ANNALES DE L'I. H. P., SECTION A

ENRICO MASSA

ENRICO PAGANI

Jet bundle geometry, dynamical connections, and the inverse problem of lagrangian mechanics

Annales de l'I. H. P., section A, tome 61, nº 1 (1994), p. 17-62

<http://www.numdam.org/item?id=AIHPA_1994__61_1_17_0>

Ann. Inst. Henri Poincaré,

Vol. 61, n° 1, 1994, p. 17-62. *Physique théorique*

Jet bundle geometry, dynamical connections, and the inverse problem of Lagrangian mechanics

by

Enrico MASSA

Dipartimento di Matematica dell'Università di Genova,
Via L. B. Alberti, 4. 16132 Genova, Italia

and

Enrico PAGANI

Dipartimento di Matematica dell'Università di Trento
38050 Povo di Trento, Italia

ABSTRACT. – Some aspects of the geometry of jet-bundles, especially relevant for the formulation of Classical Mechanics, are investigated. The main result is the construction of a tensor analysis on the first jet extension of the configuration space-time, based on a suitable linear connection, determined entirely by the dynamics of the system. The significance of this "dynamical connection" in the geometrization of Classical Mechanics is discussed, paying a special attention to two particular aspects, namely the implementation of the concept of "relative time derivative" in the Lagrangian framework, and the derivation of the Helmholtz conditions for the inverse problem of Lagrangian Dynamics.

Key-words: Lagrangian Dynamics, inverse problem, linear and affine connections.

PACS: 02.40.+m, 03.20.+i

1991 Mathematical subject Classification: 70 D 10, 34 A 55, 53 B 05

This research was partly supported by the National Group for Mathematical Physics of the Italian Research Council (CNR), and by the Italian Ministry for Public Education, through the research project "Metodi Geometrici e Probabilistici in Sistems Dinamici, Meccanica Statistica, Relatività e Teoria dei Campi".

RÉSUMÉ. – Dans cet article on étudie des aspects de la géométrie des espaces des jets qui sont importants pour une formulation géométrique de la mécanique classique. On construit une « analyse tensorielle » sur la première extension des jets de l'espace-temps des configurations moyennant une connexion linéaire déterminée complètement par la dynamique du système. On met en évidence le rôle joué par cette « connexion dynamique » dans la géométrisation de la mécanique classique ; en particulier on introduit le concept de « dérivée temporelle relative » dans le contexte lagrangien et les conditions de Helmholtz pour le problème inverse de la dynamique lagrangienne.

INTRODUCTION

Jet bundle geometry provides a natural environment for the frame-independent formulation of Classical Mechanics, especially suited to the study of systems with a finite number of degrees of freedom.

The idea is to associate to each such system a corresponding configuration space-time $\mathcal{V}_{n+1}$, viewed as a fiber bundle over $\mathbb{R}$, with projection t: $\mathcal{V}_{n+1} \to \mathbb{R}$ formalizing the concept of *absolute time*. The first jet-extension $j_1(\mathcal{V}_{n+1})$ is then naturally identified with the "evolution space" (or "velocity space") of the system ([1], [2]).

During the last decade, the theory of jet-bundles, and, in particular, the knowledge of the properties of the first jet space $j_1(\mathcal{V}_{n+1})$, has been considerably enhanced through the introduction of two important geometrical objects, namely the *fundamental tensor* of $j_1(\mathcal{V}_{n+1})$, and the *horizontal distribution* – often described in the language of the theory of connections – associated with a given dynamical flow (*see*, among others ([1], [3], [4]) and references therein).

As a consequence of this fact, the recent literature has witnessed a renewed interest towards several classical problems, like the study of dynamical symmetries, also in connection with possible generalizations of Noether's theorem ([5]-[9]), the inverse problem of Lagrangian Mechanics ([1], [10]-[19]), the theory of singular Lagrangians ([20]-[22]), the dynamics of non-holonomic systems ([2], [23]), and so on.

In this paper we discuss some aspects of the geometry of $j_1(\mathcal{V}_{n+1})$, all more or less directly related to the affine nature of the fibration $j_1(\mathcal{V}_{n+1}) \xrightarrow{t} \mathcal{V}_{n+1}$.

After a review of the main geometrical ideas involved in the construction of the fundamental tensor, in Section 1 attention is focussed on a special subalgebra of the tensor algebra over $j_1(\mathcal{V}_{n+1})$, called the *virtual algebra*, playing a role quite similar to the algebra of "free tensors" on an affine space, and endowed with a well defined law of *parallel transport* along the fibres.

The analysis includes a description of the basic algebraic and differential operations induced by the fundamental tensor: among others, we mention here the *fiber differentiation* d_v, whose action on the Grassmann algebra $\mathcal{G}(j_1(\mathcal{V}_{n+1}))$ preserves the subalgebra $\hat{\mathcal{G}}(j_1(\mathcal{V}_{n+1}))$ of virtual differential forms, satisfies the condition $d_v \cdot d_v = 0$, and reflects in a natural way the affine nature of the fibres of $j_1(\mathcal{V}_{n+1})$.

In Section 2 we discuss the construction of a *tensor analysis* over $j_1(\mathcal{V}_{n+1})$. The starting point relies on the introduction of a special class $\hat{\mathcal{S}}$ of linear connections, such that, for each $\nabla \in \hat{\mathcal{S}}$, the resulting covariant differentiation preserve all the intrinsic geometric structures of $j_1(\mathcal{V}_{n+1})$, namely the temporal 1-form dt, the fundamental tensor, and the parallel transport along the fibres. For technical reasons, to be explained in Section 2.1, the characterization is completed by adding a few additional restrictions concerning the torsion and curvature tensors.

In this way we end up with a class $\hat{\mathcal{S}}$, the elements of which have a clear dynamical significance, in the sense that for each choice of the dynamical flow $\mathbf{Z}$, the condition $\nabla\mathbf{Z} = 0$ – *i.e.* the requirement that the differential algorithm preserve not only the geometry, but also the dynamics of the system, as represented by the flow itself – singles out a unique connection $\nabla \in \hat{\mathcal{S}}$, thereby called the *dynamical connection* associated with $\mathbf{Z}$.

In particular, it is seen that the notion of horizontality induced by the dynamical connection in $j_1(\mathcal{V}_{n+1})$ agrees with the usual one, based on the distinguished role assigned to the Lie derivative of the fundamental tensor along $\mathbf{Z}$ (*see* e.g. [1], [3]).

The physical significance of the dynamical connection is further clarified in Section 2.3, through a discussion of a possible implementation of the concept of *relative time derivative* in a Lagrangian context. The analysis stems from the fact that every frame of reference $\mathfrak{F}$ determines its own dynamical flow $\mathbf{Z}_{\mathfrak{F}}$ in $j_1(\mathcal{V}_{n+1})$ (*see* e.g. [2]), and therefore also a corresponding covariant derivative, induced by the associated dynamical connection. Among other aspects, the argument illustrates the geometrical counterpart of Poisson's formulae in $j_1(\mathcal{V}_{n+1})$.

Finally, in Section 3, the entire mathematical apparatus is applied to the study of the so called *inverse problem* of Lagrangian Mechanics [24] (in this connection *see* also ([25], [26]), respectively for the $n = 1$ and $n = 2$ cases). The line of approach relies on an extension of the theory of generalized potential to virtual forms of arbitary degree. This is seen to provide a convenient geometrical setting, yielding back, on a unified basis, the different formulations of Helmholtz's conditions obtained in Refs. ([1], [12], [27]).

1. THE VIRTUAL ALGEBRA OVER $j_1(\mathcal{V}_{n+1})$

1.1. Preliminaries

For future reference, in this Subsection we review some aspects of jet-bundle theory, especially relevant to the developments of Lagrangian Mechanics. For a more exhaustive account, the reader is referred to the current literature (*see*, among others, Refs. ([2]-[4], [23], [28])).

(i) Let $\mathcal{B}$ be a mechanical system, with n degrees of freedom. Keeping the same notation as in [2], we associate with $\mathcal{B}$ an $(n+1)$-dimensional differentiable manifold $\mathcal{V}_{n+1}$, called the *configuration space-time*, carrying a natural fibration t: $\mathcal{V}_{n+1} \to \mathbb{R}$, identified with the *absolute time.*

Also, we denote by $V(\mathcal{V}_{n+1})$ and $j_1(\mathcal{V}_{n+1})$ respectively the *vertical bundle* over $\mathcal{V}_{n+1}$, and the first jet-extension of $\mathcal{V}_{n+1}$ associated with the stated fibration.

As it is well known, both spaces $V(\mathcal{V}_{n+1})$ and $j_1(\mathcal{V}_{n+1})$ may be viewed as submanifolds of the tangent space $\mathbf{T}(\mathcal{V}_{n+1})$, according to the identifications

$$V(\mathcal{V}_{n+1}) = \{\mathbf{X} \,|\, \mathbf{X} \in \mathbf{T}(\mathcal{V}_{n+1}),\ \langle \mathbf{X},\, dt\rangle = 0\} \tag{1.1a}$$

$$j_1(\mathcal{V}_{n+1}) = \{\mathbf{X} \,|\, \mathbf{X} \in \mathbf{T}(\mathcal{V}_{n+1}),\ \langle \mathbf{X},\, dt\rangle = 1\} \tag{1.1b}$$

Eqs. (1.1a, b) exhibit the nature of $V(\mathcal{V}_{n+1})$ and $j_1(\mathcal{V}_{n+1})$ respectively as a vector bundle over $\mathcal{V}_{n+1}$, and as an affine bundle over $\mathcal{V}_{n+1}$, modelled on $V(\mathcal{V}_{n+1})$.

Following the standard conventions, given any local coordinate system $t, q^1, \ldots, q^n$ on $\mathcal{V}_{n+1}$, we denote by $t, q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n$ the corresponding jet-coordinate system on $j_1(\mathcal{V}_{n+1})$. In this way, any vertical vector $\mathbf{v}$ is represented in the form

$$\mathbf{v} = v^i \left(\frac{\partial}{\partial q^i}\right)_{\pi(\mathbf{v})} \tag{1.2a}$$

while the identification between points in $j_1(\mathcal{V}_{n+1})$ and vectors in $\mathcal{V}_{n+1}$ is made explicit by the relation

$$z \quad \leftrightarrow \quad \left(\frac{\partial}{\partial t}\right)_{\pi(z)} + \dot{q}^i(z)\left(\frac{\partial}{\partial q^i}\right)_{\pi(z)} \tag{1.2b}$$

(ii) Let us now stick our attention to the fibered manifold π: $j_1(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$. For each $x \in \mathcal{V}_{n+1}$, we denote by $\Sigma_x =: \pi^{-1}(x)$ the fibre over x, viewed as submanifold of $j_1(\mathcal{V}_{n+1})$. Also, for all $z \in j_1(\mathcal{V}_{n+1})$, we indicate by $(\pi_z)_*: \mathbf{T}_z(j_1(\mathcal{V}_{n+1})) \to \mathbf{T}_{\pi(z)}(\mathcal{V}_{n+1})$ the differential of the projection π at z (the "push-forward" map), and by $(\pi_z)_*{}^*$: $\mathbf{T}^*_{\pi(z)}(\mathcal{V}_{n+1}) \to \mathbf{T}^*_z(j_1(\mathcal{V}_{n+1}))$ the corresponding adjoint map ("pull-back"), both extended – the first to arbitrary contravariant objects, and the second to arbitrary covariant ones – through the operation of tensor product.

In view of eqs. (1.2a, b), each fibre Σ_x is an affine space, modelled on $V_x(\mathcal{V}_{n+1})$, the modelling being given by the ordinary vector addition $(z, \mathbf{v}) \to z + \mathbf{v}$ in $\mathbf{T}_x(\mathcal{V}_{n+1})$.

As a result, we have a canonical identification of the tangent space $\mathbf{T}(\Sigma_x)$ with the cartesian product $\Sigma_x \times V_x(\mathcal{V}_{n+1})$.

DEFINITION 1.1. – *For each $z \in j_1(\mathcal{V}_{n+1})$, the tangent space to the fibre $\Sigma_{\pi(z)}$ at z – henceforth denoted by $V_z(j_1(\mathcal{V}_{n+1}))$ – is called the vertical space at z. The map assigning to each $z \in j_1(\mathcal{V}_{n+1})$ the vertical space $V_z(j_1(\mathcal{V}_{n+1}))$, viewed as a subspace of $\mathbf{T}_z(\mathcal{V}_{n+1})$, is called the vertical distribution over $j_1(\mathcal{V}_{n+1})$. The vector bundle $V(j_1(\mathcal{V}_{n+1})) := \bigcup_z V_z(j_1(\mathcal{V}_{n+1}))$ is called the vertical bundle over $j_1(\mathcal{V}_{n+1})$.*

In view of the identification $V_z(j_1(\mathcal{V}_{n+1})) \simeq V_{\pi(z)}(\mathcal{V}_{n+1})$, every vertical vector $\mathbf{v}$ at $\pi(z)$ determines a corresponding vertical vector $\hat{\mathbf{v}}$ at z, identified with the tangent vector to the "straight line" $\xi \to z + \xi\,\mathbf{v}$ at the point $\xi = 0$.

The correspondence $\mathbf{v} \to \hat{\mathbf{v}}$ is known as the *vertical lift* of vectors. In local coordinates, we have the explicit representation

$$\mathbf{v} = v^i\left(\frac{\partial}{\partial q^i}\right)_{\pi(z)} \quad \Rightarrow \quad \hat{\mathbf{v}} = v^i\left(\frac{\partial}{\partial \dot{q}^i}\right)_z \tag{1.3a}$$

or, in terms of bases

$$\widehat{\left(\frac{\partial}{\partial q^i}\right)}_{\pi(z)} = \left(\frac{\partial}{\partial \dot{q}^i}\right)_z \tag{1.3b}$$

(iii) The adjoint map $(\pi_z)_*{}^*$ determines a homomorphism $\sigma \to \hat{\sigma} := (\pi_z)_*{}^*(\sigma)$ of the exterior algebra $\Lambda_{\pi(z)}(\mathcal{V}_{n+1})$ at $\pi(z)$ into the exterior algebra at z. The image space $\mathcal{S}_z(j_1(\mathcal{V}_{n+1})) := (\pi_z)_*{}^*(\Lambda_{\pi(z)}(\mathcal{V}_{n+1}))$

is called the algebra of *semibasic forms* at z (*see* e.g. [29]). In local coordinates, every r-form $\hat{\sigma} \in \mathcal{S}_z(j_1(\mathcal{V}_{n+1}))$ will be represented in the form

$$\hat{\sigma} = r\,\sigma_{0\,i_2\ldots i_r}\,(dt)_z \wedge (dq^{i_2})_z \wedge \ldots \wedge (dq^{i_r})_z + \sigma_{i_1\ldots i_r}\,(dq^{i_1})_z \wedge \ldots \wedge (dq^{i_r})_z \tag{1.4}$$

the presence of the factor r being a pure matter of notational convenience. From eq. (1.4) it is easily seen that the correspondence $(\pi_z)_*{}^*$ maps $\Lambda^r_{\pi(z)}(\mathcal{V}_{n+1})$ isomorphically onto $\mathcal{S}^r_z(j_1(\mathcal{V}_{n+1}))$, for all $z \in j_1(\mathcal{V}_{n+1})$, and all $r = 0, \ldots, n+1$.

This allows to introduce a bilinear pariring $\langle\,\|\,\rangle$ between vertical vectors and semibasic 1-forms at z, different from the ordinary one – which, on the stated objects, would be identically zero – and related to the lifts $\mathbf{v} \to \hat{\mathbf{v}}$, $\sigma \to \hat{\sigma}$ through the identification

$$\langle \hat{\mathbf{v}} \,|\, \hat{\sigma} \rangle := \langle \mathbf{v}, \sigma \rangle, \qquad \forall\, \mathbf{v} \in V_{\pi(z)}(\mathcal{V}_{n+1}), \quad \sigma \in \mathbf{T}^*_{\pi(z)}(\mathcal{V}_{n+1}) \tag{1.5}$$

$\langle\ ,\ \rangle$ denoting the ordinary pairing in $\mathcal{V}_{n+1}$. Notice that, according to eq. (1.5), the functional $\langle (dt)_z \| \cdot \rangle$ vanishes identically on $V_z(j_1(\mathcal{V}_{n+1}))$.

(iv) In view of eqs. (1.2a, b), each point $z \in j_1(\mathcal{V}_{n+1})$ determines a corresponding *vertical projection* $\mathcal{P}_z$ on the tangent space $\mathbf{T}_{\pi(z)}(\mathcal{V}_{n+1})$, defined by the relation

$$\mathcal{P}_z(\mathbf{X}) := \mathbf{X} - \langle \mathbf{X}, (dt)_{\pi(z)} \rangle\, z, \qquad \forall\, \mathbf{X} \in \mathbf{T}_{\pi(z)}(\mathcal{V}_{n+1}) \tag{1.6a}$$

By duality, this gives rise to a projection $\mathcal{P}^*_z$ on the cotangent space $\mathbf{T}^*_{\pi(z)}(\mathcal{V}_{n+1})$ – and, more generally, on the entire exterior algebra $\Lambda_{\pi(z)}(\mathcal{V}_{n+1})$ – on the basis of the equation

$$\mathcal{P}^*_z(\sigma) = z \,\rfloor\, (dt \wedge \sigma) \tag{1.6b}$$

the symbol $\rfloor$ denoting right interior multiplication [30].

From eq. (1.6b) it is easily seen that the condition $\mathcal{P}^*_z(\sigma) = 0$ is necessary and sufficient in order for a 1-form $\sigma \in \mathbf{T}^*_{\pi(z)}(\mathcal{V}_{n+1})$ to satisfy $\langle \mathbf{v}, \sigma \rangle = 0$ for all vertical vectors $\mathbf{v} \in V_{\pi(z)}(\mathcal{V}_{n+1})$.

In this respect, the image space of $\mathbf{T}^*_{\pi(z)}(\mathcal{V}_{n+1})$ under the map $\mathcal{P}^*_z$ may therefore be identified with the *dual* of the vector space $V_{\pi(z)}(\mathcal{V}_{n+1})$. Equivalently, using the isomorphism between $V_{\pi(z)}(\mathcal{V}_{n+1})$ and $V_z(j_1(\mathcal{V}_{n+1}))$ given by the vertical lift, as well as the injectivity of the map $(\pi_z)_*{}^*$, we may regard the image space $V^*_z(j_1(\mathcal{V}_{n+1})) := (\pi_z)_*{}^*\,\mathcal{P}^*_z((\mathbf{T}^*_{\pi(z)}(\mathcal{V}_{n+1})))$ as the dual of $V_z(j_1(\mathcal{V}_{n+1}))$ under the pairing (1.5).

DEFINITION 1.2. – *The tensor algebra* $\hat{\mathcal{D}}_z(j_1(\mathcal{V}_{n+1}))$ *generated by the spaces* $V_z(j_1(\mathcal{V}_{n+1}))$ *and* $V_z^*(j_1(\mathcal{V}_{n+1}))$, *regarded as dual to each other under the pairing (1.5), is called the virtual algebra at* z. *The totality* $\hat{\mathcal{D}}(j_1(\mathcal{V}_{n+1}))$ *of tensor fields* $\mathbf{T}$ *defined on open domains* $U \subset j_1(\mathcal{V}_{n+1})$ *and satisfying* $\mathbf{T}(z) \in \hat{\mathcal{D}}_z(j_1(\mathcal{V}_{n+1}))\ \forall z \in U$ *is called the virtual algebra over* $j_1(\mathcal{V}_{n+1})$.

In a similar way one may define the covariant virtual algebra, the symmetric virtual algebra, the Grassmann virtual algebra, etc.

From a structural viewpoint, the significance of Definition 1.2 lies in the fact that, for each $x \in \mathcal{V}_{n+1}$, the restriction of the virtual algebra to the fibre $\Sigma_x \subset j_1(\mathcal{V}_{n+1})$ coincides with the tensor algebra of the vector space $V_x(\mathcal{V}_{n+1})$, *i.e.* with the "natural" tensor algebra over Σ_x consistent with the nature of the latter as an affine space modelled on $V_x(\mathcal{V}_{n+1})$. This provides e.g. a well defined notion of *parallel transport* of virtual objects (in particular, of vertical vectors) along the fibres. We shall return on this point in Section 2.

