ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 59, nº 2 (1993), p. 201-236

<http://www.numdam.org/item?id=AIHPA_1993__59_2_201_0>

Ann. Inst. Henri Poincaré,

Vol. 59, n° 2, 1993, p. 201-236. *Physique théorique*

N-dimensional affine Weyl-Heisenberg wavelets

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ABSTRACT. – n-dimensional coherent states systems generated by translations, modulations, rotations and dilations are described. Starting from unitary irreducible representations of the n-dimensional affine Weyl-Heisenberg group, which are not square-integrable, one is led to consider systems of coherent states labeled by the elements of quotients of the original group. Such systems can yield a resolution of the identity, and then be used as alternatives to usual wavelet or windowed Fourier analysis. When the quotient space is the phase space of the representation, different embeddings of it into the group provide different descriptions of the phase space.

RÉSUMÉ. – On décrit des systèmes d'états cohérents engendrés par des translations, dilatations et rotations en n-dimensions. Partant de représentations unitaires irréductibles du groupe de Weyl-Heisenberg affine en dimension n, qui ne sont pas de carré intégrable, on est naturellement conduit à considérer des systèmes d'états cohérents indexés par les éléments d'un espace quotient. De tels systèmes fournissent des résolutions de l'identité, et peuvent ainsi être utilisés comme des alternatives à l'analyse par ondelettes ou à l'analyse de Fourier à fenêtre glissante. Dans le cas

particulier où l'espace quotient en question est isomorphe à l'espace des phases associé à la représentation, différents plongements de celui-ci dans le groupe fournissent différentes descriptions de l'espace des phases.

I. INTRODUCTION

Wavelet analysis was originally proposed as an alternative to windowed Fourier analysis in a signal processing context. It was recognized later on that similar techniques had been used for a long time in many different communities, such as harmonic analysis and approximation theory, image analysis, optics and quantum physics.

Since the early days of wavelet analysis, the concept of "wavelet" itself has changed. The original wavelets were functions generated from a single one (the so-called mother wavelet) by dilations and translations; the mother wavelet was then a function of vanishing integral, having good localization properties in both the direct space and the Fourier space. The term wavelet has now to be understood in a more general sense. The aim of wavelet analysis is mainly to provide different representations of functions (or signals), as superpositions of elementary functions. The corresponding representation is then used for different purposes, such as for instance data compression, feature extraction or pattern recognition.

An important aspect of such a program is the determination of the possible decompositions that are adapted to a given problem. We will focus here on the continuous decompositions, otherwise stated the coherent states approach. It is well known that an elegant formulation of the theory of coherent states can be obtained through the language of group representation theory. More precisely, it was shown in [Gr.Mo.Pa], [As.Kl] that a system of coherent states is canonically associated with any square-integrable representation of a separable locally compact group. The simplest examples, namely the wavelets and Gabor functions (*i.e.* coherent states associated with windowed Fourier transform) are obtained from the affine (or $ax+b$) group and the Weyl-Heisenberg group respectively. These two examples have been particularly interesting in a signal analysis context, since they provide representations of signals in terms of time-frequency (or phase space) variables (the scale being interpreted as an inverse frequency in the affine case). Actually, in both cases the time-frequency plane corresponds to the phase space of the group representation, in the geometric sense. Other examples of groups (more specially semidirect products) are considered in [Ka].

During the last few years, efforts have been made to construct different time-frequency representations of the same nature. One of us proposed in [To. 1-2] to consider a bigger group containing both the affine and the Weyl-Heisenberg groups, to interpolate between affine wavelet analysis and Windowed Fourier analysis. In such a case, it was shown that since the usual representations of such a bigger group (called the affine Weyl-Heisenberg group) are not square-integrable, the usual construction does not apply, and must be modified in a suitable way. More precisely, the restriction of the representation to a suitable quotient space of the group (the associated phase space in that case) restores square-integrability (in a slightly modified form) and thus yields systems of coherent states.

We address here the problem of the n-dimensional generalization of such coherent states systems, and proceed as follows. We consider the extension of the n-dimensional Weyl-Heisenberg group by dilations and rotations in $\mathbb{R}^n$. The corresponding group, called the affine-Weyl-Heisenberg group $\mathbf{G}_{a\,\mathrm{WH}}$ then also contains as a subgroup the group considered in [Mu] for the construction of n-dimensional wavelets. $\mathbf{G}_{a\,\mathrm{WH}}$ essentially possesses two types of representations (called here the Stone-Von-Neumann-type and the affine-type) representations, none of which is square-integrable. Restricting to the Stone-Von-Neumann-type representations, we show that they become square-integrable when restricted to an appropriate homogeneous space (such a restriction involves the introduction of a cross section of the principal bundle $\mathbf{G}_{a\,\mathrm{WH}} \rightarrow$ *homogeneous space*, and the results of course depend on the cross section). Moreover, if the homogeneous space contains the phase space of the representation, one gets systems of coherent states directly from the group action (*i.e.* there exists strictly admissible sections in our terminology). This in turn implies the existence of an associated wavelet transform possessing good covariance properties. Our goal is to provide wavelet-type phase space description of functions of $L^2(\mathbb{R}^n)$, with overcomplete families of functions having different phase space localization properties. Potential applications include for example local frequency analysis and texture characterization in images (in the two-dimensional case), and related problems in higher dimensional situations.

The paper is organized as follows. Section II is devoted to a brief description of classical results and methods on continuous time-frequency decompositions, coherent states, and group theoretical approaches to the problem of constructing such decompositions. In section III, we study the structure of the n-dimensional affine Weyl-Heisenberg group, and describe its representation theory. We in particular exhibit two main series of representations, the phase spaces of which are two different quotient spaces of $\mathbf{G}_{a\,\mathrm{WH}}$. Finally, section IV is devoted to the description of some representation theorems associated which quotients of $\mathbf{G}_{a\,\mathrm{WH}}$. More precisely, we focus on the problem of finding strictly admissible sections, since admissible and weakly admissible ones are in general less difficult to obtain.

II. COHERENT STATES, SQUARE-INTEGRABLE GROUP REPRESENTATIONS AND WAVELETS

1. Continuous wavelet decompositions

Continuous wavelet decompositions were introduced (or re-introduced, since mathematicians and signal processers have been using similar techniques for quite a long time [Me.1]) by A. Grossmann and J. Morlet [Gr.Mo] in a signal analysis context. In the one-dimensional case, the basic idea is to decompose an arbitrary $L^2(\mathbb{R})$ function into elementary contributions (that we will generically call wavelets for simplicity), generated from a unique one (the mother wavelet) by applying simple transformations:

$$f=\int_{\Lambda} T_f(\lambda)\,\psi_\lambda\,d\mu(\lambda) \tag{II.1}$$

Here Λ stands for the above set of simple transformations, assumed to be a measured space with measure μ. In such a context, the set of coefficients $T_f(\lambda)$, as a function on Λ, is interpreted as another representation of the function f itself, in other words a transform of f. For instance, it is a very usual procedure in signal analysis to study such alternative representations to extract informations from an analyzed signal.

The most famous example was proposed in the mid fourties by D. Gabor [Ga], and has been widely studied and used since Gabor's original paper. The starting point was to introduce a notion of locality into Fourier analysis ([1]), by using appropriate windows. Letting $g\in L^1(\mathbb{R})\cap L^2(\mathbb{R})$ be such a window, to any $f\in L^2(\mathbb{R})$ is associated its windowed Fourier transform G_f:

$$G_f(b,\omega)=\int_{\mathbb{R}} f(x)\,e^{-i\omega\cdot(x-b)}\,g(x-b)^*\,dx \tag{II.2}$$

([1]) Our conventions are as follows: the Hermitian product is linear in the left argument, and the Fourier transform $\hat{f}$ of f is given by $\hat{f}(\xi)=\int f(x)\,e^{-i\xi x}\,dx$.

Simple arguments show that the map

$$f \in \mathrm{L}^2(\mathbb{R}) \to \mathrm{G}_f \in \mathrm{L}^2(\mathbb{R}^2) \tag{II.3}$$

is actually an isometry (up to a constant factor), so that f can be expressed as

$$f(x) = \frac{1}{2\pi \|g\|^2} \int_{\mathbb{R}^2} \mathrm{G}_f(b, \omega)\, e^{i\omega\cdot(x-b)} g(x-b)\, db\, d\omega \tag{II.4}$$

(the R.H.S. converging weakly to f), *i.e.* in a form similar to (II.1). Here the simple transformations are just modulations and translations, and the associated wavelets are the so-called Gabor functions [2]:

$$g_{(b,\omega)}(x) = e^{i\omega\cdot(x-b)} g(x-b), \qquad b, \omega \in \mathbb{R} \tag{II.5}$$

An alternative representation was proposed by A. Grossmann and J. Morlet [Gr.Mo], in which the modulations were replaced by dilations. This presented the advantage of enforcing the spatial localization ability of the method. Indeed, since the Gabor functions are of constant size, the spatial resolution can't be better than the window's size. Let then ψ be a mother wavelet, assumed to be a $\mathrm{L}^1(\mathbb{R})$ function such that in addition the following admissibility condition holds

$$c_\psi = \int_0^\infty |\hat{\psi}(\xi)|^2 \frac{d\xi}{\xi} < \infty \tag{II.6}$$

To ψ is associated the corresponding family of wavelets, *i.e.* dilated and shifted copies of ψ

$$\psi_{(b,a)}(x) = \frac{1}{\sqrt{a}} \psi\left(\frac{x-b}{a}\right) \tag{II.7}$$

Then any Hardy function $f \in \mathrm{H}^2(\mathbb{R}) = \{ f \in \mathrm{L}^2(\mathbb{R}), \hat{f}(\xi) = 0,\ \forall \xi \leqq 0 \}$ can be decomposed as follows

$$f(x) = \frac{1}{c_\psi} \int_{\mathbb{R}_+^* \times \mathbb{R}} \mathrm{T}_f(b, a)\, \psi_{(b,a)}(x) \frac{da}{a} \frac{db}{a} \tag{II.8}$$

where the coefficients $\mathrm{T}_f(b, a)$ are given by

$$\mathrm{T}_f(b, a) = \langle f, \psi_{(b,a)} \rangle \tag{II.9}$$

and form the wavelet transform (or affine wavelet transform) of f. It was recognized later on that similar decompositions had been used by mathematicians in the context of Littlewood-Paley theory (*see* e.g. [Fr.Ja.We]), where (II.8) was known as Calderón's identity. This aspect of wavelet decompositions has been the starting point for the construction of orthonormal bases of wavelets. We will not describe that point here, and refer to [Me.2], [Da.1] for a self-contained exposition. To describe the

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(2) Notice that $\mathrm{G}_f(b, \omega)$ is now the scalar product of f and $g_{(b,\omega)}$.

$L^2(\mathbb{R})$ space instead of the $H^2(\mathbb{R})$ space, it is necessary either to assume that $|\hat{\psi}(\xi)|^2$ is an even function (as was done by Littlewood-Paley theory specialists), or to introduce an additional simple transformation, namely the symmetry with respect to the origin (or what is equivalent to consider positive and negative dilation parameters a), in order to describe positive and negative frequencies.

The generalization of Gabor analysis to arbitrary dimensions is fairly simple. Let $g \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ be the mother window, and associate with it the following Gabor functions

$$g_{(\underline{b}, \underline{\omega})}(\underline{x}) = e^{i\underline{\omega}\cdot(\underline{x}-\underline{b})} g(\underline{x}-\underline{b}), \qquad \underline{b}, \underline{\omega} \in \mathbb{R}^n \tag{II.10}$$

Then any $f \in L^2(\mathbb{R}^n)$ can be decomposed as follows

$$f(\underline{x}) = \frac{1}{(2\pi)^n \|g\|^2} \int_{\mathbb{R}^{2n}} G_f(\underline{b}, \underline{\omega}) e^{i\underline{\omega}\cdot(\underline{x}-\underline{b})} g(\underline{x}-\underline{b})\, d\underline{b}\, d\underline{\omega} \tag{II.11}$$

with $G_f(\underline{b}, \underline{\omega}) = \langle f, g_{(\underline{b}, \underline{\omega})} \rangle$. G_f is then a function in $L^2(\mathbb{R}^{2n})$. Notice that $\mathbb{R}^{2n}$ is isomorphic to the phase space of $\mathbb{R}^n$. As we will see, this is far from being a coincidence.

The generalization of affine wavelet analysis to arbitrary dimensions is a little bit more complex, since the Hardy space $H^2(\mathbb{R})$ is typically a one-dimensional object. There are essentially two versions of the n-dimensional wavelet analysis. The first one amounts to consider ψ functions that are rotation invariant and satisfy an admissibility condition similar to (II.6) (*see* [Fr.Ja.We]). The second one is based by an extension of the set Λ of simple transformations by the rotations in $\mathbb{R}^n$ [we recall here that the rotations of $\mathbb{R}^n$ form a group denoted by $SO(n)$]. To the mother wavelet $\psi \in L^1(\mathbb{R}^n)$ are then associated the following wavelets

$$\psi_{(\underline{b}, a, \underline{r})}(\underline{x}) = \frac{1}{a^{n/2}} \psi\left(\underline{\underline{r}}^{-1} \cdot \frac{\underline{x}-\underline{b}}{a}\right) \tag{II.12}$$

where $\underline{b} \in \mathbb{R}^n$, $a \in \mathbb{R}_+^*$, $\underline{\underline{r}} \in SO(n)$. The admissibility condition then reads

$$0 < k_\psi = \mathrm{Vol}(SO(n-1)) \int_{\mathbb{R}^n} |\hat{\psi}(\underline{\xi})|^2 \frac{d\underline{\xi}}{\|\underline{\xi}\|^n} < \infty \tag{II.13}$$

[We recall here that as a compact group, $SO(n)$ has finite volume]. In such a context, any $f \in L^2(\mathbb{R}^n)$ can be decomposed as

$$f(\underline{x}) = \frac{1}{k_\psi} \int_{\mathbb{R}_+^* \times \mathbb{R}^n \times SO(n)} T_f(\underline{b}, a, \underline{\underline{r}})\, \psi_{(\underline{b}, a, \underline{r})}(\underline{x}) \frac{da}{a} \frac{d\underline{b}}{a^n} dm(\underline{\underline{r}}) \tag{II.14}$$

where $dm(\underline{\underline{r}})$ is some invariant measure on $SO(n)$ that we will specify later, and the wavelet transform is now the function of $L^2(\mathbb{R}_+^* \times \mathbb{R}^n \times SO(n))$ defined by

$$T_f(\underline{b}, a, \underline{\underline{r}}) = \langle f, \psi_{(\underline{b}, a, \underline{r})} \rangle \tag{II.15}$$

The wavelet transform is then in that case a function of $n(n+1)/2+1$ variable, and can't be considered as a function on the phase space. We will come back to that point a little bit later.

