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On Lichtenstein's analysis of rotating newtonian stars

by

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ABSTRACT. – Euler's equation of a rigidly rotating body is investigated. L. Lichtenstein's analysis of the existence problem of rotating stars is reconsidered with modern mathematical methods. Using the implicit function theorem, it is discussed under which circumstances there exists a solution of Euler's equation in the neighbourhood of a given one. The given solution must fulfill certain Hölder conditions and certain symmetry conditions. A criterion is derived which can be used to determine whether the implicit function theorem may be applied or not. The criterion is evaluated in the case of a static starting solution. This yields an existence theorem for slowly rotating stars.

RÉSUMÉ. – L'équation d'Euler d'un corps en rotation rigide est l'objet de l'étude. L'analyse de L. Lichtenstein sur le problème de l'existence des étoiles en rotation est reconsidérée avec des méthodes mathématiques modernes. En utilisant le théorème des fonctions implicites on discute dans quelles circonstances il y a des solutions dans la proximité d'une solution donnée. La solution donnée doit satisfaire à certaines conditions Hölder et à certaines conditions de symétrie. Un critère est donné, qu'on peut utiliser pour déterminer si le théorème des fonctions implicites peut être appliqué, ou non. Le critère est évalué pour une solution de départ statique. Ceci produit un théorème d'existence pour des étoiles en rotation lente.

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0. INTRODUCTION

In Newtonian gravitational theory rotating stars are usually described by matter densities ρ which are solutions of Euler's equation. This equation is a nonlinear integro-differential equation which cannot be solved in general, and few is known about existence theorems. Some authors ([1], [2], [3]) used variational methods in Sobolev spaces to prove the existence of a special class of rotating stars. But they had to restrict the equation of state and, therefore, the matter the star can consist of. In these papers, a certain behaviour of the equation of state $p(\rho)$ is required for $\rho \to 0$ and $\rho \to \infty$. Especially, the density must vanish if the pressure vanishes. Thus, a star whose density doesn't vanish on its surface is not treated by these authors. In the present paper, a method due to Liapounoff [4] and Poincaré [5] will be used to treat the problem of existence. These authors assumed that there is a given solution ρ of Euler's equation for a star which rotates rigidly with angular velocity ω_0. In order to solve Euler's equation for an angular velocity ω in the neighbourhood of ω_0, they defined a variation of the matter density ρ. A function ζ determines this variation. Then Euler's equation is written in terms of ω and ζ. The zeros (ω, ζ) of this equation determine a matter density ρ_ζ which is a solution of Euler's equation of a body, rotating rigidly with the angular velocity ω. However, Liapounoff and Poincaré only regarded the very special equation of state $\rho = const$. Furthermore, they didn't really solve Euler's equation but only regarded an expansion series up to second order. Later, Liapounoff [6] showed under which circumstances the series converges in the case $\rho = const$, i. e. under which circumstances there is a deformation ζ_ω and a corresponding density ρ_{ζ_ω} for angular velocities ω which are sufficiently close to ω_0. Lichtenstein improved the work of Liapounoff [7] and extended this method to much more general equations of state [8]. Lichtenstein's method has the advantage that in the inhomogeneous case $\rho \neq const$ the equation of state $p(\rho)$ is not explicitly needed to determine whether these solutions do exist or not.

This paper focuses on a reconsideration of Lichtenstein's analysis of the inhomogeneous case in terms of modern analysis. Following Lichtenstein's idea, the variation of the density ρ is defined in section (2), i.e. it is defined how the function ζ determines a variation of the density ρ. Then Euler's equation is rewritten in terms of ω and ζ. This equation can be regarded as an operator T which maps a scalar ω and a function ζ of the Banach space B_1 to the Banach space B_2. The Banach space B_1 and B_2 are introduced in section (3). The zeros of the operator T are the solutions of Euler's equation.

The partial derivative $D_\zeta T(\omega, \zeta)$ of the operator T with respect to ζ is determined in section (4). Because ρ is assumed to be a solution of Euler's equation of a star which rotates with angular velocity ω_0, it holds that $T(\omega_0, 0)=0$. If the partial derivative $D_\zeta T(\omega_0, 0): B_1 \rightarrow B_2$ is a bijective linear operator, the implicit function theorem [9] guarantees the existence of a $\delta>0$ such that for all $\omega \in (\omega_0-\delta, \omega_0+\delta)$ there exists a ζ_ω with $T(\omega, \zeta_\omega)=0$. Lichtenstein proved the convergence of an iteration in order to show the existence of these zeros. It is interesting that the iteration which is usually used to prove the implicit function theorem is the same as Lichtenstein's iteration; in other words, Lichtenstein has proven the implicit function theorem in a special case.

In section (5) a criterion is derived to determine whether the partial derivative $D_\zeta T(\omega_0, 0)$ is bijective and the implicit function theorem may be applied or not, or, in other words, under which circumstances solutions of Euler's equation in the neighbourhood of the given one exist. The result is proposition (5.3). Unfortunately, it is hard to verify this criterion. Thus, no general existence theorem can be presented. But if the given solution ρ is a solution of the static Euler equation, ρ is spherical [10], and for a spherical solution the criterion of section (5) can be verified. This yields an existence theorem (6.1) of slowly rotating stars.

In appendix (A) some properties of the Newtonian potential are presented which are used in this paper. Finally, in appendix (B) some relations are shown, which are needed to prove that the operator T is continuously differentiable with respect to ζ according to the norm of the Banach space B_2.

Some possible extensions of the results of this paper are presented in the following. Euler's equation of a differentially rotating body is similar to Euler's equation of a rigidly rotating body. Thus, the existence theorem of section (6) can be extended to the case of a differentially rotating star. Therefore, differentially rotating stars exist if the angular velocity $\omega(R)$ is sufficiently small and sufficiently smooth, i.e. if $|\omega(R)|$ and $|\partial_R \omega(R)|$ are sufficiently small. Here, the function $\omega(R)$ is the angular velocity of points whose distance to the axis is R.

The variation of the solution ρ can also be used to change the equation of state. Let $\{p_t(\rho);\, t\in[0, 1]\}$ be a set of equations of state and ρ_0 a solution of Euler's equation with the equation of state p_0. If the function $p_t(\rho(x)): (t, x)\in[0, 1]\times\mathbb{R}^3 \rightarrow \mathbb{R}$ fulfills certain smoothness conditions, then the implicit function theorem can be used to find solutions of Euler's equation with an equation of state p_t for sufficiently small t,

1. FORMULATION OF THE PROBLEM

Euler's equation of a rigidly rotaing body is

$$\frac{\nabla p(\rho)}{\rho}-\frac{\omega^2}{2}\nabla(xe_R(x))^2-\nabla\Phi=0. \tag{1.1}$$

Here

$$\Phi(x) := \int \frac{\rho(x')}{\|x-x'\|} d^3 x'$$

denotes the gravitational potential (¹) of the matter density ρ. ω is the angular velocity of the rotation and $e_R(x)$ the unit vector which is orthogonal to the axis and directed outside. It is assumed that the fluid is barotropic, i. e. that the pressure $p=p(\rho)$ is a given function of the density. Then, equation (1.1) can easily be integrated. The result is

$$A(\rho(x)) - \Phi(x) - \frac{\omega^2}{2}(xe_R)^2 = \lambda \tag{1.2}$$

where $A(\rho) = \int_0^\rho \frac{\partial_s p(s)}{s} ds$ and λ is an appropriate constant.

It is assumed that a solution $\rho: T \subset \mathbb{R}^3 \to \mathbb{R}^+$ of Euler's equation (1.2) with the angular velocity ω_0 is given. The bounded body of the star is called T and the surface of the star ∂T. This solution is assumed to have the following properties: $\rho(x)$ is Hölder continuously differentiable with the Hölder exponent (²) $0<\nu<1$. Furthermore $\rho(x)$ has three linearly independent symmetry planes which intersect at one point. This intersection point is the origin of the coordinate system, and the z-axis is the axis of the rigid rotation. These three symmetry planes imply $\nabla\rho(0)=0$. The surfaces $\rho=const$ are assumed to be a set of smoothly deformed sphere surfaces which intersect the positive z-axis at one point. These $\rho=const$ surfaces will now be used to introduce a radial coordinate. Any point $x \in T$ lies on a surface $\rho=const$. This surface intersects the positive z-axis at a point x_z. The length r of the vector x_z will be the radial coordinate of the point x and the corresponding surface $\rho=const$ will be denoted ∂T_r. This radial function $r(x)$ is assumed to be a norm of the points $x \in T$, i. e. there exist two constants d_1 and d_2, such that

$$d_1 \|x\| \leqq r(x) \leqq d_2 \|x\|.$$

Now we can regard ρ as a function depending only on r. Thus, $\rho(x)=\rho(r(x))$. The function $\rho(r)$ is assumed to be monotonically decreasing. The sum of gravitational and centrifugal force

$$\nabla\left[\Phi(x) + \frac{\omega_0^2}{2}(xe_R)^2\right]$$

(¹) $\|\cdot\|$ denotes the usual euclidian norm in $\mathbb{R}^3$.

(²) In other words, ρ is continuously differentiable and the derivatives $\partial_{x_i}\rho$ fulfill the following Hölder condition with an appropriate constant C:

$$|\partial_{x_i}\rho(x) - \partial_{x_i}\rho(x')| \leqq C\|x-x'\|^\nu.$$

fulfills the following inequality for all $i=1, 2, 3$, all $x \in \mathrm{T}$ and two appropriate constants c_1 and c_2.

$$c_1 \|x\| \leqq \left\| \nabla \left[\Phi(x) + \frac{\omega_0^2}{2} (x e_R)^2 \right] \right\| \leqq c_2 \|x\|$$

In order to solve Euler's equation (1.2) for an angular velocity $\omega \neq \omega_0$, a variation of the density ρ will be defined. Let $n(x)$ be the normal vector which is orthogonal to the surface $\partial T_{r(x)}$ at the point x and points towards the outside of $\partial T_{r(x)}$. For any smooth and sufficiently small (3) function $\zeta: T \to \mathbb{R}$ with $\zeta(0)=0$, the following coordinate transformation is defined.

$$g_\zeta : T \to T_\zeta,\ x \to g_\zeta(x) := x + \zeta(x) n(x)$$

If g_ζ can be inverted, $\rho_\zeta(x) := \rho(g_\zeta^{-1}(x))$ is a new distribution of matter. This new matter density is a solution of Euler's equation (2.2), if for all $x \in T_\zeta$

$$0 = \Phi_\zeta(x) - \Phi_\zeta(0) + \frac{\omega^2}{2} (x e_R)^2 - A\, \rho_\zeta(x)) + A(\rho_\zeta(0))$$

$$0 = \int_{\mathrm{T}_\zeta} \rho_\zeta(x') \left(\frac{1}{\|x - x'\|} - \frac{1}{\|x'\|} \right) d^3 x' + \frac{\omega^2}{2} (x e_R)^2 - A(\rho_\zeta(x)) + A(\rho_\zeta(0)).$$

