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# On essential self-adjointness of the relativistic hamiltonian of a spinless particle in a negative scalar potential

by

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ABSTRACT. — The relativistic quantum hamiltonian  $H$  describing a spinless particle in an electromagnetic field is considered, where  $H$  is associated with the classical hamiltonian  $c \{ m_0^2 c^2 + |p - A(x)|^2 \}^{1/2} + V(x)$  via the Weyl correspondence. We show that if  $V(x)$  is bounded below by a polynomial,  $H$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R}^n)$ . This result is quite different from that on the non-relativistic hamiltonian, *i. e.* the Schrödinger operator, and is close to that on the Dirac equation. Our proof is done by using the commutator theorem in [6].

RÉSUMÉ. — L'hamiltonien relativiste quantique  $H$  décrivant une particule sans spin dans un champ électromagnétique est considéré, où  $H$  est associé à l'hamiltonien classique  $c \{ m_0^2 c^2 + |p - A(x)|^2 \}^{1/2} + V(x)$  via la correspondance de Weyl. Nous démontrons que si  $V(x)$  est borné inférieurement par un polynôme,  $H$  est essentiellement auto-adjoint sur  $C_0^\infty(\mathbb{R}^n)$ . Ce résultat est tout à fait différent de celui sur l'hamiltonien non-relativiste, c'est-à-dire l'opérateur de Schrödinger, et est voisin de celui sur l'opérateur de Dirac. La preuve est faite en utilisant le théorème du commutateur dans [6].

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1. INTRODUCTION

In the present paper we study the problem of essential self-adjointness of the operator

$$\begin{aligned}
 H f(x) = H_A f(x) + V(x) f(x) \equiv & (2\pi)^{-n} \text{Os} - \iint e^{i(x-x') \cdot \xi} \\
 & \times h_A\left(\frac{x+x'}{2}, \xi\right) f(x') dx' d\xi + V(x) f(x) \quad (1.1)
 \end{aligned}$$

as an operator in the Hilbert space  $L^2(\mathbb{R}^n)$ , where

$$\begin{aligned}
 h_A(x, \xi) = c \{ m_0^2 c^2 + |\xi - A(x)|^2 \}^{1/2}, \\
 A(x) = (a_1(x), \dots, a_n(x)), \quad \} \quad (1.2)
 \end{aligned}$$

$V(x)$  is a real valued function and  $c, m_0$  are positive constants.

$\text{Os} - \iint \dots dx' d\xi$  means the oscillatory integral (e.g. chapter 1 in [11]).

$L^2 = L^2(\mathbb{R}^n)$  is the space of all square integrable functions on  $\mathbb{R}^n$ .  $H_A$  is called the Weyl quantized hamiltonian with a classical hamiltonian  $h_A(x, \xi)$ . When  $n=3$ , this operator  $H$  can be considered as the hamiltonian describing a relativistic spinless particle with charge one and rest mass  $m_0$  in an electromagnetic field whose scalar and vector potentials are given by  $V(x)$  and  $A(x)$  respectively. There  $c$  denotes the velocity of light ([16], [7], [4], [8] and etc.).

Let  $C_0^\infty(\mathbb{R}^n)$  be the space of all infinitely differentiable functions with compact support. We denote  $H_A$  where  $A(x) = (0, \dots, 0)$  by  $H_0$ . Essential self-adjointness and spectral properties of  $H_0 + V(x)$  where  $V(x)$  is the Coulomb potential, a Yukawa-type potential and their sum have been studied in [16], [7] and [4]. On the other hand as for general  $H_A$ , essential self-adjointness of  $H = H_A + V(x)$  has been studied in [12], [8] and [9] under the assumption that  $V(x)$  is bounded from below. Recently the author proved self-adjointness of  $H$  with domain  $\{ f(x) \in L^2; H f(x) \in L^2 \}$  as one of results in [10] under the assumptions (1.3) and (1.4) below.

