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Distribution near the real axis of scattering poles generated by a non-hyperbolic periodic ray

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ABSTRACT. – We prove lower bounds in small neighborhoods of the real axis on the number of scattering poles for a trapping obstacle with unique periodic non-hyperbolic ray. The periodic ray is such that all eigenvalues of the corresponding Poincaré map are equal to one.

RÉSUMÉ. – Nous prouvons une borne inférieure pour le nombre de pôles de diffusion dans un petit voisinage de l'axe réel pour un piège avec un seul rayon périodique non hyperbolique. Ce rayon périodique est tel que les valeurs propres de l'application de Poincaré correspondante sont toutes égales à un.

1. INTRODUCTION

In this note we consider scattering by two disjoint convex obstacles Q_1 and Q_2, which admit only one periodic ray, such that all eigenvalues of the corresponding Poincaré map are equal to one.

(1) Grants MM1/91 and MM3/91.

Denote by $S(z)$ the scattering matrix [6] for the following acoustic boundary value problem:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(x, t) - \Delta u(x, t) = 0 \quad \text{in } \Omega \times (-\infty, +\infty), \\ u(x, t) = 0 \quad \text{on } \Gamma \times (-\infty, +\infty), \\ u(x, 0) = f(x) \quad \text{in } \Omega, \\ \dfrac{\partial}{\partial t} u(x, 0) = 0 \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega = \mathbb{R}^3 \setminus \{Q_1 \cup Q_2\}$, $\Delta = \sum_{j=1}^{3} \partial^2/\partial x_j^2$ and $\Gamma = \partial\Omega$ is smooth. It is well known [6] that $S(z)$ is an operator valued function, meromorphic in the whole plane and analytic in $\{z \in \mathbb{C} : \operatorname{Im} z \leqq 0\}$. We define the *scattering poles* to be the poles of $S(z)$.

We consider the same geometric situation as in [4], where M. Ikawa proved that if $\beta > 0$ is sufficiently small, then for every $\alpha > 0$, the regions

$$G_{\alpha, \beta} = \{z \in \mathbb{C};\ \operatorname{Im} z \leqq \alpha (|\operatorname{Re} z| + 1)^{-\beta}\} \tag{1.2}$$

contain an infinite number of poles of $S(z)$ (Theorem 2 from [4]).

The purpose of this work is to obtain information about the density of the scattering poles in any of the regions $G_{\alpha, \beta}$. To our best knowledge, there are only two works, which consider a unique periodic ray, such that all eigenvalues of the corresponding Poincaré map are equal to one. The first one is the above mentioned work of Ikawa [4]. The second one [5] is also by Ikawa and it refers to a special case of two convex obstacles in $\mathbb{R}^2$. This work shows, that for such a special obstacle, the poles of the scattering matrix near the real axis accumulate to the "pseudo-poles" $(q\pi)/d$, $q \in \mathbb{Z}$, where d is the distance between the two bodies. The result of the present work shows that there are too many poles of $S(z)$ near the "pseudo-poles", in contrast to the case of two disjoint strictly convex bodies ([2], [3]), for which, roughly speaking, one pseudo-pole [1] correspond to one actual scattering pole.

In the work of Sjöstrand and Zworski [8] it is shown how some knowledge about the singularities of the trace distribution [1], allows us to obtain lower bounds on the number of poles in logarithmic neighborhoods $\{z \in \mathbb{C} : 0 \leqq \operatorname{Im} z \leqq C \ln |z|\}$ of the real axis. Their result proves in some cases, for which it is possible to investigate the singularities of the trace, a weaker version of the Modified Lax and Phillips Conjecture.

Our method is based on a careful study of the singularities of a specific representation of the trace [4]. After that we use a suitable formulation of part of the result from [8] in order to count the poles of $S(z)$ in the regions $G_{\alpha\beta}$.

The paper is organized as follows. In Section 2 we describe the geometry of the scattering obstacle and formulate the main theorem. We also state some preliminary notations used in the sequel and prove a modified version of the result from [8]. Section 3 is devoted to the investigation of the singularities of the trace and in Section 4 we prove the main theorem.

2. MAIN THEOREM, PRELIMINARY NOTATIONS AND RESULTS

We consider the same geometry as in [4]. Namely, let Q_1 and Q_2 be convex sets in $\mathbb{R}^3$ with sufficiently smooth boundaries Γ_1 and Γ_2. Let $a_j \in \Gamma_j$, $j=1, 2$ be such points that $d=|a_1-a_2|=\operatorname{dist}(Q_1, Q_2)$. We suppose that the principal curvatures $k_{jl}(x)$, $l=1, 2$ of Γ_j at $x \in \Gamma_j$ satisfy for $j=1, 2$ the estimates

$$C|x-a_j|^{s-2} \geqq k_{jl}(x) \geqq C^{-1}|x-a_j|^{s-2}, \qquad x \in \Gamma_j, \tag{2.1}$$

for some $C>0$ and $\infty > s \geqq 4$.