In local coordinates, given any $\sigma = a_0\,(dt)_{\pi(z)} + a_i\,(dq^i)_{\pi(z)}$, eqs. (1.2b), (1.6b) imply the result

$$(\pi_z)_*{}^* \cdot \mathcal{P}_z^*(\sigma) = (\pi_z)_*{}^*(\sigma - \langle z, \sigma\rangle\,(dt)_{\pi(z)}) = a_i\,((dq^i)_z - \dot{q}^i(z)\,(dt)_z)$$

From this it follows easily that the contact 1-forms

$$\omega^i := dq^i - \dot{q}^i\,dt, \qquad i = 1, \ldots, n \tag{1.7}$$

form a local basis for the covariant virtual algebra over $j_1(\mathcal{V}_{n+1})$, dual of the basis $\left\{\frac{\partial}{\partial \dot{q}^1}, \ldots, \frac{\partial}{\partial \dot{q}^n}\right\}$ under the pairing (1.5). In particular, on account of the previous discussion, it is also seen that, when restricted to a single fibre Σ_x, the fields $\frac{\partial}{\partial \dot{q}^i}$ and the 1-forms ω^i provide an *affine basis* for the algebra of "free tensors" over Σ_x, in the ordinary sense of the term.

Under a change of jet coordinates, we have the transformation laws

$$\left.\begin{aligned} \frac{\partial}{\partial \dot{\bar{q}}^i} &= \frac{\partial q^k}{\partial \bar{q}^i}\,\frac{\partial}{\partial \dot{q}^k};\\ \bar{\omega}^i = d\bar{q}^i - \dot{\bar{q}}^i\,dt &= \frac{\partial \bar{q}^i}{\partial q^k}\,dq^k + \frac{\partial \bar{q}^i}{\partial t}\,dt\\ -\left(\frac{\partial \bar{q}^i}{\partial q^k}\,\dot{q}^k + \frac{\partial \bar{q}^i}{\partial t}\right) dt &= \frac{\partial \bar{q}^i}{\partial q^k}\,\omega^k \end{aligned}\right\} \tag{1.8}$$

which involve the time variable t only parametrically, thus justifying the attribute "virtual" assigned to the algebra $\hat{\mathcal{D}}(j_1(\mathcal{V}_{n+1}))$.

Remark 1.1. – The homomorphism $z\rfloor : \Lambda_{\pi(z)}(\mathcal{V}_{n+1}) \to \Lambda_{\pi(z)}(\mathcal{V}_{n+1})$ may be lifted to an antiderivation (still denoted by $z\rfloor$) of the algebra of semibasic differential forms at z. The argument is entirely straightforward, and is summarized into the identification

$$z\rfloor\hat{\sigma} := (\pi_z)_*{}^*(z\rfloor\sigma) = \widehat{(z\rfloor\sigma)} \tag{1.9}$$

where, for each $\hat{\sigma} \in \mathcal{S}(j_1(\mathcal{V}_{n+1}))$, we are denoting by σ the unique element of $\Lambda_{\pi(z)}(\mathcal{V}_{n+1})$ defined by the condition $\hat{\sigma} = (\pi_z)_*{}^*(\sigma)$. In local coordinates, we have the explicit representation

$$z\rfloor\hat{\sigma} = \left[\left(\frac{\partial}{\partial t}\right)_z + \dot{q}^i(z)\left(\frac{\partial}{\partial q^i}\right)_z\right]\rfloor\hat{\sigma}, \qquad \forall\,\hat{\sigma} \in \mathcal{S}_z(j_1(\mathcal{V}_{n+1}))$$

i.e., expressing $\hat{\sigma}$ in components in the form (1.4), and recalling eq. (1.7)

$$z\rfloor\hat{\sigma} = r\,(\sigma_{0\,i_2\ldots i_r} + \dot{q}^{i_1}(z)\,\sigma_{i_1\,i_2\ldots i_r})\,(\omega^{i_2})_z \wedge \ldots \wedge (\omega^{i_r})_z \tag{1.10}$$

(v) Further significant conclusions may be reached by considering the effect of the projection (1.6a). By composing the latter with the vertical lift (1.3), we get a homomorphism $\hat{\mathcal{P}}_z$, sending each vector $\mathbf{w} = w^0\left(\frac{\partial}{\partial t}\right)_{\pi(z)} + w^i\left(\frac{\partial}{\partial q^i}\right)_{\pi(z)} \in \mathbf{T}_{\pi(z)}(\mathcal{V}_{n+1})$ into the vertical vector

$$\hat{\mathcal{P}}_z(\mathbf{w}) := \widehat{\mathcal{P}_z(\mathbf{w})} = (w^i - \dot{q}^i(z)\,w^0)\left(\frac{\partial}{\partial \dot{q}^i}\right)_z \in \mathbf{T}_z(j_1(\mathcal{V}_{n+1})) \tag{1.11}$$

The correspondence (1.11) is clearly non-injective, its kernel being generated by the point z itself, viewed as a vector in $\mathbf{T}_{\pi(z)}(\mathcal{V}_{n+1})$.

Further composition of the map (1.11) with the push-forward $(\pi_z)_*$ results in a homomorphism $\mathbf{J}_z := \hat{\mathcal{P}}_z \cdot (\pi_z)_* : \mathbf{T}_z(j_1(\mathcal{V}_{n+1})) \to \mathbf{T}_z(j_1(\mathcal{V}_{n+1}))$. By the "quotient law", this identifies a tensor of type (1, 1) at z. By varying z over $j_1(\mathcal{V}_{n+1})$, we end up with a tensor field $\mathbf{J} \in \mathcal{D}^1_1(j_1(\mathcal{V}_{n+1}))$, defined uniquely in terms of the intrinsic properties of $j_1(\mathcal{V}_{n+1})$ as the first jet extension of the fibration $\mathcal{V}_{n+1} \xrightarrow{t} \mathbb{R}$, and therefore endowed with a *universal* character (much in the same way as the canonical 1-form expresses an attribute of *every* cotangent space).

A detailed account of the role of the tensor $\mathbf{J}$ in Lagrangian Mechanics may be found in ([1], [3]). For a more abstract approach, valid for jet extensions of fibered manifolds over arbitrary (finite-dimensional) bases, *see* e.g. ([4], [31]-[33]). Following [1], we shall call $\mathbf{J}$ the *fundamental tensor* of $j_1(\mathcal{V}_{n+1})$.

In components, starting with the vector

$$\mathbf{X} = X^0 \left(\frac{\partial}{\partial t}\right)_z + X^i \left(\frac{\partial}{\partial q^i}\right)_z + \dot{X}^i \left(\frac{\partial}{\partial \dot{q}^i}\right)_z \in \mathbf{T}_z (j_1 (\mathcal{V}_{n+1}))$$

and recalling eq. (1.11), as well as the definition (1.7), a straightforward calculation yields the result

$$\begin{aligned} \mathbf{J}(\mathbf{X}) &= \hat{\mathcal{P}}_z \cdot (\pi_z)_* (\mathbf{X}) \\ &= (X^i - \dot{q}^i (z) X^0) \left(\frac{\partial}{\partial \dot{q}^i}\right)_z = \langle \mathbf{X}, \omega_z^i \rangle \left(\frac{\partial}{\partial \dot{q}^i}\right)_z \end{aligned} \tag{1.12}$$

showing that, in any jet-coordinate system, the fundamental tensor has the canonical representation

$$\mathbf{J} = \frac{\partial}{\partial \dot{q}^i} \otimes \omega^i \tag{1.13}$$

In particular, this points out that the tensor $\mathbf{J}$ belongs to the virtual algebra $\hat{\mathcal{D}}(j_1(\mathcal{V}_{n+1}))$.

(vi) In the dynamical aspects of the theory, an important role is played by the so called *fiber metric* expressing the inertial properties of the system in study in terms of a smooth, positive definite scalar product (,) between vertical vectors on $\mathcal{V}_{n+1}$, represented locally through the components

$$a_{ij} := \left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right) \tag{1.14}$$

In view of the previous discussion, it is easily seen that the product (1.14) assigns a Euclidean structure to each fibre $\Sigma_x \subset j_1(\mathcal{V}_{n+1})$, $x \in \mathcal{V}_{n+1}$, thus giving rise to a scalar product between vertical vectors on $j_1(\mathcal{V}_{n+1})$ and to a corresponding isomorphism $g: \hat{\mathcal{D}}^1(j_1(\mathcal{V}_{n+1})) \to \hat{\mathcal{D}}_1(j_1(\mathcal{V}_{n+1}))$, yielding a process of "rasing and lowering the tensor indices) within the virtual algebra $\hat{\mathcal{D}}(j_1(\mathcal{V}_{n+1}))$. Recalling eqs. (1.3), (1.5), (1.14), we have the natural identifications

$$\left(\frac{\partial}{\partial \dot{q}^i}, \frac{\partial}{\partial \dot{q}^j}\right)_z = \left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right)_{\pi(z)} = (a_{ij})_{\pi(z)} \tag{1.15a}$$

$$\left\langle g\left(\frac{\partial}{\partial \dot{q}^i}\right) \middle\| \frac{\partial}{\partial \dot{q}^j} \right\rangle = \left(\frac{\partial}{\partial \dot{q}^i}, \frac{\partial}{\partial \dot{q}^j}\right) = a_{ij} \tag{1.15b}$$

whence

$$\left.\begin{aligned} g\left(X^i \frac{\partial}{\partial \dot{q}^i}\right) &= a_{ij} X^i \omega^j := X_i\, \omega^i; \\ g^{-1}(Y_i\, \omega^i) &= a^{ij} Y_i \frac{\partial}{\partial \dot{q}^j} := Y^j \frac{\partial}{\partial \dot{q}^j} \end{aligned}\right\} \tag{1.16}$$

with $a^{ij} a_{jk} = \delta^i_k$.

In particular, by eqs. (1.16), the totally covariant expression for the fundamental tensor $\mathbf{J}$ reads

$$g \otimes \mathrm{id}\,(\mathbf{J}) = g\left(\frac{\partial}{\partial \dot{q}^j}\right) \otimes \omega^j = a_{ij}\,\omega^i \otimes \omega^j$$

i.e. it carries the same information as the fiber metric itself.

As a final remark, we notice that the isomorphism $g : \hat{\mathcal{D}}^1(j_1(\mathcal{V}_{n+1})) \to \hat{\mathcal{D}}_1(j_1(\mathcal{V}_{n+1}))$ based on eq. (1.15b) may also be characterized in terms of the ordinary pairing $\langle\ ,\ \rangle$, as the unique map sending each vertical vector $\mathbf{X}$ into a 1-form $g(\mathbf{X})$ satisfying

$$\langle g(\mathbf{X}),\, \mathbf{Y}\rangle = (\mathbf{X},\, \mathbf{J}(\mathbf{Y}))$$

for all vectors $\mathbf{Y}$ on $j_1(\mathcal{V}_{n+1})$. The proof follows easily from eq. (1.15b), and is left to the reader.

1.2. Fiber differentiation

To complete our preliminary scheme, we shall now outline the construction of two important operations on the tensor algebra $\mathcal{D}(j_1(\mathcal{V}_{n+1})) = \bigoplus_{r,\,s=0}^{\infty} \mathcal{D}^r_s(j_1(\mathcal{V}_{n+1}))$, namely the *verticalizer*, and the *fiber differentiation*. Both stem from the fact that, according to the quotient law, given any differentiable manifold M, the module $\mathcal{D}^1_1(M)$ of tensor fields of type (1, 1) over M may be identified with the space of $\mathcal{F}$-linear maps $\mathcal{D}^1(M) \to \mathcal{D}^1(M)$. This assigns a semig-group structure to $\mathcal{D}^1_1(M)$, based on the composition law $(\mathbf{T}\cdot\mathbf{W})(\mathbf{X}) := \mathbf{T}(\mathbf{W}(\mathbf{X}))\ \forall\,\mathbf{T},\,\mathbf{W} \in \mathcal{D}^1_1(M)$, $\mathbf{X} \in \mathcal{D}^1(M)$, expressed in component as

$$(\mathbf{T}\cdot\mathbf{W})^i{}_j = T^i{}_k\,W^k{}_j \tag{1.17}$$

In addition to this, every $\mathcal{F}$-linear map $\mathbf{W} : \mathcal{D}^1(M) \to \mathcal{D}^1(M)$ admits an obvious extension to a *derivation* $\phi_{\mathbf{W}}$ of the tensor algebra $\mathcal{D}(M)$, commuting with contractions, preserving type of tensors, and vanishing on functions [34]. As a fairly obvious example, one may consider e.g. the derivation ϕ_{id} associated with the identity operator, whose action on an arbitrary tensor field $\mathbf{u} \in \mathcal{D}^r_s(M)$ is easily recognized to be

$$\phi_{\mathrm{id}}(\mathbf{u}) = (r - s)\,\mathbf{u} \tag{1.18}$$

A simple check shows that the correspondence $\mathbf{W} \to \phi_{\mathbf{W}}$ preserves the commutation relations:

$$\phi_{\mathbf{T}}\cdot\phi_{\mathbf{W}} - \phi_{\mathbf{W}}\cdot\phi_{\mathbf{T}} = \phi_{(\mathbf{T}\cdot\mathbf{W} - \mathbf{W}\cdot\mathbf{T})}, \qquad \forall\,\mathbf{T},\,\mathbf{W} \in \mathcal{D}^1_1(M) \tag{1.19a}$$

and satisfies the identity

$$\begin{aligned}\phi_{\mathbf{T}}(\mathbf{W}) &= \mathbf{T}\cdot\mathbf{W} - \mathbf{W}\cdot\mathbf{T}\\ &= -\phi_{\mathbf{W}}(\mathbf{T}), \qquad \forall\,\mathbf{T},\,\mathbf{W}\in\mathcal{D}^1_1(M)\end{aligned} \tag{1.19b}$$

Coming back to the space $j_1(\mathcal{V}_{n+1})$, the presence of the fundamental tensor $\mathbf{J}\in\mathcal{D}^1_1(j_1(\mathcal{V}_{n+1}))$ gives rise to a distinguished derivation $\phi_{\mathbf{J}}:\mathcal{D}(j_1(\mathcal{V}_{n+1}))\to\mathcal{D}(j_1(\mathcal{V}_{n+1}))$, henceforth denoted by v, and called the *verticalizer.*

In view of the eq. (1.13), the action of the latter is completely characterized by the relations

$$v(f)=0; \qquad v(\mathbf{X})=\langle\mathbf{X},\,\omega^i\rangle\,\frac{\partial}{\partial\dot q^i}; \qquad v(\eta)=-\left\langle\eta,\,\frac{\partial}{\partial\dot q^i}\right\rangle\omega^i \tag{1.20}$$

$\forall\,f\in\mathcal{F}(j_1(\mathcal{V}_{n+1}))$, $\mathbf{X}\in\mathcal{D}^1(j_1(\mathcal{V}_{n+1}))$, $\eta\in\mathcal{D}_1(j_1(\mathcal{V}_{n+1}))$, together with $v(\mathbf{t}\otimes\mathbf{w})=v(\mathbf{t})\otimes\mathbf{w}+\mathbf{t}\otimes v(\mathbf{w})$ $\forall\,\mathbf{t},\,\mathbf{w}\in\mathcal{D}(j_1(\mathcal{V}_{n+1}))$.

Let us now consider in particular the Grassman algebra $\mathcal{G}(j_1(\mathcal{V}_{n+1}))$. A straightforward argument then shows that, if $\mathbf{A}$ denotes any derivation of *even* degree on $\mathcal{D}(j_1(\mathcal{V}_{n+1}))$, both operations $dt\wedge\mathbf{A}$ and $(d\cdot\mathbf{A}-\mathbf{A}\cdot d)$ have the nature of *antiderivations* of $\mathcal{G}(j_1(\mathcal{V}_{n+1}))$.

DEFINITION 1.3. – *The antiderivation of $\mathcal{G}(j_1(\mathcal{V}_{n+1}))$ defined by the formula*

$$d_v := d\cdot v - v\cdot d + dt\wedge\phi_{\mathrm{id}} \tag{1.21}$$

(ϕ_{id} denoting the derivation (1.18)), is called the fiber differentiation over $j_1(\mathcal{V}_{n+1})$.

On account of eqs. (1.20), (1.21), by direct computation we get the relation

$$d_v f = -v(df) = \frac{\partial f}{\partial\dot q^i}\,\omega^i \tag{1.22a}$$

as well as the anticommutation rule

$$\begin{aligned}d_v\cdot d + d\cdot d_v &= dt\wedge\phi_{\mathrm{id}}\cdot d + d\cdot dt\wedge\phi_{\mathrm{id}}\\ &= dt\wedge(\phi_{\mathrm{id}}\cdot d - d\cdot\phi_{\mathrm{id}}) = -dt\wedge d\end{aligned} \tag{1.22b}$$

Eqs. (1.22a, b) allow to evaluate the fiber differential of an arbitrary r-form. In particular, they imply the relations

$$d_v(\dot q^i)=\omega^i, \qquad d_v\,\omega^i = d_v(dq^i-\dot q^i\,dt)=0 \tag{1.23}$$

as well as the results

$$d_v(q^i)=0, \qquad d_v(t)=0, \qquad d_v(dt)=0, \qquad d_v(d\dot q^i)=2\,d\dot q^i\wedge dt$$

In addition to these, the main property of the operator (1.21) is stated by the following

THEOREM 1.1. – *The fiber differential satisfies the identity*

$$d_v \cdot d_v \equiv 0 \tag{1.24}$$

Proof. – An easy check shows that the product $d_v \cdot d_v$ is a *derivation* of the algebra $\mathcal{G}(j_1(\mathcal{V}_{n+1}))$; in order to prove the stated result, it is therefore sufficient to verify that $d_v \cdot d_v$ vanishes on functions, as well as on exact differentials. An indeed, by eqs. (1.22a, b), (1.23), we have

$$d_v \cdot d_v\, f = d_v \left(\frac{\partial f}{\partial \dot{q}^i}\, \omega^i \right) = \frac{\partial^2 f}{\partial \dot{q}^j\, \partial \dot{q}^i}\, \omega^i \wedge \omega^j = 0$$

$$\begin{aligned} d_v \cdot d_v \cdot df &= d_v\, (-d \cdot d_v\, f - dt \wedge df) \\ &= d \cdot d_v \cdot d_v\, f + dt \wedge (d \cdot d_v\, f + d_v \cdot df) = 0 \end{aligned}$$

$\forall f \in \mathcal{F}(j_1(\mathcal{V}_{n+1}))$. □

As a concluding remark we observe that, when restricted to the virtual Grassmann algebra $\hat{\mathcal{G}}(j_1(\mathcal{V}_{n+1})) \subset \mathcal{G}(j_1(\mathcal{V}_{n+1}))$, the fiber differentiation d_v has exactly the same formal properties as the usual exterior differentiation. More precisely, on the basis of eqs. (1.22a), (1.23), recalling the discussion in Section 1.1, it is easily seen that the operation d_v acts separately on each fibre $\Sigma_x = \pi^{-1}(x)$ of $j_1(\mathcal{V}_{n+1})$, giving rise to an antiderivation of the Grassmann algebra $\mathcal{G}(\Sigma_x)$ *identical* to the exterior differentiation, and consistent with the identification of the $\dot{q}^i$'s as a set of *affine coordinates* on Σ_x, and of the ω^i's as their "differentials".

Remark 1.2. – The fiber differentiation (1.21) is not the only antiderivation of $\mathcal{G}(j_1(\mathcal{V}_{n+1}))$ satisfying the conditions (1.22a), (1.23), (1.24) [*i.e.*, giving rise to a "natural" action on the fibres of $j_1(\mathcal{V}_{n+1})$]. Further comments to this point, centered on the definition of a "covariant" counterpart of eq. (1.21), will be presented in Section 2.

2. THE DYNAMICAL CONNECTION

2.1. Linear connections on $j_1(\mathcal{V}_{n+1})$

As it is well known, the geometric formulation of Dynamics in the manifold $j_1(\mathcal{V}_{n+1})$ relies on the introduction of the concept of dynamical flow (or semi-spray, as it is also called by some Authors). Once again, for a discussion of this approach, we refer to the current literature (*see* among others, [1]-[3]). For the present purposes, it is sufficient to recall that a dynamical flow on $j_1(\mathcal{V}_{n+1})$ is a vector field $\mathbf{Z} \in \mathcal{D}^1(j_1(\mathcal{V}_{n+1}))$ satisfying the condition

$$(\pi_z)_* \mathbf{Z}_z = z_{|\pi(z)} \qquad \forall z \in j_1(\mathcal{V}_{n+1}) \tag{2.1}$$

the element $z_{|\pi(z)}$ at the r.h.s. being viewed as a vector on $\mathcal{V}_{n+1}$.

