the results of these two last sections to study the dynamics of degenerate Lagrangian systems.

2. CHARACTERIZATION OF VECTOR BUNDLES

Let M be a differentiable manifold and X a vector field on M. If $x \in M$ is a singular point of X, *i.e.* $X_x = 0$, then we define the characteristic operator $(A_X)_x$ of X at x as the linear endomorphism $(A_X)_x : T_xM \to T_xM$ given by

$$(A_X)_x\,(Y) = \nabla_Y X,$$

where ∇ is an arbitrary linear connection on M. It is easy to prove that $(A_X)_x$ does not depend on ∇. In fact, choose local coordinates (x^i) on M and put

$$X = X^i\,\frac{\partial}{\partial x^i}, \qquad Y = Y^i\,\frac{\partial}{\partial x^i}, \qquad \nabla_{\frac{\partial}{\partial x^i}}\,\frac{\partial}{\partial x^j} = \Gamma^k_{ij}\,\frac{\partial}{\partial x^k}.$$

Then

$$(A_X)_x Y = Y^j\,\frac{\partial X^i}{\partial x^j}\,\frac{\partial}{\partial x^i}, \tag{2.1}$$

since x is a singular point.

Now, let M be the total space of a vector bundle $M \to N$. Then the *canonical vector field of the vector bundle* M is the infinitesimal generator C of the global flow on M induced by the scalar multiplication on each fibre. This vector field satisfies the following properties:

(i) C generates a global one-parameter transformation group on M.

(ii) For each point $x \in M$, there exists a unique $\lim_{t \to -\infty} (\exp tC)(x)$, where $\exp tC$ denotes the flow of C.

(iii) The characteristic operator $(A_C)_x$ associated to C satisfies $((A_C)_x)^2 = (A_C)_x$ for each singular point x of C.

(iv) The set S of the singular points of C is a submanifold of M of codimension = rank $(A_C)_x$ for all $x \in S$.

In fact, choose bundle coordinates (x^i, y^a) on M, where (x^i) are coordinates in N and (y^a) are coordinates in the fibre. Then C is locally expressed by

$$C = y^a \frac{\partial}{\partial y^a}.$$

Hence the singular set S of C is the zero section of M, and so, it is diffeomorphic to N.

Nagano [10] has proved the converse:

THEOREM 2.1. – *Suppose that there exists a vector field C on a manifold M satisfying the above conditions* (i)-(iv). *Then there exists a unique vector bundle structure on M such that C is the canonical vector field.*

We give a sketch of the proof. If S is the singular submanifold, we put

$$N(S)_x = \{X \in T_x M \,|\, (A_C)_x (X) = X\}, \tag{2.2}$$

for each $x \in S$. Then $N(S)$ is the normal bundle of S in M, *i.e.*,

$$(TM)|_S = TS \oplus N(S).$$

Moreover we have

$$T_x S = \{X \in T_x M \,|\, (A_C)_x (X) = 0\}. \tag{2.3}$$

Then we can define a map $\phi : N(S) \to M$ as follows. We first define the exponential map $\exp : E \to M$ with respect to some linear connection, where E is a sphere bundle $E \subset N(S)$ and then we extend ϕ to $N(S)$. This construction is possible from the properties of C. Moreover ϕ becomes a diffeomorphism and then the vector bundle structure on $N(S) \to S$ is transferred to $M \to S$ in such a way that C becomes the canonical vector field of $M \to S$.

As a direct consequence we have the following

COROLLARY 2.1. – *Two vector bundles are isomorphic if and only if there exists a diffeomorphism which preserves the canonical vector fields.*

3. CHARACTERIZATION OF TANGENT BUNDLES

Let Q be a differentiable manifold and TQ its tangent bundle. Let J be the canonical almost tangent structure on TQ. J is a (1, 1)-type tensor field locally expressed by

$$J\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial v^i}, \qquad J\left(\frac{\partial}{\partial v^i}\right) = 0,$$

where (q^i, v^i) are bundle coordinates for TQ (*see* [9]). Let C be the canonical vector field on TQ; C is usually called the *Liouville vector field* on TQ and it is locally given by

$$C = v^i \frac{\partial}{\partial v^i}.$$

A direct computation in local coordinates shows that

$$JC = 0, \qquad L_C J = -J. \tag{3.1}$$

Obviously, C satisfies the conditions (i)-(iv) since it is the canonical vector field of TQ.