We shall use the following counting function

$$N_{\alpha, \beta}(r) = \#\{\lambda_j \in G_{\alpha, \beta} : |\operatorname{Re} \lambda_j| \leqq r\},$$

where λ_j, $j=1, 2, \ldots$ are the poles of the scattering matrix associated with (1.1).

Now we state the main result of this work.

THEOREM 2.1. — *Suppose that the principal curvatures of* Γ_1 *and* Γ_2 *satisfy* (2.1). *Then there exists* $\beta_0>0$ *such that for any* β *and* β' *such that* $0<\beta<\beta'<\beta_0$ *and any* $\alpha>0$ *the following estimate holds*:

$$N_{\alpha, \beta}(r) > C_{\alpha, \beta, \beta'} r^{2(1-1/s)-\beta'(1+2/s)},$$

where the constant $C_{\alpha, \beta, \beta'}$ *is positive and independent of* r.

Remark 2.2. — Note that for any $\nu>0$ we may choose $\beta_0>0$ so small that the order $\delta=2(1-1/s)-\beta'(1+2/s)$ of the lower bound satisfies the estimates:

$$1<3/2-\nu<\delta<2.$$

Note also, that if the "flatness" of the bodies near the periodic ray increases ($s \to \infty$), and $\beta' \to 0$, then δ tends to 2.

A starting point for the proof of Theorem 2.1 is the trace formula of Bardos-Guillot-Ralston [1] and Melrose [7].

$$u(t)=\mathrm{Tr}_{L^2(\mathbb{R}^3)}(\cos t\sqrt{-\Delta}-\cos t\sqrt{-\Delta_0})=\begin{cases}\dfrac{1}{2}\sum\limits_{\text{poles}} e^{i\lambda_j t}, & t>0;\\[2ex] \dfrac{1}{2}\sum\limits_{\text{poles}} e^{-i\lambda_j t}, & t<0,\end{cases}\tag{2.2}$$

which holds in the sense of distributions. Here Δ is the Dirichlet Laplacian in $L^2(\Omega)$ and Δ_0 is the selfadjoint Laplacian in $L^2(\mathbb{R}^3)$. We shall apply $u(t)$ to functions $e^{i\lambda t}\varphi(t)$, $\varphi\in C_0^\infty(\mathbb{R}^+)$, in order to study the behaviour of $|\widehat{\varphi u}(\lambda)|$ for $\lambda\to\infty$.

In what follows we fix a function $\varphi_0(t)\in C_0^\infty(-1,1)$, such that:

$$0\leqq\varphi_0(t),\qquad \varphi_0(0)=1/(2\pi),\qquad \hat\varphi_0(-\lambda)=\hat\varphi_0(\lambda)\geqq 0\qquad\text{for }\lambda\in\mathbb{R}\tag{2.3}$$

and we pose $\varphi_{\gamma,g}(t)=\varphi_0(\gamma^{-1}(t-g))$.

Let Λ be the set of all scattering poles. We define (*see* [8])

$$\Lambda_\rho=\{\lambda_j\in\Lambda:\operatorname{Im}\lambda_j\leqq\rho\ln|\lambda_j|\};\qquad N_\rho(r)=\#\{\lambda_j\in\Lambda_\rho:|\operatorname{Re}\lambda_j|\leqq r\}.$$

In the rest of this section we follow [8], in order to prove

PROPOSITION 2.3. – *Let $g>0$ and $1>\gamma>0$. Then for any $r>2$ and $-1\neq k\in\mathbb{R}$ we have*:

$$N_\rho(r)>C\int_1^{r/2}|\widehat{(\varphi_{\gamma,g}u)}(\lambda)|\,d\lambda-C_{a,k}\gamma^{-a}r^{k+1}-C_b\gamma^{-b}r^{4-b}-C_{p,\delta}\gamma^{-p}r^{3-\delta p},$$

where $a>k$, $b, p>3$, $\delta\in(0,1)$ and $\rho>(3-k)/(g-\gamma)$ are arbitrary and the positive constants C, $C_{a,k}$, C_b and $C_{p,\delta}$ do not depend on γ, g and r.