In jet-coordinates, the requirement (2.1) gives rise to the representation

$$\mathbf{Z} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + Z^i \frac{\partial}{\partial \dot{q}^i} \tag{2.2}$$

with $Z^i = Z^i(t, q, \dot{q})$. Any integral curve of a field of the form (2.2) is automatically the jet-extension of a section $\gamma: \mathbb{R} \to \mathcal{V}_{n+1}$. In this respect, the concept of dynamical flow is directly involved in the study of the *problem of motion*, or, more generally, in the representation of a system of ordinary second order differential equations in normal form.

By eq. (2.2) it is easily seen that the difference between two dynamical flows is a vertical vector field over $j_1(\mathcal{V}_{n+1})$.

Closely related with the concept of dynamical flow is the identification of a distinguished class of *linear connection* on $j_1(\mathcal{V}_{n+1})$, whose importance will be better appreciated in the subsequent development of the theory.

To start with, we observe that, given any vertical vector field $\mathbf{V} = V^i \frac{\partial}{\partial \dot{q}^i} \in \hat{\mathcal{D}}^1(j_1(\mathcal{V}_{n+1}))$, eq. (1.7) and the definition (1.20) of the verticalizer v yield the identity

$$v([\mathbf{V} \cdot \mathbf{Y}]) = \langle \mathcal{L}_{\mathbf{V}} \mathbf{Y}, \omega^i \rangle \frac{\partial}{\partial \dot{q}^i} = \mathbf{V}(\langle \mathbf{Y}, \omega^i \rangle) \frac{\partial}{\partial \dot{q}^i} + \langle \mathbf{Y}, dt \rangle \mathbf{V} \tag{2.3a}$$

$\forall \mathbf{Y} \in \mathcal{D}^1(j_1(\mathcal{V}_{n+1}))$. In particular, this gives

$$v([v(\mathbf{X}), \mathbf{Z}]) = v(\mathbf{X}) \tag{2.3b}$$

for any choice of the dynamical flow $\mathbf{Z}$ and of the vector field $\mathbf{X} \in \mathcal{D}^1(j_1(\mathcal{V}_{n+1}))$.

In addition to this, another useful observation, already pointed out in Section 1 is that, due to the affine nature of the fibration $\pi: j_1(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$, each fibre $\Sigma_x = \pi^{-1}(x)$, $x \in \mathcal{V}_{n+1}$ is naturally endowed with a law of parallel transport, based on the fact that all vertical spaces $V_z(j_1(\mathcal{V}_{n+1}))$, $z \in \pi^{-1}(x)$ are isomorphic – through the vertical lift (1.3a) – to one and the same "modelling" vector space $V_x(\mathcal{V}_{n+1})$.

From this viewpoint, the class of vector fields "constant" along the fibres is clearly identical to the totality of vertical fields of the form

$$\mathbf{U} = U^i (t, q) \frac{\partial}{\partial \dot{q}^i} \tag{2.4}$$

i.e., to the totality of vector fields arising as vertical lifts of vertical vectors on $\mathcal{V}_{n+1}$.

With this in mind, the principle we shall adopt in order to single out a distinguished class $\mathcal{S}$ of linear connections on $j_1 (\mathcal{V}_{n+1})$ is the requirement that the covariant derivative associated with any $\nabla \in \mathcal{S}$ preserve the geometric structures already present in $j_1 (\mathcal{V}_{n+1})$, namely the 1-form dt, the fundamental tensor $\mathbf{J}$, and the parallel transport along the fibres.

All this is summarized into the conditions

$$\nabla_{\mathbf{X}}\, dt = \nabla_{\mathbf{X}}\, \mathbf{J} = 0 \tag{2.5a}$$

for any $\mathbf{X} \in \mathcal{D}^1(j_1 (\mathcal{V}_{n+1}))$, and

$$\nabla_{\mathbf{V}}\, \mathbf{U} = 0 \tag{2.5b}$$

for any vertical vector field $\mathbf{V} \in \hat{\mathcal{D}}^1 (j_1 (\mathcal{V}_{n+1}))$ and any vertical lift $\mathbf{U}$ of the form (2.4).

In jet coordinates, the requirement (2.5b) is mathematically equivalent to the n^2 conditions

$$\nabla_{\frac{\partial}{\partial \dot{q}^i}} \left(\frac{\partial}{\partial \dot{q}^j} \right) = 0 \qquad i, j = 1, \ldots, n \tag{2.6}$$

which are intrinsically significant on their own, due to their invariance under coordinate transformations. In a similar way, eqs. (2.5a) may be cast in the form

$$\left. \begin{array}{c} v (\nabla_{\mathbf{X}}\, \mathbf{Y}) = \nabla_{\mathbf{X}}\, v (\mathbf{Y}) \qquad \langle \nabla_{\mathbf{X}}\, \mathbf{Y}, d\,t \rangle = \mathbf{X} (\langle \mathbf{Y},\, dt \rangle) \\ \forall \mathbf{X},\, \mathbf{Y} \in \mathcal{D}^1 (j_1 (\mathcal{V}_{n+1})) \end{array} \right\} \tag{2.7}$$

implying e.g. the verticality of $\nabla_{\mathbf{X}}\, \mathbf{Z}$ for any $\mathbf{X} \in \mathcal{D}^1 (j_1 (\mathcal{V}_{n+1}))$ and any dynamical flow $\mathbf{Z}$, as well as the verticality of $\nabla_{\mathbf{X}}\, \mathbf{V}$ for any $\mathbf{X}$ and any vertical vector field $\mathbf{V}$.

In particular, denoting by $\mathbf{T}$ the torsion tensor field associated with ∇, by eqs. (2.3b), (2.7) we derive the identities

$$v (\mathbf{T} (\mathbf{Z},\, v (\mathbf{X}))) = v (\nabla_{\mathbf{Z}}\, v (\mathbf{X}) - \nabla_{v (\mathbf{X})}\, \mathbf{Z} - [\mathbf{Z},\, v (\mathbf{X})]) = v (\mathbf{X}) \tag{2.8a}$$

$$\langle \mathbf{T} (\mathbf{Z},\, v (\mathbf{X})), d\,t \rangle = \langle \nabla_{\mathbf{Z}}\, v (\mathbf{X}) - \nabla_{v (\mathbf{X})}\, \mathbf{Z} - [\mathbf{Z},\, v (\mathbf{X})],\, dt \rangle \tag{2.8b}$$

valid for arbitrary choice of the dynamical flow $\mathbf{Z}$ and for all $\mathbf{X} \in \mathcal{D}^1(j_1(\mathcal{V}_{n+1}))$.

THEOREM 2.1. – *Let* ∇ *be a connection in the class* $\mathcal{S}$, $\mathbf{T}$ *the associated torsion tensor field, and* $\mathbf{Z}$ *an arbitrary dynamical flow on* $j_1(\mathcal{V}_{n+1})$. *Define two* $\mathcal{F}$*-linear maps* $\mathcal{P}_H : \mathcal{D}^1(j_1(\mathcal{V}_{n+1})) \to D^1(j_1(\mathcal{V}_{n+1}))$, $\mathcal{Q}: \mathcal{D}^1(j_1(\mathcal{V}_{n+1})) \to D^1(j_1(\mathcal{V}_{n+1}))$ *on the basis of the equations*

$$\mathcal{P}_H(\mathbf{X}) := \mathbf{T}(\mathbf{Z}, v(\mathbf{X})) \tag{2.9a}$$

$$\mathcal{Q}(\mathbf{X}) := v(\mathbf{T}(\mathbf{Z}, \mathbf{X})) + \langle \mathbf{X}, dt\rangle \mathbf{Z} \tag{2.9b}$$

$\forall \mathbf{X} \in \mathcal{D}^1(j_1(\mathcal{V}_{n+1}))$. *Then*

(i) *the definition (2.9a, b) is independent of the choice of the flow* $\mathbf{Z}$.

(ii) *both operators (2.9a, b) are projection operators. For any point* $z \in j_1(\mathcal{V}_{n+1})$, *the image spaces* $\mathbf{H}_z := \mathcal{P}_H(\mathbf{T}_z(j_1(\mathcal{V}_{n+1})))$ *and* $\mathbf{Q}_z := \mathcal{Q}(\mathbf{T}_z(j_1(\mathcal{V}_{n+1})))$ *are complementary to each other in* $\mathbf{T}_z(j_1(\mathcal{V}_{n+1}))$, *in the sense clarified by the relations*

$$\mathbf{H}_z \cap \mathbf{Q}_z = \{0\} \qquad \mathbf{H}_z \oplus \mathbf{Q}_z = \mathbf{T}_z(j_1(\mathcal{V}_{n+1})) \tag{2.10}$$

Proof. – For each pair of dynamical flows $\mathbf{Z}$, $\mathbf{Z}'$, consider the vertical vector $\mathbf{V} = \mathbf{Z} - \mathbf{Z}'$. A simple calculation then yields the relation

$$T(\mathbf{Z}, v(\mathbf{X})) - \mathbf{T}(\mathbf{Z}', v(\mathbf{X})) = \mathbf{T}(\mathbf{V}, v(\mathbf{X})) = 0$$

the last step depending on eq. (2.6), and

$$\begin{aligned} &v(\mathbf{T}(\mathbf{Z}, \mathbf{X})) + \langle \mathbf{X}, dt\rangle \mathbf{Z} - v(\mathbf{T}(\mathbf{Z}', \mathbf{X})) - \langle \mathbf{X}, dt\rangle \mathbf{Z}' \\ &\quad = v(\mathbf{T}(\mathbf{V}, \mathbf{X})) + \langle \mathbf{X}, dt\rangle \mathbf{V} = v(\nabla_{\mathbf{V}} \mathbf{X} - \nabla_{\mathbf{X}} \mathbf{V} - [\mathbf{V}, \mathbf{X}]) \\ &\quad + \langle \mathbf{X}, dt\rangle \mathbf{V} = \nabla_{\mathbf{V}} v(\mathbf{X}) - \mathbf{V}(\langle \mathbf{X}, \omega^i\rangle) \frac{\partial}{\partial \dot{q}^i} = 0 \end{aligned}$$

as a consequence of eqs. (2.3a), (2.6), (2.7). This proves assertion (i).

Also, by eqs. (2.8a), (2.9a, b) it is easily seen that $\mathcal{P}_H$ and $\mathcal{Q}$ satisfy the idempotence relations

$$\begin{aligned} \mathcal{P}_H^2(\mathbf{X}) &= \mathbf{T}(\mathbf{Z}, v(\mathcal{P}_H(\mathbf{X}))) = \mathbf{T}(\mathbf{Z}, v(\mathbf{X})) = \mathcal{P}_H(\mathbf{X}) \\ \mathcal{Q}^2(\mathbf{X}) &= v(\mathbf{T}(\mathbf{Z}, \mathcal{Q}(\mathbf{X}))) + \langle \mathcal{Q}(\mathbf{X}), dt\rangle \mathbf{Z} \\ &= v(\mathbf{T}(\mathbf{Z}, \mathbf{X})) + \langle \mathbf{X}, dt\rangle \mathbf{Z} = \mathcal{Q}(\mathbf{X}) \end{aligned}$$

as well as the identity

$$\mathcal{P}_H(\mathcal{Q}(\mathbf{X})) = 0 \tag{2.11}$$

$\forall \mathbf{X} \in \mathcal{D}^1(j_1(\mathcal{V}_{n+1}))$. From these we conclude that $\mathcal{P}_H$ and $\mathcal{Q}$ are projection operators, and that their image spaces satisfy the first of eqs. (2.10).

Finally, by the definition (2.9b) itself, it follows at once that, at each point z, the image space $\mathbf{Q}_z = \mathcal{Q}(T_z(j_1(\mathcal{V}_{n+1})))$ coincides with the $(n+1)$-dimensional subspace of $\mathbf{T}_z(j_1(\mathcal{V}_{n+1}))$ generated by the totality of vertical vectors and of dynamical flows, all evaluated at z.
At the same time, the image space $\mathbf{H}_z$ has dimension not lower than n, since, for instance, the vectors $\mathcal{P}_H\left(\frac{\partial}{\partial q^i}\right)$, $i = 1, \ldots, n$ are linearly independent, as a consequence of the linear independence of the vectors $\frac{\partial}{\partial \dot{q}^i}$, $i = 1, \ldots, n$, and of the identity

$$v\left(\mathcal{P}_H\left(\frac{\partial}{\partial q^i}\right)\right) = v\left(\mathbf{T}\left(\mathbf{Z}, v\left(\frac{\partial}{\partial q^i}\right)\right)\right) = v\left(\frac{\partial}{\partial q^i}\right) = \frac{\partial}{\partial \dot{q}^i} \tag{2.12}$$

following from eqs. (2.8a), (2.9a). The second relation (2.10) then follows by a dimensionality argument. □

DEFINITION 2.1. – *The operator $\mathcal{P}_H$ will be called the horizontal projection associated with the connection ∇. The image space $\mathbf{H}_z = \mathcal{P}_H(\mathbf{T}_z(j_1(\mathcal{V}_{n+1})))$ at each point $z \in j_1(\mathcal{V}_{n+1})$ will be called the horizontal subspace at z. The correspondence $z \rightarrow \mathbf{H}_z$ will be called the horizontal distribution determined by ∇ on $j_1(\mathcal{V}_{n+1})$.*

As pointed out in the proof of Theorem 2.1, a (local) basis for the horizontal distribution is provided by the vector fields

$$\tilde{\partial}_i := \mathcal{P}_H\left(\frac{\partial}{\partial q^i}\right) = \mathbf{T}\left(\mathbf{Z}, \frac{\partial}{\partial \dot{q}^i}\right) \qquad i = 1, \ldots, n \tag{2.13}$$

In this sense, the horizontal distribution may be viewed as the image of the vertical distribution $\mathbf{V}(j_1(\mathcal{V}_{n+1}))$ under the map $\mathbf{T}(\mathbf{Z}, \cdot\,)$ [whose action on the vertical subspace is indeed independent of $\mathbf{Z}$, as a consequence of eq. (2.6)].

By eqs. (2.8b), (2.12), (2.13), recalling the definition of the verticalizer v, we get the relations

$$v(\tilde{\partial}_i) = \frac{\partial}{\partial \dot{q}^i} \tag{2.14a}$$

$$\langle \tilde{\partial}_i, \omega^j \rangle = -\langle \tilde{\partial}_i, v(d\dot{q}^j) \rangle = \langle v(\tilde{\partial}_i), d\dot{q}^j \rangle = \delta_i^j \tag{2.14b}$$

$$\langle \tilde{\partial}_i, dt \rangle = \left\langle \mathbf{T}\left(\mathbf{Z}, \frac{\partial}{\partial \dot{q}^i}\right), dt \right\rangle = 0 \tag{2.14c}$$

Together with the results pointed out in the proof of Theorem 2.1, these allow to draw the following conclusions:

a) for any choice of the dynamical flow $\mathbf{Z}$, the vector fields $\left\{\mathbf{Z},\, \tilde{\partial}_i,\, \frac{\partial}{\partial \dot{q}^i},\, i = 1, \ldots, n\right\}$ form a (local) basis for the contravariant tensor algebra over $j_1(\mathcal{V}_{n+1})$. The corresponding dual basis includes the temporal 1-form dt, the contact 1-forms ω^i, and n further 1-forms

$$\tilde{\nu}^i := d\dot{q}^i - \langle \mathbf{Z},\, d\dot{q}^i\rangle\, dt - \langle \tilde{\partial}_j,\, d\dot{q}^i\rangle\, \omega^j = \mathcal{L}_{\mathbf{Z}}\, \omega^i + \tau_j{}^i\, \omega^j \tag{2.15}$$

with

$$\tau_j{}^i := -\langle \tilde{\partial}_j,\, d\dot{q}^i\rangle \tag{2.16}$$

Under coordinate transformations, in addition to eqs. (1.8) we have the transformation law

$$\bar{\tilde{\partial}}_i = \mathbf{T}\left(\mathbf{Z},\, \frac{\partial q^j}{\partial \bar{q}^i}\, \frac{\partial}{\partial \dot{q}^j}\right) = \frac{\partial q^j}{\partial \bar{q}^i}\, \tilde{\partial}_j \tag{2.17a}$$

whence also, by duality

$$\bar{\tilde{\nu}}^i = \left\langle \bar{\tilde{\nu}}^i,\, \frac{\partial}{\partial \dot{q}^j}\right\rangle \tilde{\nu}^j = \frac{\partial \bar{q}^i}{\partial q^j}\, \tilde{\nu}^j \tag{2.17b}$$

b) in terms of the basis described in *a*), the projection $\mathcal{P}_H$ is given by the formula

$$\mathcal{P}_H\,(\mathbf{X}) = \mathbf{T}\left(\mathbf{Z},\, \langle \mathbf{X},\, \omega^i\rangle\, \frac{\partial}{\partial \dot{q}^i}\right) = \langle \mathbf{X},\, \omega^i\rangle\, \tilde{\partial}_i \tag{2.18a}$$

In a similar way, on account of the verticality of $\nabla_{\mathbf{X}}\, \mathbf{Z}$, one has the identity

$$\langle \mathbf{T}\,(\mathbf{Z},\, \mathbf{X}),\, \omega^i\rangle = \langle \nabla_{\mathbf{Z}}\, \mathbf{X} - [\mathbf{Z},\, \mathbf{X}],\, \omega^i\rangle = \langle \mathbf{X},\, (\mathcal{L}_{\mathbf{Z}} - \nabla_{\mathbf{Z}})\, \omega^i\rangle$$

which, together with eqs. (2.9b), (2.15), provides the representation

$$\mathcal{Q}\,(\mathbf{X}) = \langle \mathbf{X},\, \tilde{\nu}^i - \tau_j{}^i\, \omega^j - \nabla_{\mathbf{Z}}\, \omega^i\rangle\, \frac{\partial}{\partial \dot{q}^i} + \langle \mathbf{X},\, dt\rangle\, \mathbf{Z} \tag{2.18b}$$

All conclusions reached so far hold identically for any connection $\nabla \in \mathcal{S}$, as a consequence of the assumptions (2.5a, b). The resulting characterization of all class $\mathcal{S}$ is still quite general, and leaves room for further specializations. We now take advantage of this, and restrict the choice of the connection ∇, by adding three further requirements, namely:

(i) the operators $\mathcal{P}_H$ and $\mathcal{Q}$ form a complete set of complementary projections, in the sense that the kernel of each one coincides with the range of the other, or, what is the same, that their sum coincides with the identity operator. In view of eqs. (2.18a, b), this amounts to requiring

$$\nabla_{\mathbf{Z}}\, \omega^i + \tau_j{}^i\, \omega^j = 0 \tag{2.19a}$$

Taking the condition $\nabla_{\mathbf{Z}}\, \mathbf{J} = 0$ into account, eq. (2.19a) may be written in the equivalent form

$$\nabla_{\mathbf{Z}} \frac{\partial}{\partial \dot{q}^i} = \tau_i{}^j \frac{\partial}{\partial \dot{q}^j} \tag{2.19b}$$

(ii) the covariant derivative associated with ∇ commutes with the horizontal projection $\mathcal{P}_H$ (and thus, as a consequence of (i), also with $\mathcal{Q}$). In formulae

$$\nabla_{\mathbf{X}}\, \mathcal{P}_H\, \mathbf{Y} = \mathcal{P}_H\, \nabla_{\mathbf{X}}\, \mathbf{Y} \qquad \forall\, \mathbf{X},\, \mathbf{Y} \in \mathcal{D}^1\, (j_1\, (\mathcal{V}_{n+1})) \tag{2.20a}$$

or, more specifically, in terms of the basis $\left\{\mathbf{Z},\, \tilde{\partial}_i,\, \frac{\partial}{\partial \dot{q}^i},\, i = 1, \ldots, n\right\}$ introduced above

$$\nabla_{\mathbf{X}}\, \tilde{\partial}_i = \mathcal{P}_H\, \nabla_{\mathbf{X}}\, \tilde{\partial}_i = \langle \nabla_{\mathbf{X}}\, \tilde{\partial}_i,\, \omega^j \rangle\, \tilde{\partial}_j \qquad \forall\, \mathbf{X} \in \mathcal{D}^1\, (j_1\, (\mathcal{V}_{n+1})) \tag{2.20b}$$

(iii) the curvature tensor $\mathbf{R}$ associated with the connection ∇ satisfies

$$\mathbf{R}\, (\mathbf{Z}_1,\, \mathbf{Z}_2) = 0 \tag{2.21a}$$

for any pair of dynamical flows $\mathbf{Z}_1,\, \mathbf{Z}_2$.