Now, we prove the converse.

THEOREM 3.1. – *Let M be a 2n-dimensional manifold endowed with an integrable almost tangent structure J and C a vector field on M satisfying* (3.1), i.e.,

$$JC = 0, \qquad L_C J = -J.$$

If C also satisfies the conditions (i)-(iv), *then there exists a unique vector bundle structure on M which is isomorphic to the tangent bundle TS of the singular submanifold S of C. Moreover this isomorphism transports the canonical almost tangent structure and the Liouville vector field of TS to J and C, respectively.*

Proof. – Since J is integrable then there exist adapted local coordinates (x^i, y^i) in such a way that J is locally given by

$$J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \qquad J\left(\frac{\partial}{\partial y^i}\right) = 0,$$

Suppose that C is locally written by

$$C = A^i \frac{\partial}{\partial x^i} + B^i \frac{\partial}{\partial y^i},$$

where $A^i = A^i(x, y)$, $B^i = B^i(x, y)$. From $JC = 0$ we have $A^i = 0$, and from $L_C J = -J$ we have

$$\frac{\partial B^i}{\partial y^j} = \delta^i_j.$$

Hence, we may define a new system of local coordinates $(\overline{x}^i, \overline{y}^i)$ by

$$\overline{x}^i = x^i, \ \overline{y}^i = B^i(x, y).$$

Thus we obtain

$$C = \overline{y}^i \frac{\partial}{\partial \overline{y}^i}, \tag{3.2}$$

and moreover $(\overline{x}^i, \overline{y}^i)$ are also adapted coordinates for J. As a first consequence we deduce that the singular submanifold S defined by C has dimension n.

Now, by Nagano's theorem we obtain a unique vector bundle structure on M over S such that C is the canonical vector field. This isomorphism ϕ is the one defined from $N(S)$ to M:

$$\begin{tikzcd} N(S) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & M \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

where π is the canonical projection and π' is the induced projection via ϕ.

Note that in coordinates $(\overline{x}^i, \overline{y}^i)$ the characteristic operator A_C is given by

$$(A_C)_x \left(\frac{\partial}{\partial \overline{x}^i}\right) = 0, \qquad (A_C)_x \left(\frac{\partial}{\partial \overline{y}^i}\right) = \frac{\partial}{\partial \overline{y}^i}, \tag{3.3}$$

at each point $x \in S$. Therefore, from (2.2) and (2.3) it follows

$$N(S)_x = \left\{X \in T_x M | X = b^i \frac{\partial}{\partial \overline{y}^i}\right\},$$

$$T_x S = \left\{X \in T_x M | X = a^i \frac{\partial}{\partial \overline{x}^i}\right\},$$

and then $J : TS \to N(S)$ is a diffeomorphism. In fact, this diffeomorphism is a vector bundle isomorphism

$$\begin{tikzcd} TS \arrow[rr, "J"] \arrow[dr, "\tau_S"'] & & N(S) \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

where $\tau_S : TS \to S$ is the canonical projection. Combining both results we obtain a vector bundle isomorphism

$$\begin{tikzcd} TS \arrow[rr, "\phi \circ J"] \arrow[dr, "\tau_S"'] & & M \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

which applies the canonical vector C_S field of TS to C. Moreover we can directly check that the following diagram is commutative:

$$\begin{tikzcd} TTS \arrow[r, "J_S"] \arrow[d, "T(\phi \circ J)"'] & TTS \arrow[d, "T(\phi \circ J)"] \\ TM \arrow[r, "J"] & TM \end{tikzcd}$$

where J_S denotes the canonical almost tangent structure on TS.

Finally, the unicity is a direct consequence of Corollary 2.1. □

Remark 3.1. – It would not be necessary to assume conditions (iii) and (iv) in the statement of Theorem 3.1, because thay are a direct consequence of (3.2). In fact, the singular set S, locally defined by the vanishing of the coordinates $\overline{y}^i$, is a regular submanifold of M of dimension n and, from (3.3) it follows that rank $(A_C)_x = n$ and $(A_C)_x^2 = (A_C)_x$, for all $x \in S$.

Remark 3.2. – In the approach of Filippo *et al.* [5] these authors prove that under the hypotheses of Theorem 3.1 there exists on M a maximal tangent bundle atlas. As a direct consequence M becomes a tangent bundle, say $M = TS$. Our approach enphasizes this result.