Since $\rho_\zeta(g_\zeta(x)) = \rho(x)$ and $\rho_\zeta(0) = \rho(0)$ the evaluation of this equation at the points $g_\zeta(x)$ yields for all $x \in T$

$$T(\omega, \zeta)(x) := \int_{T_\zeta} \rho_\zeta(x') \left(\frac{1}{\|g_\zeta(x) - x'\|} - \frac{1}{\|x'\|} \right) d^3 x' + \frac{\omega^2}{2} (g_\zeta(x) e_R)^2 - A(\rho(x)) + A(\rho(0)) = 0. \quad (1.3)$$

$T(\omega, \zeta)$ can be regarded as an operator which maps the scalar ω and the function $\zeta: T \to \mathbb{R}$ to the function $T(\omega, \zeta): T \to \mathbb{R}$. The zeros of this operator are the solutions of Euler's equation. One zero is known. Because ρ is a solution of Euler's equation with angular velocity ω_0, it holds $T(\omega_0, 0)=0$. Suppose that two Banach spaces B_1 and B_2 can be found such that $T: \mathbb{R} \times B_1 \to B_2$, $\omega \times \zeta \to T(\omega, \zeta)$ is a continuously differentiable operator. If, furthermore, the partial derivative $D_\zeta T(\omega_0, 0): B_1 \to B_2$ is bijective, then the implicit function theorem [9] can be used to guarantee the existence of a $\delta > 0$ such that for all $\omega \in (\omega_0 - \delta, \omega_0 + \delta)$ there exists exactly one $\zeta_\omega \in B_1$ with $T(\omega, \zeta_\omega) = 0$. The corresponding matter density ρ_{ζ_ω} is a solution of Euler's equation with angular velocity ω.

(3) "Sufficiently small" will be defined in the next section.

2. THE BANACH SPACES

Define the Banach space

$$B_1 := \left\{ f \;\middle|\; \begin{array}{l} f \text{ is continuous, } f = O(\|x\|) \text{ for } x \to 0, \\ f \text{ is continuously differentiable for all } x \in T \setminus \{0\} \\ \text{For all } v \in \mathbb{R}^3 \text{ with } \|v\| = 1 \text{ there exists } \lim\limits_{t \to 0+} \nabla f(tv) \\ f \text{ is symmetric relative to the three symmetry planes of } T \end{array} \right\}$$

with the norm

$$\|f\|_{B_1} := \sup \left\{ \frac{f(x)}{\|x\|};\ x \in T \setminus \{0\} \right\} + \sup \{ \|\nabla f(x)\|;\ x \in T \setminus \{0\} \}.$$

It should be remarked, that the gradient ∇f of a function $f \in B_1$ might be discontinuous at the point $x = 0$.

PROPOSITION 2.1. – *There exists a $Z < 1$ such that for all $\zeta \in B_1$ with $\|\zeta\|_{B_1} \leqq Z$ the coordinate transformation*

$$g_\zeta:\quad T \to T_\zeta, \qquad x \to x + \zeta(x)\, n(x)$$

is bijective and for all $x \in T \setminus \{0\}$ continuously differentiable. There holds the inequality

$$(1 - Z)\|x\| \leqq \|g_\zeta(x)\| \leqq (1 + Z)\|x\|.$$

The Jacobi matrix $D g_\zeta^{-1}(x)$ fulfills the following inequality for all $x \in T_\zeta \setminus \{0\}$ and all $v \in \mathbb{R}^3$.

$$\|D g_\zeta^{-1}(x)\, v\| \leqq 2\|v\|.$$

Furthermore, the matter density $\rho_\zeta(x) := \rho(g_\zeta^{-1}(x))$ is continuously differentiable for all $x \in T_\zeta$ and fulfills

$$\|\nabla \rho_\zeta\|_\infty \leqq 2\|\nabla \rho\|_\infty.$$

Proof. – The existence and the differentiability of g_ζ^{-1} and the inequality $\|D g_\zeta^{-1}(x)\, v\| \leqq 2\|v\|$ are easily derived using the inverse function theorem. Applying the chain rule yields the continuous differentiability of ρ_ζ for all $x \neq 0$. Due to $\nabla \rho(0) = 0$ and the Hölder continuity of $\nabla \rho(x)$, it holds that $\|\nabla \rho(x)\| = O(\|x\|^\nu)$ for $x \to 0$. Furthermore, because

$$\nabla^T \rho_\zeta(x) = \nabla^T (\rho(g_\zeta^{-1} x))\, D g_\zeta^{-1}(x),$$

$\|\nabla \rho_\zeta(x)\| = O(\|x\|^\nu)$ for $x \to 0$. □

With this proposition, the function $T(\omega, \zeta)(x)$ of equation (1.3) is well defined for all $\zeta \in B_1$ with $\|\zeta\|_{B_1} \leqq Z$. The results of potential theory listed in the appendix (A) show that $T(\omega, \zeta)(x)$ lies in the Banach space

$$B_2 := \left\{ f \;\middle|\; \begin{array}{l} f \text{ is continuously differentiable, } \nabla f = O(\|x\|) \text{ for } x \to 0, \\ f \text{ is symmetric relative to the three symmetry planes of } T \end{array} \right\}$$

with the norm

$$\| f \|_{B_2} := \sup\{ | f(x) |;\ x \in T \} + \sup \left\{ \frac{\| \nabla f(x) \|}{\| x \|};\ x \in T \setminus \{0\} \right\}.$$

3. DETERMINATION OF THE DERIVATIVE $D_\zeta T(\omega, \zeta)$

To determine the partial derivative

$$D_\zeta T(\omega, \zeta) h = \lim_{t \to 0} \frac{T(\omega, \zeta + th) - T(\omega, \zeta)}{t},$$

define the set of coordinate transformations

$$g_t(x) := x + (\zeta(x) + t h(x)) n(x)$$

and the functions

$$f(t, x) = \int_{T_{\zeta+th}} \rho_{\zeta+th} \left(\frac{1}{\| x - x' \|} - \frac{1}{\| x' \|} \right) d^3 x' + \frac{\omega^2}{2} [x e_R]^2.$$

If the partial derivative $D_\zeta T(\omega, \zeta) h$ exists at all, then

$$\begin{aligned} &[D_\zeta T(\omega, \zeta) h](x) \\ &= \lim_{t \to 0} \frac{[f(t, g_t(x)) - A(\rho(x)) + A(\rho(0))] - [f(0, g_0(x)) - A(\rho(x)) + A(\rho(0))]}{t} \\ &= \partial_t [f(t, g_t(x))]_{t=0} \\ &= \partial_t f(0, g_0(x)) + \nabla f(0, g_0(x)) h(x) n(x). \end{aligned}$$

Here the gradient ∇f consists of the partial derivatives with respect to the coordinates x_1, x_2 and x_3. It holds that

$$\nabla f(t, x) = \int_{T_{\zeta+th}} \rho_{\zeta+th} \frac{-x + x'}{\| x - x' \|^3} d^3 x' + \omega^2 [x e_R] e_R.$$

To determine $\partial_t f(0, x)$, define

$$f_1 := \int_{T_\zeta} \rho_{\zeta+th} \left(\frac{1}{\| x - x' \|} - \frac{1}{\| x' \|} \right) d^3 x'$$

and

$$f_2 := \int_{T_{\zeta+th} \setminus T_\zeta} \rho_{\zeta+th} \left(\frac{1}{\| x - x' \|} - \frac{1}{\| x' \|} \right) d^3 x'.$$

It holds that

$$\partial_t f_1 |_{t=0} = \int_{T_\zeta} \partial_t \rho_{\zeta+th} |_{t=0} (x') \left(\frac{1}{\| x - x' \|} - \frac{1}{\| x' \|} \right) d^3 x',$$

and since $\rho_{\zeta+th}(x)=\rho(g_t^{-1}(x))$,

$$\partial_t \rho_{\zeta+th}|_{t=0}(x)=\nabla\rho(g_0^{-1}(x)) \lim_{t\to 0} \frac{g_t^{-1}(x)-g_0^{-1}(x)}{t}.$$

With

$$\begin{aligned}\lim_{t\to 0} \frac{g_t^{-1}(g_0(x'))-x'}{t} &= \lim_{t\to 0} \frac{g_t^{-1}(g_0(x')+th(x')n(x')-th(x')n(x'))-x'}{t} \\ &= \lim_{t\to 0} \frac{g_t^{-1}(g_t(x')-th(x')n(x'))-x'}{t} \\ &= -Dg_0^{-1}(g_0(x'))h(x')n(x')\end{aligned}$$

it follows that

$$\lim_{t\to 0} \frac{g_t^{-1}(x)-g_0^{-1}(x)}{t} = -Dg_0^{-1}(g_0(g_0^{-1}(x)))h(g_0^{-1}(x))n(g_0^{-1}(x))$$

and

$$\begin{aligned}\partial_t \rho_{\zeta+th}|_{t=0} &= -\nabla\rho(g_0^{-1}(x))[Dg_0^{-1}(x)h(g_0^{-1}(x))n(g_0^{-1}(x))] \\ &= -\nabla\rho_\zeta(x)h(g_0^{-1}(x))n(g_0^{-1}(x)).\end{aligned}$$

Thus,

$$\partial_t f_1|_{t=0}=\int_{T_\zeta} -\nabla\rho_\zeta(x)h(g_0^{-1}(x))n(g_0^{-1}(x))\left(\frac{1}{\|x-x'\|}-\frac{1}{\|x'\|}\right)d^3x'.$$

To determine the integral f_2, the shell $T_{\zeta+th}\setminus T_\zeta$ will be parametrized by two coordinates υ and φ which parameterize the surface ∂T_ζ and a radial coordinate $r\in[0, th(g_\zeta^{-1}(x))]$. If $d^3x=\tau(r, \upsilon, \varphi)\, dr\, d\upsilon\, d\varphi$, then

$$f_2=\int_{\partial T_\zeta} d\upsilon'\, d\varphi' \int_0^{th} dr'\, \rho_{\zeta+th}(r', \upsilon', \varphi')\left(\frac{1}{\|x-x'\|}-\frac{1}{\|x'\|}\right)\tau(r', \upsilon', \varphi').$$

Because the integral $\int_{\partial T_\zeta} d\upsilon'\, d\varphi'$ does not depend on the parameter t, it holds that

$$\begin{aligned}\partial_t f_2|_{t=0}(x)=\int_{\partial T_\zeta} d\upsilon'\, d\varphi'\, \partial_t \Bigg[\int_0^{th} dr'\, \rho_{\zeta+th}(r', \upsilon', \varphi') \\ \times\left(\frac{1}{\|x-x'\|}-\frac{1}{\|x'\|}\right)\tau(r', \upsilon', \varphi')\Bigg]_{t=0}.\end{aligned}$$

The equation

$$\partial_t\left[\int_0^{th} dr'\ldots\right]_{t=0}=\rho_\zeta(0, \upsilon', \varphi')\left(\frac{1}{\|x-x'\|}-\frac{1}{\|x'\|}\right)h(\upsilon', \varphi')\tau(r', \upsilon', \varphi')$$

leads to

$$\partial_t f_2|_{t=0}(x)=\int_{\partial T_\zeta} \rho_\zeta(x')\left(\frac{1}{\|x-x'\|}-\frac{1}{\|x'\|}\right)h(g_\zeta^{-1}(x'))\,d\sigma_\zeta(x')$$

where $d\sigma_\zeta(x')$ denotes the infinitesimal surface element of ∂T_ζ.