In terms of bases, taking eq. (2.6) into account, and expressing each $\mathbf{Z}_i$, $(i = 1,\, 2)$ as $\mathbf{Z} + \mathbf{V}_i$, $\mathbf{V}_i \in \hat{\mathcal{D}}^1\, (j_1\, (\mathcal{V}_{n+1}))$, eq. (2.21a) is summarized into the conditions

$$0 = \mathbf{R}\left(\mathbf{Z},\, \frac{\partial}{\partial \dot{q}^i}\right) = \nabla_{\mathbf{Z}}\, \nabla_{\frac{\partial}{\partial \dot{q}^i}} - \nabla_{\frac{\partial}{\partial \dot{q}^i}}\, \nabla_{\mathbf{Z}} - \nabla_{\left[\mathbf{Z},\, \frac{\partial}{\partial \dot{q}^i}\right]} \tag{2.21b}$$

As far as the meaning of the stated conditions is concerned, one may notice that the first two provide a "natural" strengthening of the properties of $\mathcal{P}_H$ and $\mathcal{Q}$ proved in Theorem 2.1; the requirement (iii), on the contrary, is more technical, even if intuitively appealing: in a sense, the ansatz (2.21a) may be thought of as the simplest conjecture concerning the quantity $\mathbf{R}\, (\mathbf{Z}_1,\, \mathbf{Z}_2)$, much in the same way as the requirement of vanishing torsion singles out the Levi-Civita connection as the "simplest" metric connection in a Riemannian manifold.

The role of the previous conditions is clarified by the following

THEOREM 2.2. – *Let $\hat{\mathcal{S}} \subset \mathcal{S}$ denote the totality of connections $\nabla \in \mathcal{S}$ satisfying the requirements (i), (ii), (iii), above. Also, let $\mathbf{Z}$ denote any given* *dynamical flow over $j_1(\mathcal{V}_{n+1})$. Then, each $\nabla \in \hat{\mathcal{S}}$ is completely determined by the knowledge of the covariant derivatives $\nabla_{\mathbf{Z}'}\mathbf{Z}$ for all dynamical flows $\mathbf{Z}'$.*

Proof. – In (local) jet coordinates once the dynamical flow $\mathbf{Z}$ has been chosen, the knowledge of $\nabla_{\mathbf{Z}'}\mathbf{Z}$ for all possible flows $\mathbf{Z}'$ relies on the assignment of the $(n+1)$ vertical vectors

$$\mathbf{A} = A^i \frac{\partial}{\partial \dot{q}^i} := \nabla_{\mathbf{Z}}\mathbf{Z} \qquad \mathbf{B}_i = B_i{}^j \frac{\partial}{\partial \dot{q}^j} := \nabla_{\frac{\partial}{\partial \dot{q}^i}}\mathbf{Z} \tag{2.22}$$

From these, in order to characterize the connection ∇ uniquely, we have to evaluate:

– the explicit representation of the horizontal vector fields $\tilde{\partial}_i$;

– the connection coefficients of ∇ in the basis $\left\{\mathbf{Z},\, \tilde{\partial}_i,\, \frac{\partial}{\partial \dot{q}^i}\right\}$.

To this end, we observe that, in view of eq. (2.13), the requirement (2.19b) gives rise to the identification

$$\begin{aligned}\tilde{\partial}_i &= \nabla_{\mathbf{Z}}\frac{\partial}{\partial \dot{q}^i} - \nabla_{\frac{\partial}{\partial \dot{q}^i}}\mathbf{Z} - \left[\mathbf{Z},\, \frac{\partial}{\partial \dot{q}^i}\right] \\ &= \left(\tau_i{}^j - B_i{}^j + \frac{\partial Z^j}{\partial \dot{q}^i}\right)\frac{\partial}{\partial \dot{q}^j} + \frac{\partial}{\partial q^i}\end{aligned} \tag{2.23}$$

Comparison with the definition (2.16) of the coefficients $\tau_i{}^j$ provides the relations

$$-\tau_i{}^j = \langle \tilde{\partial}_i,\, d\dot{q}^j\rangle = \tau_i{}^j - B_i{}^j + \frac{\partial Z^j}{\partial \dot{q}^i}$$

which, inserted once again in eq. (2.23), yield

$$\tilde{\partial}_i = \frac{\partial}{\partial q^i} - \tau_i{}^j \frac{\partial}{\partial \dot{q}^i} \tag{2.24}$$

with

$$\tau_i{}^j = \frac{1}{2}\left(B_i{}^j - \frac{\partial Z^j}{\partial \dot{q}^i}\right) \tag{2.25}$$

Also, by eqs. (2.24), (2.25), eq. (2.23) may be written in the equivalent form

$$\left[\mathbf{Z},\, \frac{\partial}{\partial \dot{q}^i}\right] = -\tilde{\partial}_i + (\tau_i{}^j - B_i{}^j)\frac{\partial}{\partial \dot{q}^j} = -\tilde{\partial}_i - \frac{1}{2}\left(B_i{}^j + \frac{\partial Z^j}{\partial \dot{q}^i}\right)\frac{\partial}{\partial \dot{q}^j} \tag{2.26}$$

Notice that, in view of eq. (2.19b), eq. (2.25) determines not only the horizontal vectors $\tilde{\partial}_i$ – through eq. (2.24) – but also the connection coefficients involved in the description of $\nabla_{\mathbf{Z}}\frac{\partial}{\partial \dot{q}^i}$.

From the first of eqs. (2.7), taking eq. (2.14a) and the definition (1.20) of the verticalizer v into account, we derive the further identity

$$\left\langle \nabla_{\mathbf{X}} \frac{\partial}{\partial \dot{q}^i}, \tilde{\nu}^j \right\rangle = \langle v(\nabla_{\mathbf{X}} \tilde{\partial}_i), \tilde{\nu}^j \rangle = \langle \nabla_{\mathbf{X}} \tilde{\partial}_i, \omega^i \rangle$$

Together with the requirement (2.20b), this provides the relation

$$\nabla_{\mathbf{X}} \tilde{\partial}_i = \left\langle \nabla_{\mathbf{X}} \frac{\partial}{\partial \dot{q}^i}, \tilde{\nu}^j \right\rangle \tilde{\partial}_j \tag{2.27}$$

thus reducing the evaluation of the quantities $\nabla_{\mathbf{X}} \tilde{\partial}_i$ to the knowledge of $\nabla_{\mathbf{X}} \frac{\partial}{\partial \dot{q}^i}$, $\forall \mathbf{X} \in \mathcal{D}^1(j_1(\mathcal{V}_{n+1}))$.

Finally, by the requirement (2.21b), taking eq. (2.26) into account, we obtain the relations

$$\nabla_{\tilde{\partial}_i} + \frac{1}{2}\left(B_i{}^j + \frac{\partial Z^j}{\partial \dot{q}^i}\right) \nabla_{\frac{\partial}{\partial \dot{q}^j}} = -\nabla_{\left[\mathbf{Z}, \frac{\partial}{\partial \dot{q}^i}\right]} = \nabla_{\frac{\partial}{\partial \dot{q}^i}} \nabla_{\mathbf{Z}} - \nabla_{\mathbf{Z}} \nabla_{\frac{\partial}{\partial \dot{q}^i}} \tag{2.28}$$

The previous results provide a representation of the whole set of connection coefficients of ∇ in the basis $\left\{\mathbf{Z}, \tilde{\partial}_i, \frac{\partial}{\partial \dot{q}^i}\right\}$. The situation is summarized into the following table

$$\nabla_{\mathbf{Z}} \frac{\partial}{\partial \dot{q}^i} = \tau_i{}^j \frac{\partial}{\partial \dot{q}^j} \qquad \nabla_{\mathbf{Z}} \tilde{\partial}_i = \tau_i{}^j \tilde{\partial}_j \tag{2.29a}$$

$$\nabla_{\frac{\partial}{\partial \dot{q}^i}} \frac{\partial}{\partial \dot{q}^j} = 0 \qquad \nabla_{\frac{\partial}{\partial \dot{q}^i}} \tilde{\partial}_j = 0 \tag{2.29b}$$

$$\nabla_{\tilde{\partial}_i} \frac{\partial}{\partial \dot{q}^j} = \nabla_{\frac{\partial}{\partial \dot{q}^i}} \nabla_{\mathbf{Z}} \frac{\partial}{\partial \dot{q}^j} = \frac{\partial \tau_j{}^k}{\partial \dot{q}^i} \frac{\partial}{\partial \dot{q}^k} \qquad \nabla_{\tilde{\partial}_i} \tilde{\partial}_j = \frac{\partial \tau_j{}^k}{\partial \dot{q}^i} \tilde{\partial}_k \tag{2.29c}$$

completed by eqs. (2.22), and by the relation

$$\nabla_{\tilde{\partial}_i} \mathbf{Z} = \nabla_{\frac{\partial}{\partial \dot{q}^i}} \mathbf{A} - \nabla_{\mathbf{Z}} \mathbf{B}_i - \frac{1}{2}\left(B_i{}^j + \frac{\partial Z^j}{\partial \dot{q}^i}\right) \mathbf{B}_j = C_i{}^k \frac{\partial}{\partial \dot{q}^k} \tag{2.30a}$$

with

$$C_i{}^k := \frac{\partial A^k}{\partial \dot{q}^i} - \mathbf{Z}(B_i{}^k) - B_i{}^j B_j{}^k + \frac{1}{2}\left(B_i{}^j \frac{\partial Z^k}{\partial \dot{q}^j} - \frac{\partial Z^j}{\partial \dot{q}^i} B_j{}^k\right) \tag{2.30b}$$

and with $\tau_i{}^j$ defined by eq. (2.25).

This proves that, once the dynamical flow $\mathbf{Z}$ has been (arbitrarily) fixed, the assignment of the quantities (2.22), together with the requirement $\nabla \in \hat{\mathcal{S}}$, determines the connection ∇ uniquely.

Conversely, for any given set of quantities (2.22), if we use eqs. (2.24), (2.25), (2.29a, b, c), (2.30a, b) in order to define a lienar connection over $j_1(\mathcal{V}_{n+1})$, a straightforward check shows that ∇ meets all the requirements involved in the definition of the class $\hat{\mathcal{S}}$. □

2.2. The dynamical connection

In the analysis developed so far, the concept of dynamical flow plays a minor role, its introduction being involved in the *representation* of an arbitrary connection $\nabla \in \hat{\mathcal{S}}$ in a suitable basis, but not in the intrinsic characterization of ∇ as a geometrical object. At the same time, however, Theorem 2.2 suggests a natural link between dynamical flows and connections, as pointed out by the following

COROLLARY 2.1. – *For a given dynamical flow* $\mathbf{Z}$ *over* $j_1(\mathcal{V}_{n+1})$, *there exists a unique connection* $\nabla \in \hat{\mathcal{S}}$ *satisfying the condition*

$$\nabla_{\mathbf{X}} \mathbf{Z} = 0 \qquad \forall \mathbf{X} \in \mathcal{D}^1(j_1 \mathcal{V}_{n+1})) \tag{2.31}$$

The proof follows at once from Theorem 2.2, by identifying ∇ with the unique connection in the class $\hat{\mathcal{S}}$ corresponding to the ansatz $\mathbf{A} = \mathbf{B}_i = 0$ ($\Leftrightarrow \nabla_{\mathbf{Z}} \mathbf{Z} = \nabla_{(\partial/\partial \dot{q}^i)} \mathbf{Z} = 0$), and observing that, as a consequence of eqs. (2.30a, b), this implies also $\nabla_{\tilde{\partial}_i} \mathbf{Z} = 0$ and, therefore, the validity of eq. (2.31).

DEFINITION 2.2. – *The connection* $\nabla \in \hat{\mathcal{S}}$ *determined by the ansatz (2.31) will be called the dynamical connection associated with the flow* $\mathbf{Z}$.

The importance of Definition 2.2 lies on the fact that, in Dynamics, the relevant geometric object is not the jet bundle $j_1(\mathcal{V}_{n+1})$ alone, but, rather, the pair $(j_1(\mathcal{V}_{n+1}), \mathbf{Z})$ formed by $j_1(\mathcal{V}_{n+1})$, and by a distinguished dynamical flow $\mathbf{Z}$ over $j_1(\mathcal{V}_{n+1})$.

In this respect, the dynamical connection associated with $\mathbf{Z}$ may be interpreted as the unique linear connection over $j_1(\mathcal{V}_{n+1})$ preserving *all* the geometrical structures of the pair $(j_1(\mathcal{V}_{n+1}), \mathbf{Z})$, including the flow $\mathbf{Z}$ itself. Once again, the situation resembles very closely what happens in Riemannian Geometry, when, starting with a differentiable manifold M, endowed with a fundamental form $\Phi = g_{ij}\, dx^i \otimes dx^j$, one singles out the *Levi-Civita* connection ∇ over (M, Φ): pursuing this analogy, the requirement (2.31) is now the equivalent of the metricity condition $\nabla \Phi = 0$, while the condition $\nabla \in \hat{\mathcal{S}}$ is the counterpart of the requirement of vanishing torsion, in the sense already pointed out in Section 2.1.

The significance of the dynamical connection is further clarified by observing that the assignment of a dynamical flow $\mathbf{Z}$ determines a corresponding *longitudinal projection* $\mathcal{P}_{\mathbf{Z}} : \mathcal{D}^1(j_1(\mathcal{V}_{n+1})) \rightarrow \mathcal{D}^1(j_1(\mathcal{V}_{n+1}))$, according to the formula

$$\mathcal{P}_{\mathbf{Z}}(\mathbf{X}) := \langle \mathbf{X}, dt \rangle \mathbf{Z} \qquad \forall \mathbf{X} \in \mathcal{D}^1(j_1(\mathcal{V}_{n+1})) \tag{2.32}$$

For any $\nabla \in \hat{\mathcal{S}}$, taking eq. (2.9b) into account, it is easily seen that the difference

$$\mathcal{P}_V(\mathbf{X}) := \mathcal{Q}(\mathbf{X}) - \mathcal{P}_{\mathbf{Z}}(\mathbf{X}) = v(\mathbf{T}(\mathbf{Z}, \mathbf{X})) \tag{2.33}$$

identifies a projection operator $\mathcal{P}_V : \mathcal{D}^1(j_1(\mathcal{V}_{n+1})) \rightarrow \mathcal{D}^1(j_1(\mathcal{V}_{n+1}))$, depending explicitly on $\mathbf{Z}$, and sending each tangent space $\mathbf{T}_z(j_1(\mathcal{V}_{n+1}))$, $z \in j_1(\mathcal{V}_{n+1})$ onto the vertical subspace $V_z(j_1(\mathcal{V}_{n+1}))$.

In this way, recalling also the definition (2.9a) of $\mathcal{P}_H$, we end up with a complete set of projections $\mathcal{P}_H$, $\mathcal{P}_V$, $\mathcal{P}_{\mathbf{Z}}$, acting on each tangent space $\mathbf{T}_z(j_1(\mathcal{V}_{n+1}))$, and canonically related to the direct sum decomposition

$$\mathbf{T}_z(j_1(\mathcal{V}_{n+1})) = \mathbf{H}_z \oplus \mathbf{V}_z \oplus L(\mathbf{Z}_z) \tag{2.34}$$

in which $\mathbf{V}_z$ is a shorthand for $V_z(j_1(\mathcal{V}_{n+1}))$, while $L(\mathbf{Z}_z)$ denotes the 1-dimensional subspace generated by $\mathbf{Z}$.

The previous conclusion holds true as a consequence of the results established in Section 2.1, no matter how we chose the connection $\nabla \in \hat{\mathcal{S}}$. A simple check, however, shows that a necessary and sufficient condition in order for the covariant derivative induced by ∇ to *preserve* the decomposition (2.34) [*i.e.*, to commute separately with each of the projection operators (2.32), (2.33), and not merely with their sum], is that ∇ coincide with the dynamical connection associated with $\mathbf{Z}$. The proof is entirely straightforward, and is left to the reader.

On the basis of the previous discussion, we may assign a precise geometrical meaning to the notion of horizontal distribution [or, what is the same, to the decomposition of each tangent space into the direct sum (2.34)] *determined* by a given dynamical flow $\mathbf{Z}$, the definition relying on the projection operators induced by $\mathbf{Z}$, either directly, or through the associated dynamical connection [1].

The following statement relates the present approach to a classical result obtained by Crampin [1] (in this respect, see also ([3], [35], [36]), mainly

[1] A closer analysis shows that, in the construction of the projection operators, one of the assumptions used in the definition of the class $\hat{\mathcal{S}}$ – namely, the requirement (2.21a) – is not explicitly involved. This fact may be used in order to generalize the present approach, e.g. by reducing (or modifying, or eliminating) the restrictions on the curvature tensor of ∇. It goes without saying that any such modifications would be reflected in the connection coefficients involved in the evaluation of the derivatives $\nabla_{\tilde{\partial}_i}$.

in connection with a different type of approach to the theory of connections on $j_1(\mathcal{V}_{n+1})$).

THEOREM 2.3. – *Given a dynamical flow* $\mathbf{Z}$ *over* $j_1(\mathcal{V}_{n+1})$, *let* $\dot{\mathbf{J}} := \mathcal{L}_{\mathbf{Z}}\mathbf{J} \in \mathcal{D}^1_1(j_1(\mathcal{V}_{n+1}))$ *denote the Lie derivative along* $\mathbf{Z}$ *of the fundamental tensor* (1.13). *Then, the direct sum decompposition* (2.34) *induced by* $\mathbf{Z}$ *reflects the spectral structure of* $\dot{\mathbf{J}}$ – *viewed as an* $\mathcal{F}$*-linear map on* $\mathcal{D}^1(j_1(\mathcal{V}_{n+1}))$ – *in the sense that at each point* $z \in j_1(\mathcal{V}_{n+1})$, *the subspaces* H_z, V_z *and* $L(\mathbf{Z}_z)$ *coincide with the eigenspaces of the operator* $\dot{\mathbf{J}}_z : \mathbf{T}_z(j_1(\mathcal{V}_{n+1})) \to \mathbf{T}_z(j_1(\mathcal{V}_{n+1}))$ *corresponding to the eigenvalues* $-1,1$ *and* 0 *respectively.*

Proof. – In terms of the projection operators $\mathcal{P}_H$, $\mathcal{P}_V$, $\mathcal{P}_Z$ involved in the splitting (2.34), the proof of the stated result relies on checking the relation (to be understood in the operatorial sense)

$$\dot{\mathbf{J}} = \mathcal{P}_V - \mathcal{P}_H \tag{2.35}$$

And indeed, by direct computation, recalling eqs. (1.13), (2.15), (2.18a, b) (2.19a), (2.26), (2.33), as well as the assumption $B_i{}^j = 0$, we get the identifications

$$\begin{aligned}\dot{\mathbf{J}} &= \mathcal{L}_{\mathbf{Z}}\left(\frac{\partial}{\partial \dot{q}^i} \otimes \omega^i\right)\\ &= \left(-\tilde{\partial}_i + \tau_i{}^j \frac{\partial}{\partial \dot{q}^j}\right) \otimes \omega^i + \frac{\partial}{\partial \dot{q}^i} \otimes (\tilde{\nu}^i - \tau_j{}^i \omega^j)\\ &= -\tilde{\partial}_i \otimes \omega^i + \frac{\partial}{\partial \dot{q}^i} \otimes \tilde{\nu}^i\end{aligned} \tag{2.36}$$

$$\mathcal{P}_V(\mathbf{X}) = \mathcal{Q}(\mathbf{X}) - \mathcal{P}_Z(\mathbf{X}) = \langle \mathbf{X}, \tilde{\nu}^i \rangle \frac{\partial}{\partial \dot{q}^i} \quad \Rightarrow \quad \mathcal{P}_V = \frac{\partial}{\partial \dot{q}^i} \otimes \tilde{\nu}^i \tag{2.37a}$$

$$\mathcal{P}_H(\mathbf{X}) = \langle \mathbf{X}, \omega^i \rangle \tilde{\partial}_i \quad \Rightarrow \quad \mathcal{P}_H = \tilde{\partial}_i \otimes \omega^i \tag{2.37b}$$

whence the result. □

Remark 2.1. – For convenience of the reader, we collect here the main bulk of formulae pertaining to the dynamical connection ∇, and to the natural basis $\left\{\mathbf{Z}, \tilde{\partial}_i, \frac{\partial}{\partial \dot{q}^i}\right\}$ associated with a given dynamical flow $\mathbf{Z}$. All expressions are special cases of previous results, or follow from these by direct computations:

– explicit representation of the vectors $\tilde{\partial}_i$, of the 1-forms $\tilde{\nu}^i$, and of the connection coefficients:

$$\tilde{\partial}_i = \frac{\partial}{\partial q^i} - \tau_i{}^j \frac{\partial}{\partial \dot{q}^j}; \qquad \tilde{\nu}^i = d\dot{q}^i - Z^i dt + \tau_j{}^i \omega^j \tag{2.38}$$

$$\nabla_{\mathbf{Z}} \frac{\partial}{\partial \dot{q}^i} = \tau_i{}^j \frac{\partial}{\partial \dot{q}^j}; \qquad \nabla_{\mathbf{Z}} \tilde{\partial}_i = \tau_i{}^j \tilde{\partial}_j \tag{2.39a}$$

$$\nabla_{\tilde{\partial}_i} \frac{\partial}{\partial \dot{q}^j} = \Gamma^k_{ij} \frac{\partial}{\partial \dot{q}^k}; \qquad \nabla_{\tilde{\partial}_i} \tilde{\partial}_j = \Gamma^k_{ij} \tilde{\partial}_k \tag{2.39b}$$

$$\nabla_{\frac{\partial}{\partial \dot{q}^i}} \frac{\partial}{\partial \dot{q}^j} = \nabla_{\frac{\partial}{\partial \dot{q}^i}} \tilde{\partial}_j = 0; \qquad \nabla_{\mathbf{Z}} = 0 \tag{2.39c}$$

with

$$\tau_i{}^j = -\frac{1}{2} \frac{\partial Z^j}{\partial \dot{q}^i}; \qquad \Gamma^k_{ij} := \frac{\partial \tau_j{}^k}{\partial \dot{q}^i} = -\frac{1}{2} \frac{\partial^2 Z^k}{\partial \dot{q}^i \partial \dot{q}^j} \tag{2.40}$$

– structure equation for the basis $\left\{\mathbf{Z},\, \tilde{\partial}_i,\, \frac{\partial}{\partial \dot{q}^i}\right\}$:

$$[\tilde{\partial}_i, \tilde{\partial}_j] = \Delta_{ij}{}^k \frac{\partial}{\partial \dot{q}^k}; \quad \left[\tilde{\partial}_i, \frac{\partial}{\partial \dot{q}^j}\right] = \Gamma^k_{ij} \frac{\partial}{\partial \dot{q}^k}; \quad \left[\frac{\partial}{\partial \dot{q}^i}, \frac{\partial}{\partial \dot{q}^j}\right] = 0 \tag{2.41a}$$

$$[\mathbf{Z}, \tilde{\partial}_j] = \tau_j{}^k \tilde{\partial}_k + Q^k{}_j \frac{\partial}{\partial \dot{q}^k}; \qquad \left[\mathbf{Z}, \frac{\partial}{\partial \dot{q}^j}\right] = -\tilde{\partial}_j + \tau_j{}^k \frac{\partial}{\partial \dot{q}^k} \tag{2.41b}$$

and similarly

$$d\omega^k = (-\tilde{\nu}^k + \tau_r{}^k \omega^r) \wedge dt \tag{2.42a}$$

$$d\tilde{\nu}^k = -\frac{1}{2} \Delta_{ij}{}^k \omega^i \wedge \omega^j + \Gamma^k_{ij} \tilde{\nu}^i \wedge \omega^j + (Q^k{}_j \omega^j + \tau_j{}^k \tilde{\nu}^j) \wedge dt \tag{2.42b}$$

with

$$Q^k{}_j := -\mathbf{Z}(\tau_j{}^k) - \tilde{\partial}_j (Z^k) + \tau_j{}^s \tau_s{}^k \tag{2.43a}$$

$$\Delta_{ij}{}^k := \tilde{\partial}_j \tau_i{}^k - \tilde{\partial}_i \tau_j{}^k = \frac{1}{3} \left[\frac{\partial}{\partial \dot{q}^i} Q^k{}_j - \frac{\partial}{\partial \dot{q}^j} Q^k{}_i\right] \tag{2.43b}$$

From eqs. (2.41) $\div$ (2.43) one gets the auxiliary relations

$$\mathcal{L}_{\mathbf{Z}} \omega^k = \mathbf{Z} \rfloor d\omega^k = \tilde{\nu}^k - \tau_r{}^k \omega^r \tag{2.44a}$$

$$\mathcal{L}_{\mathbf{Z}} \tilde{\nu}^k = -Q^k{}_r \omega^r - \tau_r{}^k \tilde{\nu}^r \tag{2.44b}$$

as well as the identities

$$d_v \tilde{\nu}^k = 2\tilde{\nu}^k \wedge dt; \qquad (d - dt \wedge \mathcal{L}_{\mathbf{Z}})\, \omega^k = 0 \tag{2.45a}$$

$$\frac{\partial}{\partial \dot{q}^i} Q^k{}_j = -\mathbf{Z}(\Gamma^k_{ij}) + \Delta_{ij}{}^k + \tilde{\partial}_j \tau_i{}^k + \tau_i{}^r \Gamma^k_{jr} - \tau_r{}^k \Gamma^r_{ij} + \tau_j{}^r \Gamma^k_{ir} \tag{2.45b}$$

Remark 2.2. – By means of the dynamical connection ∇, it is possible to define two "covariant" antiderivations, D_v and D_h of the Grassmann algebra $\mathcal{G}(j_1(\mathcal{V}_{n+1}))$, henceforth called the *vertical differentiation* and the *horizontal differentiation*, described explicitly by the relations

$$D_v \sigma := \omega^i \wedge \nabla_{\frac{\partial}{\partial \dot{q}^i}} \sigma \tag{2.46a}$$

$$D_h \sigma := \omega^i \wedge \nabla_{\tilde{\partial}_i} \sigma \tag{2.46b}$$

$\forall\, \sigma \in \mathcal{G}(j_1(\mathcal{V}_{n+1}))$. An easy check shows that both operations (2.46a, b) have indeed the nature of antiderivations, and that, although defined in terms of bases, they have an invariant behaviour under jet-coordinate transformations, so that, as geometrical objects, they depend only on the connection ∇ (and, through ∇, on the dynamical flow $\mathbf{Z}$).

By the definitions (2.46a, b), using eqs. (2.39a, b, c), we get the relations:

$$D_v f = \frac{\partial f}{\partial \dot{q}^i}\, \omega^i; \qquad D_v\, \omega^k = D_v\, dt = D_v\, \tilde{\nu}^k = 0 \tag{2.47a}$$

$$D_h\, f = \tilde{\partial}_i f\, \omega^i; \qquad D_h \omega^k = D_h\, dt = 0; \qquad D_h \tilde{\nu}^k = -\Gamma^k_{ij} \wedge \tilde{\nu}^j \tag{2.47b}$$

From these, one can easily see that the vertical differentiation D_v satisfies the identity

$$D_v \cdot D_v = 0$$

and that the restriction of D_v to the virtual Grassmann algebra $\hat{\mathcal{G}}(j_1(\mathcal{V}_{n+1}))$ has the same effect as the fiber derivative d_v introduced in Section 1.

For later use, we also observe that the antiderivations D_v, D_h and the derivation $\nabla_{\mathbf{Z}}$ are related by the identity

$$D_v \nabla_{\mathbf{Z}} - \nabla_{\mathbf{Z}} D_v = D_h \tag{2.49}$$

Indeed, since both sides of eq. (2.49) have the nature of antiderivations, it is sufficient to compare their action on functions and on 1-forms. The explicit calculations are left to the reader.

By eqs. (2.48), (2.49) we get the further identity

$$D_v D_h + D_h D_v = 0 \tag{2.50}$$

In a similar way, we may evaluate the antiderivation $\nabla_{\mathbf{Z}} D_h - D_h \nabla_{\mathbf{Z}}$. Recalling eqs. (2.41b), (2.46b), as well as the definition of the curvature tensor, a straightforward computation yields the result

$$\begin{aligned} \nabla_{\mathbf{Z}} D_h - D_h \nabla_{\mathbf{Z}} &= \nabla_{\mathbf{Z}} \omega^i \wedge \nabla_{\tilde{\partial}_i} + \omega^i \wedge (\nabla_{\mathbf{Z}} \nabla_{\tilde{\partial}_i} - \nabla_{\tilde{\partial}_i} \nabla_{\mathbf{Z}}) \\ &= \omega^i \wedge [\mathbf{R}(\mathbf{Z},\, \tilde{\partial}_i) + Q^j{}_i\, \nabla_{\frac{\partial}{\partial \dot{q}^j}}] \end{aligned} \tag{2.51a}$$

where, by comparison with eqs. (2.39), (2.41b), (2.43b), (2.45b), $\mathbf{R}(\mathbf{Z}, \tilde{\partial}_i)$ is given by the relation

$$\begin{aligned}\mathbf{R}(\mathbf{Z}, \tilde{\partial}_i) &= \left(\Delta_{ri}{}^j - \frac{\partial Q^j{}_i}{\partial \dot{q}^r}\right)\left(\tilde{\partial}_j \otimes \omega^r + \frac{\partial}{\partial \dot{q}^j} \otimes \tilde{\nu}^r\right) \\ &= -\frac{1}{3}\left(2\,\frac{\partial Q^j{}_i}{\partial \dot{q}^r} + \frac{\partial Q^j{}_r}{\partial \dot{q}^i}\right)\left(\tilde{\partial}_j \otimes \omega^r + \frac{\partial}{\partial \dot{q}^j} \otimes \tilde{\nu}^r\right) \qquad (2.51\text{b})\end{aligned}$$

[so that, strictly speaking, the term $\mathbf{R}(\mathbf{Z}, \tilde{\partial}_i)$ in eq. (2.51a) denotes the derivation associated with the tensor (2.51b) through the algorithm indicated in Section 1.2].

Remark 2.3. – The difference $\nabla_{\mathbf{Z}} - \mathcal{L}_{\mathbf{Z}}$ is a derivation of the tensor algebra $\mathcal{D}(j_1(\mathcal{V}_{n+1}))$, which commutes with contractions, vanishes on functions, and preserves types of tensors. Therefore, it admits the representation

$$\nabla_{\mathbf{Z}} - \mathcal{L}_{\mathbf{Z}} = \phi_{\mathbf{W}} \qquad (2.52)$$

in terms of a suitable tensor field $\mathbf{W} \in \mathcal{D}^1_1(j_1(\mathcal{V}_{n+1}))$ (see Section 1.2). A straightforward calculation provides the identification

$$\mathbf{W} = \tilde{\partial}_i \otimes \tilde{\nu}^i - Q^i{}_j\,\frac{\partial}{\partial \dot{q}^i} \otimes \omega^j \qquad (2.53)$$

2.3. The time derivative

A deeper insight into the role of the dynamical connection may be gained by analysing how the concept of relative time derivative (*i.e.* of time-derivative with respect to a given frame of reference) may be implemented in the Lagrangian framework. For simplicity, we shall restrict our attention to discrete mechanical systems, formed by N material points $\mathbf{P}_1, \ldots, \mathbf{P}_N$, with masses $m_1, \ldots, m_N$. In this connection, we recall the following preliminary concepts [2]:

(i) every frame of reference $\mathfrak{F}$ determines a corresponding relativization process, summarized into a set of applications

$$\mathbf{x}_i : \quad \mathcal{V}_{n+1} \to \mathcal{E}_3, \qquad i = 1, \ldots, N \qquad (2.54)$$

assigning to each admissible configuration of the system, at any instant t, the positions of the points $\mathbf{P}_1, \ldots, \mathbf{P}_N$ in the Euclidean 3-spaces $\mathcal{E}_3$ associated with $\mathfrak{F}$.

The first jet-extension of the maps (2.54) provides a further set of applications

$$\mathbf{v}_i : \quad j_1(\mathcal{V}_{n+1}) \to \mathbf{T}(\mathcal{E}_3), \qquad i = 1, \ldots, N \qquad (2.55)$$

expressing the velocities of the points of the system in the frame of reference $\mathfrak{F}$.

In jet-coordinates, we have the explicit representation

$$\mathbf{x}_i = \mathbf{x}_i\,(t,\, q^1, \ldots,\, q^n); \qquad \mathbf{v}_i = \frac{\partial \mathbf{x}_i}{\partial q^k}\,\dot{q}^k + \frac{\partial \mathbf{x}_i}{\partial t}, \quad i = 1, \ldots,\, N \tag{2.56}$$

Moreover, by the Absolute Space Axiom, the representation (1.14) of the components of the fiber metric may be written in the equivalent form—

$$a_{rs} = \sum_{i=1}^{N} m_i\,\frac{\partial \mathbf{x}_i}{\partial q^r}\cdot\frac{\partial \mathbf{x}_i}{\partial q^s} = \sum_{i=1}^{N} m_i\,\frac{\partial \mathbf{v}_i}{\partial \dot{q}^r}\cdot\frac{\partial \mathbf{v}_i}{\partial \dot{q}^s} \tag{2.57}$$

for each choice of the frame of reference $\mathfrak{F}$, the dot denoting ordinary scalar product in the associated 3-space $\mathcal{E}_3$.

(ii) every vertical vector $\mathbf{X} = X^r\,\dfrac{\partial}{\partial \dot{q}^r} \in \hat{\mathcal{D}}^1\,(j_1\,(\mathcal{V}_{n+1}))$ determines an N-tuple of maps $\mathbf{u}_i\,:\,j_1\,(\mathcal{V}_{n+1}) \to \mathbf{T}\,(\mathcal{E}_3)$, $i = 1, \ldots,\, N$, *i.e.* an N-tuple $\mathbf{u}_i\,(t,\, q,\, \dot{q})$ of vector valued functions over $j_1\,(\mathcal{V}_{n+1})$ through the definition [2]

$$\mathbf{u}_i := X^r\,\frac{\partial \mathbf{x}_i}{\partial q^r} = X^r\,\frac{\partial \mathbf{v}_i}{\partial \dot{q}^r} = \langle \mathbf{X},\, d_v \mathbf{v}_i\rangle \tag{2.58a}$$

Conversely, every N-tuple of vectors $\mathbf{u}_i\,(t,\, q, \dot{q}) \in \mathbf{T}(\mathcal{E}_3)$, associated with the points $\mathbf{P}_i$, and depending on the kinetic state of the system, determines a corresponding vertical vector $\mathbf{Y} = Y^r\,\dfrac{\partial}{\partial \dot{q}^r}$ according to the identification

$$\begin{aligned} g\,(\mathbf{Y}) :&= \left(\sum_{i=1}^{N} m_i \mathbf{u}_i \cdot \frac{\partial \mathbf{x}_i}{\partial q^r}\right)\omega^r \\ &= \left(\sum_{i=1}^{N} m_i \mathbf{u}_i \cdot \frac{\partial \mathbf{v}_i}{\partial \dot{q}^r}\right)\omega^r \\ &= \sum_{i=1}^{N} m_i \mathbf{u}_i \cdot (d_v \mathbf{v}_i) \end{aligned} \tag{2.58b}$$

g denoting the process of lowering the indices within the virtual algebra given by eq. (1.16).

---

[2] This correspondence may be extended to the vertical vectors over $\mathcal{V}_{n+1}$, through a preliminary use of the vertical lift (1.3).

As it was to be expected, by applying the process (2.58b) to the family of vectors $\mathbf{u}_i$ described by eq. (2.58a), on account of eq. (2.57) one gets the identification

$$g(\mathbf{Y}) = \left(\sum_{i=1}^{N} m_i \, X^s \, \frac{\partial \mathbf{x}_i}{\partial q^s} \cdot \frac{\partial \mathbf{x}_i}{\partial q^r} \right) \omega^r = a_{rs} X^s \omega^r = g(\mathbf{X}) \quad \Rightarrow \quad \mathbf{Y} = \mathbf{X}$$

This points out the injectivity of the correspondence $\mathbf{X} \rightarrow (\mathbf{u}_1, \ldots, \mathbf{u}_N)$ described by eq. (2.58a), as well as the surjectivity of the correspondence (2.58b). Further details on this subject may be found in [2].

After these preliminaries, let us now turn our attention to an arbitrary world-line in $j_1(\mathcal{V}_{n+1})$, *i.e.* to a curve $\hat{\gamma}$ obtained as jet extension of a section $\gamma : \mathbb{R} \rightarrow \mathcal{V}_{n+1}$, and thus representing an admissible evolution of the system.

For each vertical vector field $\mathbf{X} = X^i \dfrac{\partial}{\partial \dot{q}^i}$ defined in a neighbourhood of $\hat{\gamma}$, we indicate by $\mathbf{X}_{|\hat{\gamma}} : \mathbb{R} \rightarrow \mathbf{T}(j_1(\mathcal{V}_{n+1}))$ the restriction of $\mathbf{X}$ to the curve $\hat{\gamma}$.

On the basis of the previous discussion, a reasonable definition of *time-derivative* of $\mathbf{X}$ along $\hat{\gamma}$ *relative to the frame of reference* $\mathfrak{F}$ is then provided by the identification

$$\left(\frac{D}{Dt} \right)_{\mathfrak{F}} \mathbf{X}_{|\hat{\gamma}} := g^{-1} \left(\sum_{i=1}^{N} m_i \, \frac{d\mathbf{u}_i}{dt} \cdot (d_v \mathbf{v}_i)_{|\hat{\gamma}} \right) \tag{2.59}$$

in which the vectors $\mathbf{u}_i = \mathbf{u}_i(t)$ are determined by $\mathbf{X}_{|\hat{\gamma}}$ through eq. (2.58a), while $\dfrac{d}{dt}$ denotes the time-derivative in the usual sense.

The operation (2.59) has the nature of a directional derivative: it seems therefore natural to look for a geometric counterpart of the latter in terms of a suitable linear connection, *induced* by the choice of the frame of reference.

This viewpoint fits quite nicely into the framework developed in Section 2.2, the link being provided by the fact that every frame of reference $\mathfrak{F}$ determines its own dynamical flow $\mathbf{Z}_{\mathfrak{F}}$ over $j_1(\mathcal{V}_{n+1})$, completely characterized by the conditions

$$\sum_{i=1}^{N} m_i \, \mathbf{Z}_{\mathfrak{F}}(\mathbf{v}_i) \cdot \frac{\partial \mathbf{x}_i}{\partial q^r} = 0 \qquad r = 1, \ldots, n \tag{2.60}$$

in which $\mathbf{x}_i\ (t,\ q)$ and $\mathbf{v}_i\ (t,\ q,\ \dot{q})$ are the functions involved in the relativization process (2.56). Further details on this point may be found in [2] [3].

In coordinates, making use of the representation

$$\mathbf{Z}_{\mathfrak{F}} = \frac{\partial}{\partial t} + \dot{q}^k\,\frac{\partial}{\partial q^k} + Z^k\,\frac{\partial}{\partial \dot{q}^k}$$

and recalling the well known identities $\dfrac{\partial \mathbf{x}_i}{\partial q^k} = \dfrac{\partial \mathbf{v}_i}{\partial \dot{q}^k}$, $\mathbf{Z}_{\mathfrak{F}}\left(\dfrac{\partial \mathbf{x}_i}{\partial q^k}\right) = \dfrac{\partial \mathbf{v}_i}{\partial q^k}$, as well as the expression (2.57) for the fiber metric, by eq. (2.60) we obtain an explicit characterization of the components Z^k as solutions of the linear algebraic system [4]

$$\sum_{i=1}^{N} m_i \left(\frac{\partial \mathbf{v}_i}{\partial t} + \dot{q}^k\,\frac{\partial \mathbf{v}_i}{\partial q^k}\right)\cdot\frac{\partial \mathbf{x}_i}{\partial q^r} + g_{kr}\,Z^k = 0 \tag{2.61}$$

We can now state

PROPOSITION 2. 1. – *The time derivative relative to the frame of reference $\mathfrak{F}$ along a world-line $\hat{\gamma}$ coincides with the absolute derivative along $\hat{\gamma}$ induced by the dynamical connection ∇ associated with the flow $\mathbf{Z}_{\mathfrak{F}}$.*

Proof. – By eq. (2.56), we have the identity

$$\begin{aligned}\frac{\partial}{\partial \dot{q}^s}\left(\frac{\partial \mathbf{v}_i}{\partial t} + \dot{q}^k\,\frac{\partial \mathbf{v}_i}{\partial q^k}\right) &= \frac{\partial \mathbf{v}_i}{\partial q^s} + \left(\frac{\partial}{\partial t} + \dot{q}^k\,\frac{\partial}{\partial q^k}\right)\frac{\partial \mathbf{v}_i}{\partial \dot{q}^s}\\ &= \frac{\partial \mathbf{v}_i}{\partial q^s} + \left(\frac{\partial}{\partial t} + \dot{q}^k\,\frac{\partial}{\partial q^k}\right)\frac{\partial \mathbf{x}_i}{\partial q^s} = 2\,\frac{\partial \mathbf{v}_i}{\partial q^s}\end{aligned}$$

whence also, taking eq. (2.61) into account, and recalling the expression (2.40) for the connection coefficients $\tau_s{}^k$

---

[3] Dynamically, eqs. (2.60) characterize the integral curves of $\mathbf{Z}_{\mathfrak{F}}$ as representing a special class of motions in which the relative accelerations of the points of the system in the frame of reference $\mathfrak{F}$ are entirely due to the presence of the constraints — *i.e.* to the reactive forces – without the intervention of active forces of whatsoever type.