4. CHARACTERIZATION OF STABLE TANGENT BUNDLES

Let $\mathbb{R} \times TQ$ be a stable tangent bundle of an n-dimensional differentiable manifold Q with canonical projection $\tilde{\tau}_Q : \mathbb{R} \times TQ \to Q$. The Liouville vector field C and the canonical almost tangent structure J on TQ may be canonically extended to $\mathbb{R} \times TQ$. Then we construct a new tensor field $\overline{J}$ of type (1, 1) on $\mathbb{R} \times TQ$ by

$$\overline{J} = J + dt \otimes \frac{\partial}{\partial t}.$$

Then

$$\overline{J}\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t}, \qquad \overline{J}\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial v^i}, \qquad \overline{J}\left(\frac{\partial}{\partial v^i}\right) = 0,$$

where (t, q^i, v^i) are local coordinates on $\mathbb{R} \times TQ$. Thus

$$(1)\; dt\left(\frac{\partial}{\partial t}\right) = 1, \qquad (2)\; \overline{J}^2 = dt \otimes \frac{\partial}{\partial t}, \qquad (3)\; \operatorname{rank} \overline{J} = n + 1.$$

It is clear that C is the canonical vector field of the vector bundle $\mathbb{R} \times TQ \to \mathbb{R} \times Q$. Also the vector field $t\,(\partial/\partial t) + C$ on $\mathbb{R} \times TQ$ is the canonical vector field of the vector bundle $\mathbb{R} \times TQ \to Q$. Furthermore, we have

$$\overline{J}C = 0, \qquad L_C\overline{J} = -J, \tag{4.1}$$

Bearing in mind the properties (1)-(3) above, Oubiña [11] has introduced the notion of almost stable tangent structures as follows.

DEFINITION 4.1. – Let M be a differentiable manifold of dimension $2n+1$. A triple $(\overline{J}, \omega, \xi)$, where $\overline{J}$ is a tensor field of type (1, 1), ω is a 1–form and ξ is a vector field on M such that

$$(1)\; \omega(\xi) = 1, \qquad (2)\; \overline{J}^2 = \omega \otimes \xi, \qquad (3)\; \operatorname{rank} \overline{J} = n + 1,$$

is called an *almost stable tangent structure* and the manifold M an *almost stable tangent manifold.*

The integrability of an almost stable tangent structure was established in [8]:

PROPOSITION 4.1. – *An almost stable tangent structure* $(\overline{J}, \omega, \xi)$ *is integrable if and only if the Nijenhuis tensor* $N_{\overline{J}}$ *of* $\overline{J}$ *vanishes and* ω *is closed.*

Here the integrability means that around each point there exists a system of local coordinates (t, x^i, y^i) (called *adapted coordinates*) such that

$$\overline{J}\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t}, \quad \overline{J}\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \quad \overline{J}\left(\frac{\partial}{\partial y^i}\right) = 0, \quad \omega = dt, \quad \xi = \frac{\partial}{\partial t}.$$

In other words, $(\overline{J}, \omega, \xi)$ is integrable if and only if it is locally isomorphic to the canonical almost stable tangent structure on $\mathbb{R} \times TQ$.

THEOREM 4.1. – *Let* M *be a* $(2n+1)$*-dimensional manifold endowed with an integrable almost stable tangent structure* $(\overline{J}, \omega, \xi)$ *such that* ω *is an exact 1-form, say* $\omega = df$, *and* C *a vector field on* M *satisfying* (4.1), i.e.,

$$\overline{J}C = 0, \qquad L_C\overline{J} = -J$$

where $J = \overline{J} - \omega \otimes \xi$. *Suppose that the vector field* $\overline{C} = f\xi + C$ *satisfies the conditions* (i)-(iv). *Then there exists a unique vector bundle structure on* M *over* S, *where* S *is the singular submanifold of* $\overline{C}$, *which is isomorphic to the stable tangent bundle* $\mathbb{R} \times TS$. *Moreover this isomorphism transports the canonical almost stable tangent structure and the canonical vector field of* $\mathbb{R} \times TS$ *to* $\overline{J}$ *and* $\overline{C}$, *respectively.*

Proof. – Let (t, x^i, y^i) be a system of local coordinates adapted to $(\overline{J}, \omega, \xi)$. We have

$$J\left(\frac{\partial}{\partial t}\right) = 0, \qquad J\left(\frac{\partial}{\partial x^i}\right) = \frac{\partial}{\partial y^i}, \qquad J\left(\frac{\partial}{\partial y^i}\right) = 0. \tag{4.2}$$