$\left(\frac{\partial}{\partial x}\right)^\alpha a_j(x) \equiv \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} a_j(x) (j=1, 2, \dots, n)$  are bounded on  $\mathbb{R}^n$  for all multi-indices  $\alpha = (\alpha_1, \dots, \alpha_n)$  such that

$$|\alpha| \equiv \alpha_1 + \dots + \alpha_n \neq 0. \quad (1.3)$$

There exists a constant  $m \geq 0$  such that

$$\left| \left(\frac{\partial}{\partial x}\right)^\alpha V(x) \right| \leq C_\alpha (1 + |x|)^m \quad \text{on } \mathbb{R}^n$$

are valid for all multi-indices  $\alpha$  with constants  $C_\alpha$ . (1.4)

Our aim in the present paper is to show that the above assumption (1.4) can be replaced by a much weaker one for essential self-adjointness of  $H$  with domain  $C_0^\infty(\mathbb{R}^n)$ . For example, we can obtain the following results. We denote by  $L_{loc}^2 \equiv L_{loc}^2(\mathbb{R}^n)$  the space of all locally square integrable functions. Let  $V(x)$  be a real valued function in  $L_{loc}^2$  such that

$$-C(1+|x|)^m \leq V(x) \quad \text{on } \mathbb{R}^n \tag{1.5}$$

is valid for non-negative constants  $C$  and  $m$ . Let  $Z$  be a constant less than  $(n-2)c/2$ . Then both  $H_A + V(x)$  and  $H_0 - \frac{Z}{|x|} + V(x)$  with domain  $C_0^\infty(\mathbb{R}^n)$  are essentially self-adjoint under a slightly weaker assumption than (1.3) (Theorem 2.2 and Corollary 2.4 in the present paper).  $n \geq 3$  is assumed for the latter operator. The assumption (1.3) is not so limited, because we need such an assumption to define  $H_A$  by (1.1). But we must note that a more general definition of  $H_A$  is proposed in [8].

As for the Schrödinger operators  $-\frac{1}{2m_0}\Delta + V_S(x)$ , we know that we need for their essential self-adjointness the limitation on the decreasing rate at infinity of negative part of  $V_S(x)$  (e.g. Theorem 2 in [5] and page 157 in [1]). On the other hand as for the Dirac operator, we know from Theorem 2.1 in [3] that such a limitation is not necessary at all for its essential self-adjointness. Hence our decreasing rate (1.5) for essential self-adjointness of  $H$  lies between those of the Schrödinger and the Dirac operators.

Our proof in the present paper is quite different from that in [10]. In [10] we studied the theory of pseudo-differential operators with basic weight functions and applied it. In the present paper we use the commutator theorem in [6].

The plan of the present paper is as follows. In section 2 we will state all results. Some of results will be proved there. Sections 3 and 4 will be devoted to the proofs of main results.

## 2. THEOREMS

Let  $k(x, \xi)$  be a  $C^\infty$ -function on  $\mathbb{R}^{2n}$ . We suppose that for any multi-indices  $\alpha \neq (0, \dots, 0)$  and  $\beta$  there exists a constant  $C_{\alpha, \beta}$  satisfying

$$|k_{(\beta)}^{(\alpha)}(x, \xi)| \leq C_{\alpha, \beta} \langle x \rangle \quad \text{on } \mathbb{R}^{2n}, \tag{2.1}$$

where  $\langle x \rangle = \{1 + |x|^2\}^{1/2}$  and  $k_{(\beta)}^{(\alpha)}(x, \xi) = \left(\frac{\partial}{\partial \xi}\right)^\alpha \left(\frac{1}{i}\right)^{|\beta|} \left(\frac{\partial}{\partial x}\right)^\beta k(x, \xi)$ . It follows from the mean value theorem that

$$|k_{(\beta)}(x, \xi) - k_{(\beta)}(x, 0)| \leq C_\beta \langle x \rangle \langle \xi \rangle$$

are valid for all  $\beta$  with constants  $C_\beta$ . Hence by analogy with arguments in chapter 2 of [11] and chapter 4 of [15] we can define the pseudo-differential operator  $K(X, D_x)$  with symbol  $k(x, \xi)$  by

$$K(X, D_x) f(x) = (2\pi)^{-n} \int e^{ix \cdot \xi} k(x, \xi) \hat{f}(\xi) d\xi, \tag{2.2}$$

for  $f(x) \in \mathcal{S}$ .  $\hat{f}(\xi)$  denotes the Fourier transformation  $\int e^{-ix \cdot \xi} f(x) dx$  and  $\mathcal{S}$  the space of all rapidly decreasing functions on  $\mathbb{R}^n$ . It is easy to show that  $K(X, D_x)$  makes a continuous operator from  $\mathcal{S}$  into  $\mathcal{S}$ .

