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On the genericity of nonvanishing instability intervals in periodic Dirac systems

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ABSTRACT. — Using Floquet-Lyapunov theory, it is shown that for Baire-almost every periodic potential the Dirac system has all its instability intervals open. Consequently, one-dimensional Dirac operators with periodic potentials generically possess infinitely many spectral gaps. These results also hold true if only even potentials are admitted.

RÉSUMÉ. — Utilisant la théorie de Floquet et Liapounoff, il est démontré que pour Baire presque tous potentiels périodiques tous les intervalles d'instabilité du système Dirac sont ouverts. En conséquence, les opérateurs de Dirac unidimensionnels à potentiel périodique possèdent génériquement une infinité de lacunes spectrales. Ces résultats restent vrais quand on n'admet que des potentiels pairs.

1. INTRODUCTION

Floquet-Lyapunov theory provides a very useful tool to study the spectral properties of self-adjoint one-dimensional Schrödinger and Dirac operators h with periodic potentials: the *discriminant* $D(\lambda)$ of the corresponding eigenvalue equation,

$$(h-\lambda)u=0,$$

defined as the trace of the canonical fundamental system of this ordinary differential equation. It is a smooth function of the spectral parameter λ, and is associated to the spectrum of h by means of the following well-known properties (*cf.* [6], [16]):

A) $\sigma(h)=\{\lambda\in\mathbb{R} \mid |D(\lambda)|\leqq 2\}$;

B) D is strictly monotonous within each of the *stability intervals* $\{\lambda\in\mathbb{R} \mid |D(\lambda)|<2\}$;

C) if h_p and h_a denote the selfadjoint realizations of the operator h on a periodicity interval with periodic and antiperiodic boundary conditions, respectively, then $\sigma(h_p)=D^{-1}(2)$, and $\sigma(h_a)=D^{-1}(-2)$. Both are unbounded discrete sets; λ is a degenerate (double) eigenvalue of h_p $[h_a]$ if and only if it is a double zero of $D-2$ $[D+2]$.

It follows that the spectrum of h has *band structure* as it is the closure of the union of the stability intervals; in particular, h has no discrete spectrum. This extreme structural simplicity of the spectrum is confined to strictly periodic potentials; even moderate relaxation of this requirement, such as almost periodicity, already unfolds the whole range of spectral variability ([2], [13]).

Whenever there is a non-degenerate (closed) interval between two neighbouring stability intervals, its interior (a *nonvanishing instability interval* of the eigenvalue equation) is a gap in the spectrum. If, however, two stability intervals are separated by a single point λ, this point belongs to the spectrum, which is a closed subset of $\mathbb{R}$; by abuse of language, one then speaks of a *vanishing instability interval*, regarding the empty interior of the pointlike degenerate interval as an instability interval in statu nascendi. In this case, λ is also called a *coexistence value*, since it is characterized by the simultaneous existence of two linearly independent periodic or antiperiodic solutions of the eigenvalue equation (as λ is a double eigenvalue of h_p or h_a).

Although one-dimensional Schrödinger and Dirac operators with periodic potentials always possess infinitely many instability intervals (note that h_p and h_a are unbounded), the number of spectral gaps may actually be considerably smaller: e. g., the zero potential is clearly periodic, yet the potential-free Dirac operator has only one spectral gap, *i.e.*, one nonvanishing and infinitely many vanishing instability intervals. On the other hand, the Meissner and the Dirac-Meissner operators (with non-zero piecewise constant periodic potential) have infinitely many spectral gaps (*see* [16] 17.D), G)); however, in these examples some instability intervals can still be observed to vanish. A famous example where no instability interval vanishes is the Mathieu equation,

$$-u''(x)+(a\cos x-\lambda)\,u(x)=0, \qquad a\in\mathbb{R}\setminus\{0\}$$

(Ince [7]; *see* also [4] and the references given there). However, the conjecture that in general, coexistence values are totally absent in one-dimensional Schrödinger equations with non-zero even periodic potentials ([8], [12]) was far too optimistic, as was pointed out by Borg [3] with reference to the Meissner equation. Indeed, already the addition of a $\cos 2x$ term in the Mathieu equation produces coexistence values [11]. Another particularly striking counterexample with a smooth potential is the Lamé equation,

$$-u''(x)+(n(n+1)\mathcal{P}(x+\omega')-\lambda)u(x)=0$$

(with $2\omega'$ the imaginary period of the doubly-periodic Weierstrass $\mathcal{P}$ function), which has exactly n nonvanishing instability intervals ([9], [1]).