Suppose that C is locally given by

$$C = \tau\,\frac{\partial}{\partial t} + A^i\,\frac{\partial}{\partial x^i} + B^i\,\frac{\partial}{\partial y^i}.$$

Since $\overline{J}C = 0$, we obtain $\tau = 0$ and $A^i = 0$. Moreover, from $L_C\overline{J} = -J$ and (4.2) we deduce

$$\frac{\partial B^i}{\partial t} = 0, \qquad \frac{\partial B^i}{\partial y^j} = \delta^i_j.$$

Then we have a new adapted coordinate system $(\overline{t}, \overline{x}^i, \overline{y}^i)$ defined by

$$\overline{t} = f,\ \overline{x}^i = x^i,\ \overline{y}^i = B^i\,(x, y).$$

C and $\overline{C}$ are actually written as follows:

$$\begin{aligned} C &= \overline{y}^i \frac{\partial}{\partial \overline{y}^i}, \\ \overline{C} &= \overline{t} \frac{\partial}{\partial \overline{t}} + \overline{y}^i \frac{\partial}{\partial \overline{y}^i}. \end{aligned} \tag{4.3}$$

Hence the singular submanifold S of $\overline{C}$ has dimension n. According to Nagano's theorem we deduce that there exists a unique vector bundle structure on M over S such that $\overline{C}$ is the canonical vector field. In fact, we have the following commutative diagram:

$$\begin{tikzcd} N(S) \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & M \arrow[dl, "\pi'"] \\ & S & \end{tikzcd}$$

where $N\,(S) = \{X \in T_xM | (A_{\overline{C}})_x(X) = X,\ x \in S\}$ is the normal bundle of S. From (2.1) and (4.3) it follows that $\xi_x \in N\,(S)_x$, for each $x \in S$. On the other hand, we define a vector bundle isomorphism $\psi : \mathbb{R} \times TS \rightarrow N\,(S)$ by

$$\psi : (r,\ X_x) \in \mathbb{R} \times T_xS \rightarrow r\xi_x + \overline{J}X_x \in N\,(S)_x.$$

Thus, we have the following commutative diagram:

$$\begin{tikzcd} \mathbb{R} \times TS \arrow[rr, "\psi"] \arrow[dr, "\tilde{\tau}_S"'] & & N(S) \arrow[dl, "\pi"] \\ & S & \end{tikzcd}$$

where $\tilde{\tau}_S : \mathbb{R} \times TS \rightarrow S$ is the canonical projection. Now, $\phi \circ \psi : \mathbb{R} \times TS \rightarrow M$ is a vector bundle isomorphism which transports the canonical almost stable tangent structure and the canonical vector field of $\mathbb{R} \times TS$ to $\overline{J}$ and $\overline{C}$ respectively. $\square$

Remark 4.1. – As in section 3, it is not necessary to assume conditions (iii) and (iv). They are now a direct consequence of (4.3).

5. AN APPLICATION TO DEGENERATE LAGRAGIAN SYSTEMS

Let $L : TQ \to \mathbb{R}$ be a Lagrangian function and $\alpha_L = J^* (dL)$ (resp. $\omega_L = -d\alpha_L$) the Poincaré-Cartan 1–form (resp. 2–form). If C is the Liouville vector field on TQ then the energy associated to L is $E_L = CL - L$ and the Euler-Lagrange equations corresponding to L can be written in the intrinsic form

$$\iota_X \omega_L = dE_L. \tag{5.1}$$

We say that L is *regular* if the Hessian matrix of L with repect to the velocities $(\partial^2 L/\partial v^i \partial v^j)$ is non-singular. Hence L is regular if and only if ω_L is symplectic. In such a case there exists a unique vector field ξ_L on TQ such that

$$\iota_{\xi_L} \omega_L = dE_L.$$

ξ_L is called the *Euler-Lagrange* vector field and it has the following properties:

(1) ξ_L is a second-order differential equation (SODE), (*i.e.*, $J\xi_L = C$);

(2) the paths of ξ_L are the solutions of the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = 0, \qquad v^i = \frac{dq^i}{dt}.$$

If L is degenerate then (5.1) will not possesses a globally defined solution in general, and even if it exists it will not be unique nor SODE. We say that a degenerate Lagrangian L admits a *global dynamics* if there exists a vector field X on TQ such that

$$\iota_X \omega_L = dE_L.$$

Let $K = \ker \omega_L = \{X \in T(TQ)/\iota_X \omega_L = 0\}$ be the characteristic distribution of ω_L; K is called the *gauge distribution.* If we suppose that ω_L has constant rank $2r$ (i.e., ω_L is a presymplectic structure of rank $2r$) then K is an involutive distribution of dimension 2 dim $Q - 2r$.