**THEOREM 2.1.** — *Let  $\Phi(x)$  be a real valued function in  $L^2_{loc}(\mathbb{R}^n)$ . Assume that  $K(X, D_x)$  defined above is symmetric on  $C^\infty_0(\mathbb{R}^n)$  and that*

$$K(X, D_x) + \Phi(x) \geq 0 \quad \text{on } C^\infty_0(\mathbb{R}^n). \tag{2.3}$$

The quadratic form inequality (2.3) means that

$(\{K(X, D_x) + \Phi(x)\} f(x), f(x)) \geq 0$  for all  $f(x) \in C^\infty_0(\mathbb{R}^n)$ . Moreover we assume that for all  $W(x)$  being in  $L^2_{loc}$  with  $W(x) \geq 0$  almost everywhere (a. e.)  $K(X, D_x) + \Phi(x) + W(x)$  with domain  $C^\infty_0(\mathbb{R}^n)$  is essentially self-adjoint. Then if  $V(x) \in L^2_{loc}$  satisfies (1.5) for non-negative constants  $C$  and  $m$ , then  $K(X, D_x) + \Phi(x) + V(x)$  with domain  $C^\infty_0(\mathbb{R}^n)$  is also essentially self-adjoint.

Theorem 2.1 will be proved in section 3. We will prove the following theorem from Theorem 2.1 by using the results obtained in [8].

**THEOREM 2.2.** — *Consider  $H$  defined by (1.1) with domain  $C^\infty_0(\mathbb{R}^n)$ . We assume*

$$\left| \left( \frac{\partial}{\partial x} \right)^\alpha a_j(x) \right| \leq C_\alpha \log \{ \langle x \rangle \} \tag{2.4}$$

for all  $\alpha \neq (0, \dots, 0)$  with constants  $C_\alpha$ . Let  $V(x)$  be the same function as in Theorem 2.1. Then  $H$  is essentially self-adjoint.

*Remark 2.1.* — As was stated in introduction,  $H$  defined by (1.1) with domain  $\{f(x) \in L^2; Hf(x) \in L^2\}$  is self-adjoint under the assumptions (1.3) and (1.4). We note that this  $H$  is also self-adjoint even if (1.3) is replaced by (2.4) there. This result follows from Theorem 1 in [10] at once.

*Proof of Theorem 2.2.* — We can easily have from the assumption

$$|h^{(\alpha)}_{A(\beta)}(x, \xi)| \leq C'_{\alpha, \beta} \{ \langle x \rangle^2 + \langle \xi \rangle \}$$

for all  $\alpha$  and  $\beta$  with constants  $C'_{\alpha, \beta}$ . So it follows from the analogy with arguments in section 2 of chapter 2 in [11] that  $H_A$  makes a continuous operator from  $\mathcal{S}$  to  $\mathcal{S}$  and  $H_A$  is symmetric on  $\mathcal{S}$ . We note that the

assertion in Lemma 2.2 in [8] remains valid under our weaker assumption (2.4) than that in [8]. So Theorem 5.1 in [8] indicates  $H_A \geq 0$  on  $C_0^\infty(\mathbb{R}^n)$  and essential self-adjointness of  $H_A + W(x)$  with domain  $C_0^\infty(\mathbb{R}^n)$  for any  $W(x) \in L_{loc}^2$  such that  $W(x) \geq 0$  a. e.

We set

$$p(x, \xi) = (2\pi)^{-n} \text{Os} - \iint e^{-iy \cdot \eta} h_A(x + y/2, \xi + \eta) dy d\eta.$$

Then

$$P(X, D_x) = H_A \quad \text{on } C_0^\infty(\mathbb{R}^n) \tag{2.5}$$

follows from analogy of Theorem 2.5 in [11]. Let  $l$  be an even integer such that  $l > n + 1$ . Then taking the integration by parts, we have

$$p_{(\beta)}^{(\alpha)}(x, \xi) = (2\pi)^{-n} \text{Os} - \iint e^{-iy \cdot \eta} \langle y \rangle^{-l} (1 - \Delta_\eta)^{l/2} \times \{ \langle \eta \rangle^{-l} (1 - \Delta_y)^{l/2} h_{A(\beta)}^{(\alpha)}(x + y/2, \xi + \eta) \} dy d\eta$$

for any  $\alpha$  and  $\beta$ . We note that  $h_A(x, \xi)$  satisfies the same inequalities as (2.1) for all  $\alpha$  and  $\beta$  such that  $|\alpha + \beta| \neq 0$  with another constants  $C_{\alpha, \beta}$  under the assumption (2.4). So using  $\langle x + y \rangle^\sigma \leq \sqrt{2} \langle x \rangle^\sigma \langle y \rangle^\sigma$  ( $\sigma = 1$  and  $-1$ ,  $x, y \in \mathbb{R}^n$ ), we can see that

$$|p_{(\beta)}^{(\alpha)}(x, \xi)| \leq C''_{\alpha, \beta} \langle x \rangle \tag{2.6}$$

are valid for all  $\alpha$  and  $\beta$  such that  $|\alpha + \beta| \neq 0$  with constants  $C''_{\alpha, \beta}$ . Hence we can easily see from (2.5) and (2.6) that we can apply Theorem 2.1 to  $H_A + V(x)$  as  $K(X, D_x) = H_A = P(X, D_x)$  and  $\Phi(x) = 0$ . So Theorem 2.2 can be proved.