These examples show that it is certainly no general property of periodic one-dimensional Schrödinger operators to have all instability intervals nonvanishing. Yet it has been proven ([15], [13]) that it is a generic property, insofar as it is only for a "small" set of exceptions that some or even infinitely many instability intervals vanish.

The purpose of the present paper is to establish the corresponding result for the one-dimensional Dirac system; this is of particular interest as for this equation, no example like the Mathieu equation appears to be known. We shall show that the ordinary differential equation system

$$u'(x)=-i\sigma_2(\sigma_3+q(x)-\lambda)u(x), \qquad (*)$$

with matrices $\sigma_2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}$, $\sigma_3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$, and two-component u, has no coexistence values for "almost every" (real-valued periodic) potential q. Of course, the notion of "almost every" must be given a precise meaning; we use the Bairean topological definition (*cf.* [5] 7.1):

A statement holds *generically* in a Baire topological space X, or *for Baire-almost every* $x \in X$, if the set of exceptions is a countable union of nowhere dense subsets of X, a subset being called *nowhere dense* if its closure has empty interior.

We remark that although the "generic" case may be regarded as typical in general, obvious examples often belong to the set of exceptions, and examples for the generic case are not always easily constructed: e.g., the generic continuous function is nowhere differentiable ([5] Ex. 7.14).

The condition that X be a Baire space is crucial if "generically" is to signify "up to few exceptions". In topological spaces of the first Baire category, which are countable unions of nowhere dense sets themselves, *anything* is true "generically".

Thus a "generic" statement about periodic Dirac systems presupposes that we choose a Baire space of admissible potentials, subject to periodicity and regularity requirements. In particular, all potentials will be locally integrable, so that the solutions of (∗) are pairs (written as column vectors) of locally absolutely continuous functions that are uniquely determined by their initial value.

DEFINITION 1. — Let $m>0$. Whenever Y is a space of real-valued functions, denote by Y_m the subspace of m-periodic functions. We call a Baire topological space X a *suitable potential class of period m* if

a) X is a linear subspace of $L_m^\infty(\mathbb{R}, \mathbb{R})$ containing the constant functions,

b) $\|\cdot\|_\infty$ is continuous in the topology of X, and

c) the following consistency requirement is fulfilled: if $q \in X$, λ a coexistence value of the differential equation (∗), and (u, v) the corresponding (real-valued) canonical fundamental system, then $u^T u$, $v^T v \in X$ (by u^T we indicate the pointwise transpose of u, considered as a 2×1 matrix).

Remark 1. — Requirement *c)* may appear rather implicit but is not difficult to satisfy provided X is chosen large enough; for instance, we admit as potential classes the real Banach spaces $L_m^\infty(\mathbb{R}, \mathbb{R})$, and $C_m^k(\mathbb{R}, \mathbb{R})$ with the norm $\|f\| := \sum_{j=0}^{k} \left\| \frac{d^j}{dx^j} f \right\|_\infty$, as well as the Fréchet space $C_m^\infty(\mathbb{R}, \mathbb{R})$ considered in [15].

Remark 2. — With regard to the application of the genericity theorem in situations demanding that all potentials are even, e. g. when studying spherically symmetric three-dimensional Dirac operators (*cf.* [14]), it is of interest to note that if X_1 is a suitable potential class, so is the subspace of even potentials, $X_2 := \{ f \in X_1 \mid f(-\cdot) = f \}$.

Indeed, if $q \in X_2$, $\lambda \in \mathbb{R}$ and (u, v) the corresponding canonical fundamental system of (∗), then the functions $U := \sigma_3 u(-\cdot)$ and $V := -\sigma_3 v(-\cdot)$ also constitute a canonical fundamental system. By uniqueness it follows that $u(-\cdot)^T u(-\cdot) = (\sigma_3 U)^T (\sigma_3 U) = u^T u$, and accordingly for v, so X_2 satisfies requirement *c)*.

