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Analyticity and Borel summability of the ϕ^4 models. I. The dimension $d=1$

by

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ABSTRACT. — We prove that the Schwinger functions of the ϕ^4 models, in dimension $d=1$, are analytic functions of the coupling constant in a common domain which extends to $|\operatorname{Arg}\lambda|<\frac{3\pi}{2}$, and in which each of them is the Borel sum of its Taylor series at $\lambda=0$; these series are uniformly Borel summable of level 1 in all directions except that of negative real numbers.

RÉSUMÉ. — On démontre que les fonctions de Schwinger des modèles ϕ^4, en dimension $d=1$, sont des fonctions analytiques de la constante de couplage dans un domaine commun d'ouverture angulaire $|\operatorname{Arg}\lambda|<\frac{3\pi}{2}$, dans lequel elles sont la somme de Borel de leurs séries de Taylor en $\lambda=0$; ces séries sont uniformément Borel-sommables de niveau 1 dans toutes les directions, sauf celle des réels négatifs.

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INTRODUCTION

In this paper, we begin a study of the analyticity properties in the coupling constant λ of the ϕ^4 models, in the region $\mathrm{Re}\,\lambda<0$.

In dimension $d \leqq 3$, the aim is to show that the domain in which the Schwinger functions are given by the Borel sum of their Taylor's series has an angular extension as large as one can expect, namely that it covers all directions up to $|\mathrm{Arg}\,\lambda|<3\pi/2$ (the functions being of course 2-valued if $\mathrm{Re}\,\lambda<0$, an example of Stokes phenomenon).

A (far) more ambitious project will be an approach of the four dimensional problem, since the procedure allows to reach the region of asymptotic freedom, the only one where one can expect the existence of the (euclidean) theory; [moreover, an argument by Khuri [8], strongly suggests that the functions may be real for $\lambda<0$, whereas the cut-off approximations are not].

It is quite easy to obtain representations of the cut-off functions as integrals over an auxiliary field, which lead to a domain of analyticity in the region $|\mathrm{Arg}\,\lambda|<\pi$, [1]; to go further, the idea is to start with discretized approximations given by finite dimensional integrals, and then to make a change of integration path, multiplying the (gaussian) variables by some complex number, which allows an analytic continuation in λ.

As the complex measures obtained that way have no limit as the number of variables goes to infinity, one needs to remove the discretization by technics like "phase-space expansions".

As a first result, we show in the simplest case of dimension $d=1$, that this procedure works. We deal with fields with an arbitrary number N of components, that is with the N-dimensional anharmonic oscillator described (for $\lambda \geqq 0$), by the Schrödinger operator

$$H=-\frac{1}{2}\Delta_u+\frac{m^2}{2}|u|^2+\lambda|u|^4$$

in $\mathbf{R}^N$. In this case, we prove that all Schwinger functions are analytic, and the Borel summability of level 1 holds, in some common domain which, for any $\eta>0$, includes a sector of the form

$$\left\{\lambda\in\bar{\mathbf{C}};\ |\mathrm{Arg}\,\lambda|\leqq\frac{3\pi}{2}-\eta,\ |\lambda|\leqq R_\eta\right\}.$$

An almost similar result is known [12] for the eigenvalues of the Hamiltonian: they have "the same" property in domains of the same shape; but these domains are not uniform with respect to the order of the eigenvalue, and it is not known if (and, as suggested by numerical analysis, it is probably false that) there is some common domaine extending to

$|\operatorname{Arg}\lambda|<3\pi/2$ in which this holds simultaneously for all eigenvalues (*see* [17]).

In fact the relation between the singularities of the Schwinger functions and those of the eigenvalues is quite unclear, except the fact that (at least in the scalar case) the vacuum energy $E_0(\lambda)$ is analytic where the Schwinger functions are, (since it can be expressed as a linear combination of equal-time Schwinger functions: if $H=-\frac{1}{2}\frac{d^2}{du^2}+V(u)$, then

$$E_0=\left(\psi_0, \left[V(\phi_0)+\frac{1}{2}\phi_0 V'(\phi_0)\right]\psi_0\right),$$

where ψ_0 is the ground state and ϕ_0 the field at time 0). Nevertheless the present result is not a consequence of [12].

Technically, the main problem, (the only serious one in dimension $d=1$), is to control the factorials which arise in the estimations of the terms of the expansion; usually this is done by some "domination of the low momentum fields by the interaction term" (*see* for example [9]), which improves the gaussian estimates; in the present case no such thing seems to exist (and in fact has not to be expected, since the coupling constant has the "wrong" sign), and the suitable bounds come from some mutual "quasi independence" of the field variables located in phase-space cells.

The discrete approximations and their analytic continuation are introduced in section 1, we prove the existence and the properties of the non normalized Schwinger functions in a finite volume in sections 2 and 3, the infinite volume limit is studied in sections 4 and 5, (this presentation introduces some repetitions, but a more condensed one would have been too confuse). The appendices recall, in a model-independent form, the main features of the expansions.

1. THE FINITE DIMENSIONAL APPROXIMATION AND ITS ANALYTIC CONTINUATION

1.0. One starts with the standard situation of a phase-space analysis, one introduces the following notations: for $r\in\mathbf{N}$, let $\mathscr{D}_r$ be the set $\{[k2^{-r}l_0, (k+1)2^{-r}l_0];\ k\in\mathbf{Z}\}$ of segments of length $2^{-r}l_0$, in $\mathbf{R}$; let $\mathscr{D}=\bigcup_{r=0}^{\infty}\mathscr{D}_r$, $\mathscr{D}_+=\bigcup_{r=1}^{\infty}\mathscr{D}_r$, $\mathscr{D}^r=\bigcup_{s=0}^{r}\mathscr{D}_s$, $\mathscr{D}^r_+=\bigcup_{s=1}^{r}\mathscr{D}_s$; for $\Delta\in\mathscr{D}$, $|\Delta|$ is the length of Δ, (*i.e.* $|\Delta|=2^{-r}l_0$, if $\Delta\in\mathscr{D}_r$); O_Δ is the middle of Δ; and, if $\Delta\in\mathscr{D}_r$, $(r\geqq 1)$, $\hat{\Delta}$ is the unique element of $\mathscr{D}_{r-1}$ such that $\Delta\in\hat{\Delta}$; if $\Lambda\subset\mathbf{R}$ is a finite union of segments in $\mathscr{D}_0$, one sets $\underline{\Lambda}_r=\{\Delta\in\mathscr{D}_r;\ \Delta\subset\Lambda\}$, and defines in an obvious way $\underline{\Lambda}$, $\underline{\Lambda}_+$, $\underline{\Lambda}^r$, $\underline{\Lambda}^r_+$, $(r\in\mathbf{N})$.

Now let $\mathscr{S}' := \mathscr{S}'(\mathbf{R}, \mathbf{R})$ be the space of real tempered distributions over $\mathbf{R}$, $\nu \in \mathscr{M}^1(\mathscr{S}')$ the gaussian measure defined by

$$\hat{\nu}(f) := \int_{\mathscr{S}'} e^{i\langle \omega, f \rangle} \nu(d\omega) = e^{-1/2 \| f \|_2^2}, \qquad (f \in \mathscr{S}),$$

and Φ the process indexed by the Sobolev space $\mathscr{H}^{-1} := \mathscr{H}^{-1}(\mathbf{R}, \mathbf{R})$ such that, for $f \in \mathscr{S}$, $\Phi(f)$ is the class (mod. ν) of the function $\omega \mapsto \langle \omega, \Sigma_m^{-1/2} f \rangle$, with $\Sigma_m := (-\mathrm{D}^2 + m^2)$, for some given $m > 0$.

For each integer $\mathrm{N} > 0$ one sets $\mathscr{S}'_{\mathrm{N}} = (\mathscr{S}')^{\mathrm{N}}$, $\nu_{\mathrm{N}} = \nu^{\otimes \mathrm{N}}$, $\Phi_{\mathrm{N}} = \Phi^{\otimes \mathrm{N}}$; and $\Phi_{\mathrm{N}}^{(j)}$ is the j-th component of Φ_{N}.

1.1. For $\Delta \in \mathscr{D}$, $\chi_\Delta \in \mathscr{H}^{-1}$ being the Dirac measure δ_{0_Δ}, for $r \in \mathbf{N}$, one sets

$$\Phi_r(x) := \sum_{\Delta \in \mathscr{D}_r} 1_\Delta(x) . \Phi(\chi_\Delta), \qquad (x \in \mathbf{R}), \tag{1.1.1}$$

(where 1_Δ is the indicatrix function of Δ), $\Phi_{\mathrm{N}, r} = \Phi_r^{\otimes \mathrm{N}}$, and

$$|\Phi_{\mathrm{N}, r}(x)|^4 := \left(\sum_{j=1}^{\mathrm{N}} \Phi_{\mathrm{N}, r}^{(j)}(x)^2 \right)^2, \tag{1.1.2}$$

$$|\Phi_{\mathrm{N}, r}|^4(1_\Lambda) := \int_\Lambda |\Phi_{\mathrm{N}, r}(x)|^4 dx = \sum_{\Delta \in \Lambda_r} |\Delta| \left(\sum_{j=1}^{\mathrm{N}} \Phi_{\mathrm{N}}^{(j)}(\chi_\Delta)^2 \right)^2. \tag{1.1.3}$$

Given a family $f := \{ f_{j;k_j} \in \mathscr{H}^{-1};\ 1 \leqq j \leqq \mathrm{N},\ 1 \leqq k_j \leqq l_j \}$, and $\lambda \geqq 0$, one sets (1)

$$\mathrm{Z}_{\mathrm{N}, \lambda; \Lambda, r}(f) := \int_{\mathscr{S}'^{\mathrm{N}}} \prod_{j=1}^{\mathrm{N}} \prod_{k_j=1}^{l_j} \Phi_{\mathrm{N}, r}^{(j)}(f_{j;k_j}) . e^{-\lambda |\Phi_{\mathrm{N}, r}|^4 (1_\Lambda)} \nu_{\mathrm{N}}. \tag{1.1.4}$$

The Schwinger functions $\mathrm{S}_{\mathrm{N}, \lambda}(f)$ are then obtained from (2):

$$\mathrm{Z}_{\mathrm{N}, \lambda; \Lambda}(f) := \lim_{r \to \infty} \mathrm{Z}_{\mathrm{N}, \lambda; \Lambda, r}(f), \tag{1.1.5}$$

and

$$\mathrm{S}_{\mathrm{N}, \lambda; \Lambda}(f) := \mathrm{Z}_{\mathrm{N}, \lambda; \Lambda}(f) / \mathrm{Z}_{\mathrm{N}, \lambda; \Lambda}(\varnothing), \tag{1.1.6}$$

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(1) The Schwinger functions are in principle defined for functions $f_{j;k_j} \in \mathscr{S}$; but, on the one hand, in the particular case of the dimension $\mathrm{d} = 1$, the Dirac measures belong to $\mathscr{H}^{-1}$ and one can choose $f_j = \delta_{x_j}$; and, on the other hand it will be useful to require that each function $\Sigma_m^{-1/2} f_{j;k_j}$ has its support in some segment in $\mathscr{D}_0$ (any function in $\mathscr{S}$ is a sum, convergent in $\mathscr{H}^{-1}$, of functions having this property; this allows to "reconstruct" the Schwinger functions for $f_{j;k_j} \in \mathscr{S}$).