In order to study the reduction of degenerate Lagrangians we assume that the following conditions are satisfied:

(A1) ω_L is presymplectic;

(A2) L admits a global dynamics;

(A3) the foliation defined by K is a fibration; *i.e.*, $(TQ)_0 = TQ/K$ has a structure of quotient manifold such that the canonical projection $\pi_L : TQ \to (TQ)_0$ is a surjective submersion.

Cantrijn *et al.* [1] have obtained a classification of Lagrangians in three types accordingly to the dimension of $K \cap V\,(TQ)$, where $V(TQ)$ is the vertical distribution.

Type I: $\dim K = \dim K \cap V\,(TQ) = 0$,

Type II: $\dim K = 2 \dim K \cap V\,(TQ) \neq 0$,

Type III: $\dim K < 2 \dim K \cap V\,(TQ)$.

Lagrangians of type I are precisely regular Lagrangians. A Lagrangian L is of type II if and only if $J\,(K) = K \cap V\,(TQ)$. If L is of type II and it admits a global dynamics, then there exists a SODE ξ on TQ such that

$$\iota_\xi \omega_L = dE_L.$$

The following properties were proved in [1]:

(1) ξ projects onto $(TQ)_0$ to a vector field ξ_0;

(2) E_L projects onto $(TQ)_0$ to a function E_0;

(3) Moreover, if K is a tangent distribution (i.e., K is the natural lift of a distribution D on Q, *see* [1]) then J and C projects onto $(TQ)_0$ to an integrable almost tangent structure J_0 and to a vector field C_0.

If L satisfies (A1), (A2), (A3) and K is a tangent distribution, we deduce that $(TQ)_0$ is an integrable almost tangent manifold with almost tangent structure J_0 and a vector field C_0 such that

$$J_0 C_0 = 0, \qquad L_{C_0} J_0 = -J_0,$$

because $JC = 0$ and $L_C J = -J$. Since C is complete we deduce that C_0 is complete too, in such a way that the flow of C_0 is precisely the projection of the flow of C, *i.e.*,

$$(\exp\, tC_0)\,(\pi_L\,(x)) = \pi_L\,((\exp\, tC)\,(x)), \qquad x \in TQ,$$

Hence there exists $\lim_{t \to -\infty} (\exp\, tC_0)\,(x_0)$ and it is unique.

Thus, from Theorem 3.1, we have

PROPOSITION 5.1. – *$(TQ)_0$ has a unique structure of vector bundle which is isomorphic to the tangent bundle TS of the singular manifold S of C_0. Moreover this isomorphism transports the canonical almost tangent structure and the Liouville vector field of TS to J_0 and C_0, respectively.*

Furthermore, under the above hypothesis, Cantrijn *et al.* have proved that there exists a local regular Lagrangian L_0 on $(TQ)_0$ such that $L_0 \circ \pi_L$ and L are gauge equivalent, and hence, they give the same dynamical information. Actually, we have proved that this local Lagrangian L_0 is in fact defined on some *bona fide* tangent bundle TS.

As an illustration we can consider the dynamics of an electron in a monopole field [5]. There are no a global regular Lagrangian description, but there exists a global degenerate Lagrangian description. In fact, the configuration manifold is $Q = SU(2) \times \mathbb{R}$ and the global degenerate Lagrangian L is defined on $TQ = T(SU(2) \times \mathbb{R})$. The gauge distribution is

$$K = \{X_3^V, X_3^C\},$$

where X_3 is the fundamental vector field of $U(1) \cong S^1$ in the Hopf bundle $SU(2) \cong S^3 \to S^2$. Then a direct computation shows that

$$(TQ)_0 = T(SU(2) \times \mathbb{R})/K = T(SU(2)/U(1) \times \mathbb{R}).$$

Next, we shall give an application of Theorem 4.1 to degenerate non-autonomous Lagrangian systems.