Finally, it can be seen that

$$\begin{aligned}[D_\zeta T(\omega,\zeta)h](x) &= h(x)\Psi_\zeta(x)+\int_{T_\zeta} -h(g_\zeta^{-1}(x'))(n(g_\zeta^{-1}(x'))\nabla\rho_\zeta x')) \\ &\qquad\times\left(\frac{1}{\|g_\zeta(x)-x'\|}-\frac{1}{\|x'\|}\right)d^3x' \\ &\quad+\int_{\partial T_\zeta} h(g_\zeta^{-1}(x'))\,\rho_\zeta(x')\left(\frac{1}{\|g_\zeta(x)-x'\|}-\frac{1}{\|x'\|}\right)d\sigma_\zeta(x')\end{aligned}$$

where

$$\Psi_\zeta(x):=n(x)\int_{T_\zeta}-\rho_\zeta(x')\frac{g_\zeta(x)-x'}{\|g_\zeta(x)-x'\|^3}\,d^3x'+\omega^2[(g_\zeta(x))\,e_R](ne_R).$$

The results of potential theory which are listed in the appendix (A) show that for all $\zeta\in B_1$ with $\|\zeta\|_{B_1}\leqq Z$ (*See* proposition 2.1)

$$D_\zeta T(\omega,\zeta):B_1\to B_2,\ h(x)\to[D_\zeta T(\omega,\zeta)h](x)$$

is a bounded linear operator. To guarantee that $T(\omega,\zeta)$ is continuously differentiable with respect to ζ and that $D_\zeta T(\omega,\zeta)$ is the partial derivative, the following points must be proven:

1. For all $h\in B_1$ it holds that

$$0=\lim_{t\to 0}\left\|\frac{T(\omega,\zeta+th)-T(\omega,\zeta)}{t}-D_\zeta T(\omega,\zeta)h\right\|_{B_2}.$$

2. For all $h\in B_1$ with $\|h\|_{B_1}\leqq 1$ and all $\varepsilon>0$, there exists a $\delta>0$ so that for all $\|\zeta_1-\zeta_2\|_{B_1}<\delta$ and $|\omega_1-\omega_2|<\delta$

$$\|D_\zeta T(\omega_1,\zeta_1)h-D_\zeta T(\omega_2,\zeta_2)h\|_{B_2}\leqq\varepsilon.$$

The proofs of these points are in appendix (B).

4. EXAMINATION OF THE DERIVATIVE $D_\zeta T(\omega_0, 0)$

To apply the implicit function theorem, the partial derivative $D_\zeta T(\omega, 0)$ must be a bijective operator which maps the Banach space B_1 onto the Banach space B_2. It is often difficult to show whether an operator is bijective or not. In this section, a simple criterion will be derived which can be used to determine whether the partial derivative

$$[D_\zeta T(\omega_0, 0) h](x)$$
$$= h(x)\Psi(x) + \int_T -h(x')\partial_r \rho(r(x'))\left(\frac{1}{\|x-x'\|} - \frac{1}{\|x'\|}\right) d^3x'$$
$$+ \int_{\partial T} \rho(x') h(x')\left(\frac{1}{\|x-x'\|} - \frac{1}{\|x'\|}\right) d\sigma(x'),$$

is bijective or not. Here

$$\Psi(x) = n(x)\nabla\left[\int_T \rho(x')\frac{1}{\|x-x'\|}\,d^2x' + \frac{\omega^2}{2}(xe_R)^2\right].$$

Hence, $\Psi(x)$ is the normal component of the sum of gravitational and centrifugal force. In the following, the abbreviation

$$\int_0^\infty d(-\rho(r)) \int_{\partial T_r} d\sigma(x') h(x')\left(\frac{1}{\|x-x'\|} - \frac{1}{\|x'\|}\right)$$
$$:= \int_T -h(x')\partial_r \rho(r(x'))\left(\frac{1}{\|x-x'\|} - \frac{1}{\|x'\|}\right) d^3x'$$
$$+ \int_{\partial T} \rho(x') h(x')\left(\frac{1}{\|x-x'\|} - \frac{1}{\|x'\|}\right) d\sigma(x')$$

will be used.

DEFINITION AND PROPOSITION 4.1. — *Let $C^0_S(T)$ be the Banach space of the continuous functions $f: T \to \mathbb{R}$ which are symmetric relative to the three symmetry planes of T. Then, the operator*

$$K:\ C^0_S(T) \to C^0_S(T),$$
$$K:\ h(x) \to [Kh](x)$$
$$:= \frac{-1}{\Psi(x)} \int_0^\infty d(-\rho(r)) \int_{\partial T_r} d\sigma(x') h(x')\left(\frac{1}{\|x-x'\|} - \frac{1}{\|x'\|}\right)$$

is compact.

Proof. — First, for every continuous function $h(x)$ the function

$$f(x) = \int_0^\infty dt(-\rho(r)) \int_{\partial T_r} d\sigma(x') h(x')\left(\frac{1}{\|x-x'\|} - \frac{1}{\|x'\|}\right)$$

is continously differentiable for all $x \in T \setminus \partial T$ and $\|\nabla f(x)\| = O(\|x\|^\nu)$ for $x \to 0$. Because it was assumed that

$$|\Psi(x)| = \left|n(x)\nabla\left[\Phi(x) + \frac{\omega_0^2}{2}(xe_R)^2\right]\right| = \left\|\nabla\left[\Phi(x) + \frac{\omega_0^2}{2}(xe_R)^2\right]\right\| \geqq c_1\|x\|,$$

the function $[Kh](x) = \frac{f(x)}{\Psi(x)}$ can be extended continuously to $x=0$. It should be noted that $n(x)$ is orthogonal to the surfaces

$$\Phi(x) + \frac{w_0^2}{2}(xe_R)^2 = const.$$

An operator is called compact if, for every bounded sequence $h_i (i \in \mathbb{N})$, there exists a convergent subsequence of the sequence $[Kh_i]$. Applying the Arzelà-Ascoli theorem (4) it remains to be proven that for every bounded sequence h_i the sequence $[Kh_i](x)$ is equicontinuous, i.e. that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for every $i \in \mathbb{N}$ and all x_1, x_0 with $\|x_1 - x_0\| < \delta$ it holds that

$$|[Kh_i](x_0) - [Kh_i](x_1)| \leqq \varepsilon.$$

It only will be shown that the sequence $\frac{F_i(x)}{\Psi(x)}$ with

$$F_i(x) := \int_T -h_i(x')(n(x')\nabla\rho(x'))\left(\frac{1}{\|x-x'\|} - \frac{1}{\|x'\|}\right) d^3x'$$

is equicontinuous. The proof that

$$\frac{1}{\Psi}\int_{\partial T} \rho(x')\, h_i(x')\left(\frac{1}{\|x-x'\|} - \frac{1}{\|x'\|}\right) d\sigma(x')$$

is equicontinuous is similar.

Due to equation (A.3) there is a constant C_3 such that ∇F_i fulfills

$$\|\nabla F_i(x_1) - \nabla F_i(x_0)\| < C_3 \|h_i\|_\infty \|x_0 - x_1\|^\nu.$$

Since h_i is a bounded sequence, there exists a constant $\tilde{M}$ such that

$$\|\nabla F_i(x_1) - \nabla F_i(x_0)\| < \tilde{M} \|x_0 - x_1\|^\nu.$$

So $|F_i(x)| \leqq \frac{\tilde{M}}{1+\nu}\|x\|^{1+\nu}$ and

$$\left|\frac{F_i(x)}{\Psi(x)}\right| \leqq \frac{\tilde{M}}{c_1(1+\nu)}\|x\|^\nu.$$

For all $\frac{\varepsilon}{2} > 0$ there exists a Δ such that for all x_1, x_2 with $\|x_1\|$, $\|x_2\| < \Delta$ it holds that

$$\left|\frac{F_i(x_1)}{\Psi(x_1)} - \frac{F_i(x_2)}{\Psi(x_2)}\right| \leqq \left|\frac{F_i(x_1)}{\Psi(x_1)}\right| + \left|\frac{F_i(x_2)}{\Psi(x_2)}\right| < \varepsilon.$$

(4) *See* [11], theorem 67.2.

For all x with $\|x\|>\frac{\Delta}{2}$, there is a constant M' such that $\left\|\nabla \frac{F_i}{\Psi}(x)\right\|<M'$ for all $i\in\mathbb{N}$. So for all x_1, x_2 with $\|x_1\|$, $\|x_2\|>\frac{\Delta}{2}$ and $\|x_1-x_2\|<\frac{\varepsilon}{M'}$ it holds that

$$\left|\frac{F_i(x_1)}{\Psi(x_1)}-\frac{F_i(x_2)}{\Psi(x_2)}\right|<\varepsilon.$$

Choose $\delta\leqq\inf\left\{\frac{\Delta}{2}, \frac{\varepsilon}{M'}\right\}$, then this inequality holds for all x_1, x_2 in T with $\|x_1-x_2\|<\delta$.

PROPOSITION 4.2. – *The operator*

$$D_\zeta T(\omega_0, 0): B_1 \to B_2$$

is bijective if the operator

$$\mathbf{1}-K: C_S^0(T)\to C_S^0(T)$$

is bijective.

Proof. – Assume that $D_\zeta T(\omega_0, 0)$ is not injective. Then there exists a $h\in B_1$ with

$$0=h+\frac{1}{\Psi}\int_0^\infty d(-\rho(r))\int_{\partial T_r} d\sigma(x')\,h(x')\left(\frac{1}{\|x-x'\|}-\frac{1}{\|x'\|}\right).$$

Since $B_1\subset C_S^0(T)$, it is obvious that h is in the kernel of $\mathbf{1}-K$, a contradiction to the assumption.