Q.E.D.

*Remark 2.2.* – As will be noted in Remark 3.1 in the present paper, the assumption in Theorem 2.1 that (2.1) must hold for all  $\alpha \neq (0, \dots, 0)$  and  $\beta$  can be weakened. The assertion of Theorem 2.1 remains valid even if we replace this assumption by a weaker one that (2.1) holds for all  $\alpha \neq (0, \dots, 0)$  and  $\beta$  satisfying  $|\alpha| \leq J$  and  $|\beta| \leq J$ , where  $J$  is an integer determined from  $n$  and  $m$ . So the assumption on  $\{a_j(x)\}_{j=1}^n$  in Theorem 2.2 can be similarly replaced by a weaker one that

$$\left| \left( \frac{\partial}{\partial x} \right)^\alpha a_j(x) \right| \leq C_\alpha \langle x \rangle^\varepsilon$$

are valid for all  $0 < |\alpha| \leq J$ , where  $\varepsilon > 0$  is a sufficiently small constant and  $J$  is a sufficiently large integer.  $\varepsilon$  and  $J$  are determined from  $n$  and  $m$ .

**THEOREM 2.3.** – *Let  $H_0$  be the operator defined in introduction with domain  $C_0^\infty(\mathbb{R}^n)$ . Suppose that  $\Phi(x)$  is a real valued function in  $L_{loc}^2$  and a  $H_0$ -bounded multiplication operator with relative bound less than one. Let*

$V(x)$  be the same function as in Theorem 2.1. Then  $H_0 + \Phi(x) + V(x)$  with domain  $C_0^\infty(\mathbb{R}^n)$  is essentially self-adjoint.

Theorem 2.3 will be proved in section 4.

**COROLLARY 2.4.** — Let  $n \geq 3$  and  $Z$  be a constant less than  $(n-2)c/2$ . Let  $V(x)$  be the same function as in Theorem 2.1. Then  $H_0 - \frac{Z}{|x|} + V(x)$  with domain  $C_0^\infty(\mathbb{R}^n)$  is essentially self-adjoint.

*Proof of Corollary 2.4.* — When  $Z \leq 0$ , essential self-adjointness of  $H_0 - \frac{Z}{|x|} + V(x)$  follows from Theorem 2.2 at once. Let  $0 < Z < \left(\frac{n-2}{2}\right)c$ . We denote  $L^2$ -norm by  $\| \cdot \|$ . We know the Hardy inequality

$$\left(\frac{n-2}{2}\right)^2 \left\| \frac{\psi(x)}{|x|} \right\|^2 \leq \sum_{j=1}^n \left\| \frac{\partial \psi}{\partial x_j}(x) \right\|^2$$

for  $\psi(x) \in C_0^\infty(\mathbb{R}^n)$  (e. g. page 169 in [13] and (2.9) in [7]). So

$$\begin{aligned} \left(\frac{n-2}{2}\right)^2 \left\| \frac{\psi(x)}{|x|} \right\|^2 &\leq (2\pi)^{-n} \int |\xi|^2 |\hat{\psi}(\xi)|^2 d\xi \\ &\leq (2\pi)^{-n} \int \{m_0^2 c^2 + |\xi|^2\}^{1/2} |\hat{\psi}(\xi)|^2 d\xi \\ &= c^{-2} \|H_0 \psi(x)\|^2 \end{aligned}$$

holds for  $\psi(x) \in C_0^\infty(\mathbb{R}^n)$ . Consequently  $-\frac{Z}{|x|}$  is  $H_0$ -bounded with relative bound less than one. Hence Corollary 2.4 follows from Theorem 2.3 at once.

Q.E.D.