Let $L : \mathbb{R} \times TQ \to \mathbb{R}$ be a non-autonomous Lagrangian function and $\theta_L = \tilde{J}^*(dL) + Ldt$(resp.$\Omega_L = -d\theta_L$) the Poincaré-Cartan 1-form (resp. 2-form), where $\tilde{J} = J - C \otimes dt$. The motions equations corresponding to L are globally written as

$$\iota_X \Omega_L = 0, \qquad \iota_X dt = 1. \tag{5.2}$$

A direct computation shows that L is *regular* if and only if (Ω_L, dt) is a cosymplectic structure on $\mathbb{R} \times TQ$. In such a case there exists a unique vector field ξ_L on $\mathbb{R} \times TQ$ such that

$$\iota_{\xi_L} \Omega_L = 0, \qquad \iota_{\xi_L} dt = 1.$$

As in the autonomous case, ξ_L is called the *Euler-Lagrange* vector field and we have

(1) ξ_L is a non-autonomous SODE, (*i.e.*, $\tilde{J}\xi_L = \partial/\partial t + C$), and

(2) the solutions of ξ_L are the solutions of the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial v^i}\right) - \frac{\partial L}{\partial q^i} = 0, \qquad v^i = \frac{dq^i}{dt}.$$

As in the autonomous case, if L is degenerate then (5.2) will not possess a globally defined solution in general, and even if it exists it will be neither unique nor a non-automomous SODE.

The *gauge distribution* is defined by

$$K = \ker \Omega_L \cap \ker dt.$$

In order to study the reduction of degenerate non-autonomous Lagrangians we assume that the following conditions are satisfied:

(NA1) $(\Omega_L,\ dt)$ is a precosymplectic structure of rank $2r$ (*i.e.*, $\Omega_L^r \wedge dt \neq 0,\ \Omega_L^{r+1} = 0$);

(NA2) L admits a global dynamics X; that is, there exists a solution of (5.2).

(NA3) the foliation defined by K is a fibration.

We remark that (NA1) implies (NA2) (*see* [3], [4], [7]). The canonical projection is denoted by $\pi_L : \mathbb{R} \times TQ \to (\mathbb{R} \times TQ)/K$.

The reduction of L was studied by de León, Mello and Rodrigues (*see* [7]) and Ibort and Marín (*see* [6]). Furthermore, these last authors have obtained a classification of degenerate non-autonomous Lagrangians in three types accordingly to the dimension of $K \cap V(\mathbb{R} \times TQ)$, where $V(\mathbb{R} \times TQ)$ is the vertical subbundle of $T(\mathbb{R} \times TQ)$ corresponding to the projection $\mathbb{R} \times TQ \to \mathbb{R} \times Q$.

Type I: $\dim K = \dim K \cap V(\mathbb{R} \times TQ) = 0$,

Type II: $\dim K = 2 \dim K \cap V(\mathbb{R} \times TQ) \neq 0$,

Type III: $\dim K < 2 \dim K \cap V(\mathbb{R} \times TQ)$.

Lagrangians of type I are precisely regular Lagrangians. A Lagrangian L is of type II if and only $\overline{J}(K) = \tilde{J}(K) = J(K) = K \cap V(\mathbb{R} \times TQ)$. If L is of type II and it admits a global dynamics, then there exists a non-autonomous SODE ξ on $\mathbb{R} \times TQ$ such that

$$\iota_\xi \Omega_L = 0, \qquad \iota_\xi dt = 1.$$

Suppose that L is the type II and it satisfies (NA1), (NA2) and (NA3). Then $(\Omega_L,\ dt)$ projects onto a cosymplectic structure $(\Omega_0,\ \eta_0)$ on the quotient manifold $(\mathbb{R} \times TQ)/K$, and ξ onto a vector field ξ_0 such that

$$\iota_{\xi_0} \Omega_0 = 0, \qquad \iota_{\xi_0} \eta_0 = 1.$$

Moreover, if K is an s-tangent distribution (*i.e.*, K is the natural lift of an involutive distribution D on $\mathbb{R} \times Q$) then (*see* de León *et al.* [7]) $\overline{J}$, $\partial/\partial t$ and the Liouville vector field C project onto a (1, 1)-type tensor field $\overline{J}_0$, a vector field T_0 and a vector field C_0, respectively, in such a way that $(\overline{J}_0,\ T_0,\ \eta_0)$ is an integrable almost stable tangent structure. Also, we can prove that there exists a local regular Lagrangian L_0 defined on some open subset of $(\mathbb{R} \times TQ)/K$ such that $L_0 \circ \pi_L$ and L are gauge equivalent.