To prove that $D_\zeta T(\omega_0, 0)$ is surjective, choose any $g\in B_2$. The inequality $\left|\frac{1}{\Psi}\right|\leqq\frac{1}{c_1\|x\|}$ can be used to show that $\frac{g(x)}{\Psi(x)}\in B_1$. Because $\mathbf{1}-K$ was assumed to be surjective, there exists a continuous function h with $h=\frac{q}{\Psi}+Kh$. If it can be shown that $Kh\in B_1$, then $h\in B_1$ and $g=D_\zeta T(\omega_0, 0)h$. For every continuous h the function $[Kh](x)$ is Hölder continuous. Because $\frac{q}{\Psi}$ is Hölder continuous, it follows that h is Hölder continuous. Because $\partial_r\rho$ was assumed to be Hölder continuous, it holds that $Kh\in B_1$ for any Hölder continuous function h.

With proposition 4.2 and the fact that for any compact operator K the operator $\mathbf{1}-K$ is bijective if and only if it is injective [12], it is sufficient to prove that $\mathbf{1}-K$ is injective in order to prove that the implicit function theorem may be applied to the operator $T(\omega, \zeta)$.

PROPOSITION 4.3. — *Assume that*

$$0=\Psi(x)\,h(x)+\int_0^\infty d(-\rho(r))\int_{\partial T_r} d\sigma(x')\,h(x')\left(\frac{1}{\|x-x'\|}-\frac{1}{\|x'\|}\right)$$

if and only if $h=0$, *where* $\rho(x)$ *is a solution of Euler's equation* (2.1) *which has the properties mentioned in section* (1). *Then, there exists a* $\delta>0$, *such that for every* $\omega\in(\omega_0-\delta,\,\omega_0+\delta)$ *there exists a* $\zeta_\omega\in B_1$ *with* $T(\omega,\,\zeta_\omega)=0$. *The matter density* $\rho_{\zeta_\omega}(x)=\rho(g_{\zeta_\omega}^{-1}(x))$ *is a solution of Euler's equation of a rigidly rotating body with the angular velocity* ω *and the equation of state* $p(\rho)$. *Because* $g_{\zeta_\omega}^{-1}(0)=0$, *all stars* T_{ζ_ω} *have the same density at the center* $(x=0)$.

5. SLOWLY ROTATING STARS

The following proposition is an application of proposition (4.3).

PROPOSITION 5.1. — *Let* $p(\rho)$ *be an equation of state with* $\partial_\rho p(\rho)<0$ *such that the corresponding spherical solution* $\rho: B_{|T|}(0)\subset\mathbb{R}^3\to\mathbb{R}^+$ *of the static Euler equation*

$$\frac{\partial_r p(\rho(r))}{\rho(r)}-\partial_r\Phi(r)=0$$

is Hölder continuously differentiable. Here $r(x)=\|x\|$ *is the usual radial coordinate, and* $B_{|T|}(0)$ *is the ball with radius* $|T|$. *Assume furthermore, that* $\rho(x)=\rho(r)$ *is monotonically decreasing. Then for all sufficiently small angular velocities* ω *there exists a solution* $\rho_\omega: T\subset\mathbb{R}^3\to\mathbb{R}^+$ *of Euler's equation*

$$\frac{\nabla p(\rho_\omega)}{\rho_\omega}-\frac{\omega^2}{2}\nabla(xe_R(x))^2-\nabla\int_T\frac{\rho_\omega(x')}{\|x-x'\|}\,d^3x'=0$$

with the same equation of state $p(\rho)$ *and the angular velocity* ω. *Furthermore, it holds that* $\rho_\omega(0)=\rho(0)$.

Because $\rho(x)$ is a spherical matter density, $n(x)=\dfrac{x}{\|x\|}$ and the function $\Psi(x)=\Psi(r)$ has the simple form

$$\Psi(r)=\frac{-4\pi}{r^2}\int_0^r\rho(r')\,r'^2\,dr'.$$

Since $\nabla\Phi(x)=n(x)\,\Psi(r(x))$, there are constants c_1 and c_2 such that

$$c_1\|x\|\leqq\|\nabla\Phi(x)\|=|\Psi(r(x))|\leqq c_2\|x\|. \tag{5.1}$$

Because the density ρ is spherical, the three symmetry planes can be chosen arbitrarily. We choose the planes $x_1=0$, $x_2=0$, and $x_3=0$. Thus,

$\rho(x)$ has the properties which are mentioned in section (1). According to proposition (4.3) it is sufficient to prove that $(\mathbf{1}-K)h=0$ if and only *if* $h=0$.

Define the unbounded linear operator

$$\begin{aligned} K': &\quad C^0(T)\subset L^2(T)\to L^2(T), \\ K': &\quad h(x) \to [K'h](x) \\ &:= \frac{-1}{\Psi(x)}\int_0^\infty d(-\rho(r))\int_{\partial T_r} d\sigma(x')\,h(x')\left(\frac{1}{\|x-x'\|}-\frac{1}{\|x'\|}\right). \end{aligned}$$

Note that K' maps a subset $C^0(T)$ of the Hilbert space $L^2(T)$ into the Hilbert space $L^2(T)$ and K maps the Banach space $C^0_S(T)$ into itself. Because for any continuous function f it holds that $\|f\|_\infty=0$ if and only if $\|f\|_2=\sqrt{\int_T |f|^2\,d^3x}=0$, the kernel of the operator $\mathbf{1}-K$ consist of all $f\in \operatorname{Ker}(\mathbf{1}-K')$ which are symmetric relative to the planes $x_1=0$, $x_2=0$ and $x_3=0$. Since any function $f\in L^2(T)$ can be expanded with respect to an orthonormal basis, the analysis of the operator $\mathbf{1}-K'$ in the Hilbert space $L^2(T)$ has an advantage over the analysis of the operator $\mathbf{1}-K$ in the Banach space $C^0(T)$.

Choose any $h(r, \upsilon, \varphi)$ in the kernel of $\mathbf{1}-K'$. h can be expanded into spherical harmonics $Y_{lm}(\upsilon, \varphi)$, i. e. that

$$h(r, \upsilon, \varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l} h_{lm}(r)\,Y_{lm}(\upsilon, \varphi) \tag{5.2}$$

with the continuous functions

$$h_{lm}(r)=\int_{\|x\|=1} Y^*_{lm}(\upsilon, \varphi)\,h(r, \upsilon, \varphi)\,d\sigma.$$

Furthermore, $\dfrac{1}{\|x-x'\|}$ can be expanded into spherical harmonics.

$$\frac{1}{\|x-x'\|}=\sum_{l=0}^{\infty}\frac{4\pi}{2l+1}\sum_{m=-l}^{l}\frac{r_<^l}{r_>^{l+1}}\,Y^*_{lm}(\upsilon', \varphi')\,Y_{lm}(\upsilon, \varphi) \tag{5.3}$$

where $r_<=\inf\{r(x), r(x')\}$ and $r_>=\sup\{r(x), r(x')\}$.

If the continuous function h is in the kernel of $\mathbf{1}-K'$, then for all l, m, and all $0\leq r\leq |T|$, it holds that

$$\int_{\partial B_r(0)} d\sigma\,[(\mathbf{1}-K')h](r, \upsilon, \varphi)\,Y^*_{lm}(\upsilon, \varphi)=0. \tag{5.4}$$

Inserting (5.2) and (5.3) into equation (5.4) yields

$$0=h_{lm}(r)+\frac{1}{\Psi(r)}\int_0^\infty d(-\rho(r'))\,h_{lm}(r')\times\begin{cases}\dfrac{4\pi}{2l+1}\dfrac{r'^{(l+2)}}{r^{(l+1)}}-4\pi r'\delta_{l0} & \text{for } r'\leqq r\\[2ex] \dfrac{4\pi}{2l+1}\dfrac{r^l}{r'^{(l-1)}}-4\pi r'\delta_{l0} & \text{for } r'\geqq r\end{cases}. \tag{5.5}$$

For all $l>1$ equation (5.5) yields $h_{lm}=0$.

Proof. – Assume that $h_{lm}(r)$ is solution of equation (5.5) with $l>1$. Then

$$\begin{aligned}h_{lm}(r)=&\frac{1}{\Psi(r)}\frac{4\pi}{2l+1}\frac{1}{r^2}\int_0^r-\rho'(r')\,h_{lm}(r')\left(\frac{r'}{r}\right)^{l-1}r'^3\,dr'\\&+\frac{1}{\Psi(r)}\frac{4\pi}{2l+1}r\int_r^{|T|}-\rho'(r')\,h_{lm}(r')\left(\frac{r}{r'}\right)^{l-1}dr'\\&+\frac{1}{\Psi(r)}\frac{4\pi}{2l+1}r\rho(|T|)\,h_{lm}(|T|)\left(\frac{r}{|T|}\right)^{l-1}\end{aligned}$$

where $\rho':=\partial_r\rho$. According to $\rho'\leqq 0$ it follows that

$$\begin{aligned}|h_{lm}(r)|\leqq&\frac{1}{|\Psi(r)|}\frac{4\pi}{2l+1}\frac{1}{r^2}\int_0^r-\rho'(r')\,\|h_{lm}\|_\infty\left(\frac{r'}{r}\right)^{l-1}r'^3\,dr'\\&+\frac{1}{|\Psi(r)|}\frac{4\pi}{2l+1}r\int_r^{|T|}-\rho'(r')\,\|h_{lm}\|_\infty\left(\frac{r}{r'}\right)^{l-1}dr'\\&+\frac{1}{|\Psi(r)|}\frac{4\pi}{2\,l+1}r\rho(|T|)\,\|h_{lm}\|_\infty\left(\frac{r}{|T|}\right)^{l-1}.\end{aligned}$$

Using the inequalities $\dfrac{r'}{r}\leqq 1$ for all $r'\in[0, r]$ and $\dfrac{r}{r'}\leqq 1$ for all $r'\in[r, |T|]$ yields

$$\begin{aligned}|h_{lm}(r)|\leqq&\frac{1}{|\Psi(r)|}\frac{4\pi}{2l+1}\frac{1}{r^2}\|h_{lm}\|_\infty\int_0^r-\rho'(r')\,r'^3\,dr'\\&+\frac{1}{|\Psi(r)|}\frac{4\pi}{2l+1}r\,\|h_{lm}\|_\infty\int_r^{|T|}-\rho'(r')\,dr'\\&+\frac{1}{|\Psi(r)|}\frac{4\pi}{2l+1}r\rho(|T|)\,\|h_{lm}\|_\infty\\=&\frac{\|h_{lm}\|_\infty}{|\Psi(r)|}\frac{4\pi}{2l+1}\left[\frac{1}{r^2}\int_0^r-\rho'(r')\,r'^3\,dr'+r\rho(r)\right]\end{aligned}$$

$$= \frac{3}{2l+1} \frac{\|h_{lm}\|_\infty}{|\Psi(r)|} \left[\frac{4\pi}{r^2} \int_0^r \rho(r') r'^2 \, dr' \right]$$
$$= \frac{3}{2l+1} \|h_{lm}\|_\infty$$

where $\|h_{lm}\|_\infty := \sup \{ |h_{lm}(r)|;\ r \in [0, |T|] \}$. Because $\frac{3}{2l+1} < 1$ it follows that $h_{lm} = 0$.