(2) The existence of these limits is known, it is also a consequence of the proofs below. Independently of this construction, the functions $\mathrm{S}_{n, \lambda}$, $(\lambda \geqq 0)$, are characterized as the moments of the stationary Markov process with transition semigroup generated by the Schrödinger operator $-\frac{1}{2}\Delta_u + \frac{m^2}{2}|u|^2 + \lambda |u|^4$ in $\mathbf{R}^{\mathrm{N}}$.

$$S_{N,\lambda}(f) := \lim_{\Lambda \to \mathbf{R}} S_{N,\lambda;\Lambda}(f). \qquad (1.1.7)$$

1.2. One substitutes in (1.1.4), with $\lambda=\rho^2/8$, $(\rho \geqq 0)$, the identity

$$e^{-(\rho^2/8)\,|\Phi_{N,r}|^4(1_\Lambda)} = \int_{\mathscr{S}'} e^{-(i/2)\rho \sum_{\Delta\in\underline{\Lambda}_r} \langle \sigma, 1_\Delta\rangle \cdot \left(\sum_{j=1}^{N} \Phi_N^{(j)}(\chi_\Delta)^2\right)} \nu(d\sigma), \qquad (1.2.1)$$

one obtains (from Fubini's theorem),

$$\begin{aligned} Z_{N,(\rho);\Lambda,r}(f) &:= Z_{N,\rho^2/8;\Lambda,r}(f) \\ &= \int_{\mathscr{S}'} \prod_{j=1}^{N} \Bigg[\int_{\mathscr{S}'} \prod_{k_j=1}^{l_j} \langle \omega_j, \Sigma_m^{-1/2} f_{j;k_j} \rangle \\ &\quad \times e^{-(i/2)\rho \sum_{\Delta\in\underline{\Lambda}_r} \langle \sigma, 1_\Delta\rangle \cdot \langle \omega_j, \Sigma_m^{-1/2}\chi_\Delta\rangle^2} . \nu(d\omega_j) \Bigg] \nu(d\sigma); \end{aligned} \qquad (1.2.2)$$

then, if (for almost all $\sigma\in\mathscr{S}'$), $\mathbf{A}_{\Lambda,r}(\sigma)$ is the real, self-adjoint operator of finite rank over $L^2(\mathbf{R})$ defined by

$$\mathbf{A}_{\Lambda,r}(\sigma)\psi := \sum_{\Delta\in\underline{\Lambda}_r} \langle \sigma, 1_\Delta\rangle (\Sigma_m^{-1/2}\chi_\Delta, \psi)_{L^2} . \Sigma_m^{-1/2}\chi_\Delta, \qquad (\psi\in L^2), \qquad (1.2.3)$$

and denoted by

$$\mathbf{A}_{\Lambda,r}(\sigma) = \sum_{\Delta\in\underline{\Lambda}_r} \langle \sigma, 1_\Delta\rangle \Sigma_m^{-1/2}\chi_\Delta \times \Sigma_m^{-1/2}\chi_\Delta, \qquad (1.2.4)$$

one obtains, by integrating over the ω_j's in (1.2.2):

$$\begin{aligned} Z_{N,(\rho);\Lambda,r}(f) &= \int_{\mathscr{S}'} \prod_{j=1}^{N} \Bigg(\sum_{P_j\in\mathscr{P}_2(l_j)} \prod_{p\in P_j} (\Sigma_m^{-1/2} f_{j;p_-}, [I+i\rho\mathbf{A}_{\Lambda,r}(\sigma)]^{-1}\Sigma_m^{-1/2} f_{j;p_+}) \Bigg) \\ &\quad \times \mathrm{Det}.(I+i\rho\mathbf{A}_{\Lambda,r}(\sigma))^{-N/2}\nu(d\sigma), \end{aligned} \qquad (1.2.5)$$

where $\mathscr{P}_2(.)$ is the set of partitions in pairs of a set ([3]), and $p=\{p_-, p_+\}$.

1.3. First, the right hand side of (1.2.5) is well defined for $|\mathrm{Arg}\,\rho| < \frac{\pi}{2}$, and defines an a analytic function of ρ, [again denoted by $Z_{N,(\rho);\Lambda,r}(f)$], so that (1.2.5) is now valid as soon as $|\mathrm{Arg}\,\rho| < \frac{\pi}{2}$.

([3]) One supposes each l_j is even, otherwise the corresponding sum (and the whole function), vanishes.

Then, for $\alpha\in\mathbf{C}^*$, $|\operatorname{Arg}\alpha|<\frac{\pi}{4}$, if $|\operatorname{Arg}\alpha\rho|<\frac{\pi}{2}$, one has

$$\begin{aligned}
Z_{N,(\rho);\Lambda,r}(f) &= \left(\frac{\alpha}{\sqrt{\operatorname{Re}.\alpha^2}}\right)^{|\underline{\Lambda}_r|}\int_{\mathscr{S}'}\prod_{j=1}^{N}\Bigg(\sum_{P_j\in\mathscr{P}_2(l_j)}\prod_{p\in P_j}\\
&\quad\times\left(\Sigma_m^{-1/2}f_{j;p_-},\left[I+\left(\frac{i\alpha\rho}{\sqrt{\operatorname{Re}.\alpha^2}}\right)A_{\Lambda,r}(\sigma)\right]^{-1}\Sigma_m^{-1/2}f_{j;p_+}\right)\Bigg)\\
&\quad\times\operatorname{Det}\left(I+\left(\frac{i\alpha\rho}{\sqrt{\operatorname{Re}.\alpha^2}}\right)A_{\Lambda,r}(\sigma)\right)^{-N/2}\\
&\quad\times e^{-(i/2)(\operatorname{Im}.\alpha^2/\operatorname{Re}.\alpha^2)\sum_{\Delta\in\underline{\Lambda}_r}|\Delta|^{-1}\langle\sigma,1_\Delta\rangle^2}\nu(d\sigma); \qquad (1.3.1)
\end{aligned}$$

indeed, if one introduces the variables $\underline{s}=(s_\Delta)_{\Delta\in\underline{\Lambda}_r}:\mathscr{S}'\to\mathbf{R}^{\underline{\Lambda}_r}$ by $s_\Delta(\sigma)=|\Delta|^{-1/2}\langle\sigma,1_\Delta\rangle$, one transforms the right hand side of (1.2.5) into an integral over the finite dimensional space $\mathbf{R}^{\underline{\Lambda}_r}$, with respect to the gaussian measure $\underline{s}_*\nu$, namely: $\int_{\mathbf{R}^{\underline{\Lambda}_r}}F(\underline{s})\left(\prod_{\Delta\in\underline{\Lambda}_r}e^{-s_\Delta^2/2}\frac{ds_\Delta}{\sqrt{2\pi}}\right)$. An elementary estimate of F shows that

$$Z(u):=\int_{\mathbf{R}^{\underline{\Lambda}_r}}F(u\underline{s})\left(\prod_{\Delta\in\underline{\Lambda}_r}e^{-u^2s_\Delta^2/2}u\frac{ds_\Delta}{\sqrt{2\pi}}\right)$$

is well defined for $u\in\mathbf{C}^*$ such that $|\operatorname{Arg}u|<\frac{\pi}{4}$ and $|\operatorname{Arg}u\rho|<\frac{\pi}{2}$; then Z is a constant function, because it is holomorphic and constant over $\mathbf{R}_+^*$; last, (for $|\operatorname{Arg}\rho|<\frac{\pi}{2}$), the right hand side of (1.2.5) equals Z(1) and that of (1.3.1) equals $Z\left(\frac{\alpha}{\sqrt{\operatorname{Re}.\alpha^2}}\right)$. ■

Now, *one continues analytically* $Z_{N,(\rho);\Lambda,r}(f)$ *for* $|\operatorname{Arg}\rho|<\frac{3\pi}{4}$ *by* (1.3.1) *where* α *satisfies* $|\operatorname{Arg}\alpha|<\frac{\pi}{4}$ *and is chosen so that* $|\operatorname{Arg}\alpha\rho|<\frac{\pi}{2}$ (4).

The rest of this paper is devoted to show the existence, and the properties of Borel summability, of the limits, with respect to r and Λ, which provides analytic continuations of (1.1.5) and (1.1.7).

(4) That is one forgets the useless condition $|\operatorname{Arg}\rho|<\pi/2$.
Moreover one remarks that any ρ_0 such that $|\operatorname{Arg}\rho_0|<3\pi/4$ has a neighbourhood $\mathscr{V}$ such that one can choose α independently of $\rho\in\mathscr{V}$.

2. CONTINUATION OF THE NON NORMALIZED SCHWINGER FUNCTIONS $Z_{N,\lambda;\Lambda}(f)$

2.0. To show the existence of the limits as $r \to \infty$ of the functions defined by (1.3.1), one introduces auxiliary variables as follows:

one sets $\mathcal{T}^r_\Lambda := [0, 1]^{\Lambda^r_+}$, $(r \geqq 1)$, and $\mathcal{T}_\Lambda := \bigcup_{r\geqq 1} \mathcal{T}^r_\Lambda \subset [0, 1]^{\Lambda_+}$ (5);

for $\underline{t} \in \mathcal{T}_\Lambda$ one denotes by $\underline{\Delta}(\underline{t}) \subset \underline{\Lambda}$ the family defined by

$$\Delta \in \underline{\Delta}(\underline{t}) \quad \textit{iff} \begin{cases} \forall\, \Delta_1 \subsetneqq \Delta : \quad t_{\Delta_1} = 0, \\ \textit{and, (if } \Delta \in \underline{\Delta}_+), \\ \exists\, \Delta_2 \subsetneqq \hat{\Delta} : \quad t_{\Delta_2} \neq 0, \end{cases} \tag{2.0.1}$$

then $\underline{\Delta}(\underline{t})$ is a family of pairwise disjoint segments whose union covers Λ.

Now, one defines the vectors ψ_Δ, ψ'_Δ, $\tilde{\psi}_\Delta \in L^2(\mathbf{R}, \mathbf{R})$, $(\Delta \in \mathcal{D})$, by

$$\psi_\Delta := \Sigma_m^{-1/2} \chi_\Delta, \tag{2.0.2}$$

$$\psi'_\Delta := \psi_\Delta - \psi_{\hat{\Delta}}, \tag{2.0.3}$$

$$\tilde{\psi}_\Delta := \frac{1}{2}(\psi_\Delta + \psi_{\hat{\Delta}}), \tag{2.0.4}$$

(where, conventionnaly, for $\Delta_0 \in \mathcal{D}_0$, $\psi_{\hat{\Delta}_0} = 0$, so that $\psi'_{\Delta_0} = \psi_{\Delta_0}$, $\tilde{\psi}_{\Delta_0} = \frac{1}{2}\psi_{\Delta_0}$); and for $\underline{t} \in \mathcal{T}_\Lambda$, one denotes by $A_{\underline{t}}(\sigma)$, $(\sigma \in \mathcal{S}')$, the real selfadjoint operator over $L^2(\mathbf{R})$:

$$\begin{aligned} A_{\underline{t}}(\sigma) &:= \sum_{\Delta \in \underline{\Lambda}} t_\Delta \langle \sigma, 1_\Delta \rangle (\psi_\Delta \times \psi_\Delta - \psi_{\hat{\Delta}} \times \psi_{\hat{\Delta}}) \\ &= \sum_{\Delta \in \underline{\Lambda}} t_\Delta \langle \sigma, 1_\Delta \rangle (\psi'_\Delta \times \tilde{\psi}_\Delta + \tilde{\psi}_\Delta \times \psi'_\Delta), \end{aligned} \tag{2.0.5}$$

(with, for $\Delta_0 \in \mathcal{D}_0$, $t_{\Delta_0} = 1$, by convention).

Last, one defines $Z_{N,(\rho);\underline{t}}(f)$ substituting $A_{\underline{t}}(\sigma)$ to $A_{\Lambda,r}(\sigma)$ and $\underline{\Delta}(\underline{t})$ to $\underline{\Lambda}_r$ (6) in (1.3.1), namely:

$$Z_{N,(\rho);\underline{t}}(f) = \left(\frac{\alpha}{\sqrt{\mathrm{Re}\,.\,\alpha^2}}\right)^{|\underline{\Delta}(\underline{t})|} \int_{\mathcal{S}'} \prod_{j=1}^{N} \left(\sum_{P_j \in \mathcal{P}_2(I_j)} \prod_{p \in P_j} \right.$$

$$\left. \times \left(\Sigma_m^{-1/2} f_{j;\,p_-}, \left[I + \left(\frac{i\alpha\rho}{\sqrt{\mathrm{Re}\,.\,\alpha^2}} \right) A_{\underline{t}}(\sigma) \right]^{-1} \Sigma_m^{-1/2} f_{j;\,p_+} \right) \right)$$

(5) *See* note (38).

(6) For $\underline{t} \in \mathcal{T}^r_\Lambda$, one can make or not the substitution of $\underline{\Delta}(\underline{t})$ to $\underline{\Lambda}_r$, without changing the value of the integral; one uses both forms, as necessary.

$$\times \operatorname{Det.}\left(I+\left(\frac{i\alpha\rho}{\sqrt{\operatorname{Re}.\alpha^2}}\right)A_{\underline{t}(\sigma)}\right)^{-N/2}$$
$$\times e^{-(i/2)(\operatorname{Im}.\alpha^2/\operatorname{Re}.\alpha^2)\sum_{\Delta\in\underline{\Lambda}(t)}|\Delta|^{-1}\langle\sigma,1_\Delta\rangle^2}\nu(d\sigma); \quad (2.0.6)$$

(so that $Z_{N,(\rho);\Lambda,r}(f)$ is the value of $Z_{N,(\rho);\underline{t}}(f)$ for $\underline{t}=1_{\underline{\Lambda}^r_+}$).

2.1. One now wants to show that one can apply the theorem (A.0) ([7]) to the function $\underline{t}\mapsto Z_{N,(\rho);\underline{t}}(f)$.