[4] It goes without saying that, denoting by $T\,(t,\ q,\ \dot{q}) = \dfrac{1}{2}\displaystyle\sum_{i=1}^{N} m_i v_i^2$ the kinetic energy of the system in the frame of reference $\mathfrak{F}$, eq. (2.60) may be written in the more familiar form

$$\mathbf{Z}_{\mathfrak{F}}\left(\frac{\partial T}{\partial \dot{q}^k}\right) - \frac{\partial T}{\partial q^k} = 0$$

as efficient as eq. (2.61) in order to evaluate the components Z^k.

$$\tau_s{}^k = -\frac{1}{2}\frac{\partial Z^k}{\partial \dot q^s} = \frac{1}{2}\, a^{rk}\, \frac{\partial}{\partial \dot q^s} \sum_{i=1}^{N} \left(\frac{\partial \mathbf{v}_i}{\partial t} + \dot q^k \frac{\partial \mathbf{v}_i}{\partial q^k} \right) \cdot \frac{\partial \mathbf{x}_i}{\partial q^r}$$
$$= a^{rk} \sum_{i=1}^{N} m_i \frac{\partial \mathbf{v}_i}{\partial q^s} \cdot \frac{\partial \mathbf{x}_i}{\partial q^r} \tag{2.62}$$

With this in mind, denoting by $\hat\gamma : \mathbb{R} \to j_1(\mathcal{V}_{n+1})$ an arbitrary world-line in $j_1(\mathcal{V}_{n+1})$, and by $\mathbf{X} = X^i \dfrac{\partial}{\partial \dot q^i}$ a vertical vector field defined in a neighbourhood of $\hat\gamma$, by eqs. (2.58a), (2.59), (2.62) we obtain the result

$$\left(\frac{D}{Dt}\right)_{\mathfrak{F}} \mathbf{X}_{|\hat\gamma} = a^{rk} \left[\sum_{i=1}^{N} m_i \frac{d}{dt} \left(X^s \frac{\partial \mathbf{x}_i}{\partial q^s} \right)_{|\hat\gamma} \cdot \left(\frac{\partial \mathbf{x}_i}{\partial q^r} \right)_{|\hat\gamma} \right] \left(\frac{\partial}{\partial \dot q^k} \right)_{|\hat\gamma}$$
$$= \left[\frac{d}{dt} X^k_{|\hat\gamma} + X^s \, \tau_s{}^k \right] \left(\frac{\partial}{\partial \dot q^k} \right)_{|\hat\gamma} \tag{2.63}$$

which agrees with the expression of the absolute derivative of $\mathbf{X}$ along $\hat\gamma$ determined by the connection ∇. $\square$

Remark 2.3. – In view of eq. (2.63), the representation of the operator $\left(\dfrac{D}{Dt}\right)_{\mathfrak{F}}$ is completely determined by the knowledge of the coefficients $\tau_s{}^k$, and therefore involves only a part of the information stored in the dynamical flow $\mathbf{Z}_{\mathfrak{F}}$. From a physical viewpoint, this is exactly what one would expect on the basis of the fact that, according to Poisson's formulae, the equality between time derivatives relative to different frames of reference $\mathfrak{F}$, $\mathfrak{F}'$ does not require the vanishing of the mutual acceleration – *i.e.* the identification of the associated dynamical flows $\mathbf{Z}_{\mathfrak{F}}$, $\mathbf{Z}_{\mathfrak{F}'}$ – but only the absence of a relative angular velocity.

To put this statement on quantitative grounds, we stick to eq. (2.62), as well as to the analogous one

$$\tau'^k_s = -\frac{1}{2}\frac{\partial Z'^k}{\partial \dot q^s} = a^{rk} \sum_{i=1}^{N} m_i \frac{\partial \mathbf{v}'_i}{\partial q^s} \cdot \frac{\partial \mathbf{x}'_i}{\partial q^r} \tag{2.64}$$

written with respect to a second frame of reference $\mathfrak{F}'$.

Denoting by O a point at rest in $\mathfrak{F}'$, and resorting to the standard techniques of Relative Kinematics – in particular, to the Axiom of

Absolute Space, providing an identification, at any instant t, of the 3-spaces $\mathcal{E}_3$ and $\mathcal{E}'_3$ associated with the given frames – we have the familiar relations [5]

$$\mathbf{x}_i - \mathbf{x}_O = \mathbf{x}'_i - \mathbf{x}'_O \quad \mathbf{v}_i - \mathbf{v}_O = \mathbf{v}'_i + \omega \times (\mathbf{x}'_i - \mathbf{x}'_O)$$

whence also, applying eqs. (2.62), (2.64), and recalling that $\mathbf{x}_O$, $\mathbf{x}'_O$ and ω are all independent of the variables q^i,

$$\begin{aligned}\tau_s{}^k &= a^{rk} \sum_{i=1}^{N} m_i \left(\frac{\partial \mathbf{v}'_i}{\partial q^s} + \omega \times \frac{\partial \mathbf{x}'_i}{\partial q^s} \right) \cdot \frac{\partial \mathbf{x}'_i}{\partial q^r} \\ &= \tau'^k_r + a^{rk} \omega \cdot \sum_{i=1}^{N} m_i \frac{\partial \mathbf{x}'_i}{\partial q^s} \times \frac{\partial \mathbf{x}'_i}{\partial q^r} \end{aligned} \tag{2.65}$$

The virtual 2-form

$$\Omega = \Omega_{ab} \omega^a \wedge \omega^b := \left(\omega \cdot \sum_{i=1}^{N} m_i \frac{\partial \mathbf{x}'_i}{\partial q^a} \times \frac{\partial \mathbf{x}'_i}{\partial q^b} \right) \omega^a \wedge \omega^b \tag{2.66}$$

will be called the *angular velocity tensor* of the frame $\mathfrak{F}$ with respect to $\mathfrak{F}'$.

In terms of $\mathbf{Z}_{\mathfrak{F}}$, $\mathbf{Z}_{\mathfrak{F}'}$, introducing the vertical field $\mathbf{V} = \mathbf{Z}_{\mathfrak{F}'} - \mathbf{Z}_{\mathfrak{F}}$, and recalling eqs. (2.62), (2.64), (2.65), (2.66), we obtain the representation

$$\begin{aligned}\frac{1}{2} \frac{\partial V^k}{\partial \dot{q}^s} = a^{rk} \Omega_{sr} \quad &\Leftrightarrow \quad \Omega = \frac{1}{2} \frac{\partial}{\partial \dot{q}^s} \left(a_{rk} V^k \right) \omega^s \wedge \omega^r \\ &= \frac{1}{2} d_v \, g(\mathbf{V}) \end{aligned} \tag{2.67}$$

In a similar way, by eqs. (2.63), (2.65), (2.66), we get the analogue of the Poisson formulae in $j_1(\mathcal{V}_{n+1})$:

$$\begin{aligned}\left(\frac{D}{Dt} \right)_{\mathfrak{F}} \mathbf{X}_{|\hat{\gamma}} - \left(\frac{D}{Dt} \right)_{\mathfrak{F}'} \mathbf{X}_{|\hat{\gamma}} &= X^s \left(\tau_s{}^k - \tau'^k_s \right) \left(\frac{\partial}{\partial \dot{q}^k} \right)_{\hat{\gamma}} \\ &= X^s a^{rk} \Omega_{sr} \frac{\partial}{\partial \dot{q}^k} = \frac{1}{2} g^{-1} \left(\hat{\imath}_{\mathbf{X}} \Omega \right)_{|\hat{\gamma}} \end{aligned} \tag{2.68}$$

the notation $\hat{\imath}_{\mathbf{X}}$ denoting the interior product in the virtual Grassmann algebra, evaluated with respect to the pairing (1.5).

[5] To avoid notational ambiguities, we are denoting by $\times$ the ordinary cross product in $\mathcal{E}_3$.

3. APPLICATIONS

3.1. Generalized potential

In this Section we discuss a few applications of the concepts introduced so far. These include an introductory review of the concept of generalized potential, as well as a geometrical approach to the so called *inverse problem* of Lagrangian Dynamics, along the lines of Camprin, Prince and Thompson [1], Sarlet [12] and Henneaux [14], [27]. For a further reading on this subject, see also ([3], [10], [11], [13], [37]).

To start with we consider the relation between semi-basic and virtual differential forms over $j_1(\mathcal{V}_{n+1})$ provided by the antiderivation (1.9), focussing now our attention on the case of fields of r-forms $\hat{\sigma}$ obtained as pull-back of fields over $\mathcal{V}_{n+1}$, namely $\hat{\sigma} = \pi^*(\sigma)$, $\sigma \in \mathcal{G}^r(\mathcal{V}_{n+1})$. We will denote by $U \in \hat{\mathcal{G}}^{r-1}(j_1(\mathcal{V}_{n+1}))$ the virtual $(r-1)$-form related to σ by the condition

$$U_{|z} = z \rfloor \hat{\sigma}_{|z} \qquad z \in j_1(\mathcal{V}_{n+1}) \tag{3.1a}$$

mathematically equivalent to [6]

$$U = \mathbf{Z} \rfloor \sigma \tag{3.1b}$$

for any choice of the dynamical flow $\mathbf{Z}$ over $j_1(\mathcal{V}_{n+1})$. Using the notation

$$\sigma = r\,\sigma_{0i_2\ldots i_r}\,dt \wedge dq^{i_2} \wedge \ldots \wedge dq^{i_r} + \sigma_{i_1\ldots i_r}\,dq^{i_1} \wedge \ldots \wedge dq^{i_r} \tag{3.2}$$

(see eq. (1.4)), and taking eq. (1.10) into account, we have the explicit representation

$$U = r\,(\sigma_{0i_2\ldots i_r} + \dot{q}^{i_1}\sigma_{i_1\ldots i_r})\,\omega^{i_2} \wedge \ldots \wedge \omega^{i_r} \tag{3.3}$$

Conversely, by eq. (3.3), recalling eqs. (1.7), (1.22a), (1.23), we get the inversion formula

$$r\,\sigma = r\,(\sigma_{i_1\ldots i_r}\omega^{i_1} \wedge \ldots \wedge \omega^{i_r} + dt \wedge U) = (d_v + r\,dt\wedge)\,U \tag{3.4}$$

showing that, for any $r > 0$, the correspondence $\mathcal{G}^r(\mathcal{V}_{n+1}) \to \hat{\mathcal{G}}^{r-1}(j_1(\mathcal{V}_{n+1}))$ given by eq. (3.1a), is injective.

THEOREM 3.1. – *A necessary and sufficient condition in order for a virtual* $(r-1)$*-form* $U = U_{i_2\ldots i_r}\omega^{i_2} \wedge \ldots \wedge \omega^{i_r}$ $(r > 1)$ *over* $j_1(\mathcal{V}_{n+1})$ *to admit the representation* (3.1b) *in terms of an* r*-form* $\sigma \in \pi^*\mathcal{G}^r(\mathcal{V}_{n+1})$ *is that the*

---

[6] For simplicity, in what follows, we shall identify each differential form over $\mathcal{V}_{n+1}$ with the corresponding pull-back on $j_1(\mathcal{V}_{n+1})$, thus dropping any notational difference between σ and $\hat{\sigma}$.

indexed set of partial derivatives $\dfrac{\partial U_{i_2 \ldots i_r}}{\partial \dot{q}^{i_1}}$ *be antisymmetric with respect to the whole set of indices* $i_1, \ldots, i_r$.

Proof. – The necessity of the condition follows easily from eq. (3.3). As for sufficiency, we observe that, as a consequence of the antisymmetry requirement, the components $U_{i_2 \ldots i_r}$ satisfy the relations

$$\begin{aligned} \frac{\partial^2 U_{i_2 \ldots i_r}}{\partial \dot{q}^i \partial \dot{q}^j} &= -\frac{\partial^2 U_{j i_3 \ldots i_r}}{\partial \dot{q}^i \partial \dot{q}^{i_2}} = \frac{\partial^2 U_{i i_3 \ldots i_r}}{\partial \dot{q}^j \partial \dot{q}^{i_2}} \\ &= -\frac{\partial^2 U_{i_2 \ldots i_r}}{\partial \dot{q}^j \partial \dot{q}^i} \quad \Rightarrow \quad \frac{\partial^2 U_{i_2 \ldots i_r}}{\partial \dot{q}^i \partial \dot{q}^j} = 0 \end{aligned}$$

i.e., they depend linearly on the $\dot{q}^k$'s, thus admitting a (unique) representation of the form

$$U_{i_2 \ldots i_r} = r \left(\sigma_{0 i_2 \ldots i_r} + \dot{q}^{i_1} \sigma_{i_1 i_2 \ldots i_r} \right)$$

with $\sigma_{0 i_2 \ldots i_r}$ and $\sigma_{i_1 i_2 \ldots i_r}$ depending only on $t, q^1, \ldots, q^n$. The conclusion then follows once again from eq. (3.3). $\square$

Remark 3.1. – When $r = 1$, in place of Theorem 3.1 we have the simpler criterion: a function $f \in \mathcal{F}(j_1(\mathcal{V}_{n+1}))$ admits the representation $f = \mathbf{Z} \rfloor \sigma$, $\sigma \in \pi^* \mathcal{G}^1 (\mathcal{V}_{n+1})$ if and only if f depends at most linearly on the $\dot{q}^i$'s. The proof is self-evident, and is left to the reader.

Let us now examine the interplay between eq. (3.1b) and the operation of exterior differentiation in $\mathcal{G}(\mathcal{V}_{n+1})$. To this end, for each $\sigma \in \pi^* \mathcal{G}^r (\mathcal{V}_{n+1})$, we consider both virtual forms $U = \mathbf{Z} \rfloor \sigma$, and $Q = \mathbf{Z} \rfloor d\sigma$. In view of the inversion formula (3.4), we have then the relation

$$Q = \mathbf{Z} \rfloor d \left(\frac{1}{r} \, d_v U + dt \wedge U \right) \tag{3.5}$$

independently of the choice of the dynamical flow $\mathbf{Z}$. In local jet-coordinates, setting $U = U_{i_2 \ldots i_r} \omega^{i_2} \wedge \ldots \wedge w^{i_r}$, this gives rise to the explicit representation

$$Q = \left[\frac{1}{r} \, \mathbf{Z} \left(\frac{\partial U_{i_2 \ldots i_r}}{\partial \dot{q}^{i_1}} \right) - \frac{\partial U_{i_2 \ldots i_r}}{\partial q^{i_1}} \right] \omega^{i_1} \wedge \ldots \wedge \omega^{i_r} \tag{3.6}$$

For $r = 1$ the latter yields back – up to a sign – the usual relation between generalized potential and Lagrangian components of the active forces. In view of this fact, for each $r \geq 1$, the $(r-1)$-form $U = \mathbf{Z} \rfloor \sigma$ will be called a *generalized potential* for the r-form $Q = \mathbf{Z} \rfloor d\sigma$.

A deeper insight into the role of eq. (3.5) is gained by adopting a slightly different viewpoint. To this end, to each dynamical flow $\mathbf{Z}$ over $j_1 (\mathcal{V}_{n+1})$, we associate an operator $\delta_{\mathbf{Z}} : \mathcal{G}(j_1(\mathcal{V}_{n+1})) \to \mathcal{G}(j_1(\mathcal{V}_{n+1}))$ on the basis of the relation

$$\delta_{\mathbf{Z}} \Omega := \mathbf{Z} \rfloor d \left(\frac{1}{p+1} d_v + dt \wedge \right) \Omega, \qquad \forall\, \Omega \in \mathcal{G}^p(j_1(\mathcal{V}_{n+1})) \tag{3.7}$$

In view of the previous discussion, we have the commutative diagram

$$\begin{array}{ccc} \pi^* \mathcal{G}(\mathcal{V}_{n+1}) & \xrightarrow{d} & \pi^* \mathcal{G}(\mathcal{V}_{n+1}) \\ \mathbf{Z} \rfloor \downarrow & & \downarrow \mathbf{Z} \rfloor \\ \mathcal{G}(j_1(\mathcal{V}_{n+1})) & \xrightarrow{\delta_{\mathbf{Z}}} & \mathcal{G}(j_1(\mathcal{V}_{n+1})) \end{array} \tag{3.8}$$

In particular, the relation (3.5) may be written more synthetically as

$$Q = \delta_{\mathbf{Z}} U \tag{3.9}$$

In addition to these fairly obvious remarks, the importance of the definition (3.7) lies in the fact that, through the latter, the operation at the r.h.s of eq. (3.5) is no longer restricted to differential forms of the special type (3.1b), but is extended (in a $\mathbf{Z}$-dependent way) to the entire Grassmann algebra over $j_1(\mathcal{V}_{n+1})$. As an illustration of the usefulness of this viewpoint, we state

THEOREM 3.2. – *Let Q be a virtual r-form $(r > 0)$ over $j_1(\mathcal{V}_{n+1})$. Then:*

(i) *a necessary condition for Q to "be a potential", i.e. to admit the representation $Q = \mathbf{Z} \rfloor \omega$, $\omega \in \pi^*(\mathcal{G}^{r+1}(\mathcal{V}_{n+1}))$ is that the $(r+1)$-form $\delta_{\mathbf{Z}} Q$ be independent of the choice of the dynamical flow $\mathbf{Z}$. A sufficient condition to Q for be a potential is that $\delta_{\mathbf{Z}} Q$ be a virtual form for at least one choice of $\mathbf{Z}$.*

(ii) *a necessary condition for Q to admit the representation* (3.9) *in terms of a generalized potential U is the validity of the relation*

$$d \left(\frac{1}{r+1} d_v + dt \wedge \right) Q = 0$$

A sufficient condition for the representation (3.9) *to hold is that the equation $\delta_{\mathbf{Z}} Q = 0$ be satisfied for at least one choice of the flow $\mathbf{Z}$.*

Proof. – As pointed out at the beginning of this Subsection (*see* the comment following eq. (3.1b)), if the representation $Q = \mathbf{Z} \rfloor \omega$, $\omega \in \pi^*(\mathcal{G}^{r+1}(\mathcal{V}_{n+1}))$ holds, it holds independently of the choice of the flow $\mathbf{Z}$. Taking the commutativity of the diagram (3.8) into account if follows easily that, under the stated assumption, $\delta_{\mathbf{Z}} Q = \mathbf{Z} \rfloor d\omega$ too is independent of $\mathbf{Z}$.

Conversely, assume that, for a given dynamical flow $\mathbf{Z}$, the $(r+1)$-form $\delta_{\mathbf{Z}} Q$ belong to the virtual Grassmann algebra $\hat{\mathcal{G}}\,(j_1\,(\mathcal{V}_{n+1}))$. In view of eq. (3.7), this is mathematically equivalent to the n independent relations

$$0 = \frac{\partial}{\partial \dot{q}^h} \rfloor \delta_{\mathbf{Z}} Q = -\mathbf{Z} \rfloor \frac{\partial}{\partial \dot{q}^h} \rfloor d \left(\frac{d_v Q}{r+1} + dt \wedge Q \right) \qquad h = 1, \ldots, n$$

i.e., by explicit computation

$$\begin{aligned} 0 &= -\mathbf{Z} \rfloor \mathcal{L}_{\frac{\partial}{\partial \dot{q}^h}} \left(\frac{d_v Q}{r+1} + dt \wedge Q \right) \\ &= -\frac{r}{r+1} \left(\frac{\partial Q_{i_1 \ldots i_r}}{\partial \dot{q}^h} + \frac{\partial Q_{h i_2 \ldots i_r}}{\partial \dot{q}^{i_1}} \right) \omega^{i_1} \wedge \ldots \wedge \omega^{i_r} \end{aligned}$$

Therefore, under the stated assumption, the indexed family $\dfrac{\partial Q_{i_1 \ldots i_r}}{\partial \dot{q}^h}$ is antisymmetric with respect to the whole set of indices $h, i_1, \ldots, i_r$. The existence of a representation of the form $Q = \mathbf{Z} \rfloor \omega$, $\omega \in \pi^* \mathcal{G}_{r+1}\,(\mathcal{V}_{n+1})$ is then a consequence of Theorem 3.1. This proves statement (i).