Proof. – First we note that (*see* (3) and (4) from [8]):

$$\widehat{(\varphi_{\gamma,g}u)}(\lambda)=\sum_{\lambda_j\in\Lambda}\hat\varphi_{\gamma,g}(\lambda-\lambda_j),\qquad\lambda\in\mathbb{R};\tag{2.4}$$

$$\begin{cases}|\widehat{(\varphi_{\gamma,g}u)}(\xi)|\leqq\gamma C_{M_1}e^{(g\pm\gamma)\operatorname{Im}\xi}(1+|\gamma\xi|)^{-M_1},\\ \qquad\pm\operatorname{Im}\xi\geqq0,\quad\text{for all }M_1,\end{cases}\tag{2.5}$$

where $C_{M_1}>0$ is independent of g and γ. Using (2.5) we get for $\gamma<1$

$$\begin{aligned}\Big|\sum_{\lambda_j\in\Lambda\setminus\Lambda_\rho,\,|\lambda_j|\geqq1}\hat\varphi_{\gamma,g}(\lambda-\lambda_j)\Big|&\leqq\gamma C_{M_1}\sum_{\lambda_j\in\Lambda\setminus\Lambda_\rho,\,|\lambda_j|\geqq1}e^{-(g-\gamma)\operatorname{Im}\lambda_j}(1+\gamma|\lambda-\lambda_j|)^{-M_1}\\&\leqq C_{M_1}\gamma^{1-M_1}\sum_{\lambda_j\in\Lambda\setminus\Lambda_\rho,\,|\lambda_j|\geqq1}\frac{|\lambda_j|^{-(g-\gamma)\rho}}{(1+|\lambda-\lambda_j|)^{M_1}}\end{aligned}$$

Then following [8] we get for any $k\in\mathbb{R}$ and $M_1>k+1$:

$$\Big|\sum_{\lambda_j\in\Lambda\setminus\Lambda_\rho,\ |\lambda_j|\geqq 1}\hat{\varphi}_{\gamma,g}(\lambda-\lambda_j)\Big|\leqq C_{M_1,k}\gamma^{1-M_1}\lambda^k,\qquad \text{for}\quad \rho>\frac{3-k}{g-\gamma},\tag{2.6}$$

where $C_{M_1,k}$ is independent of γ, g and λ.

We introduce a measure $\mu(dr)$ and a maximal function ψ for $\varepsilon>0$ (*see* [8]):

$$\mu(dr)=\sum_{\lambda_j\in\Lambda_\rho}e^{-(g-\varepsilon)\operatorname{Im}\lambda_j}\delta(r-\operatorname{Re}\lambda_j)\,dr;\ \psi(\tau)=\sup_{\sigma\leqq 0}\big|e^{-(g-\varepsilon)\sigma}\hat{\varphi}_{\gamma,g}(\tau+i\sigma)\big|.$$

Note that $\mu(r)=\mathcal{O}(r^3)$ and since $\psi(\tau)=\gamma\sup_{\sigma\leqq 0}|e^{\varepsilon\sigma}\hat{\varphi}_0(\gamma[\tau+i\sigma])|$ the function $\psi(\tau)$ is independent of g. The properties of $\psi(\tau)$ found in [8] ((6) from [8]) remain valid:

$$\left\{\begin{aligned}&\psi(\tau)=\gamma\hat{\varphi}_0(\gamma\tau)+\frac{\gamma^2\mathcal{O}_M(\langle\gamma\tau\rangle^{-M})}{\varepsilon-\gamma},\\&\int_{-\infty}^{r}\psi(\tau)\,d\tau=(1+\mathcal{O}(\gamma^{1/2}))Y_+(r)+\mathcal{O}_M(\langle\gamma\tau\rangle^{-M}),\end{aligned}\right.\tag{2.7}$$

where $Y_+(r)$ denotes the Heaviside function supported on $\mathbb{R}_+$ and $\gamma<\varepsilon^2$.

Using the definition of $\psi(\tau)$, (2.4) and (2.6) we get for $\rho>(3-k)/(g-\gamma)$

$$\begin{aligned}|\widehat{(\varphi_{\gamma,g}u)}(\lambda)|\leqq\Big|\sum_{\lambda_j\in\Lambda_\rho}\hat{\varphi}_{\gamma,g}(\lambda_j-\lambda)\Big|+\Big|\sum_{\lambda_j\in\Lambda\setminus\Lambda_\rho}\hat{\varphi}_{\gamma,g}(\lambda_j-\lambda)\Big|\leqq C_{M_1,k}\gamma^{1-M_1}\lambda^k\\+\int_{-\infty}^{0}\psi(\lambda-\tau)\,\mu(d\tau)+\int_{0}^{\infty}\psi(\lambda-\tau)\,\mu(d\tau).\end{aligned}\tag{2.8}$$