2. Coherent states

The term *coherent states* refers to a quantization technique that grew up from the theory of canonical coherent states for the quantum harmonic oscillator (*see* e.g. [Al.Go], [Pe], [Kl.Sk] for a review). The notion has been widely used and generalized since its introduction, and there exist now many different versions of generalized coherent states. As stressed in the introduction of [Kl.Sk], these versions share a set of minimal properties. We will say that a family of (generalized) coherent states is a set of vectors v_λ in a Hilbert space $\mathscr{H}$, indexed by some measured space Λ with positive measure μ, such that

– v_λ is a strongly continuous function of λ.

– there is an associated resolution of the identity, *i.e.* if v_λ denotes the linear form $u \to \langle u, v_\lambda \rangle$:

$$\int v_\lambda \otimes \underline{v_\lambda}\, d\mu(\lambda) = 1 \tag{II.16}$$

in the weak sense.

Coherent states then provide a functional representation of the Hilbert space $\mathscr{H}$. Indeed, the map

$$\mathrm{T}: \quad u \in \mathscr{H} \to \mathrm{T}_u \tag{II.17}$$

where

$$\mathrm{T}_u(\lambda) = \langle u, v_\lambda \rangle \tag{II.18}$$

assigns to any finite vector $u \in \mathscr{H}$ a function on Λ which is square-integrable and continuous. Moreover, $\mathrm{T}.\mathscr{H}$ does not span the whole $\mathrm{L}^2(\Lambda, \mu)$. One easily checks that if $\mathrm{F} \in \mathrm{T}.\mathscr{H}$, then F fulfills a reproducing kernel equation

$$\mathrm{F}(\lambda) = \int_\Lambda \mathrm{K}(\lambda, \rho)\, \mathrm{F}(\rho)\, d\mu(\rho) \tag{II.19}$$

where the reproducing kernel is given by

$$\mathrm{K}(\lambda, \rho) = \langle v_\rho, v_\lambda \rangle \tag{II.20}$$

Many constructions have been proposed to generate coherent states systems. Among those, we shall be concerned with the square-integrable group representations approach, which has the advantage of explicitely implementing covariance properties of the T-transform.

3. Square-integrable group representations

It was realized by H. Moscovici and A. Verona [Mo], [Mo.Ve] and A. Grossmann, J. Morlet and T. Paul independently [Gr.Mo.Pa.1-2] that there is a deep connection between the usual wavelet decompositions, coherent states theory and the theory of square-integrable group representations. Indeed, the set of simple transformations used to generate the wavelets from a single one in general inherits the structure of a group $\mathbf{G}$ (as is the case for instance for translations, modulations or dilations). Moreover, there is in addition a very important concept in signal analysis, namely that of covariance of the representation. Let us consider the example of a position-frequency representation of an image. One may want the representation of a shifted image to be a shifted copy of the representation of the image, or in other words one may ask for translation covariance. This is actually the case for both Gabor and affine wavelet analysis. One may also ask for rotation covariance, which is now fulfilled only by affine wavelet analysis. What happens there is simply that the simple transformations used to generate the wavelets are represented in a simple way in the transform space. As we will see, this is the consequence of the fact that all such wavelets are generated from a representation of the group of simple transformations, and that the representation is unitarily equivalent to a subrepresentation of the regular representation, *i.e.* a representation of the group $\mathbf{G}$ onto $L^2(\mathbf{G})$.

The connexion between time-frequency representation theorems and square-integrable group representations was realized by A. Grossmann, J. Morlet and T. Paul in [Gr.Mo.Pa.1-2]. Let us start by briefly describing the construction of [Gr.Mo.Pa.1-2]. Let then $\mathbf{G}$ be a separable locally compact Lie group, and let π be a unitary strongly continuous representation of $\mathbf{G}$ on the Hilbert space $\mathcal{H}$. π is said to be square-integrable (or to belong to the discrete series of $\mathbf{G}$) if

– π is irreducible.

– There exists at least a vector $v \in \mathcal{H}$ such that

$$0 < \int_{\mathbf{G}} |\langle \pi(g)\cdot v, v\rangle|^2 \, d\mu(g) < \infty \qquad \text{(II.21)}$$

Such a vector is said to be *admissible*.

Square-integrable group representations have been extensively studied in the literature, in particular for compact groups [Bar], locally compact unimodular groups [God] and non-unimodular locally compact groups [Du.Moo], [Car]. The main result (from the wavelet point of view of course) is the following.

THEOREM [Du.Moo], [Car]. – *Let π be a square-integrable strongly continuous unitary representation of the locally compact group $\mathbf{G}$ on $\mathcal{H}$.*

Then there exists a positive self-adjoint operator C *such that for any admissible vectors* $v_1, v_2 \in \mathcal{H}$ *and for any* $u_1, u_2 \in \mathcal{H}$

$$\int_{\mathbf{G}} \langle u_1, \pi(g)\cdot v_1 \rangle \langle \pi(g)\cdot v_2, u_2 \rangle \, d\mu(g) = \langle \mathrm{C}^{1/2}\cdot v_2, \mathrm{C}^{1/2}\cdot v_1 \rangle \langle u_1, u_2 \rangle \quad \text{(II.22)}$$

Moreover, the set of admissible vectors coincides with the domain of C. □

We will denote as usual by λ the left-regular representation of **G**. A simple consequence of the previous theorem is that a representation π of **G** is square integrable if and only if is unitarily equivalent to a subrepresentation of the left-regular representation λ (*see* [Du.Moo] for instance). The corresponding intertwiners can be realized as follows. If v is an admissible vector in $\mathcal{H}$, and $v' \in \mathcal{H}$, one can then introduce the corresponding Schur coefficients, *i.e.* the matrix coefficients of elements of **G**:

$$c_{v,v'}(g) = \langle v', \pi(g)\cdot v \rangle, \qquad g \in \mathbf{G} \quad \text{(II.23)}$$

Denote by T the map which assigns to any $u \in \mathcal{H}$ the family of coefficients $c_{v,u}(g)$, $g \in \mathbf{G}$

$$\mathrm{T}: \quad u \in \mathcal{H} \to \mathrm{T}_u = c_{v,u}(\cdot) \in \mathrm{L}^2(\mathbf{G}) \quad \text{(II.24)}$$

T is called the left transform in [Gr.Mo.Pa.1-2]. It realizes the intertwinning between π and λ as follows.

$$\mathrm{T} \circ \pi = \lambda \circ \mathrm{T} \quad \text{(II.25)}$$

The idea of Grossmann, Morlet and Paul was to use (II.24) and (II.25) for the analysis of functions, in the case where $\mathcal{H}$ is a function space. This was the starting point of many applications, especially in a signal analysis context. The left transform T is used to obtain another representation of functions, and (II.25) expresses the covariance of the transform. Notice that the continuity assumption is fulfilled by construction. Moreover, the left transform also satisfies a reproducing kernel equation, expressing that the image of $\mathcal{H}$ by T is not the whole $\mathrm{L}^2(\mathbf{G})$.

Let us consider for example the case of the so-called one-dimensional *affine group*, or "$ax+b$" *group*

$$\mathbf{G}_{\mathrm{aff}} = \mathbb{R} \times \mathbb{R}_+^* \quad \text{(II.26)}$$

with group operation

$$(q, a)\cdot(q', a') = (q + aq', aa') \quad \text{(II.27)}$$

Simple application of the coadjoint orbits method (*see* e.g. [Gui]) shows that $\mathbf{G}_{\mathrm{aff}}$ has basically two inequivalent irreducible unitary representations on $\mathrm{H}_{\pm}^2(\mathbb{R})$, of the form

$$[\pi(b, a)\cdot f](x) = \frac{1}{\sqrt{a}} f\left(\frac{x-b}{a}\right) \quad \text{(II.28)}$$

One easily sees that such a representation is square-integrable, and that the corresponding left transform is nothing else but the affine wavelet transform described in (II.8) and (II.9). The covariance equation (II.25) simply expresses that the affine wavelet transform of a dilated and shifted copy of a function $f(x)$ is nothing else but a dilated and shifted version of the wavelet transform of $f(x)$. Finally, the admissibility of a vector $\psi \in L^2(\mathbb{R})$ reduces to (II.6). In a signal analysis context, the translation parameter is interpreted as a position (or time) parameter, and the scale parameter a as a frequency parameter (more precisely the inverse of a frequency parameter). One is then led to a *time-frequency representation.*

To generalize the previous construction to the n-dimensional case, one faces an irreducibility problem. The n-dimensional affine group $\mathbf{G}_{\text{aff}} = \mathbb{R}^n \times \mathbb{R}_+^*$ doesn't act irreducibly on $L^2(\mathbb{R}^n)$, so that no square-integrability is possible. The solution proposed by R. Murenzi [Mu] is to take the semi-direct product of the affine group by $SO(n)$. The resulting group, denoted by $IG(n)$ then yields the wavelets described in (II.12)-(II.15). Notice that in such a case, the covariance by translation and dilation is extended with a rotation covariance.

Finally, let us consider the case of the n-dimensional *Weyl-Heisenberg group* $\mathbf{G}_{\text{WH}}$. We will only consider now the case of the so-called *polarized Weyl-Heisenberg group* $\mathbf{G}_{\text{WH}}^{\text{pol}}$

$$\mathbf{G}_{\text{WH}}^{\text{pol}} = \mathbb{R}^{2n} \times S^1 \tag{II.29}$$

with group operation

$$(\underline{q}, \underline{p}, \varphi) \cdot (\underline{q}', \underline{p}', \varphi') = (\underline{q} + \underline{q}', \underline{p} + \underline{p}', \varphi + \varphi' + \underline{p} . \underline{q}' [\text{mod}\, 2\pi]) \tag{II.30}$$

By the Stone-Von-Neumann theorem, all the irreducible unitary representations are unitarily equivalent to one of the following form (*see* [Mac], [Sch]). If $f \in L^2(\mathbb{R}^n)$

$$[\pi(\underline{q}, \underline{p}, \varphi) \cdot f](x) = e^{i\mu(\varphi - \underline{p} \cdot (\underline{x} - \underline{q}))} f(\underline{x} - \underline{q}) \tag{II.31}$$

for some $\mu \in \mathbb{Z}^*$. Set $\mu = 1$ for simplicity. The representation is square-integrable, and the corresponding left transform is the Gabor transform described in (II.2)-(II.5), up to a phase factor $e^{i\varphi}$. This can be interpreted as a square-integrable projective representation of the abelian group $\mathbb{R}^{2n}$. Notice that in such a case, one does not have full covariance by position-frequency shifts, precisely because of this phase factor. One then speaks of twisted covariance.

4. Square-integrability modulo a subgroup

Let us just start this section by exhibiting simple examples for which the square-integrable group representation approach does not seem convenient. The first example is that of the *full* (*i.e.* unpolarized) *Weyl-Heisenberg group*

$$\mathbf{G}_{\text{WH}} = \mathbb{R}^{2n} \times \mathbb{R} \tag{II.32}$$

with a group operation (almost the same as before) given by:

$$(\underline{q}, \underline{p}, \varphi)\cdot(\underline{q}', \underline{p}', \varphi')=(\underline{q}+\underline{q}', \underline{p}+\underline{p}', \varphi+\varphi'+\underline{p}\cdot\underline{q}') \qquad \text{(II.33)}$$

Again, it follows from Stone-Von-Neumann theorem that all unitary irreducible representations are given by (II.31), but now any $\mu\in\mathbb{R}^*$ is convenient. Anyway, such a simple change (which does not basically change the structure of the group) dramatically turns the representation (II.31) into a non square-integrable representation (essentially because the compact subgroup S^1 has been replaced by the non-compact $\mathbb{R}$, that leads to divergent integrals).

Another interesting example is that of the n-dimensional affine wavelets, *i.e.* the wavelets associated with the so-called IG (n) group [Mu]. For $n=2$, $\mathrm{IG}(2)\cong\mathbb{R}^4$, but for $n>2$, the dimension of IG (n) equals $n(n+1)/2+1$, and then IG (n) is not isomorphic to the phase space of $\mathbb{R}^n$. Nevertheless, a simple calculation shows that the whole group is not necessary to characterize functions through their wavelet transform. Indeed, let $f\in L^2(\mathbb{R}^n)$; parametrizing SO (n) by the associated Euler angles, and setting to zero the Euler angles corresponding to the factor SO $(n-1)$, it is still possible to reconstruct f from the restricted wavelet transform. The reconstruction formula is the same as (II.14), except that the integral is now taken over the coset space $\mathrm{IG}(n)/\mathrm{SO}(n-1)\cong\mathbb{R}^{2n}\times\mathbb{R}_+^*\times S^{n-1}$, and that the admissibility constant k_ψ has now to be replaced by $k_\psi/\mathrm{Vol}(\mathrm{SO}(n-1))$. This is the simplest case of wavelets associated with an homogeneous space. Notice that is such a case, the covariance has been replaced by a twisted covariance.

From such examples arises the notion of square-integrability modulo a subgroup [Al.An.Ga]. Indeed, in both cases one is interested to drop the extra factor [here $\mathbb{R}$ or SO $(n-1)$], either to recover square-integrability or to reduce the number of variables of the representation. Given a representation π of a group **G** on a Hilbert space $\mathscr{H}$ such that all integrals of the type (II.22) diverge, one may wonder whether such integrals might converge or diverge when restricted to an appropriate homogeneous space **G/H** for some closed subgroup $\mathbf{H}\subset\mathbf{G}$. Of course π is not defined directly on **G/H**, and it is necessary to first embed **G/H** in **G**. This is realized by using the canonical fiber bundle structure of **G**.

$$\Pi:\quad \mathbf{G}\to\mathbf{G}/\mathbf{H}$$

Let σ be a Borel section of this fiber bundle (it is well known that such sections always exist [Mac]), and introduce

$$\pi_\sigma=\pi\circ\sigma \qquad \text{(II.34)}$$

Let μ be some quasi-invariant measure on **G/H**. It then makes sense to study the operator

$$\mathscr{A}:\quad u\in\mathscr{H}\to\int_{\mathbf{G}/\mathbf{H}}\langle u, \pi_\sigma(x)\psi\rangle\,\pi_\sigma(x)\psi\,d\mu(x) \qquad \text{(II.35)}$$

Depending on the properties of the $\mathscr{A}$ operator, it may be possible to associate with it an isometry

$$\mathrm{T}: \quad \mathscr{H} \to \mathrm{L}^2(\mathrm{G/H})$$

similar to wavelet transforms for $\mathscr{H} \simeq \mathrm{L}^2(\mathbb{R})$. Such a program was carried out in particular cases, namely for special groups: in particular the Poincaré group [Al.An.Ga.1-2] or he one-dimensional affine Weyl-Heisenberg group [To.1-2].