### 3. PROOF OF THEOREM 2.1

**LEMMA 3.1.** — Suppose that  $k(x, \xi)$  satisfies (2.1) for all  $\alpha \neq (0, \dots, 0)$  and  $\beta$ . Let  $\zeta$  be a non-negative constant. Then there exists a positive constant  $d = d(\zeta)$  such that

$$\| [K(X, D_x), \langle x \rangle^{\zeta/2}] f(x) \| \leq d \| \langle x \rangle^{\zeta/2} f(x) \| \tag{3.1}$$

are valid for all  $f(x) \in \mathcal{S}$ .  $[K(X, D_x), \langle x \rangle^{\zeta/2}]$  denotes the commutator of operators  $K(X, D_x)$  and  $\langle x \rangle^{\zeta/2}$ .

*Proof.* – We set

$$q(x, \xi) = (2\pi)^{-n} \text{Os} - \iint e^{-iy \cdot \eta} k(x, \xi + \eta) \langle x + y \rangle^{\xi/2} dy d\eta - \langle x \rangle^{\xi/2} k(x, \xi). \quad (3.2)$$

Then we get by analogy with arguments in chapter 2 of [11]

$$Q(X, D_x) = [K(X, D_x), \langle x \rangle^{\xi/2}] \quad \text{on } \mathcal{S}. \quad (3.3)$$

It is easy to see

$$q(x, \xi) = (2\pi)^{-n} \int_0^1 d\theta \sum_{|\alpha|=1} \text{Os} - \iint e^{-iy \cdot \eta} k^{(\alpha)}(x, \xi + \theta\eta) D_x^\alpha \langle x + y \rangle^{\xi/2} dy d\eta,$$

where  $D_x^\alpha = \left(\frac{1}{i}\right)^{|\alpha|} \left(\frac{\partial}{\partial x}\right)^\alpha$ . Let  $l_1$  and  $l_2$  be integers such that  $l_1 > n + \left\lfloor \frac{\xi}{2} - 1 \right\rfloor$  and  $l_2 > n$ . Then taking the integration by parts,

$$|q(x, \xi)| \leq (2\pi)^{-n} \int_0^1 d\theta \sum_{|\alpha|=1} \iint |\langle y \rangle^{-l_1} (1 - \Delta_\eta)^{l_1/2} \{ \langle \eta \rangle^{-l_2} (1 - \Delta_y)^{l_2/2} k^{(\alpha)}(x, \xi + \theta\eta) D_x^\alpha \langle x + y \rangle^{\xi/2} \}| dy d\eta$$

holds. So we get

$$|q(x, \xi)| \leq C_0 \langle x \rangle^{\xi/2}$$

with a constant  $C_0$  from the assumption (2.1) in the same way to the proof of (2.6). Similarly we obtain

$$|q_{(\beta)}^{(\alpha)}(x, \xi)| \leq C_{\alpha, \beta} \langle x \rangle^{\xi/2} \quad (3.4)$$

for all  $\alpha$  and  $\beta$  with constants  $C_{\alpha, \beta}$ .

Next we set

$$r(x, \xi) = (2\pi)^{-n} \text{Os} - \iint e^{-iy \cdot \eta} q(x, \xi + \eta) \langle x + y \rangle^{-\xi/2} dy d\eta. \quad (3.5)$$

Then we have

$$R(X, D_x) = Q(X, D_x) \circ \langle x \rangle^{-\xi/2}. \quad (3.6)$$

$\circ \circ \circ$  denotes the product of operators. Then we obtain from (3.4)

$$|r_{(\beta)}^{(\alpha)}(x, \xi)| \leq C'_{\alpha, \beta} \quad (3.7)$$

for all  $\alpha$  and  $\beta$  with constants  $C'_{\alpha, \beta}$  in the same way to the proof of (3.4).

We note that

$$[K(X, D_x), \langle x \rangle^{\xi/2}] = R(X, D_x) \circ \langle x \rangle^{\xi/2} \quad \text{on } \mathcal{S}$$



holds from (3.3) and (3.6). So applying the Calderón-Vaillancourt theorem in [2] to  $R(X, D_x)$ , we get Lemma 3.1.

Q.E.D.

*Proof of Theorem 2.1.* — For the sake of simplicity we denote  $C_0^\infty(\mathbb{R}^n)$  by  $\mathcal{E}$ . Let  $d=d(m)$  be the constant determined in Lemma 3.1. We can choose a constant  $M>0$  satisfying

$$M \geq 2 d(m) \quad \text{and} \quad V(x) + M \langle x \rangle^m \geq 0 \quad \text{a.e.} \quad (3.8)$$

because of the assumption (1.5). We fix this  $M$ . Set

$$T = K(X, D_x) + \Phi(x) + V(x) \quad (3.9)$$

with domain  $\mathcal{E}$ . It follows from the assumptions in Theorem 2.1 and (3.8) that  $T + 3 M \langle x \rangle^m \geq 2 M \langle x \rangle^m$  on  $\mathcal{E}$  holds and  $T + 3 M \langle x \rangle^m$  with domain  $\mathcal{E}$  is essentially self-adjoint. Let  $N$  be the self-adjoint operator defined by the closure of  $T + 3 M \langle x \rangle^m$ . Then

$$N \geq 2 M \langle x \rangle^m > 0 \quad \text{on } \mathcal{E} \quad (3.10)$$

is valid and  $\mathcal{E}$  is a core for  $N$ .