With the help of (2.7) we estimate the first integral for $\lambda>0$ and $0<\gamma<1$:

$$\begin{aligned}\left|\int_{-\infty}^{0}\psi(\lambda-\tau)\,\mu(d\tau)\right|&\leqq\sum_{\lambda_j\in\Lambda_\rho,\ \operatorname{Re}\lambda_j<0}\psi(\lambda-\operatorname{Re}\lambda_j)\\&\leqq C_{M_2}\gamma\sum_{\lambda_j\in\Lambda_\rho,\ \operatorname{Re}\lambda_j<0}(1+\gamma|\lambda-\operatorname{Re}\lambda_j|)^{-M_2}\\&\leqq C_{M_2}\gamma^{1-M_2}\sum_{\lambda_j\in\Lambda_\rho,\ \operatorname{Re}\lambda_j<0}(1+|\lambda-\operatorname{Re}\lambda_j|)^{-M_2}\\&\leqq C_{M_2}\gamma^{1-M_2}\int_{-\infty}^{0}\frac{dN^-(\tau)}{(1+|\lambda-\tau|)^{-M_2}},\end{aligned}$$

where $N^-(r)=\#\{\lambda_j\in\Lambda_\rho:-r<\operatorname{Re}\lambda_j<0\}=\mathcal{O}(r^3)$. Then we get for any $M_2>4$ and $\lambda>0$:

$$\left|\int_{-\infty}^{0}\psi(\lambda-\tau)\,\mu(d\tau)\right|\leqq C_{M_2}\gamma^{1-M_2}\lambda^{4-M_2},\tag{2.9}$$

where C_{M_2} is independent of γ, g and λ. Hence from (2.8) and (2.9) we get for any $M_1 > k+1$ and $M_2 > 4$:

$$|\widehat{\varphi_{\gamma, g} u}(\lambda)| \leqq \int \psi(\lambda-\tau)\, \mu^+(d\tau) + C_{M_1, k}\, \gamma^{1-M_1} \lambda^k + C_{M_2}\, \gamma^{1-M_2} \lambda^{4-M_2},$$

where $C_{M_1, k}$ and C_{M_2} are independent of γ, g and λ and $\mu^+ = \mu|_{\mathbb{R}_+}$. Thus, for $k \neq -1$ and $r > 1$, we have:

$$\int_1^r |\widehat{\varphi_{\gamma, g} u}(\lambda)|\, d\lambda - C_{M_1, k}\, \gamma^{1-M_1} r^{k+1} - C_{M_2}\, \gamma^{1-M_2} r^{5-M_2}$$
$$\leqq \int_{-\infty}^r \int \psi(\lambda-\tau)\, \mu^+(d\tau) = \int \psi(r-\tau)\, M^+(\tau)\, d\tau, \quad (2.10)$$

where $M^+(\tau) = \mu^+([0, \tau])$.

Using again the bound on the counting measure and (2.7) we get for any $\delta \in (0, 1)$ and for any $M_3 > 4$

$$\left| \int \psi(r-\tau)\, M^+(\tau)\, d\tau \right| \leqq \int_{-\infty}^{r-r^\delta} \frac{\gamma C_{M_3} |\tau|^3}{[1+\gamma(r-\tau)]^{M_3}}\, d\tau \leqq C_{M_3, \delta}\, \gamma^{1-M_3} r^{3+\delta(1-M_3)},$$

where $C_{M_3, \delta}$ is independent of γ, g and r. Since an analogous estimate holds for the integral $\int_{r+r^\delta}^{\infty} \psi(r-\tau)\, M^+(\tau)\, d\tau$ we get:

$$\left| \int_{\mathbb{R} \setminus [r-r^\delta,\, r+r^\delta]} \psi(r-\tau)\, M^+(\tau)\, d\tau \right| \leqq C_{M_3, \delta}\, \gamma^{1-M_3} r^{3-\delta(M_3-1)}, \quad (2.11)$$

where $C_{M_3, \delta}$ is independent of γ, r and g and $M_3 > 4$, $\delta \in (0, 1)$ are arbitrary. Using (2.10), (2.11) and (2.7) we obtain:

$$\int_1^r |\widehat{\varphi_{\gamma, g} u}(\lambda)|\, d\lambda - C_{M_1, k}\, \gamma^{1-M_1} r^{k+1} - C_{M_2}\, \gamma^{1-M_2} r^{5-M_2}$$
$$- C_{M_3, \delta}\, \gamma^{1-M_3} r^{3-\delta(M_3-1)}$$
$$< \int_{r-r^\delta}^{r+r^\delta} \psi(r-\tau)\, M^+(\tau)\, d\tau < M^+(r+r^\delta) \int_{-r^\delta}^{+r^\delta} \psi(\xi)\, d\xi < CM^+(2r),$$

where the positive constant C depends only on φ_0. The last inequality completes the proof. □

For the particular case considered in the present work we shall use the following corollary from Proposition 2.3 (we pose $k = -3/2$, $a = 1$, $b = p = 4$, $\delta = 3/4$, $\rho = 5/g$).

COROLLARY 2.4. — *Let* $10<g\in\mathbb{R}$ *be arbitrary. Then for any* $1>\gamma>0$ *and* $r>2$ *we have the estimate*:

$$\mathrm{N}_{5/g}(r)>\mathrm{C}\int_1^{r/2}|\widehat{\varphi_{\gamma,g}u}(\lambda)|\,d\lambda-\mathrm{C}'\gamma^{-4},$$

where C *and* C′ *are positive constants which do not depend on* g, r *and* γ.