It is important to notice that in such a context the covariance properties of the representation are lost. We will see that it is still possible to obtain some twisted covariance properties.

III. THE AFFINE WEYL-HEISENBERG GROUP AND ITS COADJOINT ORBITS

The main tool we will use throughout this study is the group generated by translations, modulations, dilations and rotations in $\mathbb{R}^n$. It is the n-dimensional generalisation of the one-dimensional affine Weyl-Heisenberg group considered in [To.1], and we will also call it the n-dimensional affine Weyl-Heisenberg group $\mathbf{G}_{a\,\mathrm{WH}}$. This section is devoted to the study of this group and of its representation theory.

1. Structure of $\mathbf{G}_{a\,\mathrm{WH}}$

The affine Weyl-Heisenberg group is topologically isomorphic to

$$\mathbf{G}_{a\,\mathrm{WH}} \simeq \mathbb{R}^{2n+1} \times \mathbb{R}_+^* \times \mathrm{SO}(n) \tag{III.1}$$

and has a structure of semi-direct product of the n-dimensional Weyl-Heisenberg group by $\mathbb{R}_+^* \times \mathrm{SO}(n)$.

The corresponding generic element is of the form

$$\left.\begin{array}{c} g = (\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi), \\ \underline{q}, \underline{p} \in \mathbb{R}^n, \qquad a \in \mathbb{R}_+^*, \qquad \varphi \in \mathbb{R}, \qquad \underline{\underline{r}} \in \mathrm{SO}(n) \end{array}\right\} \tag{III.2}$$

with group operation

$$\begin{aligned} (\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi) \cdot (\underline{q}', \underline{p}', a', \underline{\underline{r}}', \varphi') \\ = (\underline{q} + a\underline{\underline{r}} \cdot \underline{q}', \underline{p} + a^{-1} \underline{\underline{r}} \cdot \underline{p}', aa', \underline{\underline{r}} \cdot \underline{\underline{r}}', \varphi + \varphi' + \underline{p} \cdot (a\underline{\underline{r}} \cdot \underline{q}')) \end{aligned} \tag{III.3}$$

It is readily verified that the inverse of $g \in \mathbf{G}_{a\,\mathrm{WH}}$ is given by:

$$(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi)^{-1} = (-a^{-1} \underline{\underline{r}}^{-1} \cdot \underline{q}, -a\underline{\underline{r}}^{-1} \cdot \underline{p}, a^{-1}, \underline{\underline{r}}^{-1}, -\varphi + \underline{p} \cdot \underline{q}) \tag{III.4}$$

It is easy to see that $\mathbf{G}_{a\,\mathrm{WH}}$ is unimodular, that is that the following measure is both left and right invariant:

$$d\mu(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi) = d\underline{q}\, d\underline{p} \frac{da}{a} dm(\underline{\underline{r}})\, d\varphi \tag{III.5}$$

where $dm(\underline{\underline{r}})$ is the Haar measure on $\mathrm{SO}(n)$, normalized so that $m(\mathrm{SO}(n))=1$. The rotation group $\mathrm{SO}(n)$ is conveniently described by means of the corresponding Euler angles (*see* [Vil] for instance) as follows. Let $e_1, e_2, \ldots, e_n \in \mathbb{R}^n$ be a fixed orthonormal frame. Then the stabilizer of e_n is isomorphic to an $\mathrm{SO}(n-1)$ subgroup of $\mathrm{SO}(n)$, and the decomposition $\mathrm{SO}(n)=\mathrm{S}^{n-1}\times\mathrm{SO}(n-1)$ (where $\mathrm{S}^{n-1}\cong\mathrm{SO}(n)/\mathrm{SO}(n-1)$ is the $(n-1)$-dimensional sphere embedded into $\mathbb{R}^n$) yields the following decomposition of elements $\underline{\underline{r}}\in\mathrm{SO}(n)$:

$$\underline{\underline{r}}=\underline{\underline{r}}^{(n-1)}\cdot\underline{\underline{r}}^{(n-2)}\cdot\ldots\cdot\underline{\underline{r}}^{(1)} \tag{III.6}$$

where

$$\underline{\underline{r}}^{(k)}=\underline{\underline{r}}_1(\theta_1^k)\cdot\ldots\cdot\underline{\underline{r}}_k(\theta_k^k) \tag{III.7}$$

and $\underline{\underline{r}}_k(\theta)$ is the rotation of angle θ in the oriented plane defined by e_k and e_{k+1} if $k=1, \ldots, n-1$ and e_n and e_1 if $k=n$. Here, θ_j^k runs over $[0, \pi]$ when $j\neq k$ and over $[0, 2\pi]$ when $j=k$. The corresponding form for the Haar measure on $\mathrm{SO}(n)$ reads

$$dm(\underline{\underline{r}})=dm(\theta_1^1, \ldots, \theta_{n-1}^{n-1})=\mathrm{A}(n)^{-1}\prod_{k=1}^{n-1}\prod_{j=1}^{k}[\sin^{j-1}(\theta_j^k)\,d\theta_j^k] \tag{III.8}$$

for some constant $\mathrm{A}(n)$ only depending on the dimension (*see* [Vil]). The n-dimensional Weyl-Heisenberg group can be realized as a matrix group as follows. The generic element $g=(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi)\in\mathbf{G}_{a\,\mathrm{WH}}$ is realized as the matrix:

$$g=\begin{pmatrix} 1 & {}^t[a\underline{\underline{r}}^{-1}\cdot\underline{p}] & \varphi \\ 0 & [a\underline{\underline{r}}] & [\underline{q}] \\ 0 & 0 & 1 \end{pmatrix} \tag{III.9}$$

the group law being represented by matrix multiplications. Here, ${}^t[a\underline{\underline{r}}^{-1}\cdot\underline{p}]$ stands for the transpose of the vector $a\underline{\underline{r}}^{-1}\cdot\underline{p}$.

The Lie algebra $\mathscr{G}_{a\,\mathrm{WH}}$ of $\mathbf{G}_{a\,\mathrm{WH}}$ can also be represented as a Lie algebra of matrices, as follows. For $\lambda\in\mathbb{R}$ and $\underline{\underline{\mathrm{R}}}\in so(n)$, the Lie algebra of $\mathrm{SO}(n)$, set $\underline{\underline{\mathrm{R}}}^\lambda=\underline{\underline{\mathrm{R}}}+\lambda\underline{\underline{\mathrm{K}}}$, where $\underline{\underline{\mathrm{K}}}$ is the unit matrix. Then

$$\mathscr{G}_{a\,\mathrm{WH}}\cong\left\{\begin{pmatrix} 0 & {}^t\underline{\xi} & t \\ 0 & \underline{\underline{\mathrm{R}}}^\lambda & \underline{x} \\ 0 & 0 & 0 \end{pmatrix}, \ \underline{\xi}, \underline{x}\in\mathbb{R}^n, \lambda, t\in\mathbb{R}, \underline{\underline{\mathrm{R}}}\in so(n)\right\} \tag{III.10}$$

$\mathscr{G}_{a\,\mathrm{WH}}$ is then spanned by the following set of $(n+2)\times(n+2)$ matrices, which form a basis of $\mathscr{G}_{a\,\mathrm{WH}}$.

$$\underline{\underline{\mathrm{T}}}=\begin{pmatrix} 0 & . & . & . & 0 & 1 \\ 0 & . & . & . & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & . & 0 \end{pmatrix}, \quad \underline{\underline{\mathrm{K}}}=\begin{pmatrix} 1 & . & . & . & 0 & 0 \\ 0 & 1 & . & . & 0 & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & 1 \end{pmatrix}$$

$$\underline{\underline{P}}_i = \begin{pmatrix} 0 & . & 1 & . & 0 & 0 \\ 0 & . & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & . & . & . & . & 0 \end{pmatrix}, \qquad (\underline{\underline{P}}_i)^\nu_\mu = \delta_{\mu,0}\,\delta_{\nu,i+1}$$

$$\underline{\underline{Q}}_i = \begin{pmatrix} 0 & . & . & . & . & 0 \\ 0 & . & . & . & . & . \\ . & . & . & . & . & 1 \\ . & . & . & . & . & . \\ 0 & . & . & . & . & 0 \end{pmatrix}, \qquad (\underline{\underline{Q}}_i)^\nu_\mu = \delta_{\nu,0}\,\delta_{\mu,i+1}$$

$$\underline{\underline{J}}^i_j = \begin{pmatrix} 0 & . & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & -1 & . & . \\ . & . & 1 & . & . & . \\ 0 & . & . & . & . & 0 \end{pmatrix}, \qquad (\underline{\underline{J}}^i_j)^\nu_\mu = \delta_{\nu,j+1}\,\delta_{\mu,i+1} - \delta_{\mu,j+1}\,\delta_{\nu,i+1}$$

Then any element $X \in \mathscr{G}_{a\,\mathrm{WH}}$ can be written in the form

$$X = x^i \cdot \underline{\underline{Q}}_i + \xi^i \cdot \underline{\underline{P}}_i + \lambda K + R^j_i \cdot \underline{\underline{J}}^i_j + t\,\underline{\underline{T}} \qquad \text{(III.11)}$$

where λ, $t \in \mathbb{R}$, $\underline{x}$, $\underline{\xi} \in \mathbb{R}^n$ and the R^j_i are the coefficients of a $so(n)$ matrix, denoted in the sequel by $\underline{\underline{R}}$.

The matrix realization of $\mathbf{G}_{a\,\mathrm{WH}}$ and $\mathscr{G}_{a\,\mathrm{WH}}$ are useful to specify the form of the exponential mapping. Indeed, if X is the element of $\mathscr{G}_{a\,\mathrm{WH}}$ defined in (III.10), and expressing exp(X) as in (III.9), a simple calculation shows that:

$$\left.\begin{aligned} &a = e^\lambda \\ &\underline{\underline{r}} = \exp(\underline{\underline{R}}) \\ &\underline{q} = (\underline{\underline{R}}^\lambda)^{-1}\,[\exp(\underline{\underline{R}}^\lambda) - 1] \cdot \underline{x} \\ &\underline{p} = {}^t\underline{\underline{R}}^\lambda\,[\exp(-{}^t\underline{\underline{R}}^\lambda) - 1] \cdot \underline{\xi} \\ &\varphi = t + \underline{\xi} \cdot \underline{\underline{R}}^{\lambda^{-2}}\,[\exp(\underline{\underline{R}}^\lambda) - \underline{\underline{R}}^\lambda - 1] \cdot \underline{x} \end{aligned}\right\} \qquad \text{(III.12)}$$

2. The one-dimensional case: classification of unitary irreducible representations

The one-dimensional affine Weyl-Heisenberg group has a much simpler structure than the multidimensional ones. Indeed, in such a case the group is solvable, so that its unitary dual can be characterized by the method of orbits (*see* [Aus.Ko]; our discussion of coadjoint orbits will closely follow [Kir] and [Sch]). Moreover, the unitary representations that can be obtained in the one-dimensional case are simple prototypes of the ones encountered in the multidimensional cases. The first step is the computation of the adjoint action of $\mathbf{G}_{a\,\mathrm{WH}}$ on $\mathscr{G}_{a\,\mathrm{WH}}$. Let $(q, p, a, \varphi) \in \mathbf{G}_{a\,\mathrm{WH}}$ and (x, ξ, λ, t) be the coordinates of some $\mathrm{X} \in \mathscr{G}_{a\,\mathrm{WH}}$ in the previous basis, namely the basis $\{\underline{\mathrm{Q}}, \underline{\mathrm{P}}, \underline{\mathrm{K}}, \underline{\mathrm{T}}\}$. A simple calculation shows that

$$\begin{aligned}\mathrm{Ad}(g)\cdot \mathrm{X} &= g\cdot \mathrm{X}\cdot g^{-1}\\ &= (ax-\lambda q,\ a^{-1}\xi+\lambda p,\ \lambda,\ t+apx-a^{-1}q\xi-pq\lambda)\end{aligned} \tag{III.13}$$

so that Ad(g) has the following matrix realization in the above basis:

$$\mathrm{Ad}(g)=\begin{pmatrix} a & 0 & -q & 0\\ 0 & a^{-1} & p & 0\\ 0 & 0 & 1 & 0\\ ap & -a^{-1}q & -pq & 1\end{pmatrix} \tag{III.14}$$

It is then simple [Kir] to derive the matrix form of the coadjoint action, with respect to the contragredient basis $\{\underline{\mathrm{Q}}^*, \underline{\mathrm{P}}^*, \underline{\mathrm{K}}^*, \underline{\mathrm{T}}^*\}$

$$\mathrm{Ad}^*(g)={}^t\mathrm{Ad}(g^{-1})=\begin{pmatrix} a^{-1} & 0 & 0 & -p\\ 0 & a & 0 & q\\ a^{-1}q & -ap & 1 & -pq\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{III.15}$$

Let then

$$\mathrm{F}=x_0^*\,\underline{\mathrm{Q}}^*+\xi_0^*\,\underline{\mathrm{P}}^*+\lambda_0^*\,\underline{\mathrm{K}}^*+t_0^*\,\underline{\mathrm{T}}^*\in\mathscr{G}^*_{a\,\mathrm{WH}} \tag{III.16}$$

The associated coadjoint orbit reads:

$$\left.\begin{aligned} x_0^* &\to x^*=a^{-1}x_0^*-pt_0^*\\ \xi_0^* &\to \xi^*=a\,\xi_0^*+qt_0^*\\ \lambda_0^* &\to \lambda^*=\lambda_0^*-pqt_0^*+a^{-1}qx_0^*-ap\,\xi_0^*\\ t_0^* &\to t^*=t_0^*\end{aligned}\right\} \tag{III.17}$$

There are then three distinct families of orbits.

a) Degenerate orbits

These are the orbits corresponding to the case

$$x_0^*=\xi_0^*=t_0^*=0,\qquad \lambda^*=\lambda_0^* \tag{III.18}$$

In such a case, the only possible polarization is

$$\mathrm{H}=\mathscr{G}_{a\,\mathrm{WH}} \tag{III.19}$$

and the corresponding unitary irreducible representations are characters of $\mathbf{G}_{a\,\mathrm{WH}}$.

b) Affine-type (or hyperboloidal) orbits

The affine-type orbits are characterized by a vanishing center, *i.e.*

$$t_0^*=0$$

The orbits are then completely characterized by the following real number

$$x^*\,\xi^*=x_0^*\,\xi_0^*=\mu\in\mathbb{R} \tag{III.20}$$

and λ^* runs over the real line. Consider for instance the following element in the orbit O_F characterized by μ

$$F = \underline{Q}^* + \mu \underline{P}^* \tag{III.21}$$

They then have the shape of an hyperboloid in the three-dimensional space. Let $\mathbf{G}_F$ be the stability subgroup of F for the coadjoint representation. Clearly, since

$$\mathrm{Ad}^*(q, p, a, \varphi) \cdot F = a^{-1} \underline{Q}^* + \mu a \underline{P}^* + (qa^{-1} - a\mu p) \underline{K}^* \tag{III.22}$$

$\mathbf{G}_F$ is given by

$$\mathbf{G}_F = \{(q, q/\mu, 1, \varphi), q, \varphi \in \mathbb{R}\} \tag{III.23}$$

The only possible polarization is

$$H = \mathbb{R}\underline{Q} \oplus \mathbb{R}\underline{P} \oplus \mathbb{R}\underline{T} \tag{III.24}$$

Such a polarization fulfills the Pukanszky condition (*i. e.* $\mathrm{Ad}^*(e^H) \cdot F = F + H^\perp$). Indeed, let $X = x\underline{Q} + \xi \underline{P} + t\underline{T}$. Then

$$\mathrm{Ad}^*(e^X) \cdot F = F + (x - \mu\xi) \underline{K}^* \tag{III.25}$$

which proves the assertion. Notice that $\mathbf{H} \backslash \mathbf{G}_{a\,\mathrm{WH}} \cong \mathbb{R}_+^*$.