We will prove

$$\|T f(x)\| \leq \|N f(x)\| \quad [f(x) \in \mathcal{E}] \quad (3.11)$$

and

$$\pm i \{ (T f, N f) - (N f, T f) \} \leq 3 d(N f, f) \quad [f(x) \in \mathcal{E}]. \quad (3.12)$$

(. . .) implies the inner product in  $L^2(\mathbb{R}^n)$ . Then Corollary 1.1 in [6] shows that  $T$  is essentially self-adjoint, which completes the proof.

We will first prove (3.11). Let  $f(x) \in \mathcal{E}$ . Since each  $\Phi(x)$  and  $V(x)$  is in  $L_{loc}^2$ , we can easily have

$$(T f, \langle x \rangle^m f) = (T \circ \langle x \rangle^{m/2} f, \langle x \rangle^{m/2} f) - ([K(X, D_x), \langle x \rangle^{m/2}] f, \langle x \rangle^{m/2} f). \quad (3.13)$$

We denote by  $\text{Re}(\cdot)$  and  $\text{Im}(\cdot)$  the real part and the imaginary part of complex number respectively. Then noting  $N f = T f + 3 M \langle x \rangle^m f$ , we get by (3.13)

$$\begin{aligned} \|N f\|^2 &= \|T f\|^2 + 6 M \text{Re}(T f, \langle x \rangle^m f) + 9 M^2 \|\langle x \rangle^m f\|^2 \\ &= \|T f\|^2 + 6 M \text{Re}(\{T + M \langle x \rangle^m\} \circ \langle x \rangle^{m/2} f, \langle x \rangle^{m/2} f) \\ &\quad + 3 M^2 \|\langle x \rangle^m f\|^2 - 6 M \text{Re}([K(X, D_x), \langle x \rangle^{m/2}] f, \langle x \rangle^{m/2} f). \end{aligned} \quad (3.14)$$

It is easy to see from the assumption (2.3) and (3.8)

$$T + M \langle x \rangle^m \geq 0 \quad \text{on } \mathcal{E}. \quad (3.15)$$

Hence applying Lemma 3.1 to (3.14), we obtain by (3.8)

$$\begin{aligned} & \|Nf\|^2 \\ & \geq \|Tf\|^2 + 3M^2 \|\langle x \rangle^m f\|^2 - 6Md \|\langle x \rangle^{m/2} f\|^2 \\ & \geq \|Tf\|^2 + 3M(M-2d) \|\langle x \rangle^m f\|^2 \\ & \geq \|Tf\|^2, \end{aligned}$$

which shows (3.11).

Next we will prove (3.12). Let  $f(x) \in \mathcal{E}$ . Using  $Nf = Tf + 3M \langle x \rangle^m f$  and  $\Phi(x), V(x) \in L^2_{loc}$ , we have

$$\begin{aligned} & (Tf, Nf) - (Nf, Tf) \\ & = (Tf, 3M \langle x \rangle^m f) - (3M \langle x \rangle^m f, Tf) \\ & = 3M \{ (K(X, D_x) f, \langle x \rangle^m f) - (\langle x \rangle^m f, K(X, D_x) f) \} \\ & = 6Mi \operatorname{Im}(K(X, D_x) f, \langle x \rangle^m f). \end{aligned}$$

Apply the equality

$$\begin{aligned} (K(X, D_x) f, \langle x \rangle^m f) & = (K(X, D_x) \circ \langle x \rangle^{m/2} f, \langle x \rangle^{m/2} f) \\ & \quad - ([K(X, D_x), \langle x \rangle^{m/2}] f, \langle x \rangle^{m/2} f) \end{aligned}$$

to the above. Then since  $K(X, D_x)$  is assumed to be symmetric on  $\mathcal{E}$ ,

$$\begin{aligned} (Tf, Nf) - (Nf, Tf) & \\ & = -6Mi \operatorname{Im}([K(X, D_x), \langle x \rangle^{m/2}] f, \langle x \rangle^{m/2} f) \quad (3.16) \end{aligned}$$

is valid. Hence we obtain by Lemma 3.1

$$\begin{aligned} & \pm i \{ (Tf, Nf) - (Nf, Tf) \} \\ & \leq 6Md \|\langle x \rangle^{m/2} f\|^2 \\ & = 6Md (\langle x \rangle^m f, f) \\ & \leq 3d(Nf, f). \end{aligned}$$

Here we used (3.10) for the last inequality. Thus (3.12) could be proved. This completes the proof.