To verify statement (ii), consider first the case when Q admits a potential $U = \mathbf{Z} \rfloor \sigma$; by definition, this implies $Q = \delta_{\mathbf{Z}} U = \mathbf{Z} \rfloor d\sigma$, whence, by comparison with the inversion formula (3.4)

$$d\sigma = \left(\frac{1}{r+1} d_v + dt \wedge \right) Q \quad \Rightarrow \quad d \left(\frac{1}{r+1}\, d_v + dt \wedge \right) Q = 0$$

This establishes the first part of (ii). Conversely, assume that, for a given dynamical flow $\mathbf{Z}$, the condition $\delta_{\mathbf{Z}} Q = 0$ holds true. Then, taking statement (i) into account, we see that Q admits the representation $Q = \mathbf{Z} \rfloor \omega$, with $\omega \in \pi^* \mathcal{G}\,(\mathcal{V}_{n+1})$ subject to the condition $\mathbf{Z} \rfloor d\omega = \delta_{\mathbf{Z}} Q = 0$. By the injectivity of the map $\pi^* \mathcal{G}^{r+1}\,(\mathcal{V}_{n+1}) \to \hat{\mathcal{G}}^r\,(j_1\,(\mathcal{V}_{n+1}))$ described by eq. (3.1b), this implies $d\omega = 0$, whence, locally, $\omega = d\sigma$. To sum up, under the stated assumption, Q may be expressed as $Q = \mathbf{Z} \rfloor d\sigma$, $\sigma \in \pi^* \mathcal{G}(\mathcal{V}_{n+1})$, *i.e.* $Q = \delta_{\mathbf{Z}} U$, with $U = \mathbf{Z} \rfloor \sigma$. □

Remark 3.2. – Exactly as it happened for Theorem 3.1, Theorem 3.2 too breaks down when $r = 0$. In this case, writing F in place of Q, eq. (3.7) gives

$$\delta_{\mathbf{Z}} F = \left[\mathbf{Z} \left(\frac{\partial F}{\partial \dot{q}^k} \right) - \frac{\partial F}{\partial q^k} \right] \omega^k \tag{3.10}$$

From this we can draw, among others, the following conclusions:

(i) $\delta_{\mathbf{Z}} F$ is always a virtual 1-form for *any* choice of the dynamical flow $\mathbf{Z}$, and of the differentiable function $F \in \mathcal{F}\,(j_1\,(\mathcal{V}_{n+1}))$. A necessary and sufficient condition in order for F to admit the representation $F = Z \rfloor \omega$, $\omega \in \pi^* \mathcal{G}^1 (\mathcal{V}_{n+1})$ is that $\delta_{\mathbf{Z}} F$ be *independent* of the choice of $\mathbf{Z}$.

(ii) in a similar way, a necessary and sufficient condition for F to admit the local representation $F = \mathbf{Z} \rfloor df = \mathbf{Z}(f)$, $f \in \pi^* \mathcal{F} (\mathcal{V}_{n+1})$ is that the relation $\delta_{\mathbf{Z}} F = 0$ holds for all possible choices of the dynamical flow $\mathbf{Z}$. On the contrary, the validity of $\delta_{\mathbf{Z}} F = 0$ for a *single* dynamical flow $\mathbf{Z}$, has now an entirely different meaning, and has to be regarded as a statement on $\mathbf{Z}$, expressed in the form of a linear algebraic system for the components Z^i ("Lagrange's equations"), rather than as an effective condition on the function F, apart for the requirement of algebraic compatiblity of the system itself.

3.2. Helmholtz's conditions

The results stated in Theorem 3.2 play an important role in the so called *inverse problem* of Lagrangian Mechanics, *i.e.* in the identification of a suitable set of necessary and sufficient conditions to be satisfied in order for a given dynamical flow $\mathbf{Z}$ to be derivable – locally – from a non-singular Lagrangian L, through the Lagrange equations [7]

$$\delta_{\mathbf{Z}} L = \left[\mathbf{Z} \left(\frac{\partial L}{\partial \dot{q}^k} \right) - \frac{\partial L}{\partial q^k} \right] \omega^k = 0 \tag{3.11}$$

In view of eqs. (3.7), (3.10), eq. (3.11) may be written in the equivalent form

$$\mathbf{Z} \rfloor \Omega = 0 \tag{3.12}$$

with

$$\Omega := d\,(d_v L + L\, dt) \tag{3.13}$$

expressing the *Poincaré-Cartan 2-form* of $\mathbf{Z}$.

In comparison with the usual way of looking at Lagrange's equations, the "inverse" nature of the problem formulated above is clear: the dynamical flow $\mathbf{Z}$ is now regarded as given, and the emphasis is on the integrability of eqs. (3.11), viewed as a system of partial differential equations for the unknown L. When the system is integrable – or, more precisely, when a solution L exists within the class of non-singular functions over $j_1 (\mathcal{V}_{n+1})$ – $\mathbf{Z}$ itself is called a Lagrangian flow.

[7] Here and in the following, the "non-singularity" of a function $f \in \mathcal{F}(j_1 (\mathcal{V}_{n+1}))$ will be identified with the non singularity of the corresponding Hessian matrix $\left\| \dfrac{\partial^2 f}{\partial \dot{q}^i \partial \dot{q}^j} \right\|$.

A significant insight into the nature of the inverse problem is achieved by replacing eq. (3.11) with the iterated one

$$\delta_{\mathbf{Z}} \cdot \delta_{\mathbf{Z}} L = 0 \tag{3.14}$$

Sure enough, every solution L of eq. (3.11) is also a solution of eq. (3.14). Conversely, a straightforward application of Theorem 3.2 points out the following basic facts:

(i) any function $U(t, q, \dot{q})$ depending at most linearly on the $\dot{q}$'s – *i.e.*, any "potential" $U = \mathbf{Z} \rfloor \sigma$, $\sigma \in \pi^* (\mathcal{G}^1 (\mathcal{V}_{n+1}))$ – is automatically a solution of eq. (3.14). We shall call this a *trivial* solution.

(ii) to each solution $L^* (t, q, \dot{q})$ of eq. (3.14) one can always associate – locally – a trivial solution $U(t, q, \dot{q})$ in such a way that the sum $L = L^* + U$ satisfies eq. (3.11).

Statement (i) is a straightforward consequence of the commutativity of the diagram (3.8) ($U = \mathbf{Z} \rfloor \sigma \Rightarrow \delta_{\mathbf{Z}} U = \mathbf{Z} \rfloor d\sigma \Rightarrow \delta_{\mathbf{Z}} \cdot \delta_{\mathbf{Z}} U = \mathbf{Z} \rfloor d(d\sigma) = 0$).

The proof of statement (ii) is similar: the relation $\delta_{\mathbf{Z}} \cdot \delta_{\mathbf{Z}} L^* = 0$ is in fact sufficient in order for the 1-form $\delta_{\mathbf{Z}} L^*$ to admit, locally, a generalized potential U, *i.e.* to satisfy $\delta_{\mathbf{Z}} L^* = -\delta_{\mathbf{Z}} U$, (or, what is the same, $\delta_{\mathbf{Z}} (L^* + U) = 0$), with U depending at most linearly on the $\dot{q}$'s [8]. We have thus proved

COROLLARY 3.1. – *A necessary and sufficient condition in order that a given dynamical flow* $\mathbf{Z}$ *be a Lagrangian one is that eq.* (3.14) *admits at least one solution within the class of non-singular functions over* $j_1 (\mathcal{V}_{n+1})$.

Remark 3.3. – If we regard two solutions L, L' of eq. (3.14) as equivalent whenever they differ by a trivial solution, each equivalence class of solutions identifies a possible Lagrangian L for the flow $\mathbf{Z}$, up to a gauge transformation $L \to L + \mathbf{Z}(f)$, $f \in \pi^* (\mathcal{F} (\mathcal{V}_{n+1}))$. The number of "inequivalent" (*i.e.*, non gauge-related) Lagrangians associated with $\mathbf{Z}$ is therefore identical to the number of equivalence classes under the stated relation. In this connection, *see* e.g. ([14], [38]).

Remark 3.4. – The content of Corollary 3.1 may be extended to the context of *singular* Lagrangians, by replacing the requirement of non-singularity with the weaker one of non-triviality, understood in the technical sense indicated above.

[8] In view of the discussion in Remark 3.2, it is easily seen that the previous argument determines the potential U – and, therefore, also the Lagrangian L – only up to arbitrary transformations of the form $U \to U + \mathbf{Z}(f)$, $f \in \pi^* (\mathcal{F} (\mathcal{V}_{n+1}))$.

We shall now discuss the integrability conditions for eq. (3.14). To achieve this goal, once the dynamical flow $\mathbf{Z}$ has been given, we have at our disposal the entire machinery described in Section 2 (dynamical connection, natural bases, etc.). Within this context, we observe that, for any function $F \in \mathcal{F}(j_1(\mathcal{V}_{n+1}))$, eq. (3.10), together with the commutation relations (2.41b) and the characterization (2.38) of the field $\tilde{\partial}_i$, provides the representation

$$\begin{aligned}\delta_{\mathbf{Z}} F &= \left(\frac{\partial}{\partial \dot{q}^k} \mathbf{Z}(F) - 2\tilde{\partial}_k(F) \right) \omega^k \\ &= d_v \mathbf{Z}(F) - 2dF + 2\left(\mathbf{Z}(F)\, dt + \frac{\partial F}{\partial \dot{q}^k} \tilde{\nu}^k \right)\end{aligned}$$

whence also

$$\begin{aligned}&\left(\frac{1}{2} d_v + dt\wedge \right) \delta_{\mathbf{Z}} F \\ &= \frac{1}{2}(d_v + 2dt\wedge)\left[(d_v + 2dt\wedge)\, \mathbf{Z}(F) - 2dF + 2\frac{\partial F}{\partial \dot{q}^k} \tilde{\nu}^k \right]\end{aligned} \tag{3.15}$$

On the other hand, by eqs. (1.22b), (1.24), (2.45a), we have the identities

$$\begin{aligned}&(d_v + 2dt\wedge)\cdot(d_v + 2dt\wedge) = -2dt \wedge d_v + 2dt \wedge d_v = 0 \\ &(d_v + 2dt\wedge)\cdot d = -d \cdot (d_v + dt\wedge) \\ &(d_v + 2dt\wedge)\, \tilde{\nu}^k = 0\end{aligned}$$

Eq. (3.15) may therefore be written in the form

$$\left(\frac{1}{2} d_v + dt\wedge \right) \delta_{\mathbf{Z}} F = d\,(d_v F + F dt) + \frac{\partial^2 F}{\partial \dot{q}^h \partial \dot{q}^k}\, \omega^h \wedge \tilde{\nu}^k \tag{3.16}$$

Together with eq. (3.7), this provides the representation

$$\delta_{\mathbf{Z}}\, \delta_{\mathbf{Z}} F = \mathbf{Z} \rfloor d \left(\frac{1}{2} d_v + dt\wedge \right) \delta_{\mathbf{Z}} F = \mathcal{L}_{\mathbf{Z}} \left(\frac{\partial^2 F}{\partial \dot{q}^h \partial \dot{q}^k}\, \omega^h \wedge \tilde{\nu}^k \right) \tag{3.17}$$

valid for any function $F \in \mathcal{F}(j_1(\mathcal{V}_{n+1}))$.

Eq. (3.17) implies the following statement, rephrasing, with a few minor differences, an analogous result proved in [1].

THEOREM 3.3. – *A necessary and sufficient condition in order for eq.* (3.14) *to admit a solution in the class of non-singular functions over* $j_1(\mathcal{V}_{n+1})$ *is* *the existence of a 2-form* $\Omega \in \mathcal{G}^2(j_1(\mathcal{V}_{n+1}))$, *of maximal rank, satisfying the relations*

$$v(\Omega) = 0; \qquad \mathcal{L}_{\mathbf{Z}}(\Omega) = 0 \tag{3.18a}$$

$$v(d\Omega) = 0 \tag{3.18b}$$

Every 2-form Ω *with the stated properties is automatically a Poincaré-Cartan 2-form for the flow* $\mathbf{Z}$.

Proof. – The necessity of the condition is clear. Indeed, if F denotes an arbitrary, non singular solution of eq. (3.14), taking eqs. (1.18), (1.20), (1.21), (3.13) into account, it is easily seen that the 2-form

$$\Omega := -\frac{\partial^2 F}{\partial \dot{q}^h \partial \dot{q}^k}\,\omega^h \wedge \tilde{\nu}^k \tag{3.19}$$

satisfies the relation

$$v(\Omega) = \frac{\partial^2 F}{\partial \dot{q}^h \partial \dot{q}^k}\,\omega^h \wedge \omega^k = 0; \qquad \mathcal{L}_{\mathbf{Z}}(\Omega) = 0$$

$$v(d\Omega) = -d_v\Omega + d\,v(\Omega) - 2dt \wedge \Omega = -\frac{\partial^3 F}{\partial \dot{q}^h \partial \dot{q}^k \partial \dot{q}^r}\,\omega^r \wedge \omega^h \wedge \tilde{\nu}^k = 0$$

Conversely, suppose that a 2-form Ω of maximal rank exists, satisfying the whole set of requirements (3.18a, b). By eqs. (3.18a), recalling the definition of the verticalizer v in terms of the fundamental tensor, as well as the identification $\dot{\mathbf{J}} = \mathcal{L}_{\mathbf{Z}}\,\mathbf{J}$, we get the relations

$$\phi_{\mathbf{J}}(\Omega) = v(\Omega) = 0; \qquad \phi_{\dot{\mathbf{J}}}(\Omega) = \mathcal{L}_{\mathbf{Z}}\phi_{\mathbf{J}}(\Omega) - \phi_{\mathbf{J}}\mathcal{L}_{\mathbf{Z}}(\Omega) = 0 \tag{3.20}$$

Making use of the representation (2.36) for the tensor $\dot{\mathbf{J}}$ in the natural basis associated with the dynamical flow $\mathbf{Z}$, it is easily seen that the most general 2-form of maximal rank satisfying the conditions (3.20) is necessarily of the form

$$\Omega = a_{ij}\,\omega^i \wedge \tilde{\nu}^j$$

with $a_{ij} = a_{ji}$ and $\det a_{ij} \neq 0$. From this we get also

$$v(d\Omega) = -d_v\Omega + dv(\Omega) - 2dt \wedge \Omega = \frac{\partial a_{ij}}{\partial \dot{q}^k}\,\omega^k \wedge \omega^i \wedge \tilde{\nu}^j$$

Therefore the condition $v(d\Omega) = 0$ is mathematically equivalent to $\partial a_{ij}/\partial \dot{q}^k = \partial a_{kj}/\partial \dot{q}^i$, *i.e.* to the existence of a representation of the form $a_{ij} = \partial f_i/\partial \dot{q}^j$. Together with the symmetry property $a_{ij} = a_{ji}$, this implies also $f_i = \partial F/\partial \dot{q}^i$, whence $a_{ij} = \partial^2 F/\partial \dot{q}^i \partial \dot{q}^j$.

Collecting all previous results, and recalling eq. (3.17), we conclude that any 2-form of maximal rank satisfying eqs. (3.18a, b) is necessarily of the form (3.19), with F non-singular, and subject to the single condition

$$0 = \mathcal{L}_{\mathbf{Z}} \left(\frac{\partial^2 F}{\partial \dot{q}^i \partial \dot{q}^j} \, \omega^i \wedge \tilde{\nu}^j \right) = \delta_{\mathbf{Z}} \, \delta_{\mathbf{Z}} F$$

This establishes the sufficiency part of the Theorem. Finally, let us check that, under the stated assumptions, Ω is a Poincaré-Cartan 2-form for the dynamical flow **Z**. To this end, recalling the discussion at the beginning of this Subsection, we observe that, whenever Ω admits the representation (3.19) in terms of a non-singular solution of the equation $\delta_{\mathbf{Z}} \, \delta_{\mathbf{Z}} F = 0$, it admits also the representation

$$\Omega = \frac{\partial^2 L}{\partial \dot{q}^h \partial \dot{q}^k} \, \omega^h \wedge \tilde{\nu}^k \tag{3.22}$$

with L satisfying the stronger condition $\delta_{\mathbf{Z}} L = 0$, *i.e.*, playing the role of a Lagrangian for **Z** in the sense of eq. (3.11). The required conclusion then follows from the identity (3.16), which, together with the stated characterization of L, implies the relation

$$\Omega = d \, (d_v \, L + L \, dt)$$

identical to the definition (3.13) of the Poincaré Cartan 2-form of **Z**. □

In view of the stated result, the inverse problem is now reduced to a discussion of the solvability of eqs. (3.18a, b) within the class of exterior 2-forms of maximal rank.

The analysis is simplified by the fact, already pointed out in the proof of Theorem 3.3, that the most general 2-form consistent with the requirements (3.18a) is necessarily of the type

$$\Omega = a_{ij} \omega^i \wedge \tilde{\nu}^j \tag{3.23}$$

More precisely, taking eqs. (1.20), (2.52) into account, it may be seen that, within the class of differential forms admitting the local representation (3.23), eqs. (3.18a) are reflected into the pair of conditions

$$\nabla_{\mathbf{Z}} \, \Omega = 0; \qquad \phi_{\mathbf{W}} \, (\Omega) = 0 \tag{3.24}$$

W denoting the tensor field (2.53). Indeed, by direct computation, starting with eq. (3.23), we have

$$\nabla_{\mathbf{Z}} \, \Omega = \left[\mathbf{Z} \, (a_{ij}) - a_{rj} \tau_i{}^r - a_{ir} \tau_j{}^r \right] \omega^i \wedge \tilde{\nu}^j \tag{3.25a}$$

$$\phi_{\mathbf{W}} \, (\Omega) = -a_{ij} \, \tilde{\nu}^i \wedge \tilde{\nu}^j + a_{ir} Q^r{}_j \, \omega^i \wedge \omega^j \tag{3.25b}$$

This fact, together with the identification (2.52), shows that the condition $\mathcal{L}_{\mathbf{Z}}\Omega = 0$, alone, is equivalent to the pair of relations (3.24), while the request $v(\Omega) = 0$ – which, in the present case, reduces to the symmetry condition $a_{ij} = a_{ji}$ – follows identically from these.

In a similar way, recalling the definitions (2.46a, b) of the covariant differentials, as well as eqs. (2.47a), (3.21), it is easily seen that, within the class of differential forms of the type (3.23), eq. (3.18b) may be written more simply as

$$D_v\, \Omega = 0 \tag{3.26}$$

A better insight into the nature of eqs. (3.24) is gained by introducing the iterative notation

$$\mathbf{W}^{(0)} := \mathbf{W} \qquad \mathbf{W}^{(k)} = \nabla_{\mathbf{Z}}\, \mathbf{W}^{(k-1)} \qquad k = 1, 2, \ldots$$

and observing that in view of the algorithm pointed out in Section 1.2, this gives rise to the following relations (to be understood in the operatorial sense):

$$\nabla_{\mathbf{Z}} \cdot \phi_{\mathbf{W}^{(k)}} - \phi_{\mathbf{W}^{(k)}} \cdot \nabla_{\mathbf{Z}} = \phi_{\mathbf{W}^{(k+1)}} \tag{3.27}$$

By comparison with eqs. (3.24), we have then the inductive scheme

$$\phi_{\mathbf{W}^{(0)}}(\Omega) = 0; \qquad \phi_{\mathbf{W}^{(k+1)}}(\Omega) = \nabla_{\mathbf{Z}} \cdot \phi_{\mathbf{W}^{(k)}}(\Omega), \quad k = 0, 1, \ldots$$

mathematically equivalent to

$$\phi_{\mathbf{W}^{(k)}}(\Omega) = 0, \qquad k = 0, 1, \ldots \tag{3.28}$$

Therefore, any 2-form Ω satisfying the system (3.24) will automatically satisfy the entire algebraic system (3.28).