Let us now look for the associated representations. Let then F be defined by (III.21), and let $h = (q, p, 1, \varphi) = \exp(X) \in \mathbf{H} = \exp(H)$. Then the coordinates of X are $(q, p, 0, \varphi - pq/2)$. F defines a character χ_F of $\mathbf{H}$ as follows

$$\chi_F(h) = e^{i \langle F, X \rangle} \tag{III.26}$$

and because Pukanszky condition is satisfied, the representation $\mathcal{U}$ of $\mathbf{G}_{a\,\mathrm{WH}}$ unitarily induced from χ_F is irreducible. $\mathcal{U}$ is obtained as follows. Consider the Hilbert space

$$\mathcal{H} = \left\{ f : \mathbf{G}_{a\,\mathrm{WH}} \to \mathbb{C}, \int_{\mathbf{H} \backslash \mathbf{G}_{a\,\mathrm{WH}}} |f(x)|^2 \, d\mu(x) < \infty, \right.$$
$$\left. \textit{and } f(h \cdot g) = \chi_F(h) f(g), \ \forall h \in \mathbf{H} \right\} \tag{III.27}$$

Here μ is a quasi-invariant measure on $\mathbf{H} \backslash \mathbf{G}_{a\,\mathrm{WH}}$, for example the Lebesgue measure on the half-line (another choice would lead to unitarily equivalent representations [Mac]). Let Δ be the square root of the corresponding Radon-Nikodym derivative. Then $\mathcal{U}$ acts on $\mathcal{H}$ as follows

$$\begin{aligned}
[\mathcal{U}(q, p, a, \varphi) \cdot f](0, 0, u, 0) &= \Delta(q, p, a, \varphi) f[(0, 0, u, 0) \cdot (q, p, a, \varphi)] \\
&= \Delta(q, p, a, \varphi) f[(uq, u^{-1}p, 1, \varphi \cdot)(0, 0, au, 0)] \\
&= \Delta(q, p, a, \varphi) e^{i[uq + \mu u^{-1} p]} f(0, 0, au, 0)
\end{aligned}$$

Then, identifying $\mathcal{H}$ with $L^2(\mathbb{R}_+^*)$ and taking into account the explicit form of $\Delta(q, p, a, \varphi)$, the resulting representation is

$$[\mathcal{U}(q, p, a, \varphi) \cdot f](u) = \sqrt{a}\, e^{i[uq + \mu p u^{-1}]} f(a\underline{u}) \tag{III.28}$$

$\mathcal{U}$ is the direct generalization of the usual representation of the affine group used in [Gr.Mo.Pa.1-2].

c) Stone-Von-Neumann-type (or saddle) orbits

These are orbits characterized by a nonzero value of the t^* parameter:

$$t^* \neq 0 \tag{III.29}$$

In such a case, for any initial value x_0^*, ξ_0^*, one can find values of p, q, a such that $x^* = \xi^* = 0$. One can then choose without loss of generality

$$x_0^* = \xi_0^* = 0 \tag{III.30}$$

so that the coadjoint orbits read

$$\left.\begin{aligned} x^* &= -p t^* \\ \xi^* &= q t^* \\ \lambda^* &= \lambda_0^* + \frac{x^* \cdot \xi^*}{t^*} \end{aligned}\right\} \tag{III.31}$$

Such orbits then have a saddle shape, and are characterized by $t^* \in \mathbb{R}^*$, $\lambda_0^* \in \mathbb{R}$. Let then

$$F = t^* \underline{\underline{T}}^* + \lambda^* \underline{\underline{K}}^* \tag{III.32}$$

and let O_F be the corresponding orbit. The stability subgroup of F in the coadjoint representation reads:

$$\mathbf{G}_F = \{(0, 0, a, \varphi),\ a \in \mathbb{R}_+^*,\ \varphi \in \mathbb{R}\} \tag{III.33}$$

It is not very difficult to see that the only possible polarizations are

$$H = \left\{\begin{array}{l} \mathbb{R}\underline{\underline{Q}} \oplus \mathbb{R}\underline{\underline{K}} \oplus \mathbb{R}\underline{\underline{T}} \\ \text{or} \\ \mathbb{R}\underline{\underline{P}} \oplus \mathbb{R}\underline{\underline{K}} \oplus \mathbb{R}\underline{\underline{T}} \end{array}\right. \tag{III.34}$$

which correspond to symmetric choices. A simple calculation also shows that they both satisfy the Pukanszky condition. Let us choose

$$H = \mathbb{R}\underline{\underline{Q}} \oplus \mathbb{R}\underline{\underline{K}} \oplus \mathbb{R}\underline{\underline{T}} \tag{III.35}$$

$\mathbf{H} \diagdown \mathbf{G}_{a\,\mathrm{WH}}$ is then isomorphic to the real line. Let

$$X = x\underline{\underline{Q}} + \lambda \underline{\underline{K}} + t\underline{\underline{T}} \tag{III.36}$$

be an arbitrary element of H. Then

$$\exp\{X\} = \left(x\frac{e^\lambda - 1}{\lambda},\ 0,\ e^\lambda,\ t\right) \tag{III.37}$$

Then F defines the following character χ_F of $\mathbf{H}$:

$$\chi_F(q, 0, a, \varphi) = \exp\{i(\lambda^* \ln(a) + t^* \varphi)\} \tag{III.38}$$

The representation $\mathcal{U}$ unitarily induced from χ_F then acts on

$$\mathcal{H}=\left\{ f:\mathbf{G}_{a\,\mathrm{WH}}\to\mathbb{C},\ \int_{\mathbf{H}\backslash\mathbf{G}_{a\,\mathrm{WH}}} |f(x)|^2\,d\mu(x)<\infty,\right.$$
$$\left. \textit{and } f(h\cdot g)=\chi_F(h)f(g),\ \forall h\in\mathbf{H}\right\} \quad \text{(III.39)}$$

where μ is some quasi-invariant measure on $\mathbf{H}\backslash\mathbf{G}_{a\,\mathrm{WH}}$, for example the Lebesgue measure on the real line. $\mathcal{U}$ takes the form:

$$\begin{aligned}[\mathcal{U}(q,p,a,\varphi)\cdot f](0,\xi,1,0) &= \Delta(q,p,a,\varphi)\,f[(0,\xi,1,0)\cdot(q,p,a,\varphi)]\\ &= \Delta(q,p,a,\varphi)\,f[(q,0,a,\varphi+\xi q)\cdot(0,a(\xi+p),1,0)]\\ &= \Delta(q,p,a,\varphi)\,e^{i[\lambda^*\ln(a)+t^*(\varphi+\xi q)]}f(0,0,a(\xi+p),0)\end{aligned}$$

Then, identifying $\mathcal{H}$ with $L^2(\mathbb{R})$ and taking into account the explicit form of $\Delta(q,p,a,\varphi)$, the resulting representation is

$$[\mathcal{U}(q,p,a,\varphi)\cdot f](\xi)=\sqrt{a}\,e^{i[\lambda^*\ln(a)+t^*(\varphi+\xi q)]}f(a(\xi+p)) \quad \text{(III.40)}$$

The case $t^*=1$, $\lambda^*=0$ precisely corresponds to the representation of the affine Weyl-Heisenberg group studied in [To.1-2].

d) Summary

The results of this section can be summarized in the following theorem

THEOREM. – *The characters, together with the affine-type and the Stone-Von-Neumann-type representations listed in* (III.28) *and* (III.40) *exhaust all possible irreducible unitary representations of the one-dimensional affine Weyl-Heisenberg group. None of them is square-integrable.* □

The theorem follows from the application of Kirillov-Pukanszky coadjoint orbits method (for solvable Lie groups) [Kir], [Pu] to the affine Weyl-Heisenberg group and from the above discussion. The last assertion follows from a simple calculation.

The consequence of such a result is that the coherent states construction of [Gr.Mo.-Pa.1-2] does not apply to the considered group, and it is necessary to go to coset spaces as in [To.1-2].

3. Coadjoint orbits of $\mathbf{G}_{a\,\mathrm{WH}}$

In the case $n\geqq 2$, the affine Weyl-Heisenberg group has a much more complex structure than in the one-dimensional case. In particular, it is not solvable, due to the presence of the SO(n) subgroup. Thus the method of orbits can't be directly applied to classify the irreducible unitary representations as in the solvable case. Nevertheless, it is interesting to study carefully the coadjoint orbits, that exhibit a behaviour similar to what happens in one dimension. The orbit structure is then more complicated, but the coadjoint orbits essentially fall into two classes, n-dimensional generalizations of the affine-type and the Stone-Von-Neumann-type orbits appearing in one dimension. We then study such orbits in this section, and we will consider in more details at the end of the section the case $n=2$, which is a little bit simpler because SO (2) is abelian.

To study the orbits, we will use the matrix realization of $\mathbf{G}_{a\,\mathrm{WH}}$ and $\mathcal{G}_{a\,\mathrm{WH}}$ described in (III.9-10). Let then

$$g=\begin{pmatrix} 1 & {}^t[a\underline{\underline{r}}^{-1}\cdot \underline{p}] & \varphi \\ 0 & [a\underline{\underline{r}}] & [\underline{q}] \\ 0 & 0 & 1 \end{pmatrix} \in \mathbf{G}_{a\,\mathrm{WH}} \tag{III.41}$$

$$\mathrm{X}=\begin{pmatrix} 0 & {}^t\underline{\xi} & t \\ 0 & \underline{\underline{\mathrm{R}}}^{\lambda} & \underline{x} \\ 0 & 0 & 0 \end{pmatrix} = \underline{x}^i\cdot \underline{\mathrm{Q}}_i+\underline{\xi}^i\cdot \underline{\mathrm{P}}_i+t\,\underline{\mathrm{T}}+\lambda\,\underline{\mathrm{K}}+\mathrm{R}_i^j\cdot \underline{\underline{\mathrm{J}}}_j^i\in \mathcal{G}_{a\,\mathrm{WH}} \tag{III.42}$$

Then $\mathrm{Ad}(g)\cdot \mathrm{X}=g\cdot \mathrm{X}\cdot g^{-1}$ is given by the following matrix:

$$\begin{pmatrix} 0 & {}^t[a^{-1}\underline{\underline{r}}.\underline{\xi}+\underline{\underline{r}}.{}^t\underline{\underline{\mathrm{R}}}^{\lambda}.\underline{\underline{r}}^{-1}.\underline{p}] & t+a(\underline{\underline{r}}^{-1}.\underline{p}).\underline{x}-a^{-1}\underline{\xi}.(\underline{\underline{r}}^{-1}.\underline{q})-\underline{p}.(\underline{\underline{r}}.\underline{\underline{\mathrm{R}}}^{\lambda}.\underline{\underline{r}}^{-1}.\underline{q}) \\ 0 & [\underline{\underline{r}}.\underline{\underline{\mathrm{R}}}^{\lambda}.\underline{\underline{r}}^{-1}] & a\underline{\underline{r}}.\underline{x}-\underline{\underline{r}}.\underline{\underline{\mathrm{R}}}^{\lambda}.\underline{\underline{r}}^{-1}.\underline{q} \\ 0 & 0 & 0 \end{pmatrix} \tag{III.43}$$

Since

$$\underline{\underline{r}}\cdot \underline{\underline{\mathrm{R}}}^{\lambda}\cdot \underline{\underline{r}}^{-1}=\lambda\,\underline{\underline{\mathrm{K}}}+\mathrm{R}_i^j\,\underline{\underline{r}}\cdot \underline{\underline{\mathrm{J}}}_j^i\cdot \underline{\underline{r}}^{-1}=\lambda\,\underline{\underline{\mathrm{K}}}+(\underline{\underline{r}}\cdot \underline{\underline{\mathrm{R}}}\cdot \underline{\underline{r}}^{-1})_i^j\,\underline{\underline{\mathrm{J}}}_j^i \tag{III.44}$$

one deduces the adjoint action of $\mathbf{G}_{a\,\mathrm{WH}}$

$$\mathrm{Ad}(g)\left\{\begin{array}{l} \underline{x}_0\to \underline{x}=a\underline{\underline{r}}\cdot \underline{x}_0-\lambda_0\,\underline{q}-(\underline{\underline{\mathrm{R}}}_0)_i^j\,\underline{\underline{r}}\cdot \underline{\underline{\mathrm{J}}}_j^i\cdot \underline{\underline{r}}^{-1}\cdot \underline{q} \\ \underline{\xi}_0\to \underline{\xi}=a^{-1}\underline{\underline{r}}\cdot \underline{\xi}_0+\lambda_0\,\underline{p}-(\underline{\underline{\mathrm{R}}}_0)_i^j\,\underline{\underline{r}}\cdot \underline{\underline{\mathrm{J}}}_j^i\cdot \underline{\underline{r}}^{-1}\cdot \underline{p} \\ \lambda_0\to \lambda=\lambda_0 \\ (\underline{\underline{\mathrm{R}}}_0)_i^j\to \mathrm{R}_i^j=(\underline{\underline{r}}\cdot \underline{\underline{\mathrm{R}}}_0\cdot \underline{\underline{r}}^{-1})_i^j \\ t_0\to t=t_0+a(\underline{\underline{r}}^{-1}\cdot \underline{p})\cdot \underline{x}_0-a^{-1}\underline{\xi}_0\cdot (\underline{\underline{r}}^{-1}\cdot \underline{q}) \\ \quad -\lambda_0\,\underline{p}\cdot \underline{q}-(\underline{\underline{\mathrm{R}}}_0)_i^j\,\underline{p}\cdot (\underline{\underline{r}}\cdot \underline{\underline{\mathrm{J}}}_j^i\cdot \underline{\underline{r}}^{-1}\cdot \underline{q}) \end{array}\right. \tag{III.45}$$