3. SINGULARITIES OF THE TRACE

We denote for simplicity

$$\varphi_q(t)=\varphi_{q^{-l},\,2dq}(t).$$

This section is devoted to the proof of the following:

PROPOSITION 3.1. — *There exist constants* l_1, $l'>0$ *such that for* $l\geqq l'$, *any integer* $q>0$ *and any* $\varepsilon>0$ *we have*

$$|\widehat{\varphi_q u}(\lambda)|>\begin{cases}\mathrm{C}_0\,q^{-1-2/s}\lambda^{1-2/s}-\mathrm{C}_{\varepsilon,l}\,q^{l_1}\lambda^{1-(3-\varepsilon)/s} & \text{for}\quad \lambda\geqq q^l\\ \mathrm{C}_0\,q^{-1-2/s}\lambda^{1-2/s}-\mathrm{C}_{\varepsilon,l}\,q^{l_1}q^{l[1-(3-\varepsilon)/s]} & \text{for}\quad 1<\lambda\leqq q^l.\end{cases}$$

Here $\mathrm{C}_0>0$ *is independent of* q, λ, l *and* ε *and* $\mathrm{C}_{\varepsilon,l}>0$ *is independent of* q *and* λ.

Proof. — First we recall some results by Ikawa [4]. Note that from (2.2) it follows that

$$\widehat{\varphi_q u}(\lambda)=\mathrm{Tr}_{\mathrm{L}^2(\mathbb{R}^3)}\int_{\mathbb{R}}e^{i\lambda t}\varphi_q(t)\,(\cos t\sqrt{-\Delta}-\cos t\sqrt{-\Delta_0})\,dt.$$

In the sequel we suppose that the coordinate system is such that $a_1=(0,0,0)$ and $a_2=(0,0,d)$. In [4] it is proved (*see* (3.8) from [4]) that if $\psi(y)\in\mathrm{C}_0^\infty(\Omega)$ the kernel distribution $\mathrm{E}(t,x,y)\psi(y)$ of $\cos t\sqrt{-\Delta}\,\psi$ is given by

$$\mathrm{E}(t,x,y)\psi(y)=\int_{\mathrm{S}^2}d\omega\int_0^\infty k^2\,dk\,u(t,x,k,\omega)\,e^{-ik\langle y,\omega\rangle}\psi(y),$$

where S^2 is the unit sphere in $\mathbb{R}^3$ and $u(t,x,k,\omega)$ is a solution of (1.1) for $f(x)=\omega(x)\,e^{ik\langle x,\omega\rangle}$ ($\omega(x)\in\mathrm{C}_0^\infty(\Omega)$ and $\omega(x)=1$ on $\operatorname{supp}\psi$).

In [4], a geometric optics approximation $u^{(\mathrm{N})}$ of u is built, by means of which it is shown (*see* Lemma 3.2 from [4]) that for some $\mathrm{N}\geqq 4$ and $l'>0$ we have for any $l>l'$:

$$\left|\int_{\mathbb{R}}\int_{\Omega}\mathrm{E}(t,x,x)\psi(x)\varphi_q(t)e^{i\lambda t}\,dt\,dx-(\mathrm{I}_++\mathrm{I}_-)\right|\leqq\mathrm{C}_{l,\mathrm{N}},\qquad(3.1)$$

where the constant $C_{l,N}$ is independent of λ and q. For I_+ and I_- we have (*see* (3.22) from [4]) for any integer q and for any $l>l'$

$$\left| I_{+(-)} - \int_1^\infty k^2\,dk \sum_{j=0}^{N} (ik)^{-j} \int dt \int_{S^2} d\omega \int_\Omega dx\, e^{ik\,\Phi_{2q}(x,\omega)} \right.$$
$$\left. \times v_{2q,j}(t, x, \omega)\,\psi(x)\, e^{it(\lambda-k)}\,\varphi_q(t) \right| < C, \quad (3.2)$$

where C is a constant independent of λ and q; $\Phi_r = \phi_r(x, \omega) - \langle x, \omega \rangle$; $\phi_r(x, \omega)$ and $v_{r,j}(t, x, \omega)$ are phase functions and solutions of transport equations, appearing in the asymptotic solution $u^{(N)}$. In what follows we fix $N=4$.