For $l=0$ equation (5.5) yields $h_{00} = 0$.

Proof. – Inserting $l=0$ into equation (5.5) yields

$$0 = h_{00}(r) - \frac{1}{\Psi(r)} 4\pi \frac{1}{r} \int_0^r -\rho'(r') h_{00}(r') r' (r - r') \, dr'.$$

First, it holds that $h_{00}(0) = 0$. Define

$$m(r) := \sup \{ |h_{00}(r')| \, | \, r' \leqq r \}$$

and

$$R := \sup \{ r \, | \, h_{00}(r') = 0 \text{ for all } r' \leqq r \}.$$

With a contradiction it will be shown that $R = |T|$ so that $\|h_{00}\|_\infty = m(|T|) = 0$.

Assume that $R < |T|$. Then, for all $R \leqq r \leqq |T|$ it holds that

$$|h_{00}(r)| \leqq \frac{4\pi}{r|\Psi(r)|} \left| \int_0^r -\rho'(r') h_{00}(r') r' (r - r') \, dr' \right|$$
$$= \frac{4\pi}{r|\Psi(r)|} \left| \int_R^r -\rho'(r') h_{00}(r') r' (r - r') \, dr' \right|$$
$$\leqq \frac{4\pi}{r|\Psi(r)|} \|\rho'\|_\infty m(r) r (r - R) \int_R^r dr'.$$

With equation (5.1) and $r - R \leqq r$ it holds that

$$\frac{4\pi}{r|\Psi(r)|} r (r - R) \leqq \frac{4\pi}{c_1}.$$

Thus,

$$h_{00}(r) \leqq \frac{4\pi}{c_1} \|\rho'\|_\infty m(r) (r - R).$$

Choose $r_0 > R$ so that for all $r \in [R, r_0]$ it holds that

$$\frac{4\pi}{c_1} \|\rho'\|_\infty (r - R) < \frac{1}{2}.$$

Because $m(r)$ is monotonically increasing, it follows for all $r \in [R, r_0]$ that

$$|h_{00}(r)| \leqq \frac{4\pi}{c_1} \|\rho'\|_\infty (r-R) m(r_0) \leqq \frac{m(r_0)}{2}.$$

Since $h_{00}(r)=0$ for all $r \in [0, R]$, this inequality also holds for all $r \in [0, r_0]$. Thus, the inequality $|h_{00}(r)| \leqq \frac{1}{2} \sup \{ |h_{00}(r')| \,|\, r' \leqq r_0 \}$ holds for all $r \in [0, r_0]$. This yields $h_{00}=0$ for all $r \in [0, r_0]$ with $r_0 > R$, a contradiction to $R := \sup \{ r \,|\, h_{00}(r')=0 \text{ for all } r' \leqq r \}$.

Because the spherical harmonics $Y_{1m}(\upsilon, \varphi)$ are linear combinations of the functions $\cos \upsilon$, $(\sin \upsilon \sin \varphi)$ and $(\sin \upsilon \cos \varphi)$, it is proven so far that any solution of equation (5.5) must have the form

$$h(r, \upsilon, \varphi) = h_1(r) \cos \upsilon + h_2(r) \sin \upsilon \sin \varphi + h_3(r) \sin \upsilon \cos \varphi.$$

But the functions $\cos \upsilon$, $(\sin \upsilon \sin \varphi)$ and $(\sin \upsilon \cos \varphi)$ are antisymmetric relative to the symmetry planes $z=0$, $y=0$, or $x=0$. So $(\mathbf{1}-K)h=0$ if and only if $h=0$. To complete the analysis of the operator $\mathbf{1}-K$, it is noted without proof that $h_{1m}=const$ is the only solution of equation (5.5) for $l=1$.

APPENDIX
A. PROPERTIES OF THE NEWTONIAN POTENTIAL

The proofs of the facts mentioned in this section can be found in books about elliptic partial differential equations.

Let $T \subset \mathbb{R}^3$ be a bounded domain with a smooth surface ∂T. Then the Newtonian potential

$$U(x) = \int_T f(x') \frac{1}{\|x-x'\|} d^3 x'$$

of a bounded function $f: T \to \mathbb{R}$ is a continuously differentiable function. Furthermore, there exist constants C_1 and C_2 such that for all $x \in \mathbb{R}^3$ the following inequalities hold:

$$|U(x)| \leqq C_1 \|f\|_\infty \qquad (A.1)$$

$$\|\nabla U(x)\| \leqq C_2 \|f\|_\infty. \qquad (A.2)$$

Furthermore, for all $0 < \nu < 1$ and all $R_0 > 0$, there exist constants $C_3(\nu, R_0)$ such that for all x_1, x_2 with $\|x_1 - x_2\| < R_0$, and $x_1, x_2 \in \bar{T}$ or $x_1, x_2 \in \overline{\mathbb{R}^3 \setminus T}$ the partial derivatives fulfill the inequality

$$|\partial_{x_i} U(x_1) - \partial_{x_i} U(x_2)| \leqq C_3(\nu, R_0) \|f\|_\infty \|x_1 - x_2\|^\nu. \qquad (A.3)$$

If the function f is Hölder continuous with the Hölder exponent ν, i.e. if there is a constant M with

$$|f(x_1)-f(x_2)| \leqq M \|x_1-x_2\|^\nu,$$

then there exist the second derivatives of the Newtonian potential U. Furthermore, for all $R_0>0$ there exist constants $C_4(\nu)$, $C_5(\nu)$, $C_6(\nu, R_0)$, and $C_7(\nu, R_0)$ such that for all x_1, $x_2 \in \mathbb{R}^3$ with $\|x_1-x_2\|<R_0$, and x_1, $x_2 \in \bar{T}$ or x_1, $x_2 \in \overline{\mathbb{R}^3 \setminus T}$ the following estimates hold:

$$|\partial_{x_i}\partial_{x_j} U(x_1)| \leqq C_4(\nu)\|f\|_\infty + C_5(\nu) M, \tag{A.4}$$

$$|\partial_{x_i}\partial_{x_j} U(x_1) - \partial_{x_i}\partial_{x_j} U(x_2)| \leqq [C_6(\nu, R_0)\|f\|_\infty + C_7(\nu, R_0) M] \|x_1-x_2\|^\nu. \tag{A.5}$$

In addition, for all $x \in T$ it holds that

$$\Delta U(x) = -4\pi f(x). \tag{A.6}$$

Let S be a bounded and smooth surface. Then, the Newtonian surface potential

$$V(x) = \int_S f(x') \frac{1}{\|x-x'\|} d\sigma(x') \tag{A.7}$$

is a Hölder continuous function. $V(x)$ is even analytic for all $x \notin S$. For all $R_0>0$ there exist constants D_1 and $D_2(\nu, R_0)$ such that for all x_1, $x_2 \in \mathbb{R}^3$ with $\|x_1-x_2\|<R_0$ the following estimates hold:

$$|V(x_1)| \leqq D_1 \|f\|_\infty, \tag{A.8}$$

$$|V(x_1)-V(x_2)| \leqq D_2(\nu, R_0)\|f\|_\infty \|x_1-x_2\|^\nu. \tag{A.9}$$

If, furthermore, the function f is Hölder continuous, then for appropriate constants $D_3, \ldots, D_6$ the following inequalities hold:

$$\|\nabla V(x_1)\| \leqq D_3(\nu)\|f\|_\infty + D_4(\nu) M, \tag{A.10}$$

$$\|\nabla V(x_1) - \nabla V(x_2)\| \leqq [D_5(\nu, R_0)\|f\|_\infty + D_6(\nu, R_0) M] \|x_1-x_2\|^\nu. \tag{A.11}$$

The following proposition is not common.

PROPOSITION A.1. – *Let T be a bounded domain with three linearly independent symmetry planes which intersect at the point $x=0$. Let $f: T \to \mathbb{R}$ be a continuous and bounded function which is symmetric relative to the three symmetry planes of T. If f fulfills $|f(x)| \leqq c\|x\|$ for a constant c, then there exists a constant E_1 such that the gradient of the Newtonian potential*

$$\nabla U(x) = \int_T f(x') \frac{-x+x'}{\|x-x'\|^3} d^3x'$$

fulfills

$$\|\nabla U(x)\| \leqq E_1 c \|x\|.$$

Furthermore, for all $0<\nu<1$ *there is a constant* $E_2(\nu)$ *such that for all* $v\in\mathbb{R}^3$ *with* $\|v\|=1$ *and all* $t\in[-1, 1]$ *it holds that*

$$\|\nabla U(x+t\|x\|v)-\nabla U(x)\|\leqq E_2(\nu)\,ct^\nu\|x\|.$$

Proof. — Since there are three linearly independent symmetry planes which intersect at the point $x=0$, it holds that $\nabla U(0)=0$. So an upper bound for

$$\|\nabla U(x)\|=\left\|\int_T f(x')\left(\frac{-x+x'}{\|x-x'\|^3}-\frac{x'}{\|x'\|^3}\right)d^3x'\right\|$$

must be found. This integral can be estimated similarly as $\|\nabla U(x+t\|x\|v)-\nabla U(x)\|$. To shorten this paper, only the estimate

$$\|\nabla U(x+t\|x\|v)-\nabla U(x)\|\leqq E_2(\nu)\,ct^\nu\|x\|$$

will be proven.