Then

$$\mathrm{Ad}(g^{-1})\left\{\begin{array}{l} \underline{x}_0\to \underline{x}=a^{-1}\underline{\underline{r}}^{-1}\cdot \underline{x}_0+\lambda_0\,a^{-1}\underline{\underline{r}}^{-1}\cdot \underline{q}+(\underline{\underline{\mathrm{R}}}_0)_i^j\,a^{-1}\underline{\underline{r}}^{-1}\cdot \underline{\underline{\mathrm{J}}}_j^i\cdot \underline{q} \\ \underline{\xi}_0\to \underline{\xi}=a\underline{\underline{r}}^{-1}\cdot \underline{\xi}_0-\lambda_0\,a\underline{\underline{r}}^{-1}\cdot \underline{p}+(\underline{\underline{\mathrm{R}}}_0)_i^j\,a\underline{\underline{r}}^{-1}\cdot \underline{\underline{\mathrm{J}}}_j^i\cdot \underline{p} \\ \lambda_0\to \lambda=\lambda_0 \\ (\underline{\underline{\mathrm{R}}}_0)_i^j\to \mathrm{R}_i^j=(\underline{\underline{r}}^{-1}\cdot \underline{\underline{\mathrm{R}}}_0\cdot \underline{\underline{r}})_i^j \\ t_0\to t=t_0-\underline{p}\cdot \underline{x}_0+\underline{\xi}_0\cdot \underline{q}-\lambda_0\,\underline{p}\cdot \underline{q}-(\underline{\underline{\mathrm{R}}}_0)_i^j\,\underline{p}\cdot (\underline{\underline{\mathrm{J}}}_j^i\cdot \underline{q}) \end{array}\right. \tag{III.46}$$

and then the coadjoint action

$$\mathrm{Ad}^*(g)\left\{\begin{array}{l} \underline{x}_0^* \to \underline{x}^* = a^{-1}\underline{\underline{r}}\cdot\underline{x}_0^* - \underline{p}t_0^* \\ \underline{\xi}_0^* \to \underline{\xi}^* = a\underline{\underline{r}}\cdot\underline{\xi}_0^* + \underline{q}t_0^* \\ \lambda_0^* \to \lambda^* = \lambda_0^* + (a^{-1}\underline{\underline{r}}\cdot\underline{x}_0^*)\cdot\underline{q} - (a\underline{\underline{r}}\cdot\underline{\xi}_0^*)\cdot\underline{p} - \underline{p}\cdot\underline{q}t_0^* \\ (\underline{\underline{R}}_0^*)_i^j \to (\underline{\underline{R}}^*)_i^j = (\underline{\underline{r}}\cdot\underline{\underline{R}}_0^*\cdot\underline{\underline{r}}^{-1})_i^j + (a^{-1}\underline{\underline{r}}\cdot\underline{x}^*)\cdot(\underline{\underline{J}}_j^i\cdot\underline{q}) \\ \qquad + (a\underline{\underline{r}}\cdot\underline{\xi}^*)\cdot(\underline{\underline{J}}_j^i\cdot\underline{p}) - t_0^*\underline{p}\cdot(\underline{\underline{J}}_j^i\cdot\underline{q}) \\ t_0^* \to t^* = t_0^* \end{array}\right. \tag{III.47}$$

Again three distinct sets of coadjoint orbits can be distinguished.

a) Degenerate or semi-degenerate orbits

Such orbits correspond to the choice

$$\underline{x}_0^* = \underline{\xi}_0^* = 0, \qquad t_0^* = 0 \tag{III.48}$$

Then λ^* is constant, and the coadjoint orbits are nothing else but SO(n)-orbits. We will not consider the associated representations, since they don't fall in the category of time-frequency representation theorems we are interested in.

b) Affine-type orbits

These orbits generalize the affine-type orbits encountered in the one-dimensional case. As we will see, the structure of the orbits is now more complicated, because of the SO(n) subgroup.

The affine-type orbits are defined by

$$t_0^* = 0 \tag{III.49}$$

and are then of the form

$$\left.\begin{array}{l} \underline{x}^* = a^{-1}\underline{\underline{r}}\cdot\underline{x}_0^* \\ \underline{\xi}^* = a\underline{\underline{r}}\cdot\underline{\xi}_0^* \\ \lambda^* = \lambda_0^* + \underline{x}^*.\underline{q} - \underline{\xi}^*\cdot\underline{p} \\ (\underline{\underline{R}}^*)_i^j = (\underline{\underline{r}}\cdot\underline{\underline{R}}_0^*\cdot\underline{\underline{r}}^{-1})_i^j + x^*\cdot(\underline{\underline{J}}_j^i\cdot q) + \underline{\xi}^*\cdot(\underline{\underline{J}}_j^i\cdot\underline{p}) \end{array}\right\} \tag{III.50}$$

Let us assume for simplicity that $\underline{x}_0^*//\underline{\xi}_0^*$: set $\underline{\xi}_0^* = \mu\underline{x}_0^*$. Let $\Gamma(\underline{x}_0^*)$ be the stability SO(n) − subgroup of $\underline{x}_0^*$. Then $\Gamma(\underline{x}_0^*) \cong \mathrm{SO}(n-1)$. Let F be the element of $\mathscr{G}_{a\mathrm{WH}}^*$, of coordinates $(\underline{x}_0^*, \underline{\xi}_0^*, \lambda_0^*, \underline{\underline{R}}_0^*)$. Denote by $\Gamma(\underline{\underline{R}}_0^*)$ the stability $\Gamma(\underline{x}_0^*)$-subgroup of $\underline{\underline{R}}_0^*$. Then the stability $\mathbf{G}_{a\,\mathrm{WH}}$-subgroup of F for the coadjoint action reads:

$$\mathbf{G}_\mathrm{F} = \{ (\underline{q}, \underline{p}, 1, \underline{\underline{s}}, \varphi),\ \underline{\underline{s}} \in \Gamma(\underline{\underline{R}}_0^*),\ \underline{q}, \underline{p} \in \mathbb{R}^n,$$
$$\text{with } \underline{q} - \mu\underline{p},\ \underline{\underline{J}}_j^i\cdot(\underline{q} + \mu\underline{p}) \perp \underline{x}_0^*,\ \forall\, i, j = 1, \ldots, n \} \tag{III.51}$$

and

$$\mathrm{O}_\mathrm{F} \cong \mathbf{G}_{a\,\mathrm{WH}}/\mathbf{G}_\mathrm{F} \tag{III.52}$$

Let us now specify a little bit more the considered orbit, by assuming that $\underline{\underline{R}}_0^*$ is $\Gamma(\underline{x}_0^*)$-invariant, so that $\Gamma(\underline{\underline{R}}_0^*) = \Gamma(\underline{x}_0^*)$. It is then easier to construct the associated representation of $\mathbf{G}_{a\,\mathrm{WH}}$. Here

$$\mathbf{G}_\mathrm{F} \cong \mathbb{R}^{n+1} \times so(n-1),$$

so that the phase space is isomorphic to

$$O_F \cong \mathbb{R}^n \times \mathbb{R}_+^* \times S^{n-1} \cong \mathbb{R}^{2n} \tag{III.53}$$

We consider the following polarisation (which of course contains the Lie algebra of $\mathbf{G}_F$)

$$H = \sum^{\oplus} \mathbb{R}\,\underline{Q}_i \oplus \sum^{\oplus} \mathbb{R}\,\underline{P}_i \oplus \mathbb{R}\,\underline{T} \oplus so(n-1) \tag{III.54}$$

Let $X \in H$ and $h = \exp(X) \in \mathbf{H} = \exp(H)$.

$$h = (\underline{q}, \underline{p}, 1, \underline{\underline{s}}, \varphi) \tag{III.55}$$

and the components of X read:

$$X = (\ln(\underline{\underline{s}}) \cdot (\underline{\underline{s}} - 1)^{-1} \cdot \underline{q}, \ln(\underline{\underline{s}}) \cdot (\underline{\underline{s}} - 1)^{-1} \cdot p, 1, \underline{\underline{s}}, t) \tag{III.56}$$

Notice that, since $\underline{\underline{s}} \cdot \underline{x}_0^* = \underline{x}_0^*$, one has

$$\underline{x}_0^* \cdot \ln(\underline{\underline{s}}) \cdot (\underline{\underline{s}} - 1)^{-1} \cdot q = \underline{x}_0^* \cdot \underline{q} \tag{III.57}$$

and

$$\underline{\xi}_0^* . \ln(\underline{\underline{s}}) . (\underline{\underline{s}} - 1)^{-1} . \underline{p} = \underline{\xi}_0^* . \underline{p} \tag{III.58}$$

F then defines the following character of H

$$\chi_F(h) = e^{i\, \underline{x}_0^* \cdot (\underline{q} + \mu\, \underline{p})} \tag{III.59}$$

Introduce then the following representation space

$$\mathscr{H} = \left\{ f : \mathbf{G}_{a\,\mathrm{WH}} \to \mathbb{C}, \int_{\mathbf{H} \diagdown \mathbf{G}_{a\,\mathrm{WH}}} |f(x)|^2 \, d\mu(x) < \infty, \right.$$
$$\left. \text{and } f(h \cdot g) = \chi_F(h) f(g), \ \forall h \in \mathbf{H} \right\} \tag{III.60}$$

Here μ is a quasi-invariant measure on $\mathbf{H} \diagdown \mathbf{G}_{a\,\mathrm{WH}}$, for example the Lebesgue measure on the half-line times the quotient of the Haar measure on $SO(n)$ by the Haar measure on $SO(n-1)$. Let Δ be the square root of the corresponding Radon-Nikodym derivative. Then the representation $\mathscr{U}$ induced from χ_F acts on $\mathscr{H}$ as follows.

$$\begin{aligned}
[\mathscr{U}(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi) \cdot f](0, 0, u, \underline{\rho}, 0)
&= \Delta(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi) f[(0, 0, u, \underline{\rho}, 0) \cdot (\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi)] \\
&= \Delta(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi) f[(u\,\underline{\rho} \cdot \underline{q}, u^{-1}\,\underline{\rho} \cdot \underline{p}, 1, \underline{\underline{s}}_1, \varphi) \cdot (0, 0, au, \underline{\underline{r}}_1, 0)] \\
&= \Delta(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi) e^{i [u\underline{\rho} \cdot \underline{q} + \mu u^{-1} \underline{\rho} \cdot \underline{p}]} f(0, 0, au, \underline{\underline{r}}_1, 0)
\end{aligned}$$

Here, $\underline{\underline{r}}_1$ is a representative element of its equivalence class in $SO(n-1) \diagdown SO(n)$, and one has written $\mathrm{ad}(\underline{\underline{r}}) \cdot \underline{\rho} = \underline{\underline{s}}_1 \cdot \underline{\underline{r}}_1$. Then, identifying $\mathscr{H}$ with $L^2(\mathbb{R}_+^* \times S^{n-1})$ and taking into account the explicit form of $\Delta(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi)$, the resulting representation is

$$[\mathscr{U}(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi) \cdot f](u, \underline{\rho}) = a^{n/2} e^{i [u\underline{\rho} \cdot \underline{q} + \mu u^{-1} \underline{\rho} \cdot \underline{p}]} f(au, \underline{\underline{r}}_1) \tag{III.61}$$

The case where $\underline{x}_0$ is not parallel to $\underline{\xi}_0$ and $\underline{\underline{R}}_0^*$ is not invariant can also be handled explicitely. The difference is that it is there necessary to carefully study the stability group $\mathbf{G}_F$.

c) Stone-Von-Neumann-type orbits

Unlike the affine-type orbits, the Stone-Von-Neumann-type orbits have a nonvanishing t^* element. As in the one-dimensional case, we are then free to choose $\underline{x}_0^* = \underline{\xi}_0^* = 0$, so that the coadjoint orbits read:

$$\left.\begin{aligned} \underline{x}^* &= -\underline{p}t^* \\ \underline{\xi}^* &= \underline{q}t^* \\ \lambda^* &= \lambda_0^* + \frac{\underline{x}^* \cdot \underline{\xi}^*}{t^*} \\ (\underline{\underline{R}}^*)_i^j &= (\underline{\underline{r}} \cdot \underline{\underline{R}}_0^* \cdot \underline{\underline{r}}^{-1})_i^j + \frac{\underline{x}^* \cdot \underline{\underline{J}}_j^i \cdot \underline{\xi}^*}{t^*} \end{aligned}\right\} \tag{III.62}$$

The coadjoint orbits are then fully characterized by t^* and $\underline{\underline{R}}_0^{\lambda^*} = \underline{\underline{R}}_0^* + \lambda \underline{\underline{K}}$.