2. THE GENERICITY THEOREM

THEOREM 1. — *If* X *is a suitable potential class, then for Baire-almost every* $q \in X$, Eq. (∗) *possesses no coexistence values.*

COROLLARY. — *Periodic one-dimensional Dirac operators generically possess infinitely many spectral gaps. This also holds true if only even potentials are considered.*

In order to prove Theorem 1, we proceed as follows.

First we use eigenvalue perturbation theory to show that it is possible to keep track of individual instability intervals as the (essentially bounded) potential varies (Corollary 1 to Lemma 1), and that nonvanishing instability intervals are stable, *i. e.* do not suddenly vanish under potential perturbations (Corollary 2 to Lemma 1).

The central part of the proof is devoted to establishing that, on the other hand, a vanishing instability interval is unstable in the sense that if an appropriate arbitrarily small perturbing potential is added, the instability interval opens up to a non-degenerate interval containing the former coexistence value. To this end, we expand the discriminant of the perturbed equation in a Taylor series around 0 with respect to the perturbation coupling parameter. This procedure has been applied to the one-dimensional Schrödinger equation with periodic potential by Moser [13]. In contrast, Simon [15] (*see* also [2]) applied degenerate eigenvalue perturbation theory to the coexistence value, seen as a double eigenvalue of h_p or h_a: since the canonical fundamental system (u, v) of the one-dimensional Schrödinger equation satisfies $u^2 \approx 1$, but $v^2 \approx 0$ near the origin, it is easy to find a perturbation that removes the degeneracy of the coexistence value to first order. However, the canonical fundamental system of a Dirac system has $u^{\mathrm{T}} u \approx 1 \approx v^{\mathrm{T}} v$ near the origin, so we cannot argue in the same way.

We find that at a coexistence value λ, where $|\mathrm{D}(\lambda)|=2$, the first derivative with respect to the perturbation always vanishes, whereas the second derivative provides a quadratic form for the perturbing potential, whose parameters are determined by the solutions of the unperturbed equation (Proposition 1). This result formally coincides with the findings of [13] for the one-dimensional Schrödinger equation; however, while it is obvious that the quadratic form is not actually nonpositive on the whole of X in the Schrödinger case, we here have to enter a more detailed analysis of the properties of the solutions in the case of coexistence (Lemma 2). Having thus verified the existence of a perturbation for which the quadratic form takes a positive value (Proposition 2), we find that this perturbation lifts $|\mathrm{D}(\lambda)|$ above 2 and thus produces a nonvanishing instability interval as desired (Proposition 3).

Finally, we collect the information gained about each of the infinitely many instability intervals to obtain the general genericity result.

The first step of the proof of Theorem 1 is a direct consequence of the following special case of the well-known stability theorem for eigenvalues, *cf.* [10] Theorem IV 3.16:

LEMMA 1. — *Let* X *be a suitable potential class of period* m, $q \in \mathrm{X}$, $h_p\,[h_a]$ *the self-adjoint realization of the symmetric ordinary differential operator*, $-i\sigma_2 \dfrac{d}{dx} + \sigma_3 + q$, *on* $[0, m]$ *with periodic* [*antiperiodic*] *boundary conditions*, λ *an eigenvalue of multiplicity* $\nu \in \{1, 2\}$ *of the operator* h_p $[h_a]$, *and* Γ *a* *closed curve in* $\mathbb{C}$ *which encloses* λ, *but no other parts of* $\sigma(h_p)$ $[\sigma(h_a)]$. *Then there is* $\delta>0$ *such that for every* $g\in X$, $\|g\|_\infty<\delta$, *the part of* $\sigma(h_p)$ $[\sigma(h_a)]$ *enclosed by* Γ *has total multiplicity* ν.

COROLLARY 1. — *The instability intervals may be indexed, using* $\mathbb{Z}$ *as index set, in such a way that the n-th instability interval behaves continuously under bounded perturbations of the potential, for each* $n\in\mathbb{Z}$.

This follows immediately from the fact that every coexistence value is a double eigenvalue, and the end points of any nonvanishing instability interval are simple eigenvalues of either h_p or h_a.

COROLLARY 2. — *For each* $n\in\mathbb{Z}$,

$$\Xi_n:=\{\, q\in X \mid \textit{the n-th instability interval of } (*) \textit{ does not vanish}\,\}$$

is an open subset of X.