One gives first a suitable expression of the successive derivatives $\left(\prod_{\Delta\in\underline{\Lambda}_+} D_\Delta^{q_\Delta}\right)Z_{N,(\rho);\underline{t}}(f)$, (where $D_\Delta=\frac{\partial}{\partial t_\Delta}$) under the hypothesis that $\underline{q}\in Q_\Lambda := \bigcup_{r\geq 1} N^{\underline{\Lambda}^r_+}$ and $\underline{t}\in\mathcal{T}_\Lambda$ satisfies

$$q_\Delta=0 \textit{ and } t_\Delta=1, \quad \textit{if } \Delta \textit{ contains strictly an element of } \underline{\Lambda}(\underline{t}). \quad (2.1.1)$$

First, one notes that

$$\mathbf{R}_{\underline{t}}(\sigma) := \left[I+\left(\frac{i\alpha\rho}{\sqrt{\operatorname{Re}.\alpha^2}}\right)A_{\underline{t}}(\sigma)\right]^{-1} \quad (2.1.2)$$

satisfies

$$(g_1, \mathbf{R}_{\underline{t}}(\sigma)g_2)=(g_2, \mathbf{R}_{\underline{t}}(\sigma)g_1), \qquad \forall g_1, g_2\in L^2(\mathbf{R},\mathbf{R}), \quad (2.1.3)$$

[indeed, $\mathbf{R}_{\underline{t}}(\sigma)^*=\overline{\mathbf{R}_{\underline{t}}(\sigma)}$ since $A_{\underline{t}}(\sigma)=A_{\underline{t}}(\sigma)^*=\overline{A_{\underline{t}}(\sigma)}$].

Next, given a family $g=\{g_j\}_{1\leq j\leq 2n}$ of real functions $g_j\in L^2(\mathbf{R},\mathbf{R})$, one sets

$$S_g(\underline{t};\sigma) := \sum_{P\in\mathcal{P}_2(g)}\prod_{p\in P}(g_{p_-}, \mathbf{R}_{\underline{t}}(\sigma)g_{p_+}), \quad (2.1.4)$$

where $\mathcal{P}_2(g)$ is the set of partition in pairs of the family g, [the choice of p_-, p_+ in the pair $p=\{p_-,p_+\}$ is irrelevant, according to (2.1.3)]; then, from (2.1.1) and (2.0.5), if g^Δ is the family $\{g, \tilde{\psi}_\Delta, \psi'_\Delta\}$, one has ([8]).

$$D_\Delta(S_g(\underline{t};\sigma)\operatorname{Det}.\mathbf{R}_{\underline{t}}(\sigma)^{1/2})$$
$$=\frac{-i\rho\alpha}{\sqrt{\operatorname{Re}.\alpha^2}}\langle\sigma,1_\Delta\rangle.S_{g^\Delta}(\underline{t};\sigma)\operatorname{Det}.\mathbf{R}_{\underline{t}}(\sigma)^{1/2}. \quad (2.1.5)$$

([7]) *See* appendix A.

([8]) One supposes that the functions g_j do not depend on $\underline{t}\in\mathcal{T}_\Lambda$.

But if $\tilde{f}_j=\{\Sigma_m^{-1/2} f_{j,k_j}\}_{1\leq k_j\leq l_j}$, $(1\leq j\leq N)$, one can write (2.0.6) in the new form

$$Z_{N,(\rho);\underline{t}}(f)=\left(\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\right)^{|\underline{\Lambda}_r|}\int_{\mathscr{S}'}\prod_{j=1}^{N}(S_{\tilde{f}_j}(\underline{t};\sigma).\mathrm{Det}.\mathbf{R}_{\underline{t}}(\sigma)^{1/2})\times e^{-(i/2)(\mathrm{Im}.\alpha^2/\mathrm{Re}.\alpha^2)\sum_{\Delta\in\underline{\Lambda}_r}|\Delta|^{-1}\langle\sigma,1_\Delta\rangle^2}.\nu(d\sigma),\quad(2.1.6)$$

so one computes the successive derivatives of $Z_{N,(\rho);\underline{.}}(f)$ (by derivation under the integral sign), using Leibniz formula, by repeated use of (2.1.5), starting with the $S_{\tilde{f}_j}$'s.

Given a family of functions g and $\underline{q}=\{q_\Delta\}_{\Delta\in\underline{\Lambda}_+}$, one denotes by $g^{\underline{q}}$ *the family containing* g *and, for each* $\Delta\in\underline{\Lambda}_+$, q_Δ *functions equal to* $\tilde{\psi}_\Delta$ *and* q_Δ *functions equal to* ψ'_Δ, then

$$\left(\prod_{\Delta\in\underline{\Lambda}+}D_{\Delta}^{q_\Delta}\right)Z_{N,(\rho);\underline{t}}(f)=\sum_{\{\underline{q}_j\}_{1\leq j\leq N};\sum_j\underline{q}_j=\underline{q}}\left[\left(\prod_{\Delta\in\underline{\Lambda}+}\frac{q_\Delta!}{\prod_{j=1}^{N}q_{j\Delta}!}\right)\times\left(\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\right)^{|\underline{\Lambda}_r|}\int_{\mathscr{S}'}\prod_{\Delta\in\underline{\Lambda}+}\left(\frac{-i\rho\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\langle\sigma,1_\Delta\rangle\right)^{q_\Delta}\times\prod_{j=1}^{N}\left(S_{\tilde{f}_j^{q_j}}\underline{q}(\underline{t};\sigma).\mathrm{Det}.\mathbf{R}_{\underline{t}}(\sigma)^{1/2}\right)\times e^{-(i/2)(\mathrm{Im}.\alpha^2/\mathrm{Re}.\alpha^2)\sum_{\Delta\in\underline{\Lambda}_r}|\Delta|^{-1}\langle\sigma,1_\Delta\rangle^2}.\nu(d\sigma)\right].\quad(2.1.7)$$

Now, after the derivations have been done and $\underline{t}$ fixed to its definitive value, $\mathbf{R}_{\underline{t}}(\sigma)$ depends on σ only through the variables $\{\langle\sigma,1_\Delta\rangle\}_{\Delta\in\underline{\Lambda}(\underline{t})}$, because then

$$\mathbf{A}_{\underline{t}}(\sigma)=\sum_{\Delta\in\underline{\Lambda}(\underline{t})}\langle\sigma,1_\Delta\rangle(t_\Delta\psi_\Delta\times\psi_\Delta+(1-t_\Delta)\psi_\Delta^\wedge\times\psi_\Delta^\wedge);\quad(2.1.8)$$

it is therefore possible to "suppress" the multiplication by $\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}$ on the variables orthogonal to the $\{\langle\sigma,1_\Delta\rangle\}_{\Delta\in\underline{\Lambda}(\underline{t})}$'s, this leads to substitute

$$\left[\left\langle\sigma,\left(1_\Delta-\frac{|\Delta|}{|\tilde{\Delta}|}1_{\tilde{\Delta}}\right)\right\rangle+\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\frac{|\Delta|}{|\tilde{\Delta}|}\langle\sigma,1_{\tilde{\Delta}}\rangle\right]$$

$$\text{to}\quad\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\langle\sigma,1_\Delta\rangle,$$

(where $\tilde{\Delta} \in \underline{\Lambda}(\underline{t})$ and $\tilde{\Delta} \supset \Delta$) [9], and $\underline{\Lambda}(\underline{t})$ to $\underline{\Lambda}_r$ in (2.1.7), it gives

$$\Big(\prod_{\Delta \in \underline{\Lambda}+} D_{\Delta}^{q_{\Delta}}\Big) Z_{N,(\rho);\underline{t}}(f) = \sum_{\{\underline{q}_j\}_{1 \leq j \leq N};\ \sum_j \underline{q}_j = \underline{q}} \Bigg[\Bigg(\prod_{\Delta \in \underline{\Lambda}+} \frac{q_{\Delta}!}{\prod_{j=1}^{N} q_{j\Delta}!} \Bigg) \times \left(\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\right)^{|\underline{\Lambda}(\underline{t})|} \int_{\mathscr{S}'} \prod_{\Delta \in \underline{\Lambda}+}$$

$$\times \left(-i\rho \left[\langle \sigma, \left(1_{\Delta} - \frac{|\Delta|}{|\tilde{\Delta}|} 1_{\tilde{\Delta}} \right) \rangle + \frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}} \frac{|\Delta|}{|\tilde{\Delta}|} \langle \sigma, 1_{\tilde{\Delta}} \rangle \right] \right)^{q_{\Delta}}.$$

$$\times \prod_{j=1}^{N} S_{f_j}^{q_j}(\underline{t}; \sigma).\mathrm{Det}.\mathbf{R}_{\underline{t}}(\sigma)^{N/2}$$

$$\times e^{-(i/2)(\mathrm{Im}.\alpha^2/\mathrm{Re}.\alpha^2) \sum_{\Delta \in \underline{\Lambda}(\underline{t})} |\Delta|^{-1} \langle \sigma, 1_{\Delta} \rangle^2}.\nu(d\sigma)\Bigg]. \quad (2.1.9)$$

One now wants to derive from (2.1.9) a first estimate [10] of these derivatives.

2.2. The estimation of the terms $S_{f_j}^{q_j}(\underline{t}; \sigma)$ in the right hand side of (2.1.9) mainly relies on the inequalities (2.2.1) and (2.2.2) below.

LEMMA. – *Let* H *be an Hilbert space, and, for each integer* i, $(1 \leq i \leq k)$, $u_i \in \mathrm{H}$ *and* $n_i \in \mathbf{N}_+$, *one sets* $n = \sum_{i=1}^{k} n_i$, *and denotes by* $\vee$ *the symmetric tensor product, then, for any* $\xi_i > 0$, $(1 \leq i \leq k)$, *one has*

$$\left\| \bigvee_{i=1}^{k} u_i^{\vee n_i} \right\|_{\mathrm{H}^{\vee n}} \leq \prod_{i=1}^{k} (n_i!)^{1/2}. \sup_{1 \leq i \leq k} \left(\frac{1}{\xi_i} \sum_{j=1}^{k} \xi_j |(u_i, u_j)| \right)^{n/2}. \quad (2.2.1)$$

Proof. – Let A_i, $(1 \leq i \leq k)$, be a set with n_i elements, $\mathrm{A} = \bigcup_{i=1}^{k} \mathrm{A}_i$ their disjoint union, and $\mathscr{B}(\mathrm{A})$ the set of permutations of A, one has

$$\left\| \bigvee_{i=1}^{k} u_i^{\vee n_i} \right\|_{\mathrm{H}^{\vee n}}^2 = \sum_{\tau \in \mathscr{B}(\mathrm{A})} \prod_{a \in \mathrm{A}} (u_{i_{\tau(a)}}, u_{i_a})$$
$$=^{(\mathrm{i})} \sum_{\tau \in \mathscr{B}(\mathrm{A})} \prod_{a \in \mathrm{A}} \left(\frac{1}{\xi_{i_{\tau(a)}}} u_{i_{\tau(a)}}, \xi_{i_a} u_{i_a} \right)$$

---

(9) With the hypothesis (2.1.1) each Δ which occurs in (2.1.7) – that is such that $q_{\Delta} \neq 0$ – is included in a (unique) element $\tilde{\Delta}$ of $\underline{\Lambda}(\underline{t})$.

(10) *See* (2.8.1), (2.8.3); the useful estimate will be a geometric mean of (2.8.3) and of an other one, (2.9.10), obtained after an integration by parts in (2.1.7).

$$\leqq \sum_{\tau \in \mathscr{B}(A)} \prod_{a \in A} \left| \left(\frac{1}{\xi_{i_{\tau(a)}}} u_{i_{\tau(a)}}, \xi_{i_a} u_{i_a} \right) \right|$$
$$\leqq^{(ii)} \prod_{i=1}^{k} (n_i!) \cdot \prod_{i=1}^{k} \left(\sum_{j=1}^{k} \left| \left(\frac{1}{\xi_i} u_i, \xi_j u_j \right) \right| \right)^{n_i}$$
$$\leqq \prod_{i=1}^{k} (n_i!) \cdot \sup_{1 \leqq i \leqq k} \left(\frac{1}{\xi_i} \sum_{j=1}^{k} \xi_j \left| (u_i, u_j) \right| \right)^n,$$

($^{(i)}$ because $\prod_{a \in A} \left(\frac{1}{\xi_{i_{\tau(a)}}} \cdot \xi_{i_a} \right) = 1$, $^{(ii)}$ by inspection). ■

One deduces the

COROLLARY. — *For* $1 \leqq i \leqq k$, $u_i \in H$, $n_i \in \mathbf{N}_+$ *and the sets* A_i *are as in the lemma; one supposes* n *is even; on the other hand let* $R \in L(H)$ *be an operator such that* $(u_i, R u_j) = (u_j, R u_i)$, $1 \leqq i, j \leqq k$; *then if* $\mathscr{P}_2(A)$ *is the set of partitions in pairs of* $A = \bigcup_{i=1}^{k} A_i$, *one has*

$$\left| \sum_{P \in \mathscr{P}_2(A)} \prod_{p \in P} (u_{p_-}, R u_{p_+}) \right|$$
$$\leqq 2^{n/2} \cdot \| R \|^{n/2} \cdot \prod_{i=1}^{k} (n_i!)^{1/2} \cdot \sup_{1 \leqq i \leqq k} \left(\frac{1}{\xi_i} \sum_{j=1}^{k} \xi_j \left| (u_i, u_j) \right| \right)^{n/2}. \quad (2.2.2)$$

Proof. — Let $\vee^p(R) \in L(H^{\vee p})$ be the operator defined by

$$\vee^p(R) \bigvee_{j=1}^{p} \varphi_j = \bigvee_{j=1}^{p} R \varphi_j, \qquad (\varphi_j \in H;\ 1 \leqq j \leqq p),$$

one has $\| \vee^p(R) \| \leqq \| R \|^p$.