Furthermore, suppose that the distribution D on $\mathbb{R} \times Q$ is tangent to Q, *i.e.*,

$$D_{(t,\, x)} = 0_t \oplus \tilde{D}_x,$$

where $\tilde{D}$ is an involutive distribution on Q. Notice that $dt(X) = 0$ for any $X \in K$. In such a case it is easy to see that $(\mathbb{R} \times TQ)/K \cong \mathbb{R} \times (TQ/K)$ and further the almost tangent structure J projects onto an almost tangent structure J_0 on TQ/K. Also, $\eta_0 = dt$, $T_0 = \partial/\partial t$ and $\overline{J}_0 = J_0 + \eta_0 \otimes T_0$. Then, we can apply Theorem 4.1 and conclude that

COROLLARY 5.1. – $(\mathbb{R} \times TQ)/K \cong \mathbb{R} \times (TQ/K)$ *has a unique structure of vector bundle over* S *which is isomorphic to* $\mathbb{R} \times TS$, *where* S *is the singular submanifold of* C_0. *This isomorphism transports the canonical almost stable tangent structure and the canonical vector field of* $\mathbb{R} \times TS$ *to* $\overline{J}_0$ *and* $\overline{C}_0$, *where* $\overline{C}_0 = T_0 + C_0$.

Consequently, the local Lagrangians L_0 *are in fact defined on some* bone fide *stable tangent bundle* $\mathbb{R} \times TS$.

REFERENCES

[1] F. Cantrijn, J. F. Cariñena, M. Crampin and L. A. Ibort, Reduction of Degenerate Lagrangian Systems. *J. Geom. Phys*, **3**, 1986, (3) p. 353-400.

[2] M. Crampin and G. Thompson, Affine Bundles and Integrable Almost Tangent Structures, *Math. Proc. Camb. Phil. Soc.*, **98**, 1985, p. 61-71.

[3] D. Chinea M. de León and J. C. Marrero, The Constraints Algorithm for Degenerate Non-Autonomous Lagrangian Systems, *Proceedings Workshop Recent Topics in Differential Geometry*, Puerto de la Cruz (Tenerife, Spain), *Publ. Univ. La Laguna*, Serie Informes, **32**, 1991, p. 13-29.

[4] D. Chinea M. de León and J. C. Marrero, The Constraints Algorithm for Time-dependent Lagrangians, to appear in *Journal of Mathematical Physics*, 1994.

[5] S. De Filippo, G. Landi, G. Marmo and G. Vilasi, Tensor fields defining a Tangent Bundle Structure, *Ann. Inst. Henri Poincaré*, **50**, 1989 (2), p. 205-218.

[6] L. A. Ibort and J. Marín-Solano, A Geometric Classification of Lagrangian Functions and the Reduction of Evolution Space, *J. Phys. A: Math. Gen*, **25**, 1992, p. 3353-3367.

[7] M. de León, M. H. Mello and P. R. Rodrigues, Reduction of Degenerate Non-autonomous Lagrangian Systems, *Mathematical Aspects of Classical Field Theory*, Seatle, Washington (USA), July 21-25, 1991. Ed. by Mark J. Gotay, Jerrold E. Marsden and Vincent Moncrief, *Contemporary Mathematics*, **132**, 1992, p. 275-305.

[8] M. de León, J. A. Oubiña and M. Salgado, Integrable Almost s-Tangent Structures, to appear in *Rendiconti di Matematica*, **14**, 1994.

[9] M. de León and P. R. Rodrigues, Methods of Differential Geometry in Analytical Mechanics, *North-Holland Math.*, Ser. 152, Amsterdam, 1989.

[10] T. Nagano 1-Forms with the Exterior Derivative of Maximal Rank, *J. Differential Geometry*, **2**, 1968, p. 253-264

[11] J. A. Oubiña, Almost S-Tangent Structures, *Geometriae Dedicata* **14**, 1983, p. 395-403.

[12] G. Thompson, Integrable Almost Cotangent Structures and Legendrian Bundles, *Math. Proc. Camb. Phil. Soc.*, **101**, 1987, p. 61-78.

[13] G. Thompson and U. Schwardmann, Almost Tangent and Cotangent Structure in the Large, *Transactions of the Amer. Math. Soc.*, **327**, 1991 (1), p. 313-328.

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