Indeed, for $q\in\Xi_n$ there are two disjoint closed curves Γ_1, Γ_2 in $\mathbb{C}$ which enclose each precisely one end point of the n-th instability interval, but no other parts of the spectrum of h_p or h_a. By Lemma 1, the end points remain within their respective curves under small perturbations, so they cannot coincide.

For the second step of the proof of Theorem 1, we now assume throughout that X is a suitable potential class of period m, q, $g\in X$ and $\lambda\in\mathbb{R}$.

We perturb the original Dirac system $(*)$ by the additional potential term μq, $\mu\in\mathbb{R}$. Writing the perturbed equation as a matrix differential equation, the canonical fundamental system $\Psi(.,\mu)$, for some fixed value of the coupling parameter μ, is the solution of the initial value problem

$$\frac{\partial}{\partial x}\Psi(x,\mu)=-i\sigma_2(\sigma_3+q(x)+\mu g(x)-\lambda)\Psi(x,\mu),\qquad \Psi(0,\mu)=1. \quad (0)$$

(1 here denotes the 2×2 unit matrix.) The discriminant is

$$D(\lambda,\mu):=\operatorname{tr}\Psi(m,\mu)\ (\mu\in\mathbb{R});$$

note that we study the properties of the discriminant as a function of the perturbation coupling parameter μ, whereas λ, though arbitrary, is always kept fixed in the following. The canonical fundamental system of the unperturbed equation will be denoted by $\Phi:=\Psi(.,0)=(u,v)$.

Now we calculate the first and second partial derivative of the discriminant with respect to μ, using the well-known fact that the n-th derivative (with respect to the parameter) of a solution is a solution of the differential equation obtained by formally differentiating n times the original equation. Since in our case, the initial conditions are independent of μ, the initial value of the derivatives will be the zero matrix.

For the first derivative, we obtain the differential equation

$$\frac{\partial}{\partial x}\frac{\partial}{\partial \mu}\Psi(x,\mu)=(-i\sigma_2(\sigma_3+q(x)-\lambda)-i\sigma_2\mu g(x)) \times \frac{\partial}{\partial \mu}\Psi(x,\mu)-i\sigma_2 g(x)\Psi(x,\mu),$$

and thus for $\mu=0$ the initial value problem

$$\frac{\partial}{\partial x}\frac{\partial}{\partial \mu}\Psi(x,0)=-i\sigma_2(\sigma_3+q(x)-\lambda) \times \frac{\partial}{\partial \mu}\Psi(x,0)-i\sigma_2 g(x)\Phi(x), \qquad \frac{\partial}{\partial \mu}\Psi(0,0)=0. \quad (1)$$

The solution Φ of the unperturbed initial value problem (0) occurs as a (fixed) inhomogeneity; the homogeneous equation for $\frac{\partial}{\partial \mu}\Psi$ is identical to the unperturbed equation (0). Thus we may solve (1) by variation of constants, finding

$$\frac{\partial}{\partial \mu}\Psi(t,0)=\Phi(t)\int_0^t \Phi^{-1}(s)(-i\sigma_2)g(s)\Phi(s)\,ds \qquad (t\in[0,m]),$$

and for the discriminant,

$$\frac{\partial}{\partial \mu}\mathrm{D}(\lambda,0)=\mathrm{tr}\frac{\partial}{\partial \mu}\Psi(m,0)=\int_0^m \mathrm{tr}(\Phi(m)\,\mathrm{S}(t))\,g(t)\,dt,$$

where $\mathrm{S}(t):=\Phi^{-1}(t)(-i\sigma_2)\,\Phi(t)$ $(t\in[0,m])$.

The initial value problem for the second derivative with respect to μ,

$$\frac{\partial}{\partial x}\frac{\partial^2}{\partial \mu^2}\Psi(x,0)=-i\sigma_2(\sigma_3+q(x)-\lambda) \times \frac{\partial^2}{\partial \mu^2}\Psi(x,0)-2i\sigma_2 g(x)\frac{\partial}{\partial \mu}\Psi(x,0), \qquad \frac{\partial^2}{\partial \mu^2}\Psi(0,0)=0, \quad (2)$$

again has the unperturbed differential equation (0) as its homogeneous part, so iterating the variation of constants we find