Let us focus first on the case $\underline{\underline{R}}_0^* = 0$. In such a simple case, the stability group $\mathbf{G}_F$ is given by

$$\mathbf{G}_F = \{(0, 0, a, \underline{\underline{r}}, \varphi),\ a \in \mathbb{R}_+^*,\ \underline{\underline{r}} \in SO(n),\ \varphi \in \mathbb{R}\} \tag{III.63}$$

and the phase space reads

$$O_F \cong \mathbf{G}_{a\,\mathrm{WH}}/\mathbf{G}_F \cong \mathbb{R}^{2n} \tag{III.64}$$

Choose the following polarization

$$H = \mathbb{R}\,\underline{\underline{T}} \oplus \mathbb{R}\,\underline{\underline{K}} \oplus so(n) \oplus \sum^{\oplus} \mathbb{R}\,\mathbf{Q}_i \tag{III.65}$$

F defines a character of $\mathbf{H} = \exp(H)$ as follows. If $X \in H$, and if $h = \exp(X)$ then

$$\chi_F(h) = e^{i(t^* t + \lambda^* \lambda)} \tag{III.66}$$

The representation space is then the Hilbert space

$$\mathscr{H} = \left\{ f : \mathbf{G}_{a\,\mathrm{WH}} \to \mathbb{C},\ \int_{\mathbf{H}\backslash\mathbf{G}_{a\,\mathrm{WH}}} |f(x)|^2 \, d\mu(x) < \infty, \right.$$
$$\left. \text{and } f(h \cdot g) = \chi_F(h) f(g),\ \forall h \in \mathbf{H} \right\} \tag{III.67}$$

for some quasi-invariant measure μ on the coset space $\mathbf{H}\backslash\mathbf{G}_{a\,\mathrm{WH}}$. We will take μ to be the Lebesgue measure on $\mathbb{R}^n$. Let Δ be the square root of the corresponding Radon-Nikodym derivative. Then the representation $\mathscr{U}$ unitarily induced from χ_F acts on $\mathscr{H}$ as follows.

$$\begin{aligned} &[\mathscr{U}(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi) \cdot f](0, \underline{\xi}, 1, 1, 0) \\ &\quad = \Delta(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi) f[(0, \underline{\xi}, 1, 1, 0) \cdot (\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi)] \\ &\quad = \Delta(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi) f[(\underline{q}, 0, a, \underline{\underline{r}}, \varphi + \underline{p} \cdot \underline{\xi}) \cdot (0, a\underline{\underline{r}}^{-1} \cdot (\underline{\xi} + \underline{p}), 1, 1, 0)] \\ &\quad = \Delta(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi) e^{i[\lambda^* \ln(a) + t^*(\varphi + \underline{\xi} \cdot \underline{q})]} f(0, a\underline{\underline{r}}^{-1} \cdot (\underline{\xi} + \underline{p}), 1, 1, 0) \end{aligned}$$

Using the isomorphism $\mathscr{H} \cong L^2(\mathbb{R}^n, d\mu)$ and the explicit form of the modulus Δ, we then get the following unitary representation on $L^2(\mathbb{R}^n, d\mu)$:

$$[\mathscr{U}(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi) \cdot f](\underline{\xi}) = a^{n/2} e^{i[\lambda^* \ln(a) + t^*(\varphi + \underline{\xi} \cdot \underline{q})]} f(a\underline{\underline{r}}^{-1} \cdot (\underline{\xi} + \underline{p})) \tag{III.68}$$

The case $\underline{\underline{R}}_0^* \neq 0$ is also interesting, because the representation space involves additional internal degrees of freedom. Indeed, one then has to take into account the normalizer $\mathcal{N}(\underline{\underline{R}}_0^*)$ of $\underline{\underline{R}}_0^*$ in $\mathrm{SO}(n)$. Then

$$O_F \cong \mathbb{R}^{2n} \times \mathrm{SO}(n)/\mathcal{N}(\underline{\underline{R}}_0^*) \tag{III.69}$$

It turns out that because $\mathrm{SO}(n)$ is a simple compact group, the normalizers $\mathcal{N}(\underline{\underline{R}}_0^*)$ can be completely classified by means of Dynkin diagram techniques (*see* [Bo.Fo.Ro]). Then the stability group $\mathbf{G}_F$ reads

$$\mathbf{G}_F = \{(0, 0, a, \underline{\underline{n}}, \varphi), a \in \mathbb{R}_+^*, \underline{\underline{n}} \in \mathcal{N}(\underline{\underline{R}}_0^*), \varphi \in \mathbb{R}\} \tag{III.70}$$

The corresponding quotient is an homogeneous Kahler manifold, that takes the form

$$\mathrm{SO}(n)/\mathcal{N}(\underline{\underline{R}}_0^*) \cong \mathbf{T}^k \times \mathrm{SU}(l_1) \times \ldots \times \mathrm{SU}(l_m) \times \mathrm{SO}(p) \tag{III.71}$$

for some integral numbers k, $l_1, \ldots, l_m$, p. The phase space is then isomorphic to

$$O_F \cong \mathbb{R}^{2n} \times \mathrm{SO}(n)/\mathcal{N}(\underline{\underline{R}}_0^*) \tag{III.72}$$

and is of course of even dimension, as a symplectic manifold.

d) The two-dimensional case

Although the two-dimensional affine Weyl-Heisenberg group is not solvable, the two-dimensional case is much simpler than the general case because SO(2) is one-dimensional and then abelian. Then the above discussion can still be performed, and the main difference is that the phase space possesses here a group structure.

Indeed, let $\theta \in [0, 2\pi[$ parametrize SO(2). Then as an application of the previous formulas, one is led to the following coadjoint orbits:

$$\mathrm{Ad}^*(\underline{q}, \underline{p}, a, \underline{\underline{r}}, \varphi)
\left\{\begin{array}{l}
\underline{x}^* \to a^{-1}\underline{\underline{r}}^{-1}\cdot\underline{x}^* - \underline{p}t^* \\
\underline{\xi}^* \to a\underline{\underline{r}}^{-1}\cdot\underline{\xi}^* + \underline{q}t^* \\
\lambda^* \to \lambda^* - \underline{p}\cdot\underline{q}t^* + a^{-1}\underline{x}^*\cdot\underline{\underline{r}}^{-1}\cdot\underline{q} - a\underline{\xi}^*\cdot\underline{\underline{r}}^{-1}\cdot\underline{p} \\
\theta^* \to \theta^* + \underline{p}\times\underline{q}t^* - a^{-1}\underline{x}^*\times(\underline{\underline{r}}^{-1}\cdot\underline{q}) - a\underline{\xi}^*\times(\underline{\underline{r}}^{-1}\cdot\underline{p}) \\
t^* \to t^*
\end{array}\right. \tag{III.73}$$

The coadjoint orbits structure is then the same as before.

– Degenerate orbits: $\underline{x}^* = \underline{\xi}^* = 0$, $t^* = 0$, $\lambda^* = \lambda_0^*$, $\theta^* = \theta_0^*$. Such orbits lead to extensions of representations of SO(2) to $\mathbf{G}_{a\,\mathrm{WH}}$.

– Affine-type orbits: Such orbits are characterized by $t^* = 0$, $\underline{\xi}^* = \mu\,\underline{\underline{r}}_\alpha\cdot\underline{x}^*/\|\underline{x}^*\|^2$ where $\underline{x}^* \in \mathbb{R}$, and $\lambda^*, \theta^* \in \mathbb{R}$, for some constants $\mu \neq 0$, $\alpha \in [0, 2\pi[$. The associated phase space is as follows. Let F be an element of the considered coadjoint orbit.

$$\mathbf{G}_F = \{(\underline{q}, \mu^{-1}\underline{\underline{r}}_\alpha\cdot\underline{\underline{s}}_1\cdot\underline{q}, 1, 1, \varphi), \underline{q} \in \mathbb{R}, \varphi \in \mathbb{R}\}$$

where $\underline{\underline{s}}_1$ is the matrix $\underline{\underline{s}}_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

– Stone-Von-Neumann-type orbits: now $t^* \neq 0$, and the orbits are characterized by

$$t^* \in \mathbb{R},\ \underline{x}^*,\ \underline{\xi}^* \in \mathbb{R}^2 \quad \text{and} \quad \lambda^* = \lambda_0^* + (\underline{x}^* . \underline{\xi}^*/t^*),$$
$$\theta^* = \theta_0^* - (\underline{x}^* \times \underline{\xi}^*/t^*).$$

The corresponding phase space has the following structure. If F is an element of a given coadjoint orbit, then

$$\mathbf{G}_F = \{ (0, 0, a, \underline{\underline{r}}, \varphi),\ a \in \mathbb{R}_+^*,\ \underline{\underline{r}} \in SO(2),\ \varphi \in \mathbb{R} \}.$$

The phase space is the quotient of $\mathbf{G}_{a\,\mathrm{WH}}$ by $\mathbf{G}_F$.

IV. TIME-FREQUENCY REPRESENTATION THEOREMS

In this section, we will describe representation theorems of the type (II.1) that can be obtained from group representations restricted to homogeneous spaces. As we will see, one can derive a lot of such theorems, and we will then not try to get any classification or partial classification. We will only focus on a particular representation among those we obtained in the previous chapter, namely the Stone-Von-Neumann type representation. In other words, we will restrict our attention to the representation of $\mathbf{G}_{a\,\mathrm{WH}}$ obtained in (III.68). Moreover, we choose to set $\lambda^* = 0$ and $t^* = 1$ (since the λ^*-parameter does not seem relevant to us in the context of time-frequency representations). In addition, we will not consider the central extension of $\mathbf{G}_{a\,\mathrm{WH}}$, and then consider $\mathcal{U}$ as a projective representation. Finally, to make the connection with standard notations in wavelet analysis and Gabor analysis, it is convenient to change the signs of the $\underline{p}$ and $\underline{q}$ variables. Summarizing, we will then consider the following family of functions: if $f \in L^2(\mathbb{R}^n)$:

$$f_{(\underline{q},\, \underline{p},\, a,\, \underline{\underline{r}})}(x) = [\pi(\underline{q}, \underline{p}, a, \underline{\underline{r}}, 0) \cdot f](\underline{x}) = a^{-n/2}\, e^{i\underline{p}\cdot(\underline{x}-\underline{q})} f\left(\underline{\underline{r}}^{-1} \cdot \frac{\underline{x}-\underline{q}}{a}\right) \qquad \text{(IV.1)}$$

One trivially checks that this is indeed a projective irreducible representation of $\mathbf{G}_{a\,\mathrm{WH}}/\mathbb{R}$. One also easily obtain the following.

LEMMA. – *π is not square-integrable.* □

It is then necessary to consider restrictions of π to coset spaces. We will then work in the following general setting. Let $\Gamma \in \mathbf{G}_{a\,\mathrm{WH}}$ be a closed subgroup of $\mathbf{G}_{a\,\mathrm{WH}}$, and let

$$X = \mathbf{G}_{a\,\mathrm{WH}}/\Gamma \qquad \text{(IV.2)}$$

$\mathbf{G}_{a\,\mathrm{WH}}$ then canonically inherits a structure of principal bundle

$$\Pi:\quad \mathbf{G}_{a\,\mathrm{WH}} \to X \qquad \text{(IV.3)}$$

Let then σ be a Borel section of this principal bundle. We will denote by $\pi_\sigma : X \to \mathcal{U}(L^2(\mathbb{R}^n))$ the restriction of π to the homogeneous space defined by

$$\pi_\sigma = \pi \circ \sigma \tag{IV.4}$$

DEFINITION. — *The section σ is said to be admissible if there exists a bounded positive invertible operator $\mathscr{A}$, with bounded inverse, and a function $\psi \in L^2(\mathbb{R}^n)$ such that for all $f \in L^2(\mathbb{R}^n)$*

$$\int_X |\langle \pi_\sigma(x)\cdot\psi, f\rangle|^2 \, d\mu(x) = \langle f, \mathscr{A}\cdot f\rangle \tag{IV.5}$$

σ is said to be strictly admissible if there exists a function $\psi \in L^2(\mathbb{R}^n)$ such that for all $f \in L^2(\mathbb{R}^n)$

$$\int_X |\langle \pi_\sigma(x)\cdot\psi, f\rangle|^2 \, d\mu(x) = K\,\|f\|^2 \tag{IV.6}$$

for some positive constant K.

σ is said to be weakly admissible if there exists a continuous field of operators $\mathscr{T}(x)$, $x \in X$, and a function $\psi \in L^2(\mathbb{R}^n)$ such that for all $f \in L^2(\mathbb{R}^n)$

$$\int_X |\langle \mathscr{T}(x)\cdot\pi_\sigma(x)\cdot\psi, f\rangle|^2 \, d\mu(x) = K\,\|f\|^2 \tag{IV.7}$$

for some positive constant K. □

Otherwise stated, the section σ is strictly admissible if and only if σ is admissible and the $\mathscr{A}$ operator is a multiple of the identity. Admissible sections generate a continuous frame in the terminology of [Al.An.Ga.1-2]. In any case, we will denote by ψ_x the function $\pi_\sigma(x).\psi$ when σ is strictly admissible, and the function $\mathscr{A}\cdot\pi_\sigma(x)\cdot\psi$ [resp. $\mathscr{T}(x)\cdot\pi_\sigma(x)\cdot\psi$] in the admissible (resp. weakly admissible) case. Given a weakly admissible section, there is then an associated resolution of the identity if and only if the orbit of X through ψ is total in $L^2(\mathbb{R}^n)$. In such cases, one can then construct an associated wavelet transform. However, depending on whether the section is weakly admissible, strictly admissible or just admissible, the covariance properties of the wavelet representation will be different.

Indeed, let us assume that we are given a resolution of the identity associated with a group representation, as described in [Gr.Mo.Pa.1-2]. Then $X = \mathbf{G}$, and π is a true group representation. Let $h \in \mathbf{G}$, and let $f \in \mathscr{H} \cong L^2(\mathbb{R}^n)$. Then clearly

$$T_{\pi(h)\cdot f}(g) = T_f(h^{-1}\cdot g), \qquad g \in \mathbf{G} \tag{IV.8}$$

which is nothing else but a paraphrase of (II.20), *i.e.* a consequence of the embedding of the representation π as an irreducible summand in the left regular representation of **G**.

Consider now the case of a representation π of **G**, square integrable modulo a subgroup **H**. Then $\sigma(\mathbf{G})$ needs not be a group, and (IV.8) no longer holds. However, if the section σ of **G** is strictly admissible, there remains a partial covariance, expressed as:

$$T_{\pi_\sigma(x)\cdot\pi_\sigma(y)^{-1}.f}(x)=T_f(y), \qquad x, y\in X \tag{IV.9}$$

Of course, in the simple case where $\sigma(\mathbf{G})$ is a subgroup of $\mathbf{G}$, the partial covariance reduces to a full covariance with respect to the action of $\sigma(\mathbf{G})$.

In the case of an admissible section or a weakly admissible section σ, the situation is a little bit different. Consider the weakly admissible case for instance. Then the covariance equation can be written as

$$T_{\mathcal{T}(x)^{-1}\cdot\pi_\sigma(x)\cdot\pi_\sigma(y)^{-1}\mathcal{T}(x)^{-1}.f}(x)=T_f(y), \qquad x, y\in X \tag{IV.10}$$

If σ is admissible, the covariance is a *twisted covariance*, with *global twist*. If not, the covariance is a *twisted covariance*, with *local twist*.