Now let $\mathscr{J}$ be the set of parts of A with $n/2$ elements, and for $J \in \mathscr{J}$, let $\mathscr{B}(J, J')$ be the set of bijections from J to its complement $J' = A \setminus J$, one has

$$\sum_{P \in \mathscr{P}_2(A)} \prod_{p \in P} (u_{p_-}, R u_{p_+}) = 2^{-n/2} \sum_{J \in \mathscr{J}} \sum_{\tau \in \mathscr{B}(J, J')} \prod_{a \in J} (u_{i_{\tau(a)}}, R u_{i_a})$$
$$= 2^{-n/2} \sum_{J \in \mathscr{J}} \left(\bigvee_{i=1}^{k} u_i^{\vee n_i(J')}, \vee^{n/2}(R) \bigvee_{j=1}^{k} u_j^{\vee n_j(J)} \right)_{H^{\vee n/2}},$$

where $n_i(J) = | A_i \cap J |$,

[so that $n_i(J) + n_i(J') = n_i$ and $(n_i(J)!) \cdot (n_i(J')!) \leqq n_i!$],

one deduces (2.2.2) from (2.2.1) and the Schwarz inequality, because $|\mathscr{J}| \leqq 2^n$. ■

2.3. The following lemma gives an improved estimate, which will be useful to control the "adiabatic" limit ($\Lambda \to \mathbf{R}$).

LEMMA. — *Suppose the family* $\varphi_j \in L^2(\mathbf{R})$, $(1 \leqq j \leqq n)$, *is such that some distributions* $\Sigma_m^{1/2}\varphi_j \in \mathcal{H}^{-1}$ *have their support included in some segment in* $\mathcal{D}_0$; *for each* $\Delta \in \mathcal{D}_0$, *one sets* $J_\Delta = \{j;\ 1 \leqq j \leqq n,\ \text{supp.}\ \Sigma_m^{1/2}\varphi_j \subset \Delta\}$ *and* $n_\Delta = |J_\Delta|$; J' *is the set of the indices for which the property is not assumed, and* $n' = |J'|$, *then*

$$\|(\vee_{i \in J'} \varphi_i) \vee (\vee_{\Delta \in \mathcal{D}_0} \vee_{j \in J_\Delta} \varphi_j)\|_{L^{2\vee n}} \leqq C^n (n'!)^{1/2} \prod_{\Delta \in \mathcal{D}_0} (n_\Delta !)^{1/2} \prod_{j=1}^{n} \|\varphi_j\|_{L^2}. \quad (2.3.1)$$

Proof. — Let $\mu_m \in \mathcal{M}^1(\mathcal{S}')$ the gaussian measure of mean 0 and covariance Σ_m^{-1} and, for $k \in \mathbf{N}$, let $\Pi_k \in L(L^2(\mathcal{S}', \mu_m))$ the orthogonal projection onto the "k-particles space" $\mathcal{F}_k$ (11), one has

$$\left\| \vee_{j=1}^{n} \varphi_j \right\|_{L^2(\mathbf{R})^{\vee n}} = \left\| \Pi_n \left(\prod_{j=1}^{n} \langle \cdot, \Sigma_m^{1/2}\varphi_j \rangle \right) \right\|_{L^2(\mathcal{S}', \mu_m)} \leqq \left\| \prod_{j=1}^{n} \langle \cdot, \Sigma_m^{1/2}\varphi_j \rangle \right\|_{L^2(\mathcal{S}', \mu_m)},$$

therefore

$$\begin{aligned}
&\|(\vee_{i \in J'} \varphi_i) \vee (\vee_{\Delta \in \mathcal{D}_0} \vee_{j \in J_\Delta} \varphi_j)\|_{L^2(\mathbf{R})^{\vee n}} \\
&\leqq \| \prod_{i \in J'} \langle \cdot, \Sigma_m^{1/2}\varphi_i \rangle \cdot \prod_{\Delta \in \mathcal{D}_0} \prod_{j \in J_\Delta} \langle \cdot, \Sigma_m^{1/2}\varphi_j \rangle \|_{L^2(\mathcal{S}', \mu_m)} \\
&\leqq \| \prod_{i \in J'} \langle \cdot, \Sigma_m^{1/2}\varphi_i \rangle \|_{L^4(\mathcal{S}', \mu_m)} \cdot \| \prod_{\Delta \in \mathcal{D}_0} \prod_{j \in J_\Delta} \langle \cdot, \Sigma_m^{1/2}\varphi_j \rangle \|_{L^4(\mathcal{S}', \mu_m)},
\end{aligned}$$

on has

$$\begin{aligned}
&\| \prod_{i \in J'} \langle \cdot, \Sigma_m^{1/2}\varphi_i \rangle \|_{L^4(\mathcal{S}', \mu_m)} \leqq^{(i)} \prod_{i \in J'} \| \langle \cdot, \Sigma_m^{1/2}\varphi_i \rangle \|_{L^{4n'}(\mathcal{S}', \mu_m)} \\
&\leqq^{(ii)} \prod_{j \in J'} \left((4n')^{1/2} \| \langle \cdot, \Sigma_m^{1/2}\varphi_i \rangle \|_{L^2(\mathcal{S}', \mu_m)} \right) \leqq C^{n'} (n'!)^{1/2} \prod_{i \in J'} \|\varphi_i\|_{L^2},
\end{aligned}$$

((i) from Hölder's inequality, (ii) from Nelson's "hypercontractivity" inequality [10]).

$$\begin{aligned}
&\| \prod_{\Delta \in \mathcal{D}_0} \prod_{j \in J_\Delta} \langle \cdot, \Sigma_m^{1/2}\varphi_j \rangle \|_{L^4(\mathcal{S}', \mu_m)} \\
&\leqq \prod_{\Delta \in \mathcal{D}_0} \| \prod_{j \in J_\Delta} \langle \cdot, \Sigma_m^{1/2}\varphi_j \rangle \|_{L^{4\beta}(\mathcal{S}', \mu_m)} \\
&\leqq^{(i)} \prod_{\Delta \in \mathcal{D}_0} \prod_{j \in J_\Delta} \| \langle \cdot, \Sigma_m^{1/2}\varphi_j \rangle \|_{L^{4\beta n_\Delta}(\mathcal{S}', \mu_m)}
\end{aligned}$$

(11) Otherwise stated, the "k-th Wiener's chaos".

$$\overset{\text{(ii)}}{\leqq} \prod_{\Delta \in \mathcal{D}_0} \prod_{j \in J_\Delta} \left((4\beta n_\Delta)^{1/2} \| \langle \cdot, \Sigma_m^{1/2} \varphi_j \rangle \|_{L^2(\mathcal{S}', \mu_m)}\right)$$
$$\leqq C^{n-n'} \prod_{\Delta \in \mathcal{D}_0} (n_\Delta !)^{1/2} \prod_{j \in \bigcup_\Delta J_\Delta} \| \varphi_j \|_{L^2(\mathbf{R})},$$

[(i) from Hölder's inequality, (ii) from Nelson's "hypercontractivity" inequality]. ■

On deduces the

Corollary. — *With the hypothesis of the lemma, and* $R \in L(L^2(\mathbf{R}))$ *an operator such that* $(\varphi_i, R\varphi_j) = (\varphi_j, R\varphi_i)$, $(1 \leqq i, j \leqq n)$, *one has*

$$\Big| \sum_{P \in \mathcal{P}_2(n)} \prod_{p \in P} (\varphi_{i_{p_-}}, R\varphi_{i_{p_+}}) \Big|$$
$$\leqq C^n (n'!)^{1/2} \prod_{\Delta \in \mathcal{D}_0} (n_\Delta !)^{1/2} \| R \|^{n/2} \prod_{j=1}^{n} \| \varphi_j \|_{L^2}. \quad (2.3.2)$$

[The proof is similar to that of corollary (2.2).]

2.4. Now, one shows the main estimate:

Lemma. — *One supposes that* $\underline{q} \in Q_\Lambda$ *and* $\underline{t} \in \mathcal{T}_\Lambda$ *satisfy* (2.1.1), *then each term* $S_{\mathcal{T}_j}^{\underline{q}_j}(\underline{t}; \sigma)$ *in the formula* (2.1.9) *is bounded by*

$$| S_{\mathcal{T}_j}^{\underline{q}_j}(\underline{t}; \sigma) | \leqq \mathcal{N}(f_j) C_\varepsilon^{|\underline{q}_j|} \prod_{\Delta_0 \in \underline{\Delta}_0} (|\underline{q}_j \lceil_{\Delta_0}|!)^{1/2}$$
$$\times \prod_{\Delta \in \underline{\Delta}_+} ((q_{j\Delta}!)^{1/2} |\Delta|^{(1/2-\varepsilon) q_{j\Delta}}), \quad (2.4.1)$$

where

$$\mathcal{N}(f_j) = C^{l_j} (l_j !)^{1/2} \prod_{k_j=1}^{l_1} \| f_{j, k_j} \|_{\mathcal{H}^{-1}},$$
$$|\underline{q}_j| = \sum_{\Delta \in \underline{\Delta}_+} q_{j\Delta}, \qquad |\underline{q}_j \lceil_{\Delta_0}| = \sum_{\Delta \in \underline{\Delta}_+;\, \Delta \subset \Delta_0} q_{j\Delta}.$$

Proof. — First, from (2.0.2), (2.0.3), (2.0.4), with $\chi_\Delta = \delta_{O_\Delta}$, so that $\Sigma_m^{-1/2} \psi_\Delta(x) = \dfrac{1}{2m} e^{-m|x - O_\Delta|}$,

$$\| \psi_\Delta \|_{L^2}^2 = \frac{1}{2m}, \qquad (\Delta \in \underline{\Lambda}), \quad (2.4.2)$$

$$\| \psi'_\Delta \|_{L^2}^2 = \frac{1}{m}(1 - e^{-m|\Delta|/2}) \leqq \frac{|\Delta|}{2}, \qquad (\Delta \in \underline{\Lambda}_+), \quad (2.4.3)$$

$$\| \tilde{\psi}_\Delta \|_{L^2} \leqq \sqrt{\frac{2}{m}}, \qquad (\Delta \in \underline{\Lambda}_+), \quad (2.4.4)$$

then, from (2.0.3), if $\varphi \in L^2$ is such that $\Sigma_m^{-1/2}\varphi$ has a bounded variation,

$$\sum_{\Delta \in \underline{\Delta}_r} |(\varphi, \psi'_\Delta)| \leqq \text{var.}\,[\Sigma_m^{-1/2}\varphi] = \|(\Sigma_m^{-1/2}\varphi)'\|_{L^1}, \qquad (2.4.5)$$

but

$$\text{var.}\,[\Sigma_m^{-1/2}\psi'_\Delta] = \frac{2}{m}(1-e^{-m|\Delta|/2}) \leqq |\Delta|, \qquad (\Delta \in \underline{\Delta}_+), \qquad (2.4.6)$$

therefore, if one sets

$$u_\Delta \leqq |\Delta|^{-1/2+\varepsilon}\psi'_\Delta, \qquad (\Delta \in \underline{\Delta}_+), \qquad (2.4.7)$$

one has, with $\xi_\Delta = |\Delta|^{1/2}$,

$$\sup_{\Delta_1 \in \underline{\Delta}_+} \left(\frac{1}{\xi_{\Delta_1}} \sum_{\Delta_2 \in \underline{\Delta}_+} \xi_{\Delta_2} |(u_{\Delta_1}, u_{\Delta_2})| \right) \leqq C_\varepsilon. \qquad (2.4.8)$$

Now, let $V \subset \mathcal{f}_j^{q_j}$ be the set of vectors φ_i of the form either $\Sigma_m^{-1/2} f$, $(f \in f_j)$, or $\tilde{\psi}_\Delta$, $(\Delta \in \underline{\Delta}_+)$ ([12]), one notes $V' = \mathcal{f}_j^{q_j} \setminus V$, from (2.1.4) and (2.4.7), one has

$$S_{\mathcal{f}_j^{q_j}}(\underline{t};\sigma) = \Big(\prod_{\Delta \in \underline{\Delta}_+} |\Delta|^{(1/2-\varepsilon)q_{j\Delta}}\Big)$$

$$\times \sum_{k=0}^{|V'|/2} \sum_{X \subset V;\,|X|=2k} \sum_{Y \subset V';\,|Y|=2k} \Big(\sum_{P_1 \in \mathscr{P}_2(V \setminus X)} \prod_{p_1 \in P_1} (\varphi_{p_1 -}, \mathbf{R}_{\underline{t}}(\sigma)\varphi_{p_1 +}) \Big)$$

$$\times \Big(\sum_{\tau \in \mathscr{B}(Y,X)} \prod_{y \in Y} (\varphi_{\tau(y)}, \mathbf{R}_{\underline{t}}(\sigma) u_{\Delta_y}) \Big)$$

$$\times \Big(\sum_{P_2 \in \mathscr{P}_2(V' \setminus Y)} \prod_{p_2 \in P_2} (u_{\Delta p_2 -}, \mathbf{R}_{\underline{t}}(\sigma) u_{\Delta p_2 +}) \Big). \qquad (2.4.9)$$

One bounds the factors under the sign sum: the first one by (2.3.2), the third by (2.2.2), and the second by (2.2.1) and (2.3.1), because

$$\sum_{\tau \in \mathscr{B}(Y,X)} \prod_{y \in Y} (\varphi_{\tau(y)}, \mathbf{R}_{\underline{t}}(\sigma) u_{\Delta_y}) = \Big(\bigvee_{x \in X} \varphi_x, \vee^{2k}(\mathbf{R}_{\underline{t}}(\sigma)) \bigvee_{y \in Y} u_{\Delta_y} \Big)_{(L^2)^{\vee 2k}}$$

Then, using (2.4.3) to (2.4.8), the inequality

$$\|\mathbf{R}_{\underline{t}}(\sigma)\| = \left\| \left[I + \left(\frac{i\alpha\rho}{\sqrt{\mathrm{Re}.\alpha^2}} \right) A_{\underline{t}}(\sigma) \right]^{-1} \right\|$$

$$\leqq |\cos \operatorname{Arg} \alpha\rho|^{-1} \leqq C, \qquad (2.4.10)$$

([12]) Plus, if $|q_j|$ is odd, one arbitrary vector of $\mathcal{f}_j^{q_j}$ of the form ψ'_Δ so that $|V|$, (also $|V'|$), is even.