We denote

$$J_{q,j}(x_3, t, k) = \int_{S^2} d\omega \int_{\mathbb{R}^2} dx'\, e^{ik\,\Phi_{2q}(x,\omega)}\, v_{2q,j}(t, x, \omega)\,\psi(x)$$

and state Proposition 3.3 from [4]:

PROPOSITION 3.2.

$$\left| J_{q,j} - k^{-1-2/s} e^{i\,2\,dqk} \left\{ c^0_{q,j}(x_3, t) + \sum_{h=1}^{[3s/2]} \sum_{m=1}^{m_h} c^{h,m}_{q,j}(x_3, t)\, k^{-h/s} (\log k)^{m-1} \right\} \right|$$
$$\leqq C q^{l_1} k^{-4},$$

where l_1 *is a constant,* $c^{h,m}_{q,j}(x_3, t)$ *are determined by* Φ_{2q} *and* $v_{2q,j}$ *and they satisfy*

$$\sum_{l=0}^{2} \left| \partial_t^l c^{h,m}_{q,j}(x_3, t) \right| < C q^{l_1} \quad \textit{for all } x_3 \in (0, d) \quad \textit{and} \quad t>0,$$

where C *is independent of* q *and especially*

$$c^0_{q,j}(x_3, t) = c v_{q,j}(t, 0, x_3; 0, 0, 1)\,\psi(0, 0, x_3)\, q^{-1-2/s}$$

for some fixed non zero constant c *determined by the shape of the obstacle near* a_1 *and* a_2.

From Proposition 3.2, (3.1) and (3.2), we get:

$$\left| \int_{\mathbb{R}} \int_\Omega E(t, x, x)\,\psi(x)\,\varphi_q(t)\, e^{i\lambda t}\, dt\, dx - I^{(\psi)}_{0,0}(\lambda) \right|$$
$$\leqq \left| \sum_{j=0}^{4} \left(I^{(\psi)}_{0,j+1} + \sum_{h=1}^{[3s/2]} \sum_{m=1}^{m_h} I^{(\psi)}_{h,m,j}(\lambda) \right) \right| + C_l q^{l_1}$$

Here $I^{(\psi)}_{0,5} = 0$ and we use the notations:

$$I^{(\psi)}_{0,j}(\lambda) = \int_0^d dx_3 \int_1^\infty dk \int dt\, c^0_{q,j}(x_3, t)\, e^{-i(k-\lambda)t}\, k^{1-2/s}\, e^{ik2dq}\, \varphi_q(t),$$

$$I^{(\psi)}_{h,m,j} = \int_0^d dx_3 \int_1^\infty dk \int dt\, c^{h,m}_{q,j}(x_3, t)\, e^{-i(k-\lambda)t}\, k^{1-j-(2+h)/s} (\ln k)^{m-1}\, e^{ik2dq}\, \varphi_q(t).$$

We use Lemma 3.4 and Lemma 3.5 from [4] to remove the condition on the support of $\psi(x)$ and using a partition of the unity over Ω we obtain:

$$|\widehat{\varphi_q u}(\lambda)-\sum_{\psi} \mathrm{I}_{0,0}^{(\psi)}|$$

$$\leqq\left|\sum_{\psi}\left[\sum_{j=0}^{4}\left(\mathrm{I}_{0,j+1}^{(\psi)}+\sum_{h=1}^{[3s/2]}\sum_{m=1}^{m_h}\mathrm{I}_{h,m,j}^{(\psi)}(\lambda)\right)\right]\right|+\mathrm{C}_l q^{l_1}. \quad (3.3)$$

In the sequel we shall omit the index "(ψ)".

Obviously, the proof of Proposition 3.1 follows from (3.3) and from:

LEMMA 3.3. – *The integrals* $\mathrm{I}_{0,j}$ *and* $\mathrm{I}_{h,m,j}$, $h, m\geqq 1$, $j\geqq 0$ *satisfy the estimates for any* $\varepsilon>0$:

$$|\mathrm{I}_{0,0}(\lambda)|\geqq \mathrm{C}_0 q^{-1-2/s}\lambda^{1-2/s}, \qquad \lambda>1,$$

$$|\mathrm{I}_{0,j}(\lambda)|\leqq \mathrm{C}_j q^{l_1}, \qquad \lambda>1, \qquad j=1,2,3,4,$$

$$|\mathrm{I}_{h,m,j}(\lambda)|\leqq\begin{cases}\mathrm{C}_\varepsilon q^{l_1}\lambda^{1-(3-\varepsilon)/s} & \text{for } \lambda\geqq q^l,\\ \mathrm{C}_\varepsilon q^{l_1} q^{l[1-(3-\varepsilon)/s]} & \text{for } 1<\lambda\leqq q^l,\end{cases}$$

where $\mathrm{C}_0, \ldots, \mathrm{C}_4>0$ *are independent of* q, λ, l *and* ε *and* $\mathrm{C}_\varepsilon>0$ *is independent of* q, l *and* λ.

Proof of Lemma 3.3. – First note that from the reasoning prior to (3.25) from [4] it follows that:

$$\int_0^d c_{q,0}^0(x_3, t)\,dx_3=c' q^{-1-2/s} \qquad \text{for} \quad t\in\operatorname{supp}\varphi_q,$$

where $c'>0$ is independent of t and q.