Q.E.D.

*Remark 3.1.* – We can easily see in the proof of Theorem 2.1 from the Calderón-Vaillancourt theorem that if (3.7) holds for  $|\alpha| \leq 3n$  and  $|\beta| \leq 3n$ , (3.1) is valid. Hence as was stated in Remark 2.2, we can weaken the assumption in Theorem 2.1 that (2.1) hold for all  $\alpha \neq (0, \dots, 0)$  and  $\beta$ . This can be easily verified by following the proof of Theorem 2.1.

4. PROOF OF THEOREM 2.3

We denote  $C_0^\infty(\mathbb{R}^n)$  by  $\mathcal{E}$  as in section 3. It is easy to see  $H_0 \geq m_0 c^2 > 0$  on  $\mathcal{S}$ .  $\Phi(x)$  was assumed to be  $H_0$ -bounded with relative bound less than one. So it follows from Theorem X.18 in [13] that  $\Phi(x)$  is form-bounded with the same relative bound with respect to  $H_0$ . That is, there exists a constant  $b \geq 0$  such that

$$|(\Phi(x)f, f)| < (H_0 f, f) + b(f, f)$$

are valid for all  $f(x) \in \mathcal{E}$ . Hence we see

$$\{H_0 + \Phi(x) + b\} \geq 0 \quad \text{on } \mathcal{E}. \tag{4.1}$$

We will show that  $H_0 + \Phi(x) + b + W(x)$  with domain  $\mathcal{E}$  are essentially self-adjoint for all  $W(x)$  being in  $L^2_{loc}$  with  $W(x) \geq 0$  a. e. Then the proof of Theorem 2.3 can be completed by Theorem 2.1. We will prove essential self-adjointness of  $H_0 + \Phi(x) + b + W(x)$  by analogy with arguments in the proof of Theorem X.29 in [13] where Schrödinger operators are studied. There we will use the Kato-type inequality obtained in [8].

Let  $W(x) \geq 0$  a. e. be in  $L^2_{loc}$ . Noting (4.1), it follows from Theorem X.26 in [13] that iff  $H_0 + \Phi(x) + b + W(x)$  with domain  $\mathcal{E}$  is essentially self-adjoint, the range of  $\lambda + H_0 + \Phi(x) + b + W(x)$  is dense in  $L^2$  for a constant  $\lambda > 0$ .

We may assume  $b=0$  without the loss of generality. Let  $\lambda > 0$  be a constant and  $u(x)$  be in  $L^2$  such that

$$(u(x), \{\lambda + H_0 + \Phi(x) + W(x)\} f(x)) = 0 \tag{4.2}$$

hold for all  $f(x) \in \mathcal{E}$ . (4.2) indicates that

$$(\lambda + H_0 + \Phi + W) u(x) = 0 \tag{4.2}'$$

holds in a distribution sense. Since  $u(x)$  is in  $L^2$  and  $\Phi(x) + W(x)$  is in  $L^2_{loc}$ ,  $H_0 u(x)$  is in  $L^1_{loc}$ . Hence we get from Theorem 4.1 in [8] the distribution inequality

$$\text{Re}[(\text{sgn } u(x)) H_0 u(x)] \geq H_0 |u(x)| \quad \text{in } \mathcal{D}', \tag{4.3}$$

where  $\text{sgn } u(x)$  is a bounded measurable function defined by  $\overline{u(x)}/|u(x)|$  for a point  $x$  such that  $u(x) \neq 0$  and zero for a point  $x$  such that  $u(x) = 0$ .  $\overline{u(x)}$  is the complex conjugate of  $u(x)$ . (4.3) means that

$$(\text{Re}[(\text{sgn } u(x)) H_0 u(x)], f(x)) \geq (H_0 |u(x)|, f(x))$$

hold for all  $f(x) \in \mathcal{E}$  with  $f(x) \geq 0$ . Inserting  $H_0 u(x) = -(\lambda + \Phi + W) u(x)$  into (4.3),

$$\begin{aligned} (\lambda + H_0) |u(x)| &\leq -(\Phi + W) |u(x)| \\ &\leq -\Phi(x) |u(x)| \quad \text{in } \mathcal{D}' \end{aligned} \tag{4.4}$$

is obtained. Here we used  $W(x) \in L^2_{loc}$  and  $W(x) \geq 0$  a.e. for the last inequality.