With equation (A.3) it can easily be seen that there exists a constant m such that

$$\|\nabla U(x+t\|x\|v)-\nabla U(x)\|\leqq mct^\nu\|x\|$$

holds for all x with $\|x\|>\frac{1}{9}$. Without loss of generality it can be assumed that [5] $B_{1/q}(0)\subset T$. Choose any $x\in B_{1/9}(0)$ and define $\varepsilon:=\sqrt{x}$, then $B_{2t\|x\|}(x)\subset B_\varepsilon(0)$ and

$$\begin{aligned}
&\frac{|\partial_{x_i}U(x+t\|x\|v)-\partial_{x_i}U(x)|}{\|x\|}\\
&\quad=\frac{1}{\|x\|}\left|\int_T f(x')[K(x+t\|x\|v, x')-K(x, x')]\,d^3x'\right|\\
&\quad\leqq\frac{1}{\|x\|}\left|\int_{B_\varepsilon(0)\setminus B_{2t\|x\|}(x)} f(x')[K(x+t\|x\|v, x')-K(x, x')]\,d^3x'\right|\\
&\qquad+\frac{1}{\|x\|}\left|\int_{B_{2t\|x\|}(x)} f(x')[K(x+t\|x\|v, x')-K(x, x')]\,d^3x'\right|\\
&\qquad+\frac{1}{\|x\|}\left|\int_{T\setminus B_\varepsilon(0)} f(x')[K(x+t\|x\|v, x')-K(x, x')]\,d^3x'\right|
\end{aligned}$$

where

$$K(x, x'):=\frac{-x_i+x'_i}{\|x-x'\|^3}.$$

[5] $B_r(x)$ denotes the ball of radius r with center x.

In the following, these integrals will be estimated. With $\|\nabla K(x, x')\| \leqq \frac{4}{\|x-x'\|^3}$ and $|f(x)| \leqq c\varepsilon$ for all $x \in B_\varepsilon(0)$ it follows that

$$\begin{aligned}
&\frac{1}{\|x\|}\left|\int_{B_\varepsilon(0)\setminus B_{2t\|x\|}(x)} f(x')[K(x+t\|x\|v, x')-K(x, x')]\,d^3x'\right| \\
&\qquad =\left|\int_{B_\varepsilon(0)\setminus B_{2t\|x\|}(x)} f(x')\nabla K(x+t\Theta\|x\|v, x')\,tv\,d^3x'\right| \\
&\qquad \leqq \int_{B_\varepsilon(0)\setminus B_{2t\|x\|}(x)} c\varepsilon\frac{4}{\|x+t\Theta\|x\|v-x'\|^3}\,td^3x'
\end{aligned}$$

for a $\Theta \in (0, 1)$. Since $\|(x-x')+t\Theta\|x\|v\| \geqq \|x-x'\|-t\|x\|$ and $B_\varepsilon(0) \subset B_{2\varepsilon}(x)$, it holds that

$$\begin{aligned}
&\frac{1}{\|x\|}\left|\int_{B_\varepsilon(0)\setminus B_{2t\|x\|}(x)} f(x')[K(x+t\|x\|v, x')-K(x, x')]\,d^3x'\right| \\
&\qquad \leqq \int_{B_{2\varepsilon}(x)\setminus B_{2t\|x\|}(x)} c\varepsilon\frac{4}{(\|x-x'\|-t\|x\|)^3}\,td^3x' \\
&\qquad = 16\pi ct\varepsilon\int_{2\|x\|t}^{2\varepsilon} r^2\frac{1}{(r-t\|x\|)^3}\,dr \\
&\qquad = 16\pi ct\varepsilon\int_{2\|x\|t}^{2\varepsilon} \frac{1}{r}\frac{1}{(1-(t\|x\|/r))^3}\,dr \\
&\qquad \leqq 16\pi ct\varepsilon\int_{2\|x\|t}^{2\varepsilon} \frac{1}{r}\frac{1}{(1-(1/2))^3}\,dr \\
&\qquad = 128\pi ct\varepsilon\ln\frac{\varepsilon}{\|x\|t} \\
&\qquad = 128\pi c(\sqrt{\|x\|}t)^\nu(\sqrt{\|x\|}t)^{1-\nu}|\ln\sqrt{\|x\|}t|.
\end{aligned}$$

Because the function $x^{1-\nu}\ln x$ is bounded for all $x \in (0, 1)$, there exists a constant M_1 with

$$\frac{1}{\|x\|}\left|\int_{B_\varepsilon(0)\setminus B_{2t\|x\|}(x)} f(x')[K(x+t\|x\|v, x')-K(x, x')]\,d^3x'\right| \leqq M_1 ct^\nu.$$

Choose $|T|$ large enough; then $T \subset B_{|T|}(0)$ and it holds that

$$\frac{1}{\|x\|}\left|\int_{T\setminus B_\varepsilon(0)} f(x')[K(x+t\|x\|v, x')-K(x, x')]\,d^3x'\right|$$

$$=\left|\int_{T\setminus B_\varepsilon(0)} f(x')\nabla K(x+\Theta t\|x\|v,x')\,tv\,d^3x'\right|$$

$$\leqq t\int_{B_{|T|}(0)\setminus B_\varepsilon(0)} c\|x'\|\frac{4}{\|x+\Theta t\|x\|v-x'\|^3}\,d^3x'.$$

The estimate $\|x+\Theta t\|x\|v-x'\|\geqq\|x'-x\|-t\|x\|\geqq\|x'\|-\|x\|-t\|x\|$ yields

$$\frac{1}{\|x\|}\left|\int_{T\setminus B_\varepsilon(0)} f(x')[K(x+t\|x\|v,x')-K(x,x')]\,d^3x'\right|$$

$$\leqq ct\,16\pi\int_\varepsilon^{|T|}\frac{1}{(1-((1+t)\|x\|/r))^3}\,dr$$

$$\leqq ct\,16\pi\int_\varepsilon^{|T|}\frac{1}{(1-((1+t)\|x\|/\varepsilon))^3}\,dr$$

$$\leqq ct\,16\pi\;|T|\frac{1}{(1-2\sqrt{\|x\|})^3}$$

$$\leqq ct\,16\pi|T|\frac{1}{(1-(2/3))^3}$$

$$\leqq M_2\,ct^\nu.$$

Furthermore, there exists a constant M_3 with

$$\frac{1}{\|x\|}\left|\int_{B_{2t\|x\|}(x)} f(x')[\mathrm{K}(x+t\|x\|v,x')-\mathrm{K}(x,x')]\,d^3x'\right|$$

$$\leqq\frac{1}{\|x\|}\left|\int_{B_{2t\|x\|}(x)} f(x')\,K(x+t\|x\|v,x')\,d^3x'\right|$$

$$+\frac{1}{\|x\|}\left|\int_{B_{2t\|x\|}(x)} f(x')K(x,x')\,d^3x'\right|$$

$$\leqq c\frac{\varepsilon}{\|x\|}\left(\int_{B_{4t\|x\|}(0)}\frac{1}{\|x'\|^2}\,d^3x'+\int_{B_{2t\|x\|}(0)}\frac{1}{\|x'\|^2}\,d^3x'\right)$$

$$=c\frac{\varepsilon}{\|x\|}\,24\,\pi\,t\|x\|$$

$$\leqq M_3\,ct^\nu.$$

Choose $E_2\geqq\sup\{m,\sqrt{3}\,(M_1+M_2+M_3)\}$, then for all $x\in\mathbb{R}^3$ it holds that

$$\|\nabla U(x+t\|x\|v)-\nabla U(x)\|\leqq\tilde{M}\,ct^\nu.$$

It must be noted that all constants in this section depend on the volume of the body T, its surface ∂T and the greatest circuit of the surface. In this paper occurs a whole set T_ζ of such bodies. But since ζ was restricted by $\|\zeta\|_{B_1} \leqq Z$ (see proposition (2.1)), an upper bound of the volume of all bodies T_ζ, all surfaces ∂T_ζ, and the greatest circuit of all surfaces can be found.

B. THE REMAINING PROOFS OF SECTION (3)

Because

$$\frac{\omega^2}{2}[(x+\zeta n)e_R]^2$$

only depends quadratically on ω and ζ, it easy to see that it continuously differentiable with respect to ζ and that its derivative has the form

$$h\,\omega^2 [(x+\zeta\, n)\, e_R](ne_R).$$

It remains to be proven that the Newtonian potential

$$\Phi:\quad B_1 \to B_2,$$

$$\Phi:\quad \zeta(x) \to [\Phi(\zeta)](x) := \int_{T_\zeta} \rho_\zeta \left(\frac{1}{\|g_\zeta(x)-x'\|} - \frac{1}{\|x'\|}\right) d^3 x'$$

is continuously differentiable with respect to ζ and that its derivative has the form

$$\begin{aligned}[D_\zeta \Phi(\zeta)]\, h(x) &= h(x)\left[n(x)\int_{T_\zeta} -\rho_\zeta(x')\,\frac{g_\zeta(x)-x'}{\|g_\zeta(x)-x'\|^3}\, d^3 x'\right] \\ &+ \int_{T_\zeta} -h(g_\zeta^{-1}(x'))\,(n(g_\zeta^{-1}(x'))\,\nabla\rho_\zeta(x'))\left(\frac{1}{\|g_\zeta(x)-x'\|} - \frac{1}{\|x'\|}\right) d^3 x' \\ &+ \int_{\partial T_\zeta} \rho_\zeta(x')\left(\frac{1}{\|g_\zeta(x)-x'\|} - \frac{1}{\|x'\|}\right) h(g_\zeta^{-1}(x'))\, d\sigma_\zeta(x').\end{aligned}$$

B.1. Préliminaries

In this section some propositions are presented to prove that $\Phi(\zeta)$ is continuously differentiable.

Proposition B.1. – *Let* $f_i \in B_1$ $(i \in \mathbb{N})$ *be a sequence with* $\|f_i\| \leqq Z$ *(see proposition* (2.1)). *Let* $f \in B_1$ *with* $\lim_{i\to\infty} \|f-f_i\|_{B_1} = 0$. *Then, the set of coordinate transformations*

$$g_i:\quad T \to T_i,\ x \to g_i(x) := x + n(x)\, f_i(x)$$

fulfills for $i \to \infty$ the relations

1. $\sup \{ \| g_i(x) - g(x) \|; x \in T \} \to 0$

2. The Jacobi-matrices $D g_i$ fulfill

$$\sup \{ \| D g_i(x) - D g(x) \|; x \in T \} \to 0.$$

Here $\| D g(x) \| := \sup \{ \| D g(x) v \|; \| v \| = 1 \}$.

3. $\sup \{ \| (D g_n)^{-1}(x) - (D g)^{-1}(x) \|; x \in T \} \to 0$

4. Let $U := \bigcap_{i \in \mathbb{N}} g_i(T) \cap g(T)$, *then*

$$\sup \{ \| g_n^{-1}(x) - g^{-1}(x) \| \mid x \in U \} \to 0.$$

5. $\sup \{ \| D g_n^{-1}(x) - D g^{-1}(x) \| \mid x \in U \} \to 0.$

Proof. – Because $\| f_i - f \|_{B_1} \to 0$ implies $\| f_i - f \|_\infty \to 0$ and

$$\| \nabla f_i - \nabla f \|_\infty \to 0,$$

this proposition can be proven by straightforward calculations.