(because $\mathbf{A}_{\underline{t}}(\sigma)$ is self adjoint), and $|\mathrm{V}| \leqq l_j + |\underline{q}_j| + 1$, $|\mathrm{V}'| \leqq |\underline{q}_j|$, it becomes

$$
\begin{aligned}
|\mathrm{S}\tilde{f}_j^{\underline{q}_j}(\underline{t};\sigma)| \leqq & \prod_{k_j=1}^{l_j} \|f_{j,k_j}\|_{\mathscr{H}^1} \cdot \Big(\prod_{\Delta \in \underline{\Lambda}_+} |\Delta|^{(1/2-\varepsilon)\, q_{j\Delta}} \Big) \\
& \times \sum_{k=0}^{|\underline{q}_j|/2} \binom{l_j + |\underline{q}_j|}{2k} \binom{|\underline{q}_j|}{2k} \mathrm{C}^{l_j + |\underline{q}_j|} \cdot \Big((l_j^{\mathrm{V}\setminus\mathrm{X}} !)^{1/2} \prod_{\Delta_0 \in \underline{\Lambda}_0} (|\underline{q}_{j\restriction\Delta_0}^{\mathrm{V}\setminus\mathrm{X}}|!)^{1/2} \Big) \\
& \times \Big((l_j^{\mathrm{X}} !)^{1/2} \prod_{\Delta_0 \in \underline{\Lambda}_0} (|\underline{q}_{j\restriction\Delta_0}^{\mathrm{X}}|!)^{1/2} \prod_{y \in \mathrm{Y}} (q_{j\Delta_y}!)^{1/2} \mathrm{C}_\varepsilon^k \Big) \\
& \times \Big(\prod_{x \in \mathrm{V}\setminus\mathrm{Y}} (q_{j\Delta_x}!)^{1/2} \mathrm{C}_\varepsilon^{(1/2\,|\underline{q}_j| - k)} \Big), \qquad (2.4.11)
\end{aligned}
$$

where l_j^{X}, $l_j^{\mathrm{V}\setminus\mathrm{X}}$, ($\underline{q}_j^{\mathrm{X}}$, $\underline{q}_j^{\mathrm{V}\setminus\mathrm{X}}$), is the number of elements of $\tilde{f}_j$, ($\underline{q}_j$), which belong to X, V∖X; then (2.4.1) follows easily. ■

One deduces elementarily the

COROLLARY. — *With the hypothesis of the lemma above, one has*

$$
\left| \sum_{\{\underline{q}_j\}_{1 \leqq j \leqq \mathrm{N}};\ \sum_j \underline{q}_j = \underline{q}_+} \Big(\prod_{\Delta \in \Delta_+} \frac{q_\Delta !}{\prod_{j=1}^{\mathrm{N}} q_{j\Delta}!} \Big) \prod_{j=1}^{\mathrm{N}} \mathrm{S}\tilde{f}_j^{\underline{q}_j}(\underline{t};\sigma) \right|
$$

$$
\leqq \mathscr{N}(\underline{f})\, \mathrm{C}_\varepsilon^{|\underline{q}|} \prod_{\Delta_0 \in \underline{\Lambda}_0} (|\underline{q}\restriction_{\Delta_0}|!)^{1/2} \prod_{\Delta \in \underline{\Lambda}_+} \Big((q_\Delta !)^{1/2} |\Delta|^{(1/2-\varepsilon)\, q_\Delta} \Big), \qquad (2.4.12)
$$

(*where* $\mathscr{N}(\underline{f}) = \prod_{j=1}^{\mathrm{N}} \mathscr{N}(\underline{f}_j)$; $|\underline{q}\restriction_{\Delta_0}| = \sum_{\Delta \in \underline{\Lambda}_+;\, \Delta \subset \Delta_0} q_\Delta$; $|\underline{q}| = \sum_{\Delta \in \underline{\Lambda}_+} q_\Delta$).

2.5. Now one estimates the factor Det. $\mathbf{R}_{\underline{t}}^{\mathrm{N}/2}$ of (2.1.9):

LEMMA. — *If* $\underline{t} \in \mathscr{T}_\Lambda$ *satisfies the condition* (2.1.1), *one has*

$$
\|\mathrm{Det.}\ \mathbf{R}_{\underline{t}}\|_{\mathrm{L}^p} \leqq \mathrm{C}_p^{|\Lambda| + |\underline{\Delta}(\underline{t})|}, \qquad (2.5.1)
$$

[*where* $|\Lambda|$ *is the length of* Λ *and* $|\underline{\Delta}(\underline{t})|$ *is the number of elements of* $\underline{\Delta}(\underline{t})$].

Proof. — If $\mathrm{R} = [\mathrm{I} + \tau \mathrm{A}]^{-1} = \mathrm{I} - \tau \mathrm{RA}$, according to [13], one has $|\mathrm{Det.}\ \mathrm{R}| \leqq e^{\|\tau \mathrm{RA}\|_{\mathscr{T}^1}} \leqq e^{|\tau| \cdot \|\mathrm{R}\| \cdot \|\mathrm{A}\|_{\mathscr{T}^1}}$, where $\|\,.\,\|_{\mathscr{T}^1}$ is the trace norm.

Now, if $\underline{t} \in \mathscr{T}_\Lambda$ satisfies (2.1.1),

$$
\mathbf{A}_{\underline{t}}(\sigma) = \sum_{\Delta \in \underline{\Delta}(\underline{t})} \langle \sigma, 1_\Delta \rangle (t_\Delta \psi_\Delta \times \psi_\Delta + (1 - t_\Delta) \psi_\Delta^\wedge \times \psi_\Delta^\wedge), \qquad (2.5.2)
$$

so that, from (2.4.2),

$$
\begin{aligned}
\|\mathbf{A}_{\underline{t}}(\sigma)\|_{\mathscr{T}^1} \leqq & \sum_{\Delta \in \underline{\Delta}(\underline{t})} |\langle \sigma, 1_\Delta \rangle| (\|\psi_\Delta\|^2 + \|\psi_\Delta^\wedge\|^2) \\
& \leqq \mathrm{C}' \sum_{\Delta \in \underline{\Delta}(\underline{t})} |\langle \sigma, 1_\Delta \rangle|, \qquad (2.5.3)
\end{aligned}
$$

thus, from (2.4.10),

$$\| \text{Det. } \mathbf{R}_{\underline{t}} \|_{L^p} \leqq \| e^{C'' p \sum_{\Delta \in \underline{\Lambda}(t)} |\langle \cdot, 1_\Delta \rangle|} \|_{L^1}^{1/p} = \prod_{\Delta \in \underline{\Lambda}(\underline{t})} \| e^{C |\langle \cdot, 1_\Delta \rangle|} \|_{L^1}^{1/p},$$

because the variables $\langle \sigma, 1_\Delta \rangle$'s are independent, then from

$$e^{|Cx|} \leqq e^{Cx} + e^{-Cx} \quad \text{and} \quad \int_{\mathscr{S}'} e^{C \langle \sigma, 1_\Delta \rangle} \nu(d\sigma) = e^{1/2\, C^2 |\Delta|},$$

one has

$$\| \text{Det. } \mathbf{R}_{\underline{t}} \|_{L^p} \leqq \prod_{\Delta \in \underline{\Lambda}(\underline{t})} (2 e^{1/2\, C^2 |\Delta|})^{1/p},$$

and (2.5.1) follows since $\sum_{\Delta \in \underline{\Lambda}(\underline{t})} |\Delta| = |\Lambda|$. ■

2.6. One will use repeatedly the following elementary inequality:

LEMMA. – *Let* $X \subset \mathscr{D}_r$, $(r \geqq 0)$, *and* $\underline{n} = (n_\Delta)_{\Delta \in X}$ *a family of integer (almost all vanishing); with* $n := \sum_{\Delta \subset X} n_\Delta$, *one has*

$$|\underline{n}|! \prod_{\Delta \in X} |\Delta|^{(1+\varepsilon) n_\Delta} \leqq C_\varepsilon^{|\underline{n}|} |X|^{|\underline{n}|} \prod_{\Delta \subset X} (n_\Delta !). \tag{2.6.1}$$

Proof. – With $n_s = \sum_{\Delta \subset X;\, \Delta \in \mathscr{D}_s} n_\Delta$, $(s \geqq r)$, one has $|\underline{n}| = \sum_{s \geqq r} n_s$, so that

$$|X|^{|\underline{n}|} = \prod_{s \geqq r} \Big(\sum_{\Delta \subset X;\, \Delta \in \mathscr{D}_s} |\Delta| \Big)^{n_s} \geqq \prod_{s \geqq r} \left(\frac{n_s !}{\prod_{\Delta \subset X;\, \Delta \in \mathscr{D}_s} (n_\Delta !)} \prod_{\Delta \subset X;\, \Delta \in \mathscr{D}_s} |\Delta|^{n_\Delta} \right),$$

thus

$$\prod_{s \geqq r} (n_s !) \prod_{\Delta \subset X} |\Delta|^{n_\Delta} \leqq |X|^{|\underline{n}|} \prod_{\Delta \subset X} (n_\Delta !); \tag{2.6.2}$$

on the other hand, if $|\Delta_s|$ is the length of a segment in $\mathscr{D}_s$,

$$\frac{|\underline{n}|!}{\prod_{s \geqq r} (n_s !)} \prod_{\Delta \subset X} |\Delta|^{\varepsilon n_\Delta} = \frac{|\underline{n}|!}{\prod_{s \geqq r} (n_s !)} \prod_{s \geqq r} |\Delta_s|^{\varepsilon n_s} \leqq \Big(\sum_{s \geqq 0} |\Delta_s|^{\varepsilon} \Big)^{|\underline{n}|} := C_\varepsilon |\underline{n}|, \tag{2.6.3}$$

and (2.6.1) follows from (2.6.2) and (2.6.3). ■

2.7. Now one estimates the factor

$$\prod_{\Delta \in \underline{\Lambda}_+} \left[\left\langle \sigma, \left(1_\Delta - \frac{|\Delta|}{|\tilde{\Delta}|} 1_{\tilde{\Delta}} \right) \right\rangle + \frac{\alpha}{\sqrt{\text{Re}.\alpha^2}} \frac{|\Delta|}{|\tilde{\Delta}|} \langle \sigma, 1_{\tilde{\Delta}} \rangle \right]^{q_\Delta}$$

of (2.1.9).