Now

$$\begin{aligned} |(\Phi(x)|u(x)|, f(x))| &\leq \|u(x)\| \|\Phi(x)f(x)\| \\ &\leq C_1 \|u(x)\| \|(\mathbf{H}_0 + 1)f(x)\| \end{aligned}$$

follow from  $H_0$ -boundedness of  $\Phi(x)$  for all  $f(x) \in \mathcal{E}$ , where  $C_1$  is a constant. It is easy to see that the same inequalities remain valid for all  $f(x) \in \mathcal{S}$ . So  $\Phi(x)|u(x)|$  belongs to  $\mathcal{S}'$ .  $\mathcal{S}'$  is the dual space of  $\mathcal{S}$ . It is also easy to see  $(\lambda + H_0)|u(x)| \in \mathcal{S}'$ . Hence we obtain by (4.4)

$$-(\Phi(x)|u(x)|, f(x)) \geq ((\lambda + H_0)|u(x)|, f(x)) \tag{4.5}$$

for all  $f(x) \in \mathcal{S}$  with  $f(x) \geq 0$ . Let  $\psi(x) \geq 0$  on  $\mathbb{R}^n$  be an arbitrary function in  $\mathcal{S}$  and set  $\varphi(x) = (\lambda + H_0)^{-1} \psi(x)$ . Then  $\varphi(x)$  belongs to  $\mathcal{S}$ .  $\varphi(x) \geq 0$  on  $\mathbb{R}^{2n}$  follows from (3.3) and (3.4) in [8] or Theorems XIII.52, 54 and the example on page 220 in [14]. So inserting this  $\varphi(x)$  into (4.5) as  $f(x)$ , we get

$$-(\Phi(x)|u(x)|, (\lambda + H_0)^{-1} \psi(x)) \geq (|u(x)|, \psi(x)). \tag{4.6}$$

Now  $\Phi(x)$  is assumed to be  $H_0$ -bounded with relative bound less than one. So there exist constants  $0 \leq a' < 1$  and  $0 \leq b'$  such that

$$\begin{aligned} \|\Phi(x)f(x)\| &< a' \|H_0 f(x)\| + b' \|f(x)\| \\ &< a' \|(\lambda + H_0)f(x)\| + b' \|f(x)\| \end{aligned}$$

are valid for all  $f(x) \in \mathcal{E}$ . We can easily see that these inequalities remain valid for all  $f(x) \in \mathcal{S}$ . Consequently we get for all  $g(x) \in \mathcal{S}$

$$\begin{aligned} \|\Phi(x)(\lambda + H_0)^{-1} g(x)\| &< a' \|g(x)\| + b' \|(\lambda + H_0)^{-1} g(x)\| \\ &< \left(a' + \frac{b'}{\lambda}\right) \|g(x)\|, \end{aligned}$$

which also remain valid for all  $g(x) \in L^2$ . Hence  $\Phi(x)(\lambda + H_0)^{-1}$  is a bounded operator from  $L^2$  to  $L^2$  and its operator norm is bounded by a less constant than  $\left(a' + \frac{b'}{\lambda}\right)$ . Therefore we see that  $\{\Phi(x)(\lambda + H_0)^{-1}\}^* |u(x)|$  belongs to  $L^2$  and

$$\|\{\Phi(x)(\lambda + H_0)^{-1}\}^* |u(x)|\| < \left(a' + \frac{b'}{\lambda}\right) \|u(x)\| \tag{4.7}$$

is valid, because  $u(x)$  belongs to  $L^2$ . Moreover (4.6) indicates

$$-\{\Phi(x)(\lambda + H_0)^{-1}\}^* |u(x)| \geq |u(x)| \quad \text{a.e.} \tag{4.8}$$

as the inequality between functions, because  $\psi(x) \geq 0$  is arbitrary. Hence we get by (4.7) and (4.8)

$$\|u(x)\| < \left(a' + \frac{b'}{\lambda}\right) \|u(x)\|. \quad (4.9)$$

This shows  $u(x) = 0$  a.e. when  $\lambda > 0$  is large. Thus we see that if  $\lambda > 0$  is large, the range of  $\lambda + H_0 + \Phi(x) + W(x)$  is dense in  $L^2$ . This completes the proof of Theorem 2.3.

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