PROPOSITION B.2. – *Let $\| f_i - f \|_{B_1} \to 0$ be a convergent sequence in $B_Z(0) \subset B_1$. Let $g_i(x) := x + n(x) f_i(x)$ be the set of coordinate transformations $g_i : T \to T_i$, $g : T \to T_0$. For any $h \in B_1$ with $\| h \| \leqq 1$ define the sequences*

$$F_i(x) := \int_{T_i} h(g_i^{-1}(x')) \nabla \rho(g_i^{-1}(x')) \times [(D g_i)^{-1}(g_i^{-1}(x')) n(g_i^{-1}(x'))] \left(\frac{1}{\| x - x' \|} - \frac{1}{\| x' \|} \right) d^3 x'.$$

Then all $\dfrac{\nabla F_i(x)}{\| x \|}$ *converge uniformly for all* $x \in \overline{\bigcup_i T_i}$ *and all* $h \in B_1$ *with* $\| h \| \leqq 1$. *With proposition* (A.1) *and proposition* (B.1) *if follows that the sequences* $\dfrac{\nabla F_i(g_i(x))}{\| x \|}$ *converge uniformly for all* $x \in T$ *and all* $h \in B_1$ *with* $\| h \| \leqq 1$.

Proof. – This proof is too long to be given in every detail. The main tool for the proof is that any equicontinuous sequence which converges pointwise on a dense subset, converges uniformly ([6]).

Equation (A.3) shows that the sequence $\nabla F_i(x)$ is equicontinuous for all h with $\| h \|_{B_1} \leqq 1$. Because it converges on a dense subset, it converges uniformly. So it is sufficient to prove, that $\dfrac{\nabla F_i}{\| x \|}$ converges uniformly in a ball $B_{R/2}(0)$. Choose $R > 0$ such that $B_R(0) \subset T_i$ (for all sufficiently large i),

([6]) *See* [11], proposition 67.4.

then

$$\left\| \frac{\nabla F_i - \nabla F}{\|x\|} \right\| \leqq \left\| \frac{1}{\|x\|} \nabla \int_{B_R} (h_i(g_i^{-1}(x')) - h(g^{-1}(x'))) K(x, x') d^3 x' \right\|$$
$$+ \left\| \frac{1}{\|x\|} \nabla \int_{T_i \setminus B_R} h_i(g_i^{-1}(x')) K(x, x') d^3 x' \right.$$
$$\left. - \frac{1}{\|x\|} \nabla \int_{T_0 \setminus B_R} h(g^{-1}(x')) K(x, x') d^3 x \right\|$$

where

$$K(x, x') := \frac{1}{\|x - x'\|} - \frac{1}{\|x'\|}$$

and

$$h_i(x) := h(x) \nabla \rho(x) [(D g_i)^{-1}(x) n(x)].$$

With proposition (A.1) the first integral converges uniformly to 0 in $B_{R/2}(0)$. To estimate the second part, define

$$G_i(x) := \int_{T_i \setminus B_R} h_i(g_i^{-1}(x')) K(x, x') d^3 x' - \int_{T_0 \setminus B_R} h(g^{-1}(x')) K(x, x') d^3 x.$$

The second partial derivatives of G_i converge uniformly to 0 in $B_{R/2}(0)$. Since $\nabla G_i(0) = 0$, $\frac{\nabla G_i}{\|x\|}$ converges uniformly to 0 in $B_{R/2}(0)$. Because $\|h\|_\infty$ and $\|\nabla h\|_\infty$ can be estimated by 1, the convergence is uniform for all $h \in B_1$ with $\|h\|_{B_1} \leqq 1$. □

Proposition B.3. – *Let $\|f_i - f\|_{B_1} \to 0$ be a convergent sequence in $B_z(0) \subset B_1$. Let $g_i(x) := x + n(x) f_i(x)$ be the set of coordinate transformations $g_i : T \to T_i$, $g : T \to T_0$. For any $h \in B_1$ with $\|h\| \leqq 1$ define the sequences*

$$F_i(x) := \int_{\partial T_i} h(g_i^{-1}(x')) \rho(g_i^{-1}(x)) \left(\frac{1}{\|x - x'\|} - \frac{1}{\|x'\|} \right) d\sigma(x').$$

Then all $\frac{\nabla F_i(x)}{\|x\|}$ converge uniformly for all $x \in \overline{\bigcup_i T_i}$ and all $h \in B_1$ with $\|h\| \leqq 1$. With proposition (B.1) *it follows that the sequences $\frac{\nabla F_i(g_i(x))}{\|x\|}$ converge uniformly for all $x \in T$ and all $h \in B_1$ with $\|h\| \leqq 1$.*

The proof of this proposition is very similar to the proof of proposition (B.2).

B.2. $\Phi(\zeta)$ IS DIFFERENTIABLE

Choose ζ and h such that for all sufficiently small t it holds that $\zeta + th \in B_Z(0)$. Define the coordinate transformations

$$g_t: \quad T \to T_t, \qquad x \to g_t(x) := x + (\zeta(x) + th(x))\, n(x)$$

with Jacobi matrices

$$D g_t(x)\, v = \mathbf{1}\, v + (\nabla(\zeta + th)\, v) + (\zeta + th)(v \nabla)\, n.$$

Let

$$f(t, x) = \int_{T_t} \rho(g_t^{-1}(x')) \left(\frac{1}{\|x - x'\|} - \frac{1}{\|x'\|} \right) d^3 x',$$

then $\Phi(\zeta + th)(x) = f(t, g_t(x))$ and

$$D_\zeta \Phi(\zeta)\, h(x) = \partial_t f(0, g_0(x)) + \nabla f(0, g_0(x))\, h(x)\, n(x).$$

It will now be shown that in the norm $\| \cdot \|_{B_2}$

$$\frac{f(0, g_t(x)) - f(0, g_0(x))}{t} \to \nabla f(0, g_0(x))\, h(x)\, n(x) \tag{B.1}$$

and

$$\frac{f(t, g_t(x)) - f(0, g_t(x))}{t} \to \partial_t f(0, g_0(x)). \tag{B.2}$$

Addition of equation (B.1) and equation (B.2) yields the desired result

$$\frac{f(t, g_t(x)) - f(0, g_0(x))}{t} \to \partial_t f(0, g_0(x)) + \nabla f(0, g_0(x))\, h(x)\, n(x).$$

To prove equation (B.1), the uniform convergence of

$$\frac{f(0, g_t(x)) - f(0, g_0(x))}{t} \to \nabla f(0, g_0(x))\, h(x)\, n(x) \tag{B.3}$$

and of

$$\frac{\nabla f(0, g_t(x)) - \nabla f(0, g_0(x))}{\|x\|\, t} \to \frac{\nabla[\nabla f(0, g_0(x))\, h(x)\, n(x)]}{\|x\|} \tag{B.4}$$

has to be shown.

Because T is a bounded domain and the equations $f(0, g_t(0) = 0$ and $\nabla f(0, g_0(0)) = 0$ hold for all t, the integration of equation (B.4) yields equation (B.3). Equation (B.4) can be proven in the following way:

Using sum convention, it holds that

$$\frac{\partial_{x_i}[f(0, g_t(x)) - f(0, g_0(x))]}{\|x\| t} = \frac{\partial_{x_j} f(0, g_t(x))}{\|x\|} \frac{(D g_t(x))_{ij} - (D g_0(x))_{ij}}{t} + \frac{\partial_{x_j}[f(0, g_t(x)) - f(0, g_0(x))]}{\|x\| t} (D g_0(x))_{ij}$$

where

$$(D g_t(x))_{ij} = \delta_{ij} + \partial_{x_i}[\zeta(x) n_j(x) + t h(x) n_j(x)].$$

With the continuity of $\partial_{x_i}\partial_{x_j} f(0, x)$ and the equation $\nabla f(0, 0) = 0$ it follows that $\frac{\partial_{x_j} f(0, g_t(x))}{\|x\|}$ converges uniformly to $\frac{\partial_{x_j} f(0, g_0(x))}{\|x\|}$, so the first part converges uniformly to

$$\frac{\partial_{x_j} f(0, g_0(x))}{\|x\|} \partial_{x_i} [h(x) n_j(x)].$$

Now the second part will be estimated. There exists a $\Theta \in (0, 1)$ such that

$$\frac{\partial_{x_j}[f(0, g_t(x)) - f(0, g_0(x))]}{\|x\| t} = \nabla \partial_{x_j} f(0, g_{\Theta t}(x)) \frac{h(x)}{\|x\|} n(x).$$

Because $\partial_{x_i}\partial_{x_j} f(0, x)$ is Hölder continuous, there is a constant M such that

$$\|\nabla \partial_{x_j} f(0, g_{\Theta t}(x)) - \nabla \partial_{x_j} f(0, g_0(x)\| \leq M \|h\|_\infty^\nu \Theta^\nu t^\nu \leq \tilde{M} t^\nu.$$

With the estimate $\left|\frac{h(x)}{\|x\|}\right| \leq \|h\|_{B_1}$, it follows that

$$\frac{\partial_{x_j} f(0, g_t(x))}{\|x\|} \frac{(D g_t(x))_{ij} - (D g_0(x))_{ij}}{t} + \frac{\partial_{x_j}[f(0, g_t(x)) - f(0, g_0(x))]}{\|x\| t} (D g_0(x))_{ij}$$

converges uniformly to

$$\frac{\partial_{x_j} f(0, g_0(x)) \partial_{x_i}[h n_j]}{\|x\|} + \frac{\partial_{x_j} \nabla f(0, g_0(x)) h(x) n(x)}{\|x\|} (D g_0(x))_{ij} = \frac{\partial_{x_i}[\nabla f(0, g_0(x)) h(x) n(x)]}{\|x\|}.$$

The first step to prove equation (B.2) is to prove the uniform convergence of

$$\frac{\nabla [f(t, x)-f(0, x)]}{t\|x\|} \to \frac{\nabla \partial_t f(0, x)}{\|x\|}. \tag{B.5}$$

In section (3) it was shown that

$$\partial_t f(t, x)=\int_{T_t} -h(g_t^{-1}(x'))(\nabla \rho(g_t^{-1}(x'))[D g_t^{-1}(x) n(g_t^{-1}(x'))]) \times\left(\frac{1}{\|x-x'\|}-\frac{1}{\|x'\|}\right) d^3 x' + \int_{\partial T_t} h(g_t^{-1}(x')) \rho(g_t^{-1}(x'))\left(\frac{1}{\|x-x'\|}-\frac{1}{\|x'\|}\right) d\sigma(x').$$

Thus, for all x there exists a $\Theta \in (0, 1)$ with

$$\frac{\nabla f(t, x)-\nabla f(0, x)}{t\|x\|}=\frac{\nabla \partial_t f(\Theta t, x)}{\|x\|}.$$