LEMMA. — *For any* $p \geqq 1$, $\varepsilon > 0$, *there is a constant* $C_{\varepsilon, p}$ *such that*

$$\left\| \prod_{\Delta \in \underline{\Lambda}+} [\langle ., 1_\Delta \rangle + \beta \frac{|\Delta|}{|\tilde{\Delta}|} \langle ., 1_{\tilde{\Delta}} \rangle]^{q_\Delta} \right\|_{L^p} \leqq C_{\varepsilon, p}^{|\underline{q}|} \prod_{\Delta \in \underline{\Lambda}+} \left((q_\Delta !)^{1/2} |\Delta|^{(1/2 - \varepsilon) q_\Delta} \right). \quad (2.7.1)$$

Proof. — First, from Hölder's inequality:

$$\left\| \prod_{\Delta \in \underline{\Lambda}+} \left[\langle ., 1_\Delta \rangle + \beta \frac{|\Delta|}{|\tilde{\Delta}|} \langle ., 1_{\tilde{\Delta}} \rangle \right]^{q_\Delta} \right\|_{L^p} \leqq (1 + |\beta|)^{|\underline{q}|} \sup_{\underline{0} \leqq \underline{h} \leqq \underline{q}} . \left(\| \prod_{\Delta \in \underline{\Lambda}+} \langle ., 1_\Delta \rangle^{(q_\Delta - h_\Delta)} \|_{L^{2p}} \times \left\| \prod_{\Delta \in \underline{\Lambda}+} \left(\frac{|\Delta|}{|\tilde{\Delta}|} \langle ., 1_{\tilde{\Delta}} \rangle \right)^{h_\Delta} \right\|_{L^{2p}} \right). \quad (2.7.2)$$

Then,

$$\| \prod_{\Delta \in \underline{\Lambda}+} \langle ., 1_\Delta \rangle^{(q_\Delta - h_\Delta)} \|_{L^{2p}} \leqq C_{\varepsilon, p}^{|\underline{q} - \underline{h}|} \prod_{\Delta \in \underline{\Lambda}+} \left(([q_\Delta - h_\Delta]!)^{1/2} |\Delta|^{(1/2 - \varepsilon)(q_\Delta - h_\Delta)} \right), \quad (2.7.3)$$

indeed,

$$\begin{aligned} \| \prod_{\Delta \in \underline{\Lambda}+} \langle ., 1_\Delta \rangle^{n_\Delta} \|_{L^{2p}} &= \| \prod_{s \geqq 1} \prod_{\Delta \in \underline{\Lambda}_s} \langle ., 1_{\Delta_s} \rangle^{n_{\Delta_s}} \|_{L^{2p}} \\ &\leqq^{(i)} \prod_{s \geqq 1} \| \prod_{\Delta \in \underline{\Lambda}_s} \langle ., 1_{\Delta_s} \rangle^{n_{\Delta_s}} \|_{L^{4ps^2}} \\ &=^{(ii)} \prod_{s \geqq 1} \prod_{\Delta \in \underline{\Lambda}_s} \| \langle ., 1_{\Delta_s} \rangle^{n_{\Delta_s}} \|_{L^{4ps^2}} = \prod_{s \geqq 1} \prod_{\Delta \in \underline{\Lambda}_s} (\| \langle ., 1_{\Delta_s} \rangle \|_{L^{4ps^2 n_{\Delta_s}}})^{n_{\Delta_s}} \\ &\leqq^{(iii)} \prod_{s \geqq 1} \prod_{\Delta \in \underline{\Lambda}_s} (4ps^2 n_{\Delta_s})^{(1/2) n_{\Delta_s}} (\| \langle ., 1_{\Delta_s} \rangle \|_{L^2})^{n_{\Delta_s}}, \end{aligned}$$

[$^{(i)}$ from Hölder's inequality, since $\sum_{s \geqq 1} \frac{1}{2s^2} \leqq 1$, $^{(ii)}$ because the variables $\langle \sigma, 1_{\Delta_s} \rangle$ are independent, and $^{(iii)}$ from Nelson's "hypercontractivity" inequality], then (2.7.3) follows easily because $\| \langle ., 1_{\Delta_s} \rangle \|_{L^2} = |\Delta|^{1/2}$ and $s \leqq C_\varepsilon |\Delta|^\varepsilon$.

Last,

$$\left\| \prod_{\Delta \in \underline{\Lambda}+} \left(\frac{|\Delta|}{|\tilde{\Delta}|} \langle ., 1_{\tilde{\Delta}} \rangle \right)^{h_\Delta} \right\|_{L^{2p}} \leqq C_{\varepsilon, p}^{|\underline{h}|} \prod_{\Delta \in \underline{\Lambda}+} \left((h_\Delta !)^{1/2} |\Delta|^{(1/2 - \varepsilon) h_\Delta} \right). \quad (2.7.4)$$

indeed

$$\Big\| \prod_{\Delta \in \underline{\Lambda}+} \Big(\frac{|\Delta|}{|\tilde{\Delta}|} \langle ., 1_{\tilde{\Delta}} \rangle \Big)^{h_\Delta} \Big\|_{L^{2p}}$$

$$= \Big\| \prod_{\Delta' \in \underline{\Lambda}(\underline{t})} \prod_{\Delta \in \underline{\Lambda}+; \Delta \subset \Delta'} \Big(\frac{|\Delta|}{|\Delta'|} \langle ., 1_{\Delta'} \rangle \Big)^{h_\Delta} \Big\|_{L^{2p}}$$

$$=^{(i)} \prod_{\Delta' \in \underline{\Lambda}(\underline{t})} \Big(\prod_{\Delta \in \underline{\Lambda}+; \Delta \subset \Delta'} \Big(\frac{|\Delta|}{|\Delta'|} \Big)^{h_\Delta} \Big) \Big(\| \langle ., 1_{\Delta'} \rangle \|_{L^{2p \sum_{\Delta \subset \Delta'} h_\Delta}} \Big)^{\sum_{\Delta \subset \Delta'} h_\Delta}$$

$$\leqq^{(ii)} \prod_{\Delta' \in \underline{\Lambda}(\underline{t})} \Big(\prod_{\Delta \in \underline{\Lambda}+; \Delta \subset \Delta'} \Big(\frac{|\Delta|}{|\Delta'|} \Big)^{h_\Delta} \Big) \Big(2p \sum_{\Delta \subset \Delta'} h_\Delta \Big)^{(1/2) \sum_{\Delta \subset \Delta'} h_\Delta} |\Delta'|^{(1/2) \sum_{\Delta \subset \Delta'} h_\Delta},$$

[$^{(i)}$ because the variables $\langle \sigma, 1_{\Delta'} \rangle$ are independant, $^{(ii)}$ from Nelson's "hypercontractivity" inequality], one then deduces (2.7.4) using (2.6.1); and obviously (2.7.1) follows from (2.7.3) and (2.7.4). ■

2.8. An estimation of the right hand side of (2.1.9), after Hölder's inequality, according to (2.4.12), (2.5.1), (2.7.1), (and of course $|e^{-(i/2)(\mathrm{Im}\,.\,\alpha^2/\mathrm{Re}\,.\,\alpha^2) \sum_{\Delta \in \underline{\Lambda}(\underline{t})} |\Delta|^{-1} \langle \sigma, 1_\Delta \rangle^2}| = 1$), gives

LEMMA. — *If* $\underline{q} \in Q_\Lambda$ *and* $\underline{t} \in \mathscr{T}_\Lambda$ *satisfies* (2.1.1), *then for any* $\varepsilon > 0$ *there exists a constant* C_ε, (*depending on the parameters* N, ρ, *and on the auxiliary parameter* α, *uniformly on any compact set in the domain defined by* $|\mathrm{Arg}\,\alpha| < \frac{\pi}{4}$, $|\mathrm{Arg}\,\alpha\rho| < \frac{\pi}{2}$, *and not depending on* $\underline{t}$, Λ, *neither on m bounded away from* 0) ([13]), *such that*

$$|(\prod_{\Delta \in \underline{\Lambda}+} D_\Delta^{(q_\Delta)}) Z_{N,(\rho);\underline{t}}(f)|$$

$$\leqq \mathcal{N}(f) C_\varepsilon^{|\Lambda| + |\underline{\Lambda}(\underline{t})| + |\underline{q}|} \prod_{\Delta_0 \in \underline{\Lambda}_0} (|\underline{q} \restriction_{\Delta_0}|!)^{1/2} \prod_{\Delta \in \underline{\Lambda}+} \big((q_\Delta !) |\Delta|^{(1-\varepsilon) q_\Delta} \big), \qquad (2.8.1)$$

[*where* $\mathcal{N}(f)$, $|\Lambda|$, $|\underline{\Lambda}(\underline{t})|$, $|\underline{q}|$, $|\underline{q} \restriction_{\Delta_0}|$, *have the same meaning as in* (2.4.12), (2.5.1)].

In view to apply the theorem (A.0), one sets

$$\tilde{D}_\Delta^r = \sum_{\Delta_r \in \underline{\Lambda}_r;\, \Delta_r \subset \Delta} D_{\Delta_r}, \qquad (r \geqq 1,\ \Delta \in \underline{\Lambda}_r), \qquad (2.8.2)$$

one has

([13]) Constants introduced in the proofs above have this property, so it will be for those introduced afterwards.

PROPOSITION. — *Let* $\underline{k}=\{k_\Delta^r;\ r\geqq 1,\ \Delta\in\underline{\Lambda}_+^r\}$ *be a family of (almost all vanishing) integers, and* $\underline{t}\in\mathcal{T}_\Lambda$, *one supposes* (2.1.1) *is satisfied by* $\underline{t}$ *and each* $\underline{q}\in\mathcal{Q}(\underline{k})$, [*where* $\mathcal{Q}(\underline{k})$ *is such that* $\prod_{r\geqq 1}\prod_{\Delta\in\underline{\Lambda}_+^r}(\tilde{D}_\Delta^r)^{k_\Delta^r}=\sum_{\underline{q}\in\mathcal{Q}(\underline{k})}\prod_{\Delta\in\underline{\Lambda}_+^r}D_\Delta^{q_\Delta}$],

then

$$|(\prod_{r\geqq 1}\prod_{\Delta\in\underline{\Lambda}_+^r}(\tilde{D}_\Delta^r)^{k_\Delta^r})Z_{N,(\rho);\underline{t}}(f)|$$
$$\leqq\mathcal{N}(f)C_\varepsilon^{|\Lambda|+|\underline{\Lambda}(t)|+|\underline{k}|}\prod_{\Delta_0\in\underline{\Lambda}_0}(|\underline{k}\lceil_{\Delta_0}|!)^{-1/2}$$
$$\times(\prod_{r\geqq 1}\prod_{\Delta\in\underline{\Lambda}_+^r}(k_\Delta^r!)^2|\Delta_r|^{-\varepsilon k_\Delta^r}),\qquad(2.8.3)$$

(*where* $|\underline{k}\lceil_{\Delta_0}|=\sum_{r\geqq 1}\sum_{\Delta\in\underline{\Lambda}_+^r;\,\Delta\subset\Delta_0}k_\Delta^r$, $|\underline{k}|=\sum_{\Delta_0\in\underline{\Lambda}_0}|\underline{k}\lceil_{\Delta_0}|=\sum_{r\geqq 1}\sum_{\Delta\in\underline{\Lambda}_+^r}k_\Delta^r$, *and* $|\Delta_r|$ *is the length of any element of* $\underline{\Lambda}_r$).

Proof. — From (2.8.2)

$$(\prod_{r\geqq 1}\prod_{\Delta\in\underline{\Lambda}_+^r}(\tilde{D}_\Delta^r)^{k_\Delta^r})Z_{N,(\rho);\underline{t}}(f)$$
$$=\sum_{\underline{p}\in\mathcal{P}(\underline{k})}\left(\prod_{r\geqq 1}\prod_{\Delta\in\underline{\Lambda}_+^r}\frac{k_\Delta^r!}{\prod_{\Delta_r\in\underline{\Lambda}_r;\,\Delta_r\subset\Delta}p_\Delta^{\Delta_r}!}\right)$$
$$\times\Big[(\prod_{\Delta'\in\underline{\Lambda}_+^r}(D_{\Delta'})^{\sum_{\Delta\supset\Delta'}p_\Delta^{\Delta'}})Z_{N,(\rho);\underline{t}}(f)\Big],\qquad(2.8.4)$$

where

$$\mathcal{P}(\underline{k}):=\{\underline{p}=(p_\Delta^{\Delta'})_{\Delta'\subset\Delta};\ \sum_{\Delta_r\in\underline{\Lambda}_r;\,\Delta_r\subset\Delta}p_\Delta^{\Delta_r}=k_\Delta^r,\ (r\geqq 1)\}.\qquad(2.8.5)$$