We will examine in this section the problem of existence of admissible and strictly admissible sections in the case of the following subgroups

$$\Gamma_1=\{(0, 0, a, 1, \varphi)\in\mathbf{G}_{a\,\mathrm{WH}}\} \tag{IV.11}$$

$$\Gamma_2=\{(0, \underline{p}, 1, 1, \varphi)\in\mathbf{G}_{a\,\mathrm{WH}}\} \tag{IV.12}$$

$$\Gamma_3=\{(0, \bar{0}, a, \underline{\underline{r}}, \varphi)\in\mathbf{G}_{a\,\mathrm{WH}}\} \tag{IV.13}$$

In any case, we will proceed as follows (our method was refered to as the direct approach in [To.1-2]). We first choose a Borel section of the fiber bundle (IV.3), then write the associated wavelet transform, and examine whether it is admissible, strictly admissible or not. Let us start with the description of strictly admissible sections.

1. Strictly admissible sections

We show here the existence of strictly admissible sections in the three above mentioned cases. Throughout this section, we will need the following technical result

LEMMA. – *Let $\underline{k}$, $\underline{p}$ be respectively a covector and a vector in $\mathbb{R}^n$, and consider their tensor product $\underline{k}\otimes\underline{p}$. Then*

$$\mathrm{Det}[\underline{k}\otimes\underline{p}-1]=(-1)^n(1-\langle\underline{k}, \underline{p}\rangle) \quad \square \tag{IV.14}$$

Proof. – Assume first that ${}^t\underline{k}$ and $\underline{p}$ are linearly independent, and restrict to the 2-dimensional subspace spanned by ${}^t\underline{k}$ and $\underline{p}$. Then a simple computation shows that the corresponding restricted determinant equals $(1-\langle\underline{k}, \underline{p}\rangle)$. On the complementary subspace, the determinant equals $(-1)^{n-2}$, which yields (IV.14). If ${}^t\underline{k}$ and $\underline{p}$ are proportional, then the determinant is the product of $\langle\underline{k}, \underline{p}\rangle-1$ by $(-1)^{n-1}$, still leading to (IV.14). $\square$

a) $\Gamma=\Gamma_1$

We then consider the case $\Gamma=\Gamma_1\cong\mathbb{R}_+^*\times\mathbb{R}$. Let

$$X_1=\mathbf{G}_{a\,\mathrm{WH}}/\Gamma_1 \tag{IV.15}$$

X_1 can be parametrized by elements of the form $(\underline{q}, \underline{p}, \underline{\underline{r}}) \in \mathbb{R}^{2n} \times SO(n)$. X_1 is provided with the following left and right invariant measure

$$d\mu(\underline{q}, \underline{p}, \underline{\underline{r}}) = d\underline{q}\, d\underline{p}\, dm(\underline{\underline{r}}) \tag{IV.16}$$

Let σ_0 be the flat section of the fiber bundle $\Pi_1 : \mathbf{G}_{a\,\mathrm{WH}} \to X_1$

$$\sigma_0(\underline{q}, \underline{p}, \underline{\underline{r}}) = (\underline{q}, \underline{p}, 1, \underline{\underline{r}}, 0) \in \mathbf{G}_{a\,\mathrm{WH}} \tag{IV.17}$$

and let σ_β be the Borel section defined by

$$\sigma_\beta(\underline{q}, \underline{p}, \underline{\underline{r}}) = (\underline{q}, \underline{p}, \beta(\underline{p}, \underline{q}, \underline{\underline{r}}), \underline{\underline{r}}, 0) \tag{IV.18}$$

where β is a piecewise differentiable Borel mapping of X_1 into Γ_1.

Let $\psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, and set

$$\psi_{(\underline{p}, \underline{q}, \underline{\underline{r}})}(\underline{x}) = |\beta(\underline{p}, \underline{q}, \underline{\underline{r}})|^{-n/2} e^{i\underline{p}\cdot(\underline{x}-\underline{q})} \psi(\beta(\underline{p}, \underline{q}, \underline{\underline{r}})^{-1} \underline{\underline{r}}^{-1} \cdot (\underline{x}-\underline{q})) \tag{IV.19}$$

Let $f \in L^2(\mathbb{R}^n)$; associate with it its transform

$$T_f(\underline{p}, \underline{q}, \underline{\underline{r}}) = \langle f, \psi_{(\underline{p}, \underline{q}, \underline{\underline{r}})} \rangle \tag{IV.20}$$

T_f is bounded, and by Plancherel equality

$$T_f(\underline{p}, \underline{q}, \underline{\underline{r}}) = (2\pi)^{-n} |\beta(\underline{p}, \underline{q}, \underline{\underline{r}})|^{n/2} \int_{\mathbb{R}^n} \hat{f}(\underline{\xi})\, e^{i\underline{\xi}\cdot\underline{q}}\, \hat{\psi}[\underline{k}_\xi(\underline{p}, \underline{q}, \underline{\underline{r}})]^*\, d\underline{\xi} \tag{IV.21}$$

where

$$\underline{k}_\xi(\underline{p}, \underline{q}, \underline{\underline{r}}) = \beta(\underline{p}, \underline{q}, \underline{\underline{r}})\, \underline{\underline{r}}^{-1} \cdot (\underline{\xi} - \underline{p}) \tag{IV.22}$$

Since our aim is just to provide examples of strictly admissible sections, we will only focus on the case $\beta(\underline{p}, \underline{q}, \underline{\underline{r}}) = \beta(\underline{p}, \underline{\underline{r}})$ (*i.e.* no dependence on $\underline{q}$). Let now

$$F(\underline{x}) = \int_{X_1} T_f(\underline{p}, \underline{q}, \underline{\underline{r}})\, \psi_{(\underline{p}, \underline{q}, \underline{\underline{r}})}(\underline{x})\, d\mu(\underline{p}, \underline{q}, \underline{\underline{r}}) \tag{IV.23}$$

$$\hat{F}(\underline{\xi}) = \hat{f}(\underline{\xi}) \int_{SO(n)} dm(\underline{\underline{r}}) \int_{\mathbb{R}^n} d\underline{p}\, |\beta(\underline{p}, \underline{\underline{r}})|^n\, |\hat{\psi}[\underline{k}_\xi]|^2 \tag{IV.24}$$

Let

$$J_\xi = |\mathrm{Det}[\underline{\nabla}_p \cdot \underline{k}]| \tag{IV.25}$$

denote the Jacobian of the transformation $\underline{p} \to \underline{k}_\xi$. Then a straightforward calculation leads to

$$\underline{\nabla}_p \cdot \underline{k} = \beta\, \underline{\underline{r}}^{-1} [\beta^{-2} (\underline{\nabla} \cdot \beta) \otimes (\underline{\underline{r}} \cdot \underline{k}_\xi) - 1] \tag{IV.26}$$

so that

$$J_\xi = |\beta(\underline{p}, \underline{\underline{r}})|^n\, |\mathrm{Det}[\beta^{-2} (\underline{\nabla} \cdot \beta) \otimes (\underline{\underline{r}} . \underline{k}_\xi) - 1] \tag{IV.27}$$

Let $\underline{\Xi}$ be a constant $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ tensor. The second determinant does not depend on $\underline{p}$ and $\underline{\xi}$ explicitely if

$$\frac{1}{\beta(\underline{p}, \underline{\underline{r}})^2} \underline{\nabla} . \beta = -\underline{\Xi} \tag{IV.28}$$

otherwise stated

$$\beta(\underline{p}, \underline{r}) = \frac{1}{\langle \Xi, \underline{p} \rangle + f(\underline{r})} \tag{IV.29}$$

where f is some (regular) function of $\underline{r}$. In such a case, it follows from the previous technical lemma that

$$J_\xi = |\beta(\underline{p}, \underline{r})|^n \, |1 - \langle \Xi, \underline{r} \cdot \underline{k} \rangle| \tag{IV.30}$$

and then

$$\hat{F}(\underline{\xi}) = \hat{f}(\underline{\xi}) \int_{SO(n)} dm(\underline{r}) \int_{\mathbb{R}^n} |\hat{\psi}(\underline{k})|^2 \frac{d\underline{k}}{|1 - \langle \Xi, \underline{r} \cdot \underline{k} \rangle|} \tag{IV.31}$$

Such sections are strictly admissible. Indeed, if the constant multiplying $\hat{f}(\underline{\xi})$ is finite and nonzero, we are in the case describe by (IV.6). Moreover, we directly have a resolution of the identity. The convergence of the corresponding integral is obtained as soon as ψ has sufficient decay at infinity, and vanishes on the sphere $\underline{k} = \underline{r}^{-1t} \Xi$, $\underline{r} \in SO(n)$. This generalizes the admissibility condition of the usual wavelet analysis. Notice that $\Xi = 0$ (*i.e.* constant β) corresponds to N-dimensional Gabor analysis, and the admissibility condition reduces to $\psi \in L^2(\mathbb{R}^n)$, which we have by assumption.

This result generalizes the one obtained in [To.1] in the one-dimensional case. In particular, assuming that ψ is radial suppresses the angular dependence, and yields a radial n-dimensional wavelet analysis close to that used by Littlewood-Paley specialists (*see* [Fr.Ja.We] for instance).

b) $\Gamma = \Gamma_2$

Consider now the case $\Gamma = \Gamma_2 \simeq \mathbb{R}^n$. Let then

$$X_2 = \Gamma_2 \backslash \mathbf{G}_{a\,WH} \tag{IV.32}$$

be parametrized by elements of the form $(\underline{q}, a, \underline{r}) \in \mathbb{R}^n \times \mathbb{R}_+^* \times SO(n)$. Notice that $\mathbf{G}_{a\,WH}$ now acts on the coset space X_2 on the right. X_2 is provided with the following right invariant measure

$$d\mu(\underline{q}, a, \underline{r}) = d\underline{q} \frac{da}{a} dm(\underline{r}) \tag{IV.33}$$

Consider the flat section σ_0 of the fiber bundle $\Pi_2 : \mathbf{G}_{a\,WH} \to X_2$ defined by

$$\sigma_0(\underline{q}, a, \underline{r}) = (\underline{q}, 0, a, \underline{r}, 0) \in \mathbf{G}_{a\,WH} \tag{IV.34}$$

and let σ_β be the Borel section associated with the piecewise differentiable Borel mapping $\beta : X_2 \to \Gamma_2$.

$$\sigma_\beta(\underline{q}, a, \underline{r}) = (\underline{q}, \beta(\underline{q}, a, \underline{r}), a, \underline{r}, 0) \tag{IV.35}$$

We will use the parametrization of $SO(n)$ with Euler angles as in (III.6-8): set

$$\underline{r} \simeq (\Theta, \Psi) \tag{IV.36}$$

where Θ and Ψ are sets of angular variables, globally representing Euler angles for $\underline{r}_0 \in S^{n-1}$ and $\underline{\underline{s}} \in SO(n-1)$ respectively. We will denote by $dm'(\underline{\underline{s}})$ a Haar measure on $SO(n-1)$ and by $d\nu(\underline{r}_0)$ the quotient measure on the sphere:

$$dm(\underline{\underline{r}}) = d\nu(\underline{r}_0)\, dm'(\underline{\underline{s}}) \tag{IV.37}$$

Let also $\underline{k}$ be the element of $\mathbb{R}^n$ of spherical coordinates (a^{-1}, Θ). If $d\underline{k}$ is the Lebesgue measure on $\mathbb{R}^n$, then

$$\frac{da}{a} dm(\underline{\underline{r}}) = a^n\, d\underline{k}\, dm'(\underline{\underline{s}}) \tag{IV.38}$$

Let $\psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, and set

$$\psi_{(\underline{q}, a, \underline{\underline{r}})}(\underline{x}) = a^{-n/2} e^{i\beta(\underline{q}, a, \underline{\underline{r}})\cdot(\underline{x}-\underline{q})} \psi\left(\underline{\underline{r}}^{-1} \cdot \frac{\underline{x}-\underline{q}}{a}\right) \tag{IV.39}$$

Remark. – Observe that the usual n-dimensional wavelets developed by R. Murenzi [Mu] do not appear as a particular case of the above analysis. This is due to the fact that in our case, $\mathbf{G}_{a\,\mathrm{WH}}$ acts of the coset space X_2 on the right, contrary to what happens on Murenzi's analysis. We are then naturally led to use the right-invariant measure on X_2 instead of the left-invariant one. As a result, the flat section σ_0, that would lead to usual n-dimensional wavelets, is not strictly admissible.