Then we estimate $\mathrm{I}_{0,0}(\lambda)$ for $\lambda>1$ with the aid of (2.3):

$$|\mathrm{I}_{0,0}(\lambda)|=c'\left|q^{-1-2/s}\int_1^\infty k^{1-2/s}\hat{\varphi}_0\left(\frac{\lambda-k}{q^l}\right)q^{-l}e^{i\lambda 2dq}\,dk\right|$$

$$=c' q^{-1-2/s}\int_1^\infty k^{1-2/s}\hat{\varphi}_0\left(\frac{\lambda-k}{q^l}\right)q^{-l}\,dk$$

$$=c' q^{-1-2/s}q^{l(1-2/s)}\int_{-\infty}^{(\lambda-1)q^{-l}}(-\tau+\lambda q^{-l})^{1-2/s}\hat{\varphi}_0(\tau)\,d\tau$$

$$\geqq c' q^{-1-2/s}q^{l(1-2/s)}\int_{-1}^{0}(-\tau+\lambda q^{-l})^{1-2/s}\hat{\varphi}_0(\tau)\,d\tau\geqq \mathrm{C}_0 q^{-1-2/s}\lambda^{1-2/s},$$

where $\mathrm{C}_0>0$ is independent of q, λ and l.

Next we consider the integrals $\mathrm{I}_{h,m,j}(\lambda)$ for $h, m\geqq 1$ and $j\geqq 0$. We have

$$\mathrm{I}_{h,m,j}(\lambda)=\int_0^d\int_1^\infty e^{ik2dq}(k^{1-j-(2+h)/s}(\ln k)^{m-1}\,\mathrm{J}_{h,m,j}(x_3, k, \lambda, q))\,dk\,dx_3,$$

where we denote

$$J_{h,m,j}(x_3, k, \lambda, q) = \int e^{i(\lambda-k)t} c_{q,j}^{h,m}(x_3, t)\,\varphi_q(t)\,dt.$$

For this integral we have:

$$\begin{aligned} J_{h,m,j}(x_3, k, \lambda, q) &= \int e^{i(\lambda-k)t} c_{q,j}^{h,m}(x_3, t)\,\varphi_0(q^l(t-2dq))\,dt \\ &= \int e^{iq^{-l}(\lambda-k)q^l(t-2dq)} e^{i(\lambda-k)2dq} c_{q,j}^{h,m}(x_3, t)\,\varphi_0(q^l(t-2dq))\,dt \\ &= q^{-l} e^{i(\lambda-k)2dq} \int e^{iq^{-l}(\lambda-k)\tau} c_{q,j}^{h,m}(x_3, 2dq+\tau q^{-l})\,\varphi_0(\tau)\,d\tau. \end{aligned}$$

Hence, we get with the aid of Proposition 3.2

$$\begin{aligned} |J_{h,m,j}| &\leqq \frac{C q^{-l}}{(1+|\lambda q^{-l}-kq^{-l}|)^2} \int_{-1}^{1} \left| \frac{\partial^2}{\partial\tau^2}(c_{q,j}^{h,m}(x_3, 2dq+\tau q^{-l})\,\varphi_0(\tau)) \right| d\tau \\ &\leqq \frac{C' q^{l_1-l}}{(1+|\lambda q^{-l}-kq^{-l}|)^2}, \end{aligned}$$

where C' is independent of λ, k, l and q.

We substitute this estimate in the expression for $I_{h,m,j}$ and obtain for any $\varepsilon>0$:

$$\begin{aligned} |I_{h,m,j}(\lambda)| &\leqq C_\varepsilon \int_1^\infty \frac{k^{1-(3-\varepsilon)/s} q^{l_1-l}}{(1+|\lambda q^{-l}-kq^{-l}|)^2}\,dk \\ &= C_\varepsilon q^{l_1} q^{l[1-(3-\varepsilon)/s]} \int_{q^{-l}}^\infty \frac{\tau^{1-(3-\varepsilon)/s}}{(1+|\lambda q^{-l}-\tau|)^2}\,d\tau \\ &= C_\varepsilon q^{l_1} q^{l[1-(3-\varepsilon)/s]} \int_{q^{-l}}^\infty \frac{(\tau-\lambda q^{-l}+\lambda q^{-l})^{1-(3-\varepsilon)/s}}{(1+|\lambda q^{-l}-\tau|)^2}\,d\tau, \end{aligned}$$

with a constant $C_\varepsilon>0$ independent of λ, l and q.