Therefore, from (2.8.1), written down for $\varepsilon_1<\varepsilon$, one has

$$|(\prod_{r\geqq 1}\prod_{\Delta\in\underline{\Lambda}_+^r}(\tilde{D}_\Delta^r)^{k_\Delta^r})Z_{N,(\rho);\underline{t}}(f)|\leqq\mathcal{N}(f)\,C_{\varepsilon_1}^{|\Lambda|+|\underline{\Lambda}(t)|+|\underline{k}|}$$
$$\times\sum_{\underline{p}\in\mathcal{P}(\underline{k})}\left[\prod_{r\geqq 1}\left(\prod_{\Delta\in\underline{\Lambda}_+^r}\left(\frac{k_\Delta^r!}{\prod_{\Delta_r\in\underline{\Lambda}_r;\,\Delta_r\subset\Delta}p_\Delta^{\Delta_r}!}|\Delta_r|^{k_\Delta^r}\right)\cdot\prod_{\Delta_r'\in\underline{\Lambda}_r}\left(\Big[\sum_{\Delta'\in\underline{\Lambda}_+^r;\,\Delta'\supset\Delta_r'}p_{\Delta'}^{\Delta_r'}\Big]!\right)\right)\right]$$
$$\times\prod_{\Delta_0\in\underline{\Lambda}_0}(|\underline{k}\lceil_{\Delta_0}|!)^{1/2}\cdot(\prod_{r\geqq 1}\prod_{\Delta\in\underline{\Lambda}_+^r}|\Delta_r|^{-\varepsilon_1 k_\Delta^r}),\qquad(2.8.6)$$

then, on the one hand,

$$\prod_{\Delta'_r \in \underline{\Lambda}_r} \Big(\Big[\sum_{\Delta' \in \underline{\Lambda}^r_+;\ \Delta' \supset \Delta'_r} p^{\Delta'_r}_{\Delta'}\Big]!\Big)$$

$$= \prod_{\Delta'_r \in \underline{\Lambda}_r} \frac{\Big[\sum_{\Delta' \in \underline{\Lambda}^r_+;\ \Delta' \supset \Delta'_r} p^{\Delta'_r}_{\Delta'}\Big]!}{\prod_{\Delta' \in \underline{\Lambda}^r_+;\ \Delta' \supset \Delta_r} (p^{\Delta'_r}_{\Delta'}!)} \prod_{\Delta \in \underline{\Lambda}^r_+} \left(\frac{\prod_{\Delta_r \in \underline{\Lambda}_r;\ \Delta_r \subset \Delta} (p^{\Delta_r}_{\Delta}!)}{k^r_\Delta!} k^r_\Delta!\right)$$

$$\leqq \prod_{\Delta_r \in \underline{\Lambda}_r} r^{[\sum_{\Delta;\ \Delta \supset \Delta_r} p^{\Delta_r}_{\Delta}]} \prod_{\Delta \in \underline{\Lambda}^r_+} k^r_\Delta! = \prod_{\Delta \in \underline{\Lambda}^r_+} r^{k^r_\Delta}(k^r_\Delta!),$$

but, for any $\xi_1 > 0$, $r \leqq C_{\xi_1} |\Delta_r|^{-\xi_1}$, (because $|\Delta_r| = l_0 2^{-r}$), thus

$$\prod_{\Delta'_r \in \underline{\Lambda}_r} \Big(\Big[\sum_{\Delta' \in \underline{\Lambda}^r_+;\ \Delta' \supset \Delta'_r} p^{\Delta'_r}_{\Delta'}\Big]!\Big) \leqq \prod_{\Delta \in \underline{\Lambda}^r_+} \Big(C^{k^r_\Delta}_{\xi_1} |\Delta_r|^{-\xi_1 k^r_\Delta}(k^r_\Delta!)\Big); \tag{2.8.7}$$

on the other hand, for $r \geqq 1$, $\Delta \in \underline{\Lambda}_r$,

$$\sum_{(p^{\Delta_r}_{\Delta})_{\Delta_r \subset \Delta};\ \sum_{\Delta_r \subset \Delta} p^{\Delta_r}_{\Delta} = k^r_\Delta} \left(\frac{k^r_\Delta!}{\prod_{\Delta_r \in \underline{\Lambda}_r;\ \Delta_r \subset \Delta} p^{\Delta_r}_{\Delta}!} |\Delta_r|^{k^r_\Delta}\right) = |\Delta|^{k^r_\Delta},$$

so that

$$\sum_{\underline{p} \in \mathscr{P}(\underline{k})} \left(\prod_{r \geqq 1} \prod_{\Delta \in \underline{\Lambda}^r_+} \frac{k^r_\Delta!}{\prod_{\Delta_r \in \underline{\Lambda}_r;\ \Delta_r \subset \Delta} p^{\Delta_r}_{\Delta}!} |\Delta_r|^{k^r_\Delta}\right) \leqq \prod_{r \geqq 1} \prod_{\Delta \in \underline{\Lambda}^r_+} |\Delta|^{k^r_\Delta}; \tag{2.8.8}$$

then, one chooses $\xi_2 > 0$, and sets $\beta_r = b|\Delta_r|^{-\xi_2}$, (with b such that $\sum_{r=1}^{\infty} \beta_r^{-1} = 1$), one has

$$|\underline{k} \lceil \Delta_0|! \leqq \prod_{r \geqq 1} \Big(\Big[\sum_{\Delta \in \underline{\Lambda}^r_+;\ \Delta \subset \Delta_0} k^r_\Delta\Big]!\ \beta_r^{(\sum_{\Delta \in \underline{\Lambda}^r_+;\ \Delta \subset \Delta_0} k^r_\Delta)}\Big).$$

so that

$$\prod_{\Delta_0 \in \underline{\Delta}_0} (|\underline{k} \lceil \Delta_0|!)$$

$$\leqq C^{|\underline{k}|}_{\xi_2} \prod_{\Delta_0 \in \underline{\Delta}_0} \prod_{r \geqq 1} \Big(\Big[\sum_{\Delta \in \underline{\Lambda}^r_+;\ \Delta \subset \Delta_0} k^r_\Delta\Big]!\Big) \prod_{r \geqq 1} \prod_{\Delta \in \underline{\Lambda}^r_+} |\Delta_r|^{-\xi_2 k^r_\Delta}, \tag{2.8.9}$$

last, as $|\Delta_r| \leqq |\Delta|$, $\forall \Delta \in \Lambda^r_+$, one obtains from (2.6.1)

$$\prod_{\Delta_0 \in \underline{\Delta}_0} \prod_{r \geqq 1} \Big(\Big[\sum_{\Delta \in \underline{\Lambda}^r_+;\ \Delta \subset \Delta_0} k^r_\Delta\Big]! \prod_{\Delta \in \underline{\Lambda}^r_+;\ \Delta \subset \Delta_0} |\Delta|^{k^r_\Delta}\Big)$$

$$\leqq C^{|\underline{k}|}_{\xi_3} |\Delta_0|^{|\underline{k}|} \prod_{r \geqq 1} \prod_{\Delta \in \underline{\Lambda}^r_+} (k^r_\Delta! |\Delta_r|^{-\xi_3 k^r_\Delta}). \tag{2.8.10}$$

Then one obtains (2.8.3), with $\varepsilon=\varepsilon_1+\xi_1+\xi_2+\xi_3$, by setting (2.8.7), (2.8.8), (2.8.9) and (2.8.10) into (2.8.6). ■

2.9. Now one wants to give a second estimate of

$$(\prod_{r\geq 1}\prod_{\Delta\in\underline{\Lambda}^r_+}(\tilde{D}^r_\Delta)^{k^r_\Delta})\,Z_{N,(\rho);\underline{t}}(f);$$

one starts with the formula (2.1.7), in which one forces the factors $\langle\sigma, 1_\Delta\rangle$ to "disappear" by iterated integrations by parts.

One uses the formula

$$\int_{\mathscr{S}'}\langle\sigma, g\rangle\,F(\sigma)\,\nu(d\sigma)=\int_{\mathscr{S}'}\partial_g F(\sigma)\,\nu(d\sigma),$$

where for $g\in L^2(\mathbf{R},\mathbf{R})$, $\partial_g F(\sigma)=\frac{d}{du}\Big\lceil_{u=0} F(\sigma+ug)$, one deduces for $\Delta\in\underline{\Lambda}^r_+$:

$$\int_{\mathscr{S}'}\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\langle\sigma, 1_\Delta\rangle F(\sigma)\,e^{-(i/2)(\mathrm{Im}.\alpha^2/\mathrm{Re}.\alpha^2)\sum_{\Delta'\in\underline{\Lambda}_r}|\Delta'|^{-1}\langle\sigma,1_{\Delta'}\rangle^2}\nu(d\sigma)$$

$$=\int_{\mathscr{S}'}\left(\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\right)^{-1}\partial_{1_\Delta}F(\sigma)\,e^{-(i/2)(\mathrm{Im}.\alpha^2/\mathrm{Re}.\alpha^2)\sum_{\Delta'\in\underline{\Lambda}_r}|\Delta'|^{-1}\langle\sigma,1_{\Delta'}\rangle^2}\nu(d\sigma). \quad (2.9.1)$$

One has obviously

$$\partial_{1_{\Delta_1}}\langle\sigma, 1_{\Delta_2}\rangle=|\Delta_1\cap\Delta_2|, \quad (2.9.2)$$

therefore if $E_{\underline{q}}$ is a set containing, for each $\Delta\in\underline{\Lambda}_+$, q_Δ elements equal to Δ, one obtains, from (2.1.7) and (2.9.1)

$$(\prod_{\Delta\in\underline{\Lambda}_+}D^{q_\Delta}_\Delta)\,Z_{N,(\rho);\underline{t}}(f)=(-i\rho)^{|\underline{q}|}\sum_{\{\underline{q}_j\}_{1\leq j\leq N};\ \sum_j\underline{q}_j=\underline{q}}\ \sum_{X\subset E_{\underline{q}}.\ |X|\in 2\mathbf{N}}$$

$$\times\Bigg[\prod_{\Delta\in\underline{\Lambda}_+}\frac{q_\Delta!}{\prod_{j=1}^N q_{j\Delta}!}.\Big(\sum_{P\in\mathscr{P}_2(X)}\prod_{p\in P}|\Delta_{p_-}\cap\Delta_{p_+}|\Big)$$

$$\times\left(\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\right)^{|\underline{\Lambda}(t)|}\int_{\mathscr{S}'}\left(\prod_{y\in E_{\underline{q}}\setminus X}\left(\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\right)^{-1}\partial_{1_{\Delta_y}}\right)$$

$$\times\left[\prod_{j=1}^N(S\mathcal{T}^{\underline{q}_j}_j(\underline{t};\sigma).\mathrm{Det}.\,\mathbf{R}_{\underline{t}}(\sigma)^{1/2})\right]$$

$$\times\, e^{-(i/2)(\mathrm{Im}.\alpha^2/\mathrm{Re}.\alpha^2)\sum_{\Delta\in\underline{\Lambda}(t)}|\Delta|^{-1}\langle\sigma,1_\Delta\rangle^2}.\,\nu(d\sigma)\Bigg]. \quad (2.9.3)$$

[after integrations by parts have been done, one has "suppressed" the complex multiplication on the variables orthogonal to the $\{\langle\sigma, 1_\Delta\rangle\}_{\Delta\in\underline{\Lambda}(t)}$'s on which the integrand does not depend anymore].

An estimation of the right hand side of (2.9.3) gives

LEMMA. — *Under the hypothesis of lemma* (2.8), *one has*

$$|(\prod_{\Delta\in\underline{\Lambda}_+} D_\Delta^{q_\Delta}) Z_{N,(\rho);\underline{t}}(f)|$$

$$\leqq \mathcal{N}(f) C_\varepsilon^{|\Lambda|+|\underline{\Lambda}(t)|+|\underline{q}|} \prod_{\Delta_0\in\underline{\Lambda}_0} (|\underline{q}\lceil_{\Delta_0}|!)^{1/2} \prod_{\Delta\in\underline{\Lambda}_+} \left((q_\Delta!)^{1/2}|\Delta|^{(1/2-\varepsilon)q_\Delta}\right)$$

$$\times \sum_{X\subset E_{\underline{q}};\,|X|\in 2N} [\prod_{\Delta_0\in\underline{\Lambda}_0} (|\underline{q}^{X'}_{\lceil\Delta_0}|!)$$

$$\times(\sum_{P\in\mathscr{P}_2(X)} \prod_{p\in P} |\Delta_{p_-}\cap\Delta_{p_+}|) \prod_{y\in E_{\underline{q}}\setminus X} |\Delta_y|], \quad (2.9.4)$$

(*where* $q_\Delta^{X'}$ *is the number of elements equal to* Δ *in* $X'=E_{\underline{q}}\setminus X$).

Proof. — First, according to (2.1.2), (2.1.8), if for $\Delta\in\underline{\Lambda}_+$ (with $q_\Delta\neq 0$) $\tilde\Delta\in\underline{\Lambda}(\underline{t})$ is defined by the condition $\Delta\subset\tilde\Delta$, one has

$$\left(\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\right)^{-1} \partial_{1_\Delta} \mathbf{R}_{\underline{t}}(\sigma)$$
$$= -i\rho\,\mathbf{R}_{\underline{t}}(\sigma)[t_{\tilde\Delta}\psi_{\tilde\Delta}\times\psi_{\tilde\Delta}+(1-t_{\tilde\Delta})\psi_{\hat{\tilde\Delta}}\times\psi_{\hat{\tilde\Delta}}]\mathbf{R}_{\underline{t}}(\sigma), \quad (2.9.5)$$

Therefore, from (2.1.4), for any family g of L^2-functions,

$$\left(\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\right)^{-1} \partial_{1_\Delta}(S_g(\underline{t};\sigma)\ \mathrm{Det.}\ \mathbf{R}_{\underline{t}}(\sigma)^{1/2})$$
$$= \frac{-i\rho}{2}|\Delta|([t_{\tilde\Delta}S_{g_{\tilde\Delta}}(\underline{t};\sigma)+(1-t_{\tilde\Delta})S_{g_{\hat{\tilde\Delta}}}(\underline{t};\sigma)]\ \mathrm{Det.}\ \mathbf{R}_{\underline{t}}(\sigma)^{1/2}), \quad (2.9.6)$$

where g_Δ is the family $\{g, \psi_\Delta, \psi_\Delta\}$.