Let $f \in L^2(\mathbb{R}^n)$; associate with it its transform

$$T_f(\underline{q}, a, \underline{\underline{r}}) = \langle f, \psi_{(\underline{q}, a, \underline{\underline{r}})} \rangle \tag{IV.40}$$

T_f is bounded, and it follows from the Plancherel equality that

$$T_f(\underline{q}, a, \underline{\underline{r}}) = (2\pi)^{-n} a^{n/2} \int_{\mathbb{R}^n} \hat{f}(\underline{\xi})\, e^{i\underline{\xi}\cdot\underline{q}}\, \hat{\psi}[\underline{u}_\xi(\underline{q}, a, \underline{\underline{r}})]^*\, d\underline{\xi} \tag{IV.41}$$

where

$$\underline{u}_\xi(\underline{q}, a, \underline{\underline{r}}) = a\underline{\underline{r}}^{-1} \cdot (\underline{\xi} - \beta(\underline{q}, a, \underline{\underline{r}})) \tag{IV.42}$$

We now restrict to the case where β is only a function of a and $\underline{\underline{r}}$: $\beta(\underline{q}, a, \underline{\underline{r}}) = \beta(a, \underline{\underline{r}})$. At that point, let us notice that the flat sections $\beta(\underline{q}, \bar{a}, \underline{\underline{r}}) = C^{st}$, among which lies the section $\beta = 0$, are not strictly admissible. From now on, we will assume that $\nabla . \beta \neq 0$. Let

$$J_\xi = |\operatorname{Det}[\underline{\nabla}_k \cdot \underline{u}_\xi]| \tag{IV.43}$$

Introduce for simplicity the $\begin{pmatrix}1\\1\end{pmatrix}$-tensor

$$\underline{\underline{R}} = a\underline{\underline{r}}^{-1} \tag{IV.44}$$

and the $\begin{pmatrix}1\\2\end{pmatrix}$-tensor $\underline{\underline{S}}$, defined by

$$S^{\nu}_{\mu\rho} = \partial_\mu R^{\nu}_{\rho} \tag{IV.45}$$

Then

$$J_\xi = |\operatorname{Det}\{\underline{\underline{S}} \cdot (\underline{\xi} - \beta) - \underline{\underline{R}} \cdot \underline{\nabla} \cdot \beta\}| \tag{IV.46}$$

$$= |\operatorname{Det}[\underline{\underline{R}} \cdot (\underline{\nabla} \cdot \beta)]| \cdot |\operatorname{Det}[(\underline{\nabla} \cdot \beta)^{-1} . \underline{\underline{S}}' \cdot u_\xi - 1]| \tag{IV.47}$$

where

$$S'^{\nu}_{\mu\rho} = \partial_\mu (\underline{\underline{R}}^{-1})^\nu_\rho \tag{IV.48}$$

Then, following the same procedure as in the last section, assume that

$$[\underline{\nabla} \cdot \beta] \cdot \underline{v} = \underline{\underline{S}}' \tag{IV.49}$$

for some fixed nonzero $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$-tensor $\underline{v}$ (for simplicity we will also denote by $\underline{v}$ the transpose tensor). A simple calculation shows that the corresponding β function is of the form

$$\beta(a, \underline{r}) = \frac{\underline{\underline{R}}^{-1} \cdot \underline{v}}{\|v\|^2} + f(\underline{\underline{s}}) \tag{IV.50}$$

for some (smooth) function f on $SO(n-1)$. Then, it follows from (IV.14) that

$$J_\xi = |\mathrm{Det}(\underline{\underline{R}})| \cdot |\mathrm{Det}(\underline{\nabla} \cdot \beta)| \cdot |\underline{v} \cdot \underline{u}_\xi - 1| \tag{IV.51}$$

Clearly $\mathrm{Det}(\underline{\underline{R}}) = a^n$. Moreover, let us choose the coordinate system so that $\underline{v} = \|v\| \underline{e}_n$, the unit vector $SO(n-1)$-invariant. Then $\underline{\underline{R}}^{-1} \cdot \underline{v} = \|v\| \underline{k}$, and

$$\mathrm{Det}[\underline{\nabla} \cdot \beta] = \|v\|^{-n} \tag{IV.52}$$

Summarizing, the sections defined by (IV.50) are strictly admissible. Indeed, if ψ is such that

$$0 < \int_{\mathbb{R}^n} |\hat{\psi}(\underline{u})|^2 \frac{d\underline{u}}{|1 - \underline{v} \cdot \underline{u}|} < \infty \tag{IV.53}$$

i.e. if $\hat{\psi}$ has sufficient decay at infinity, and vanishes continuously on the affine hyperplane defined by $\langle \underline{v}, \underline{u} \rangle = 1$, then there is an associated resolution of the identity. Of course, the choice of β in (IV.49) is far from unique. A more general choice would lead to an admissibility condition similar to (IV.60) (*see* below).

c) $\Gamma = \Gamma_3$

Let us finally consider the case of the Γ_3 subgroup. Notice that Γ_3 is nothing but the phase space associated with the considered representation. Let then $X_3 = \mathbf{G}_{a\,\mathrm{WH}}/\Gamma_3$. X_3 is provided with a left and right invariant measure, which reads

$$d\mu(\underline{p}, \underline{q}) = d\underline{p}\, d\underline{q} \tag{IV.54}$$

Let $\beta : X_3 \to \mathbb{R}^*_+$ and $\underline{\underline{\rho}} : X_3 \to SO(n)$ be two piecewise differentiable Borel functions, and denote by σ the corresponding Borel section of $\mathbf{G}_{a\,\mathrm{WH}}$. For simplicity we will assume that both funtions only depend on $\underline{p}$.

$$\sigma(\underline{q}, \underline{p}) = (\underline{q}, \underline{p}, \beta(\underline{p}), \underline{\underline{\rho}}(\underline{p})) \tag{IV.55}$$

Let $\psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, and set

$$\psi_{(\underline{q}, \underline{p})}(\underline{x}) = \beta(\underline{p})^{-n/2} e^{i\underline{p} \cdot (\underline{x} - \underline{q})} \psi\left(\underline{\underline{\rho}}(\underline{p})^{-1} \cdot \frac{\underline{x} - \underline{q}}{\beta(\underline{p})}\right) \tag{IV.56}$$

To any $f \in L^2(\mathbb{R}^n)$ associate its transform

$$T_f(\underline{q}, \underline{p}) = \langle f, \psi_{(\underline{q}, \underline{p})} \rangle \tag{IV.56}$$

T_f is bounded, and

$$T_f(\underline{q}, \underline{p}) = (2\pi)^{-n} \beta(\underline{q}, \underline{p})^{n/2} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{i \underline{\xi} \cdot \underline{q}} \hat{\psi}[\underline{u}_\xi(\underline{p})]^* d\underline{\xi} \tag{IV.57}$$

where

$$\underline{u}_\xi(\underline{p}) = \beta(\underline{p}) \underline{\underline{\rho}}(\underline{p})^{-1} \cdot (\underline{\xi} - \underline{p}) = \underline{\underline{R}}(\underline{p}) \cdot (\underline{\xi} - p) \tag{IV.58}$$

Again by arguments similar to the previous ones, it can be shown that if β and $\underline{\underline{\rho}}$ are such that

$$\beta(\underline{p})^{-1} \underline{\underline{\rho}}(\underline{p}) = \underline{\underline{\underline{K}}} \cdot p + \underline{\underline{F}} \tag{IV.59}$$

for some fixed $\binom{1}{2}$-tensor $\underline{\underline{\underline{K}}}$ and some fixed $\binom{1}{1}$-tensor $\underline{\underline{F}}$ then one gets a resolution of the identity for any ψ such that

$$\int_{\underline{u}_\xi(\mathbb{R}^n)} |\hat{\psi}(\underline{u})|^2 \frac{d\underline{u}}{|\operatorname{Det}(\underline{\underline{\underline{K}}} \cdot \underline{u} - 1)|} \tag{IV.60}$$

is bounded from below and above as a function of $\underline{\xi}$. There does not seem to exist a simpler formula for the determinant (at least we don't know how to get such a formula). The strict admissibility of the section depends on the choices of $\underline{\underline{\underline{K}}}$ and $\underline{\underline{F}}$ through the integration domain $\underline{u}_\xi(\mathbb{R}^n)$. Nevertheless, there clearly exist choices for which strict admissibility holds *i.e.* for which $u_\xi(\mathbb{R}^n) = \mathbb{R}^n$. For instance, the case $\beta = 1$, $\underline{\underline{\rho}} = 1$ (*i.e.* $\underline{\underline{\underline{K}}} = 0$, $\underline{\underline{F}} \neq 0$) yields the usual n-dimensional Gabor analysis. More generally, the sections corresponding to constant $\underline{\underline{\rho}}$'s are strictly admissible, and are the direct generalizations of the one described in [To.1] in the one-dimensional case.

2. Admissible sections

If one does not ask for strict admissibility, there is of course much more possible choices for the sections of $\mathbf{G}_{a\,\mathrm{WH}}/\Gamma$. Indeed, consider anyone of the previously considered homogeneous spaces (or other ones) X, and let σ be a Borel section of $\mathbf{G}_{a\,\mathrm{WH}} \to \mathrm{X}$.

Then let $\psi \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$, and consider the operator

$$\mathscr{A}: \quad f \in L^2(\mathbb{R}^n) \to \mathscr{A} \cdot f = \int_{\mathrm{X}} \langle f, \pi_\sigma(x) \cdot \psi \rangle \pi_\sigma(x) \cdot \psi \, d\mu(x) \tag{IV.61}$$

Then the section σ is admissible only if one can find a ψ such that $\mathscr{A}$ is bounded from below and above.

The general structure of admissible sections has been discussed in great details in [Al.An.Ga.1]. In particular, it is shown there that admissibility implies the existence of reproducing kernel subspaces of $L^2(X)$, to which the wavelet transform belongs.

One of the difficulties of the method comes from the fact that the $\mathscr{A}$ operator defined by (IV.61) might be difficult to control. Nevertheless, let us assume that $\mathscr{A}$ is bounded from below and above. Then it is theoretically possible to invert the associated wavelet transform, that is to recover $f(x)$ from its transform $T_f(x) = \langle f, \pi_\sigma(x) \rangle$, by using the inverse of $\mathscr{A}$. Unfortunately there does not exist in general an explicit formula for $\mathscr{A}^{-1}$, so that explicit computations are difficult to perform. The best that can be expected is that $\mathscr{A}^{-1}$ can be expressed as a small perturbation of the identity (up to a multiplicative constant). Then an approximate inversion is possible, the precision of which can be controlled.

There is actually a quite simple way to exhibit weakly admissible sections, proposed in [To.2]. The idea is the following. The previous discussions were based on a careful study of the Jacobian of the map

$$x \to u_\xi(x)$$

where x globally represents a set of variables in $X = G_{a\,WH}/\Gamma$. A possibility is to absorb such a Jacobian in a redefinition of the wavelets (corresponding to the introduction of the field of operators $\mathscr{T}(x)$ described in the last definition). Let us consider in more details the example of the X_1 coset space discussed in section IV.1.*a*. To the wavelet $\psi_{(\underline{p}, \underline{q}, \underline{r})}(\underline{x})$ associate the function $\Psi_{(\underline{p}, \underline{q}, \underline{r})}(\underline{x})$, defined as follows by its Fourier transform

$$\Psi_{\underline{p}, \underline{q}, \underline{r})}(\underline{\xi}) = \psi_{(\underline{p}, \underline{q}, \underline{r})}(\underline{\xi}) \cdot \sqrt{\frac{J_\xi}{\beta(\underline{p}, \underline{r})}} \tag{IV.62}$$

where J_ξ is the Jacobian introduced in Eq. (IV.25). Using the same arguments as before, it is not difficult to see that if the section $\beta(\underline{p}, \underline{r})$ is such that

$$\chi(\underline{\xi}) = \int_{\underline{k}_\xi(\mathbb{R}^n)} |\hat{\psi}(\underline{k})|^2 \, d\underline{k} \tag{IV.63}$$

is strictly positive and bounded almost everywhere as a function of $\underline{\xi}$ (*i.e.* is the multiplier of an invertible convolution operator $\mathscr{C}_\chi$), then Eq. (IV.7) holds, with $\mathscr{A} = \mathscr{C}_\chi$.

The interesting point in such a procedure is that since ψ is square-integrable by assumption, the admissibility of the section now only depends on β, through the set $\underline{k}_\xi(\mathbb{R}^n)$, and can be analyzed in a rather simple way. Such an analysis was done in the one-dimensional case in [To.2].

The construction can of course be applied to the other coset spaces previously studied.

3. Remark: the case of coadjoint orbits

It turns out that in the case where the coset space X is isomorphic to the phase space associated with the $\mathbf{G}_{a\,\mathrm{WH}}$ action on $L^2(\mathbb{R}^n)$, there is a simple corresponding geometrical picture. Indeed, let us consider a separable locally compact Lie group $\mathbf{G}$, assumed to be exponential and solvable for simplicity. Consider also an irreductible unitary representation π of $\mathbf{G}$, then associated with a coadjoint orbit O_F. Pick $F \in O_F$, and let H be a corresponding polarization, fulfilling the Pukanszky condition. Then by general results (*see* [Kir], [Gui] for a review) the coadjoint orbit (or phase space) O_F is given by

$$O_F = \mathrm{Ad}^*(\mathbf{G}) \cdot F \cong \mathbf{G}/\mathbf{G}_F \tag{IV.64}$$

where $\mathbf{G}_F$ is the stability $\mathbf{G}$-subgroup of F.

$$\mathbf{G}_F = \{ g \in \mathbf{G},\ \mathrm{Ad}^*(g) \cdot F = F \} \tag{IV.65}$$

The isomorphism is given by the map $[\varphi_F] : \mathbf{G}/\mathbf{G}_F \to O_F$, quotient of

$$\varphi_F : \quad g \in \mathbf{G} \to \mathrm{Ad}^*(g) . F \in O_F \tag{IV.66}$$

by its radical. Under this isomorphism, the coadjoint action of $\mathbf{G}$ on O_F is then equivalent to

$$[\mathrm{Ad}^*(g) \cdot [\varphi_F]]([h]) = [\varphi_F]([\lambda(g) \cdot h]) \tag{IV.67}$$

(here $[h] \in \mathbf{G}/\mathbf{G}_F$ denotes the equivalence class of $h \in \mathbf{G}$).

The corresponding wavelet transform

$$T : \quad L^2(\mathbf{G}/\mathrm{H}) \to L^2(\mathbf{G}/\mathbf{G}_F) \tag{IV.68}$$

then maps functions on the configuration space (or wave functions in a quantum mechanical terminology) into functions on the phase space (*i.e.* classical objects). In such a particular case, the wavelet transform appears as the inverse of a quantization map, in the language of geometric quantization. There is here an interesting aspect, that should be carefully studied in the general case. The special case where $\mathbf{G}$ is a semidirect product has already been investigated in [DB], in which a method is described for finding strictly admissible sections.

V. CONCLUSIONS

We have studied in this paper various constructions leading to time-frequency representations theorems, generalizing the approach of [To.1-2] to the n-dimensional situation. The basic tool was the representation theory of the affine Weyl-Heisenberg group $\mathbf{G}_{a\,\mathrm{WH}}$, defined as the extension of the n-dimensional Weyl-Heisenberg group by $\mathbb{R}_+^*$ (scale parameter) and $SO(n)$ (rotations).

The main goal was to provide decompositions of functions in $L^2(\mathbb{R}^n)$ into elementary contributions that are "well localized" in some parameter space (in general the phase space), and to control their localization properties.

Since the considered representations of the affine Weyl-Heisenberg group are not square-integrable, the corresponding Schur coefficients do not directly provide families of coherent states. Nevertheless, their restriction to adequate homogeneous spaces can be considered as coherent states (except that the covariance with respect to the action of the group is lost).

We have investigated in this paper coherent states systems associated with the Stone-Von-Neumann-type representation of the affine-Weyl-Heisenberg group, for three different coset spaces. When the considered coset space contains the phase space of the representation (*i.e.* in cases 1 and 3), there is no problem to find strictly admissible sections. The trivial section is actually strictly admissible. However, this is no longer true when the coset space does not contain the phase space of the representation. It would be interesting to know if such a property generalizes to arbitrary representations.

ACKNOWLEDGEMENTS

We are very indebted to J. P. Antoine and A. Grossmann for fruitful discussions. Thanks are also due to the Centre de Physique théorique de Marseille and the FYMA, Université Catholique de Louvain, where this work has been done, for kind hospitality.

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(*Manuscript received September* 23, 1992
revised version received January 22, 1993.)