For $\lambda q^{-l} \leqq 1$ we have

$$\begin{aligned} |I_{h,m,j}(\lambda)| &\leqq C_\varepsilon q^{l_1} q^{l[1-(3-\varepsilon)/s]} \int_{q^{-l}}^\infty (1+|\tau-\lambda q^{-l}|)^{-1-(3-\varepsilon)/s}\,d\tau \\ &\leqq C'_\varepsilon q^{l_1} q^{l[1-(3-\varepsilon)/s]}. \end{aligned}$$

For $\lambda q^{-l} \geqq 1$ we have

$$\begin{aligned} |I_{h,m,j}(\lambda)| &\leqq C_\varepsilon q^{l_1} q^{l[1-(3-\varepsilon)/s]} (\lambda q^{-l})^{1-(3-\varepsilon)/s} \int_{q^{-l}}^\infty (1+|\tau-\lambda q^{-l}|)^{-1-(3-\varepsilon)/s}\,d\tau \\ &\leqq C'_\varepsilon q^{l_1} \lambda^{1-(3-\varepsilon)/s}. \end{aligned}$$

Using the same argument we obtain the desired estimate for $I_{0,j}(\lambda)$, $j=1, \ldots, 4$.

Thus the Lemma is proved. □

4. PROOF OF THEOREM 2.1

First we note that from Corollary 2.4 and Proposition 3.1 it follows an estimate for some l', $l_1>0$ and for any $\varepsilon>0$ and $l \geqq l'$:

$$N_{5/(2dq)}(r) > C q^{-1-2/s} r^{2(1-1/s)} - C_{\varepsilon, l} q^{l_1} r^{2-(3-\varepsilon)/s} - C' q^{4l}. \quad (4.1)$$

Here the constants C, $C_{\varepsilon, l}$, C' are positive and do not depend on r and q.

We fix $G_{\alpha,\beta}$ for $\alpha>0$, $0<\beta<\beta_0$ (the constant β_0 we shall choose later). Let β' be such that $\beta<\beta'<\beta_0$. We choose $C_{\alpha,\beta,\beta'}>0$ such that if we pose

$$G_{\alpha,\beta,\beta'} = \{ z \in \mathbb{C} : 0 \leqq \operatorname{Im} z \leqq C_{\alpha,\beta,\beta'} |\operatorname{Re} z|^{-\beta'} \ln |z|, \ |\operatorname{Re} z| \geqq 1 \}$$

we have the inclusion

$$G_{\alpha,\beta,\beta'} \subset G_{\alpha,\beta}.$$

Hence it is sufficient to prove the theorem for $G_{\alpha,\beta,\beta'}$ instead of $G_{\alpha,\beta}$.

We denote:

$$S_\rho = \{ z \in \mathbb{C} : \operatorname{Im} z = \rho \ln |z|, \ |\operatorname{Re} z| \geqq 1 \};$$

$$S_{\alpha,\beta,\beta'} = \{ z \in \mathbb{C} : \operatorname{Im} z = C_{\alpha,\beta,\beta'} |\operatorname{Re} z|^{-\beta'} \ln |z|, \ |\operatorname{Re} z| \geqq 1 \}.$$

Then for the real part of the intersection point of $S_{\alpha,\beta,\beta'}$ and $S_{5/(2dq)}$ we get:

$$|\operatorname{Re}(S_{\alpha,\beta,\beta'} \cap S_{5/(2dq)})| = C'_{\alpha,\beta,\beta'} q^{1/\beta'}.$$

Hence

$$\{ z \in \mathbb{C} : \operatorname{Im} z \leqq [5/(2dq)] \ln |z|, \ 1 \leqq |\operatorname{Re} z| \leqq C'_{\alpha,\beta,\beta'} q^{1/\beta'} \} \subset G_{\alpha,\beta,\beta'}. \quad (4.2)$$

We apply (4.1) for $r = C'_{\alpha,\beta,\beta'} q^{1/\beta'}$ and using (4.2) we obtain:

$$N_{\alpha,\beta}(C'_{\alpha,\beta,\beta'} q^{1/\beta'}) > C''_{\alpha,\beta,\beta'} q^{-1-2/s} q^{(2/\beta')(1-1/s)} - C_{\alpha,\beta,\beta',l,\varepsilon} q^{l_1} q^{(1/\beta')[2-(3-\varepsilon)/s]} - C' q^{4l},$$

where the positive constants $C''_{\alpha,\beta,\beta'}$, $C_{\alpha,\beta,\beta',l,\varepsilon}$ and C' do not depend on q.

We rewrite the last estimate as follows

$$N_{\alpha,\beta}(r) > C'''_{\alpha,\beta,\beta'} r^{2(1-1/s)-\beta'(1+2/s)} - C'_{\alpha,\beta,\beta',l,\varepsilon} r^{[2-(3-\varepsilon)/s]+\beta' l_1} - C^0_{\alpha,\beta,\beta'} r^{\beta' 4l}.$$

Then we fix $l \geqq l'$ and $0<\varepsilon<1$ and we choose $\beta_0>0$ such that if $0<\beta<\beta'<\beta_0$ an estimate holds:

$$N_{\alpha,\beta}(r) > C_{\alpha,\beta,\beta'} r^{2(1-1/s)-\beta'(1+2/s)}.$$

Thus the theorem is proved. □

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