$$\frac{\partial^2}{\partial \mu^2}\Psi(t,0)=\Phi(t)\int_0^t \Phi^{-1}(s)(-2i\sigma_2)g(s)\frac{\partial}{\partial \mu}\Psi(s,0)\,ds$$
$$=\Phi(t)\int_0^t \Phi^{-1}(s)(-2i\sigma_2)g(s)\Phi(s)\int_0^s \Phi^{-1}(r)(-i\sigma_2)g(r)\Phi(r)\,dr\,ds,$$

and

$$\frac{\partial^2}{\partial \mu^2}\mathrm{D}(\lambda,0)=\mathrm{tr}\frac{\partial^2}{\partial \mu^2}\Psi(m,0)=2\int_0^m\int_0^t \mathrm{tr}(\Phi(m)\,\mathrm{S}(t)\,\mathrm{S}(s))\,g(t)\,g(s)\,ds\,dt.$$

The following propositions, which constitute the remainder of the second part of the proof of Theorem 1, restrict our attention to the situation in which λ is a coexistence value of (*); in this case, all solutions of (*) are periodic [if $D(\lambda, 0)=2$] or antiperiodic [if $D(\lambda, 0)=-2$] with period m.

PROPOSITION 1. — *If λ is a coexistence value of (*), then*

$$\frac{\partial}{\partial\mu}D(\lambda, 0)=0, \qquad \textit{and} \qquad \frac{\nu(\lambda)}{2}\frac{\partial^2}{\partial\mu^2}D(\lambda, 0) = -\left(\int_0^m \varphi_1 g\right)^2+\left(\int_0^m \varphi_2 g\right)^2+\left(\int_0^m \varphi_3 g\right)^2,$$

with $\nu=\operatorname{sgn} D(\lambda, 0)$, *and the coefficient functions* φ_j *obtained from* $\Phi=(u, v)$ *as* $\varphi_1:=1/2\,(u^T u+v^T v)$, $\varphi_2:=1/2\,(u^T u-v^T v)$ and $\varphi_3:=u^T v$.

Remark. — Though formally similar and of comparable significance, the φ_j differ from those of [13] in that both components of u and v are involved, while in the Schrödinger case only the functions, but not their derivatives come in.

Proof. — By (anti-) periodicity, $\varphi(m)=\nu(\lambda)\,1$, and as the Wronskian $\det\Phi\equiv 1$, we obtain setting $\Phi(s)=:\begin{pmatrix} a & b\\ c & d\end{pmatrix}(s)$:

$$S(s)=\begin{pmatrix} d & -b\\ -c & a\end{pmatrix}(s)\begin{pmatrix} -c & -d\\ a & b\end{pmatrix}(s) \qquad (s\in[0, m]).$$

Thus $\operatorname{tr} S\equiv 0$, which implies $\frac{\partial}{\partial\mu}D(\lambda, 0)=0$. Concerning the second derivative, we observe that

$$\frac{\nu(\lambda)}{2}\frac{\partial^2}{\partial\mu^2}D(\lambda, 0)=\int_0^m\int_0^t \operatorname{tr}(S(t)\,S(s))\,g(t)\,g(s)\,ds\,dt = \frac{1}{2}\int_0^m\int_0^m \operatorname{tr}(S(t)\,S(s))\,g(t)\,g(s)\,ds\,dt,$$

as the integrand is symmetric under permutation of s and t. If we now insert

$$\operatorname{tr}(S(t)\,S(s))=2\,(c(t)\,d(t)+a(t)\,b(t))\,(c(s)\,d(s)+a(s)\,b(s)) - (d^2(t)+b^2(t))\,(c^2(s)+a^2(s))-(d^2(s)+b^2(s))\,(c^2(t)+a^2(t)),$$

and remember $u=\begin{pmatrix} a \\ c \end{pmatrix}$, $v=\begin{pmatrix} b \\ d \end{pmatrix}$, it turns out that

$$\begin{aligned}\frac{\nu(\lambda)}{2}\frac{\partial^2}{\partial\mu^2}D(\lambda, 0) &= \left(\int_0^m (ab+cd)g\right)^2 - \left(\int_0^m (a^2+c^2)g\right)\left(\int_0^m (b^2+d^2)g\right) \\ &= \left(\int_0^m u^T vg\right)^2 - \left(\int_0^m u^T ug\right)\left(\int_0^m v^T vg\right) \\ &= \left(\int_0^m \varphi_3 g\right)^2 - \left(\left(\int_0^m \varphi_1 g\right)^2 - \left(\int_0^m \varphi_2 g\right)^2\right). \quad \square\end{aligned}$$

Proposition 1 shows that the second variation of the discriminant with respect to the perturbing potential g is given by a quadratic form for g. If we can find some $g \in X$ for which this form takes a positive value, the absolute value of the discriminant at λ will rise above 2 for small values of $|\mu|$, so we will have removed the degeneracy, the coexistence value broadening to a nonvanishing instability interval. In order to prove that the quadratic form is not altogether nonpositive, we have to take a closer look at the coefficient function φ_2.