This yields

$$\begin{aligned}
&\frac{\nabla f(t, x)-\nabla f(0, x)}{t\|x\|}-\frac{\nabla \partial_t f(0, x)}{\|x\|} \\
&=\frac{\nabla \partial_t f(\Theta t, x)}{\|x\|}-\frac{\nabla \partial_t f(0, x)}{\|x\|} \\
&=\frac{1}{\|x\|} \nabla \int_{T_{\Theta t}} -h(g_{\Theta t}^{-1}(x'))(\nabla \rho(g_{\Theta t}^{-1}(x')) D g_{\Theta t}^{-1}(x') n(g_{\Theta t}^{-1}(x'))) \\
&\qquad \times\left(\frac{1}{\|x-x'\|}-\frac{1}{\|x'\|}\right) d^3 x' \\
&\quad -\frac{1}{|x\|} \nabla \int_{T_0} -h(g_0^{-1}(x'))(\nabla \rho(g_0^{-1}(x')) D g_0^{-1}(x') n(g_0^{-1}(x'))) \\
&\qquad \times\left(\frac{1}{\|x-x'\|}-\frac{1}{\|x'\|}\right) d^3 x' \\
&\quad +\frac{1}{\|x\|} \nabla \int_{\partial T_{\Theta t}} h(g_{\Theta t}^{-1}(x')) \rho(g_{\Theta t}^{-1}(x'))\left(\frac{1}{\|x-x'\|}-\frac{1}{\|x'\|}\right) d\sigma(x') \\
&\quad -\frac{1}{\|x\|} \nabla \int_{\partial T_0} h(g_0^{-1}(x')) \rho(g_0^{-1}(x'))\left(\frac{1}{\|x-x'\|}-\frac{1}{\|x'\|}\right) d\sigma(x').
\end{aligned}$$

With the propositions (B.2) and (B.3) this sum converges uniformly to 0.

The integration of equation (B.5) yields the uniform convergence

$$\frac{f(t, x)-f(0, x)}{t} \to \partial_t f(0, x). \tag{B.6}$$

This equation and the continuity of $\partial_t f(0, x)$ lead to the uniform convergence of

$$\frac{f(t, g_t(x)) - f(0, g_t(x))}{t} \to \partial_t f(0, g_0(x)). \tag{B.7}$$

It holds that

$$\begin{aligned}
&\left\| \frac{\nabla f(t, g_t(x)) - \nabla f(0, g_t(x))}{\|x\| t} - \frac{\nabla \partial_t f(0, g_0(x))}{\|x\|} \right\| \\
&\leqq \left\| \frac{\nabla f(t, g_t(x)) - \nabla f(0, g_t(x))}{\|x\| t} - \frac{\nabla \partial_t f(0, g_t(x))}{\|x\|} \right\| \\
&\quad + \left\| \frac{\nabla \partial_t f(0, g_t(x))}{\|x\|} - \frac{\nabla \partial_t f(0, g_0(x))}{\|x\|} \right\|.
\end{aligned}$$

With equation (B.5) and $\frac{\|g_\zeta\|}{\|x\|} \leqq (1+Z)$ it follows that the first part converges uniformly to 0. Proposition (A.1) and the properties of the Newtonian potential (A.7) imply that there is a constant M such that for v with $\|v\|=1$ it holds that

$$\|\nabla \partial_t f(0, x + t\|x\| v) - \nabla \partial_t f(0, x)\| \leqq M \|x\| t^\nu.$$

Thus, the second part can be estimated as follows:

$$\begin{aligned}
\frac{\nabla \partial_t f(0, g_t(x)) - \nabla \partial_t f(0, g_0(x))}{\|x\|} &\leqq M t^\nu \left(\frac{\|h\|_\infty}{\|x\|} \right)^\nu \\
&\leqq M t^\nu (\|h\|_{B_1})^\nu \to 0.
\end{aligned}$$

This completes the proof of equation (B.2).

B.3. THE CONTINUITY OF THE DERIVATIVE

In this section it will be shown that the map

$$B_1 \to L(B_1, B_2),\ \zeta \to D_\zeta \Phi(\zeta)$$

is continuous. Here $L(B_1, B_2)$ denotes the space of all bounded linear maps $B_1 \to B_2$.

With the propositions (B.2) and (B.3) it follows that the map $B_1 \to L(B_1, B_2)$,

$$\begin{aligned}
\zeta \to \int_{T_\zeta} &- h(g_\zeta^{-1}(x'))(n(g_\zeta^{-1}(x'))\nabla \rho_\zeta(x')) \\
&\times \left(\frac{1}{\|g_\zeta(x) - x'\|} - \frac{1}{\|x'\|} \right) d^3 x' \\
+ \int_{\partial T_\zeta} & h(g_\zeta^{-1}(x'))\rho_\zeta(x') \left(\frac{1}{\|g_\zeta(x) - x'\|} - \frac{1}{\|x'\|} \right) d\sigma_\zeta(x')
\end{aligned}$$

is continuous. It remains to prove the continuity of the map $B_1 \to L(B_1, B_2)$,

$$\zeta \to \left[n(x) \int_{T_\zeta} -\rho_\zeta(x') \frac{g_\zeta(x)-x'}{\| g_\zeta(x)-x' \|^3} d^3 x' \right] h(x). \tag{B.8}$$

Choose any $h \in B_1$ with $\| h \|_{B_1} \leq 1$ and any sequence $\zeta_i (i \in \mathbb{N})$ which converges to ζ, i.e. that $\| \zeta_i - \zeta \|_{B_1} \to 0$ for $i \to \infty$. Define the functions

$$f_j(\zeta_i, x) = \int_{T_{\zeta_i}} -\rho(g_i^{-1}(x')) \frac{x_j - x_j'}{\| x-x' \|^3} d^3 x'.$$

With the inequalities (A.4) and (A.5) it follows that the sequences $f_j(\zeta_i, x)$ and $\nabla f_j(\zeta_i, x)$ are equicontinuous. Because they converge pointwise on a dense subset of a ball which contains all $g_{\zeta i}(T)$, they converge uniformly (7) to $f_j(\zeta, x)$ and $\nabla f_j(\zeta, x)$. Because $f_j(\zeta_i, 0) = 0$, an integration yields the uniform convergence of

$$\frac{f_j(\zeta_i, x)}{\| x \|} \to \frac{f_j(\zeta, x)}{\| x \|}.$$

Since

$$n(x) \left[\int_{T_\zeta} -\rho_\zeta(x') \frac{g_\zeta(x)-x'}{\| g_\zeta(x)-x' \|^3} d^3 x' - \int_{T_{\zeta_i}} -\rho_{\zeta_i}(x') \frac{g_{\zeta_i}(x)-x'}{\| g_{\zeta_i}(x)-x' \|^3} d^3 x' \right] h(x)$$
$$= n_j(x) [f_j(\zeta, g_\zeta(x)) - f_j(\zeta_i, g_{\zeta_i}(x))] h(x),$$

the uniform convergence of

$$| h(x) n_j(x) [f_j(\zeta_i, g_{\zeta_i}(x)) - f_j(\zeta, g_\zeta(x))] | \to 0 \tag{B.9}$$

and of

$$\left\| \frac{\nabla [h(x) n_j(x) (f_j(\zeta_i, g_{\zeta_i}(x)) - f_j(\zeta, g_\zeta(x)))]}{\| x \|} \right\| \to 0 \tag{B.10}$$

remain to be proven in order to prove equation (B.8).

The uniform convergence of $f_j(\zeta_i, x)$ and the differentiability of $f_i(\zeta, x)$ lead to equation (B.9). Equation (B.10) will be proven in components. It

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(7) *See* [11], proposition 67.4.

holds that

$$\left|\frac{\partial_{x_l}[h(x)\,n_j(x)\,(f_j(\zeta_i,\,g_{\zeta_i}(x))-f_j(\zeta,\,g_\zeta(x)))]}{\|x\|}\right|$$
$$\leqq|\partial_{x_l}[h(x)\,n_j(x)]|\left|\frac{f_j(\zeta_i,\,g_{\zeta_i}(x))-f_j(\zeta,\,g_\zeta(x))]}{\|x\|}\right|$$
$$+\left|\frac{h(x)\,n_j(x)}{\|x\|}\right||\partial_{x_l}[f_j(\zeta_i,\,g_{\zeta_i}(x))-f_j(\zeta,\,g_\zeta(x))]|.$$

Because $f_j(\zeta,\,x)$ is differentiable, there is a constant A such that

$$\left|\frac{f_j(\zeta,\,g_{\zeta_i}(x))-f_j(\zeta,\,g_\zeta(x))}{\|x\|}\right|$$
$$\leqq \mathrm{A}\,\frac{\|g_{\zeta_i}(x))-g_\zeta(x))\|}{\|x\|}\leqq A\,\|\zeta_i-\zeta\|_{B_1}.$$

Furthermore, all $\partial_{x_l}[h(x)\,n_j x)]$ are bounded functions and $\dfrac{f_j(\zeta_i,\,g_{\zeta_i}(x))}{\|x\|}$ converges uniformly to $\dfrac{f_j(\zeta,\,g_{\zeta_i}(x))}{\|x\|}$. Thus,

$$|\partial_{x_l}[h(x)\,n_j(x)]|\left|\frac{f_j(\zeta_i,\,g_{\zeta_i}(x))-f_j(\zeta,\,g_\zeta(x))}{\|x\|}\right|$$

converges uniformly to 0. Because all $\left|\dfrac{h(x)\,n_j(x)}{\|x\|}\right|$ are bounded functions, the estimate

$$|\partial_{x_l}[f_j(\zeta_i,\,g_{\zeta_i}(x))-f_j(\zeta,\,g_\zeta(x))]|$$
$$=|\partial_{x_k}f_j(\zeta_i,\,g_{\zeta_i}(x))\,(D\,g_{\zeta_i}(x))_{lk}-\partial_{x_k}f_j(\zeta,\,g_\zeta(x))\,(D\,g_\zeta(x))_{lk}|$$
$$\leqq|[\partial_{x_k}f_j(\zeta_i,\,g_{\zeta_i}(x))-\partial_{x_k}f_j(\zeta,\,g_\zeta(x))]\,(D\,g_{\zeta_i}(x))_{lk}|$$
$$+|\partial_{x_k}f_j\,(\zeta,\,g_\zeta(x))\,[(D\,g_{\zeta_i}(x))_{lk}-(D\,g_\zeta(x))_{lk}]|$$
$$\to 0$$

leads to the uniform convergence of

$$\left|\frac{\partial_{x_l}[h(x)\,n_j(x)\,(f_j(\zeta_i,\,g_{\zeta_i}(x))-f_j(\zeta,\,g_\zeta(x)))]}{\|x\|}\right|\to 0.$$

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