To formalize this fact, we denote by $\mathcal{T} \xrightarrow{\pi} j_1(\mathcal{V}_{n+1})$ the vector bundle formed by the totality of 2-forms Ω_z, $z \in j_1(\mathcal{V}_{n+1})$ admitting the representation $\Omega_z = a_{ij}\,(\omega^i)_z \wedge (\tilde{\nu}^j)_z$. Also, at each $z \in j_1(\mathcal{V}_{n+1})$ we denote by $\mathcal{K}_z \subset \mathcal{T}_z$ the simultaneous kernel of the family of operators $\{\phi_{\mathbf{W}^{(k)}},\ k = 0, 1, \ldots\}$ in $\mathcal{T}_z$.

A dynamical flow will be said to be *non-singular* on an open domain $U \subset j_1(\mathcal{V}_{n+1})$ if and only if the set $\mathcal{K}(U) := \bigcup_{z \in U} \mathcal{K}_z$ may be given the structure of a vector bundle over U, *i.e.*, roughly speaking, if and only if the dimension of the subspaces $\mathcal{K}_z \subset \mathcal{T}_z$ is constant on U.

In general, requiring $\mathbf{Z}$ to be non singular over the whole of $j_1(\mathcal{V}_{n+1})$ may turn out to be too restrictive. In most cases of real physical interest, however, $\mathbf{Z}$ happens to be non singular "almost everywhere", *i.e.* up to a subset *nowhere dense* in $j_1(\mathcal{V}_{n+1})$.

In any case, if $\mathbf{Z}$ is non-singular on U, the totality of sections $U \to \mathcal{K}(U)$ is a module – henceforth denoted by $\chi(U)$ – over the ring $\mathcal{F}(U)$.

An important feature of $\chi(U)$ is its *closure* under the operator $\nabla_{\mathbf{Z}}$. Indeed, by the very definition of the dynamical connection, the module of section $\Omega : U \to \mathcal{T}$ is closed under $\nabla_{\mathbf{Z}}$. In particular, if Ω belongs to the submodule $\chi(U)$, we have also $\phi_{\mathbf{W}^{(k)}}(\Omega) = 0 \, \forall \, k = 0, 1, \ldots$, *i.e.*, by eq. (3.27)

$$\begin{aligned} 0 &= \phi_{\mathbf{W}^{(k+1)}}(\Omega) = (\nabla_{\mathbf{Z}} \cdot \phi_{\mathbf{W}^{(k)}} - \phi_{\mathbf{W}^{(k)}} \cdot \nabla_{\mathbf{Z}})\,\Omega \\ &= -\phi_{\mathbf{W}^{(k)}} \cdot \nabla_{\mathbf{Z}}(\Omega) \qquad k = 0, 1, \ldots \end{aligned}$$

so that, by definition, $\nabla_{\mathbf{Z}}\,\Omega$ is still an element of $\chi(U)$.

Following Sarlet [12] and Henneaux ([14], [27]), we now take advantage of the fact that, once the 2-form Ω is specified on an initial hypersurface $i_0 : \Sigma_0 \to j_1(\mathcal{V}_{n+1})$, identified with (an open subset of) the slice $t = t_0$, the equation $\nabla_{\mathbf{Z}}\Omega = 0$ determines Ω uniquely in a neighbourhood of Σ_0 (namely, in the "world-tube" obtained as image space of $(-\varepsilon, \varepsilon) \times \Sigma_0$ under the local 1-parameter group of diffeomorphisms generated by $\mathbf{Z}$). Therefore, locally, the study of the solvability of the system (3.18a, b) in $j_1(\mathcal{V}_{n+1})$ may be "pulled-back" to the hypersurface Σ_0, and reduced to the identification of a suitable set of necessary and sufficient conditions to be imposed on the 2-form $\Omega_{|\Sigma_0} \in \mathcal{G}^2(\Sigma_0)$, in order that the subsequent solution of the Cauchy problem

$$\nabla_{\mathbf{Z}}\,\Omega = 0, \quad i_0^*(\Omega) = \Omega_{|\Sigma_0} \tag{3.29}$$

satisfy the whole set of equations (3.18a, b).

As a preliminary step in this analysis, we observe two basic facts:

(i) all the operators $\phi_{\mathbf{W}^{(k)}}$, $k = 0, 1, \ldots$, as well as the covariant differentials D_v, D_h defined in Section 2 have the nature of "internal operators" in Σ_0, in the sense that their action on an arbitrary 2-form Ω at a point $z \in \Sigma_0$ depends uniquely on the values of Ω on Σ_0;

(ii) in addition to eqs. (3.25a, b), (3.26), the system (3.18a, b) implies also

$$D_h(\Omega) = D_v \nabla_{\mathbf{Z}}\,\Omega - \nabla_{\mathbf{Z}} D_v\,\Omega = 0 \tag{3.30}$$

In view of assertions (i), (ii) above, we obtain the following set of necessary conditions, satisfied by the initial value $\Omega_{|\Sigma_0}$ of any solution Ω of the system (3.18a, b):

$$\Omega_{|\Sigma_0}(z') \in \mathcal{K}_{z'} \qquad \forall \, z' \in \Sigma_0 \tag{3.31a}$$

[equivalent to $\phi_{\mathbf{W}^{(k)}}(\Omega_{|\Sigma_0}) = 0 \,\forall\, k = 0, 1, \ldots$, with $\Omega_{|\Sigma_0}$ of the form (3.23)], and

$$D_v\, \Omega_{|\Sigma_0} = 0; \qquad D_h\, \Omega_{|\Sigma_0} = 0 \tag{3.31b}$$

Conversely we have the following.

THEOREM 3.4. – *Let the dynamical flow* $\mathbf{Z}$ *be non singular on the open world-tube* U *generated by an initial slice* Σ_0. *Then, for each choice of the 2-form* $\Omega_{|\Sigma_0}$ *consistent with the requirements* (3.31a, b), *the solution* Ω *of the Cauchy problem* (3.29) *satisfies the whole system* (3.18a, b) *in a neighbourhood of* Σ_0.

Proof. – Let r denote the dimension of the fibres of the vector bundle $\mathcal{K}(U)$ described above. If $r = 0$, there is nothing to prove, since the only 2-form satisfying the whole set of conditions (3.31a) is then $\Omega_{|\Sigma_0} = 0$.

If $r > 0$, for each $\zeta \in \Sigma_0$ we can find an open neighbourhood $V \ni \zeta$, and r sections $\sigma_{(\alpha)} : V \to \mathcal{K}(V)$ such that

(i) V is a world tube, generated by dragging the intersection $V \cap \Sigma_0$ along the integral curves of $\mathbf{Z}$;

(ii) the 2-forms $\sigma_{(\alpha)|z}$, $\alpha = 1, \ldots, r$ form a basis for $\mathcal{K}_z$, $\forall\, z \in V$.

In view of (ii), every section $\Omega \in \chi(V)$ may be expressed locally as

$$\Omega = \sum_{\alpha=1}^{r} f^\alpha\, \sigma_{(\alpha)}, \qquad f^\alpha \in \mathcal{F}(V) \tag{3.32}$$

Noting further that, as pointed out in the previous discussion, the fields $\nabla_{\mathbf{Z}}\, \sigma_{(\alpha)}$ are still in the class $\chi(V)$, we have a representation of the form

$$\nabla_{\mathbf{Z}}\, \sigma_{(\alpha)} = M_\alpha{}^\beta\, \sigma_{(\beta)}, \qquad M_\alpha^\beta \in \mathcal{F}(V)$$

whence, going back to eq. (3.32)

$$\nabla_{\mathbf{Z}}\, \Omega = \sum_{\alpha,\, \beta} (\mathbf{Z}(f^\alpha)\, \delta_\alpha^\beta + f^\alpha\, M_\alpha{}^\beta)\, \sigma_{(\beta)}$$

From this it follows at once that, for each choice of the initial values $f^\alpha_{|\Sigma_0}$ – *i.e.*, for each choice of $\Omega_{|\Sigma_0}$ consistent with the requirement (3.31a) – there exists a unique solution $\Omega \in \chi(V)$ of the Cauchy problem (3.29), defined in a neighbourhood of $V \cap \Sigma_0$. By the very definition of the class $\chi(V)$, the 2-form Ω satisfies

$$\phi_W(\Omega) = 0 \tag{3.33}$$

Therefore, all is left to prove is that, if $\Omega_{|\Sigma_0}$ is subject to the further requirements (3.31b), the 2-form Ω satisfies $D_v\, \Omega = 0$. To this end, recalling the definition of the covariant differentials, we observe that the validity of $\nabla_{\mathbf{Z}}\, \Omega = 0$ implies

$$D_h\,(\Omega) = D_v \nabla_{\mathbf{Z}}\, \Omega - \nabla_{\mathbf{Z}} D_v\, \Omega = -\nabla_{\mathbf{Z}} D_v\, \Omega \tag{3.34a}$$

$$(D_h \nabla_{\mathbf{Z}} - \nabla_{\mathbf{Z}} D_h)\, \Omega = -\nabla_{\mathbf{Z}} D_h\,(\Omega) = \nabla_{\mathbf{Z}} \nabla_{\mathbf{Z}} D_v\,(\Omega) \tag{3.34b}$$

Moreover, in view of eqs. (2.51a, b), (2.53), (3.25a, b), (3.29), (3.33), a straightforward but tedious calculation yields the representation

$$(D_h \nabla_{\mathbf{Z}} - \nabla_{\mathbf{Z}} D_h)\, \Omega = \langle W_{ijk} | D_v\, \Omega \rangle\, \omega^h \wedge \omega^i \wedge \tilde{\nu}^j \tag{3.35}$$

with

$$W_{hij} := \frac{1}{9} \left(2\, Q^r{}_h\, \tilde{\partial}_j \wedge \tilde{\partial}_r \wedge \frac{\partial}{\partial \dot{q}^i} + Q^r{}_i\, \tilde{\partial}_r \wedge \tilde{\partial}_h \wedge \frac{\partial}{\partial \dot{q}^j} + Q^r{}_j\, \tilde{\partial}_r \wedge \tilde{\partial}_i \wedge \frac{\partial}{\partial \dot{q}^h} \right)$$

By comparison of the latter with eqs. (3.34a, b), we conclude that, as a consequence of eqs. (3.29), (3.31a, b), the 3-form $D_v\,(\Omega)$ satisfies the linear homogeneous differential equation

$$\nabla_{\mathbf{Z}} \nabla_{\mathbf{Z}} D_v\,(\Omega) = \langle W_{hij} | D_v\,(\Omega) \rangle\, \omega^h \wedge \omega^i \wedge \tilde{\nu}^j$$

with initial data

$$(D_v\, \Omega)_{|\Sigma_0} = D_v\, \Omega_{|\Sigma_0} = 0 \qquad (\nabla_{\mathbf{Z}} D_v\, \Omega)_{|\Sigma_0} = -D_h\, \Omega_{|\Sigma_0} = 0$$

By Cauchy theorem for ordinary differential equations this implies $D_v\,(\Omega) \equiv 0$ in a neighbourhood of Σ_0. □

The previous arguments yield back – again with some minor differences – a classical result on the inverse problem, already pointed out in ([12], [14], [27]), namely:

PROPOSITION 3.1. – *Let* $\mathbf{Z}$ *be a given dynamical flow. Then, if* $\mathbf{Z}$ *is non singular on an open world-tube* U, *a necessary and sufficient condition in order for* $\mathbf{Z}$ *to be locally derivable for a Lagrangian* L *in a neighbourhood of the slice* Σ_0 *is the existence of at least one non singular matrix* $a_{ij} = a_{ij}\,(t_0,\, q,\, \dot{q})$ *satisfying the requirements*

$$(a_{ij} - a_{ji})_{|\Sigma_0} = 0; \quad (\tilde{\partial}_k\, a_{ij} - \tilde{\partial}_i\, a_{kj})_{|\Sigma_0} = 0; \quad \left(\frac{\partial a_{ij}}{\partial \dot{q}^k} - \frac{\partial a_{kj}}{\partial \dot{q}^i} \right)_{|\Sigma_0} = 0$$

$$(a_{ir}\, Q^{(k)r}{}_j - a_{jr}\, Q^{(k)r}{}_i)_{|\Sigma_0} = 0 \qquad k = 0, 1, \ldots$$

with the quantities $Q^{(k)r}{}_j$ *defined inductively in terms of the tensor* (2.43a) *by*

$$Q^{(0)r}{}_j = Q^r{}_j, \quad Q^{(k+1)r}{}_j = \nabla_{\mathbf{Z}} Q^{(k)r}{}_j = \mathbf{Z}(Q^{(k)r}{}_j) + \tau_s{}^r Q^{(k)s}{}_j - \tau_j{}^s Q^{(k)r}{}_s$$
$$k = 0, 1, \ldots$$

The proof is easily obtained by expressing in components the results stated in Theorem 3.4, making use of the representation (2.53) for the tensor W.

REFERENCES

[1] M. Crampin, G. E. Prince and G. Thompson, A Geometrical Version of the Helmholtz Conditions in Time-Dependent Lagrangian Dynamics, *J. Phys. A: Math. Gen.*, Vol. **17**, 1984, pp. 1437-1447.

[2] E. Masssa and E. Pagani, Classical Dynamics of Non-holonomic Systems: a Geometric Approach, *Ann. Inst. H. Poincaré*, Vol. **55**, 1991, pp. 511-544.

[3] M. de León and P. R. Rodrigues, *Methods of Differential Geometry in Analytical Mechanics*, North Holland, Amsterdam, 1989.

[4] D. J. Saunders, *The Geometry of Jet Bundles*, London Mathematical Society, Lecture Note Series 142, Cambridge University Press, 1989.

[5] W. Sarlet, Symmetries, First Integrals and the Inverse Problem of Lagrangian Mechanics, *J. Phys. A: Math. Gen.*, Vol. **14**, 1981, pp. 2227-2238.

[6] W. Sarlet and F. Cantrijn, Symmetries, First Integrals and the Inverse Problem of Lagrangian Mechanics: II, *J. Phys. A: Math. Gen.*, Vol. **16**, 1983, pp. 1383-1396.

[7] M. Crampin, F. Cantrijn and W. Sarlet, Pseudo-Symmetries, Noether's Theorem and the Adjoint Equation, *J. Phys. A: Math. Gen.*, Vol. **20**, 1987, pp. 1365-1376.

[8] M. Crampin, G. E. Prince and W. Sarlet, Adjoint Symmetries for Time-Dependent Second Order Equations, *J. Phys. A: Math. Gen.*, Vol. **23**, 1990, pp. 1335-1347.

[9] G. Marmo, E. J. Saletan, A. Simoni and B. Vitale, *Dynamical Systems*, John Wiley & Sons, Chichester, 1985.

[10] M. Crampin, On the Differential Geometry of the Euler-Lagrange Equations, and the Inverse Problem of Lagrangian Dynamics, *J. Phys. A: Math. Gen.*, Vol. **14**, 1981, pp. 2567-2575.

[11] M. Crampin, Tangent Bundle Geometry for Lagrangian Dynamics, *J. Phys. A: Math. Gen.*, Vol. **16**, 1983, pp. 3755-3772.

[12] W. Sarlet, The Helmholtz Condition Revisited. A New Approach to the Inverse Problem of Lagrangian Dynamics, *J. Phys. A: Math. Gen.*, Vol. **15**, 1982, pp. 1503-1517.

[13] G. Morandi, C. Ferrario, G. Lo Vecchio, G. Marmo and C. Rubano, The Inverse Problem in the Calculus of Variations and the Geometry of the Tangent Bundle, *Phys. Reports*, Vol. **188**, 1990, pp. 147-284.

[14] M. Henneaux and L. C. Shepley, Lagrangians for Spherically Symmetric Potentials, *J. Math. Phys.*, Vol. **23**, 1982, pp. 2101-2107.

[15] G. Caviglia, Dynamical Symmetries, First Integrals and the Inverse Problem of Lagrangian Dynamics, *Inverse Problems*, Vol. **1**, 1985, pp. 13-17.

[16] G. Caviglia, Helmholtz Conditions, Covariance, and Invariance Identities, *Int. J. Theor. Phys.*, Vol. **24**, 1985, pp. 377-390.

[17] F. Cantrijn, M. Crampin and W. Sarlet, Evading the Inverse Problem for Second Order Ordinary Differential Equations by Using Additional Variables, *Inverse Problems*, Vol. **3**, 1987, pp. 51-63.

[18] J. F. CARIÑENA and E. MARTINEZ, Symmetry Theory and Lagrangian Inverse Problem for Time-Dependent Second Order Differential Equations, *J. Phys. A: Math. Gen.*, Vol. **22**, 1989, pp. 2659-2665.

[19] W. SARLET, Note on Equivalent Lagrangians and Symmetries, *J. Phys. A: Math. Gen.*, Vol. **16**, 1983, pp. 229-233.

[20] G. MARMO, N. MUKUNDA and J. SAMUEL, Dynamics and Symmetries for Constrained Systems: A Geometric Analysis, *La Rivista del Nuovo Cimento*, Vol. **6**, 1983, 2, pp. 1-62.

[21] F. CANTRIJN, J. F. CARIÑENA, M. CRAMPIN and L. A. IBORT, Reduction of Degenerate Lagrangian Systems, *J. G. P.*, Vol. **3**, 1986, pp. 353-400.

[22] J. F. CARIÑENA and L. A. IBORT, Geometric Theory of the Equivalence of Lagrangians for Constrained Systems, *J. Phys. A: Math. Gen.*, Vol. **18**, 1985, pp. 3335-3341.

[23] G. GIACHETTA, Jet Methods in Non-holonomic Mechanics, *J. Math. Phys.*, Vol. **33**, 1992, pp. 1652-1665.

[24] H. HELMHOLTZ, Über die Physikalische Bedeutung des Princips der Kleisten Wirkung, *J. Reine Angew. Math.*, Vol. **100**, 1887, p. 137.

[25] G. DARBOUX, *Leçons sur la théorie générale des surfaces*, Gauthier-Villars, Paris, 1894.

[26] J. DOUGLAS, Solution of the Inverse Problem of the Calculus of Variations, *Trans. Amer. Math. Soc.*, Vol. **30**, 1941, pp. 71-128.

[27] M. HENNEAUX, Equations of Motion, Commutation Relations and Ambiguities in the Lagrangian Formalism, *Ann. Phys.*, Vol. **140**, 1982, pp. 45-64.

[28] J. F. POMMARET, *Systems of Partial Differential Equations and Lie Pseudogroups*, Gordon and Breach, New York, 1978.

[29] C. GODBILLON, *Géométrie différentielle et mécanique analytique*, Hermann, Paris, 1969.

[30] S. STERNBERG, *Lectures on Differential Geometry*, Prentice Hall, Englewood Cliffs, New Jersey, 1964.

[31] L. MANGIAROTTI and M. MODUGNO, Fibered Spaces, Jet Spaces and Connections for Field Theories, *Proceedings of the International Meeting on Geometry and Physics*, Florence, October 12-15, 1982, Pitagora, Bologna, 1983, pp. 135-165.

[32] P. LIBERMANN and C. M. MARLE, *Symplectic Geometry and Analytical Mechanics*, D. Reidel Publ. Comp., Dordrecht, 1987.

[33] P. L. GARCIÁ, The Poincaré-Cartan Invariant in the Calculus of Variations, *Symposia Mathematica*, Vol. **14**, 1974, pp. 219-246.

[34] S. KOBAYASHI and K. NOMIZU, *Foundations of Differentiel Geometry*, J. Wiley & Sons, New York, 1963.

[35] J. GRIFONE, Estructure presque tangente et connexions I, *Ann. Inst. Fourier*, Grenoble, Vol. **22**, 3, 1972, pp. 287-334.

[36] J. GRIFONE, Estructure presque tangente et connexions II, *Ann. Inst. Fourier* Grenoble, Vol. **22**, 4, 1972, pp. 291-338.

[37] W. SARLET, F. CANTRIJN and M. CRAMPIN, A New Look at Second Order Equations and Lagrangian Mechanics, *J. Phys. A: Math. Gen.*, Vol. **17**, 1984, pp. 1999-2009.

[38] M. CRAMPIN and G. E. PRINCE, Generalizing Gauge Variance for Spherically Symmetric Potentials, *J. Phys. A: Math. Gen.*, Vol. **18**, 1985, p. 2167-2175

(Manuscript received March 2, 1993;
revised version received January 6, 1994.)