LEMMA 2. — *$\varphi_2 \equiv 0$ implies $q \equiv \lambda$.*

Remark. — If $q \equiv \lambda$, then λ sits in the middle of the only nonvanishing instability interval and therefore cannot be a coexistence value. By contraposition, if λ is a coexistence value, then $\varphi_2 \not\equiv 0$.

It can also be shown that $\varphi_3 \equiv 0$ implies $q \equiv \lambda$. From the uniqueness of the solutions of (∗) it follows immediately that $\varphi_1 \neq 0$ even pointwise.

Proof. — We apply Prüfer's transformation substituting the two components of u and v by the polar functions r_j, θ_j, $j \in \{1, 2\}$: $u = r_1 \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \end{pmatrix}$, $v = r_2 \begin{pmatrix} \cos\theta_2 \\ \sin\theta_2 \end{pmatrix}$. As u and v are solutions of the differential equation (∗), we find

$$0 = r_j' \begin{pmatrix} -\sin\theta_j \\ \cos\theta_j \end{pmatrix} - r_j\theta_j' \begin{pmatrix} \cos\theta_j \\ \sin\theta_j \end{pmatrix} + r \begin{pmatrix} \cos\theta_j \\ -\sin\theta_j \end{pmatrix} + (q-\lambda) r_j \begin{pmatrix} \cos\theta_j \\ \sin\theta_j \end{pmatrix}.$$

Taking the $\mathbb{R}^2$ scalar product by $\begin{pmatrix} \cos\theta_j \\ \sin\theta_j \end{pmatrix}$, and $\begin{pmatrix} -\sin\theta_j \\ \cos\theta_j \end{pmatrix}$, respectively, we find the differential equations for θ_j and r_j:

$$\theta_j' = (q-\lambda) + \cos 2\theta_j, \qquad r_j' = r_j \sin 2\theta_j.$$

The initial values are $r_1(0) = r_2(0) = 1$, $\theta_1(0) = 0$, $\theta_2(0) = \pi/2$.

Now $\varphi_2 \equiv 0$ implies $u^T u \equiv v^T v$, so $r_1 \equiv r_2$. It follows that $r_1' \equiv r_2'$, so $\sin 2\theta_1 \equiv \sin 2\theta_2$, which implies $\theta_2 \in \{\theta_1(t), \pi/2 - \theta_1(t)\} \bmod 2\pi$ $(t \in [0, m])$.

Uniqueness of the solutions rules out the first possibility, and as the solutions are continuous, $2\theta_2 \equiv \pi - 2\theta_1$, and thus both $\theta_2' \equiv -\theta_1'$ and $\cos 2\theta_2 \equiv -\cos 2\theta_1$. Using the equations for θ_j, we find

$$(q-\lambda)+\cos 2\theta_1 = \theta_1' = -\theta_2' = -(q-\lambda)-\cos 2\theta_2 = -(q-\lambda)+\cos 2\theta_1.$$

Consequently, $(q-\lambda)\equiv 0$. □

PROPOSITION 2. — *If* λ *is a coexistence value of* $(*)$, *there exists* $g \in X$ *such that* $g \notin \operatorname{span}_{\mathbb{R}}\{1, N\}$, *where* $N := \{\varphi_1, \varphi_2\}^{\perp}$. *g may be chosen orthogonal to* φ_1.

Remark. — In $\operatorname{span}_{\mathbb{R}}\{1, N\}$, 1 represents the constant function taking the value 1; by Definition 1.*a*) the constant functions belong to X. Moser [13] considers the smaller $N := \{\varphi_1, \varphi_2, \varphi_3\}^{\perp}$ so that only $\varphi_2 \not\equiv 0$ *or* $\varphi_3 \not\equiv 0$ is needed. With this choice, however, one has to strengthen Definition 1.*c*), demanding $\varphi_3 \in X$ as well. This would destroy the possibility pointed out in Remark 2 to Definition 1, of taking for X a space of *even* potential functions, since in that case φ_1 and φ_2 are even, but φ_3 is odd. Therefore we prefer to leave φ_3 out of consideration.

Proof. — If we assume that for every $f \in X$ there is a $c \in \mathbb{R}$ such that $f - c \in N$, *i.e.* $\int_0^m (f-c)\varphi_j = 0\ (j \in \{1, 2\})$, this holds in particular for φ_1 and φ_2, which belong to X by Definition 1.*c*). Thus we find real numbers c_1, c_2 such that $\int_0^m (\varphi_j - c_j)\varphi_k = 0\ (j, k \in \{1, 2\})$. Consequently,

$$\left(\int_0^m \varphi_1^2\right)\left(\int_0^m \varphi_2^2\right) = \left(c_1 \int_0^m \varphi_1\right)\left(c_2 \int_0^m \varphi_2\right)$$
$$= \left(c_2 \int_0^m \varphi_1\right)\left(c_1 \int_0^m \varphi_2\right) = \left(\int_0^m \varphi_1 \varphi_2\right)^2.$$

Regarding X as a subset of the Hilbert space $L^2([0, m])$ in the obvious way, and using the properties of the Cauchy-Schwarz inequality, we find some $\alpha \in \mathbb{R}$ with $\varphi_2 \equiv \alpha \varphi_1$. As $\varphi_1(0) = 1$ and $\varphi_2(0) = 0$, it follows that $\alpha = 0$, *i.e.* $\varphi_2 \equiv 0$ which contradicts Lemma 2. Therefore there is some $f \in X, f \notin \operatorname{span}_{\mathbb{R}}\{1, N\}$. Since a function from $\operatorname{span}_{\mathbb{R}}\{1, N\}$ remains in this space upon the addition of a constant, the function

$$g := f - \left(\int_0^m \varphi_1\right)^{-1} \int_0^m f\varphi_1,$$

which is orthogonal to φ_1, has the desired properties. □

The following proposition concludes the second part of the proof of Theorem 1.

PROPOSITION 3. — *If* X *is a suitable potential class, then for every* $n \in \mathbb{Z}$, Ξ_n *is dense in* X.

Proof. — Let $q \in X$, $n \in \mathbb{Z}$ such that the n-th instability interval vanishes (*i.e.* $q \notin \Xi_n$), λ the corresponding coexistence value and $\nu := \operatorname{sgn} D(\lambda, 0)$. By Proposition 2, there is a $g \in X$ such that $\int_0^m g\,\varphi_1 = 0$, $\int_0^m g\,\varphi_2 \neq 0$. Thus by Proposition 1,

$$C_0 := \frac{\nu}{2} \frac{\partial^2}{\partial \mu^2} D(\lambda, 0) > 0.$$

The discriminant is also three times continuously differentiable with respect to μ. Hence there are constants $M > 0$ and $C_1 > 0$ such that

$$\left| \frac{1}{6} \frac{\partial^3}{\partial \mu^3} D(\lambda, \mu) \right| < C_1 \qquad (|\mu| < M).$$

Expanding $D(\lambda, .)$ in a Taylor series around 0 up to third order, we therefore have

$$|D(\lambda, \mu)| \geqq 2 + C_0 \mu^2 - C_1 |\mu|^3 > 2 \qquad (0 < |\mu| < \min\{M, C_0/C_1\}).$$

Consequently, for sufficiently small non-zero μ, the degeneracy is removed, and the former coexistence value λ lies within a nonvanishing instability interval. In other words: $q + \mu g \in \Xi_n$. □

Now we conclude the *proof* of Theorem 1. Corollary 2 and Proposition 3 have shown that Ξ_n is a dense open subset of X, for each $n \in \mathbb{Z}$. This means that, for every $n \in \mathbb{Z}$, X is the only closed subset of X containing Ξ_n, so the interior of $X \setminus \Xi_n$ is empty. Thus the set $\bigcup_{n \in \mathbb{Z}} X \setminus \Xi_n$ of all potentials from X for which at least one instability interval vanishes, is a countable union of nowhere dense sets. □

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