One introduces the families $g_{\underline{h}}$, ($\underline{h}=\{h_\Delta\}_{\Delta\in\underline{\Lambda}}$), *containing the elements of* g, *and for each* $\Delta\in\underline{\Lambda}$, $2h_\Delta$ *vectors equal to* ψ_Δ;
and for $Y\subset E_{\underline{q}}$ one denotes by $H_{\underline{t}}^{\underline{q}^Y}$ the set of the $\underline{h}=\{h_\Delta\}_{\Delta\in\underline{\Lambda}}$ of which the only non vanishing elements are, for each $\Delta\in\underline{\Lambda}(\underline{t})$, h_Δ and $h_{\hat\Delta}$ which satisfies

$$0\leqq h_\Delta\leqq|\{y\in Y;\ \Delta_y\subset\Delta\}|$$

and

$$h_{\hat\Delta}=|\{y\in Y;\ \Delta_y\subset\hat\Delta\}|-\sum_{\Delta';\,\hat\Delta'=\hat\Delta} h_{\Delta'};$$

then, from (2.9.6),

$$\left(\prod_{y\in Y}\left(\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\right)^{-1}\partial_{1_{\Delta_y}}\right)(S\mathcal{J}_j^{q_j}(\underline{t};\sigma)\ \mathrm{Det}.\,\mathbf{R}_{\underline{t}}(\sigma)^{1/2})$$
$$=\left(\frac{-i\rho}{2}\right)^{|Y|}(\prod_{y\in Y}|\Delta_y|)\sum_{\underline{h}\in H^{q_{\underline{t}}^Y}}\left(c_{\underline{t}}(\underline{h})\,S\mathcal{J}_{j\underline{h}}^{q_j}(\underline{t};\sigma)\right)\ \mathrm{Det}.\,\mathbf{R}_{\underline{t}}(\sigma)^{1/2},\qquad(2.9.7)$$

with constants $c_{\underline{t}}(\underline{h})$ satisfying $0\leqq c_{\underline{t}}(\underline{h})\leqq 1$.

Then one computes the right hand side of (2.9.3), according to (2.9.7),

$$(\prod_{\Delta\in\underline{\Delta}_+}D_\Delta^{q_\Delta})Z_{N,(\rho);\underline{t}}(f)$$
$$=\sum_{\{\underline{q}_j\}_{1\leqq j\leqq N};\ \sum_j\underline{q}_j=\underline{q}}\ \sum_{X\subset E_{\underline{q}};\ |X|\in 2\mathbf{N}}$$
$$\times\sum_{\{Y_j\}_{1\leqq j\leqq N};\ \bigcup_j Y_j=E_{\underline{q}}\setminus X,\,Y_i\cap Y_j=\varnothing}\ \sum_{\{\underline{h}_j\in H_{\underline{t}}^{\underline{q}_j^{Y_j}}\}_{1\leqq j\leqq N}}$$
$$\times\frac{(-i\rho)^{2|\underline{q}|-|X|}}{2^{|\underline{q}|-|X|}}\prod_{\Delta\in\underline{\Delta}_+}\frac{q_\Delta!}{\prod_{j=1}^N q_{j\Delta}!}\prod_{j=1}^N\left(c_{\underline{t}}(\underline{h}_j)\prod_{\Delta\in\underline{\Delta}(\underline{t})}\binom{q_{j\Delta}^{Y_j}}{h_{j\Delta}}\right)$$
$$\times(\sum_{P\in\mathscr{P}_2(X)}\prod_{p\in P}|\Delta_{p_-}\cap\Delta_{p_+}|)\prod_{y\in E_{\underline{q}}\setminus X}|\Delta_y|$$
$$\times\left(\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\right)^{|\underline{\Delta}(\underline{t})|}\int_{\mathscr{S}'}\prod_{j=1}^N S\mathcal{J}_{j\underline{h}_j}^{q_j}(\underline{t};\sigma).\,\mathrm{Det}.\,\mathbf{R}_{\underline{t}}(\sigma)^{N/2}$$
$$\times e^{-(i/2)(\mathrm{Im}.\alpha^2/\mathrm{Re}.\alpha^2)\sum_{\Delta\in\Delta_r}|\Delta|^{-1}\langle\sigma,1_\Delta\rangle^2}.\,\nu(d\sigma)\qquad(2.9.8)$$

One estimates each $S\mathcal{J}_{j\underline{h}_j}^{q_j}(\underline{t};\sigma)$ from (2.2.1), (2.2.2), (2.3.1), (2.3.2), as in (2.4), [with now $V\subset\mathcal{J}_{j\underline{h}_j}^{q_j}$ the set of the vectors of the form $\Sigma_m^{-1/2}f$, $(f\in f)$, or $\tilde{\psi}_\Delta,(\Delta\in\underline{\Delta}_+)$, or ψ_Δ, $(\Delta\in\underline{\Delta})$],

$$|S\mathcal{J}_{j\underline{h}_j}^{q_j}(\underline{t};\sigma)|\leqq\mathcal{N}(f_j)\,C_\varepsilon^{|\underline{q}_j|+|\underline{h}_j|}$$
$$\times\prod_{\Delta_0\in\underline{\Delta}_0}([|\underline{q}_j\restriction\Delta_0|!]^{1/2}[|\underline{h}_j\restriction\Delta_0|!])$$
$$\times\prod_{\Delta\in\underline{\Delta}_+}\left((q_{j\Delta}!)^{1/2}|\Delta|^{((1/2)-\varepsilon)q_{j\Delta}}\right),\qquad(2.9.9)$$

and one easily deduces (2.9.4) from (2.9.3), (2.9.7), (2.9.9) and (2.5.1). ■

Now one deduces from (2.9.4) the

PROPOSITION. — *With the hypothesis of proposition* (2.8), *one has*

$$|(\prod_{r\geqq 1}\prod_{\Delta\in\underline{\Delta}^r_+}(\tilde{D}^r_\Delta)^{k^r_\Delta})Z_{N,(\rho);\underline{t}}(f)|$$
$$\leqq\mathcal{N}(f)\,C_\varepsilon^{|\Lambda|+|\underline{\Delta}(t)|+|\underline{k}|}\prod_{\Delta_0\in\underline{\Delta}_0}(|\underline{k}\lceil_{\Delta_0}|!)^{1/2}$$
$$\times(\prod_{r\geqq 1}\prod_{\Delta\in\underline{\Delta}^r_+}(k^r_\Delta!)^{3/2}|\Delta_r|^{(1/2-\varepsilon)k^r_\Delta}). \quad (2.9.10)$$

Proof. — According to (2.8.4), (2.8.5), (2.9.4) written down for $\varepsilon_1<\varepsilon$, and (2.8.7), one has

$$|(\prod_{r\geqq 1}\prod_{\Delta\in\underline{\Delta}^r_+}(\tilde{D}^r_\Delta)^{k^r_\Delta})Z_{N,(\rho);\underline{t}}(f)|\leqq\mathcal{N}(f)\,C_{\varepsilon_1}^{|\Lambda|+|\underline{\Delta}(t)|+|\underline{k}|}C_{\xi_1}^{|\underline{k}|}.$$
$$\times\prod_{\Delta_0\in\underline{\Delta}_0}(|\underline{k}\lceil_{\Delta_0}|!)^{1/2}.(\prod_{r\geqq 1}\prod_{\Delta\in\underline{\Delta}^r_+}(k^r_\Delta!)^{3/2}|\Delta_r|^{((1/2)-\varepsilon_1-\xi_1)k^r_\Delta})$$
$$\times\sum_{X\subset F_{\underline{q}};\,|X|\in 2N}[\prod_{\Delta_0\in\underline{\Delta}_0}(|\underline{k}^{X'}_{\lceil\Delta_0}|!)\prod_{y\in F_{\underline{q}}\setminus X}(\sum_{\Delta\in\underline{\Delta}_{r_y};\,\Delta\subset\Delta_y}|\Delta|)$$
$$\times\sum_{P\in\mathcal{P}_2(X)}\prod_{p\in P}(\sum_{\Delta'\in\underline{\Delta}_{r_{p_-}};\,\Delta'\subset\Delta_{p_-}}\sum_{\Delta''\in\underline{\Delta}_{r_{p_+}};\,\Delta''\subset\Delta_{p_+}}|\Delta'\cap\Delta''|)], \quad (2.9.11)$$

where F_k is a set containing k^r_Δ elements equal to $\{\Delta, r\}, (r\leqq 1, \Delta\in\underline{\Delta}_r)$, and $k^{X'}{}^r_\Delta$ is the number of elements of $X'=F_{\underline{k}}\setminus X$ equal to $\{\Delta, r\}$.

But

$$\sum_{\Delta'\in\underline{\Delta}_{r_1};\,\Delta'\subset\Delta_1}\sum_{\Delta''\in\underline{\Delta}_{r_2};\,\Delta''\subset\Delta_2}|\Delta'\cap\Delta''|=|\Delta_1\cap\Delta_2|\leqq(|\Delta_1||\Delta_2|)^{1/2},$$

and obviously

$$\sum_{\Delta'\in\underline{\Delta}_r;\,\Delta'\subset\Delta}|\Delta'|=|\Delta|,$$

therefore

$$\prod_{\Delta_0\in\underline{\Delta}_0}(|\underline{k}^{X'}_{\lceil\Delta_0}|!)\prod_{y\in F_{\underline{q}}\setminus X}(\sum_{\Delta\in\underline{\Delta}_{r_y};\,\Delta\subset\Delta_y}|\Delta|)$$
$$\times\sum_{P\in\mathcal{P}_2(X)}\prod_{p\in P}(\sum_{\Delta'\in\underline{\Delta}_{r_{p_-}};\,\Delta'\subset\Delta_{p_-}}\sum_{\Delta''\in\underline{\Delta}_{r_{p_+}};\,\Delta''\subset\Delta_{p_+}}|\Delta'\cap\Delta''|)$$
$$\leqq C^{|\underline{k}|}\prod_{\Delta_0\in\underline{\Delta}_0}([|\underline{k}^X_{\lceil\Delta_0}|!]^{1/2}[|\underline{k}^{X'}_{\lceil\Delta_0}|!])(\prod_{r\geqq 1}\prod_{\Delta\in\underline{\Delta}^r_+}|\Delta|^{(1/2\,k^X{}^r_\Delta+k^{X'}{}^r_\Delta)})$$
$$\leqq C_{\xi_2}^{|\underline{k}|}\prod_{r\geqq 1}\prod_{\Delta\in\Lambda^r_+}([|k^X{}^r_\Delta|!]^{1/2}[|k^{X'}{}^r_\Delta|!]|\Delta_r|^{-\xi_2 k^r_\Delta}), \quad (2.9.12)$$

The last inequality coming from (2.6.1); one obtains easily (2.9.10), with $\varepsilon=\varepsilon_1+\xi_1+\xi_2$, be setting (2.9.12) in (2.9.11). ■

2.10. Taking the geometric mean of (2.8.3) and (2.9.10) gives

$$|(\prod_{r \geq 1} \prod_{\Delta \in \underline{\Lambda}^r_+} (\tilde{D}^r_\Delta)^{k^r_\Delta}) Z_{N, (\rho); \underline{t}}(f)|$$
$$\leqq \mathcal{N}(f) C_\varepsilon^{|\Lambda| + |\underline{\Delta}^{(t)}| + |\underline{k}|} (\prod_{r \geq 1} \prod_{\Delta \in \underline{\Lambda}^r_+} (k^r_\Delta !)^{7/4} |\Delta_r|^{(1/4 - \varepsilon) k^r_\Delta}), \quad (2.10.1)$$

then, according to the theorem (A.0), one can state

THEOREM. – *If* $|\operatorname{Arg} \rho| < \frac{3\pi}{4}$, *the functions* $Z_{N, (\rho); \Lambda, r}(f)$ *defined by* (1.3.1) – *where* α *is chosen so that* $|\operatorname{Arg} \alpha| < \frac{\pi}{4}$ *and* $|\operatorname{Arg} \alpha\rho| < \frac{\pi}{2}$ – *converge, (uniformly for* ρ *in any compact set), as* $r \to \infty$ *towards analytic functions* $Z_{N, (\rho); \Lambda}(f)$ *which coninue, (if* $\lambda = \frac{\rho^2}{8}$*), the functions defined by* (1.1.5).

Moreover, there exists a constant K, *(depending on* N *and* ρ*-uniformly on any compact set-but not depending on* Λ*), such that*

$$|Z_{N, (\rho); \underline{t}}(f)| \leqq \mathcal{N}(f) e^{K|\Lambda|}. \quad (2.10.2)$$

3. BOREL SUMMABILITY FOR THE NON NORMALIZED SCHWINGER FUNCTION

3.1. The aim of this chapter is to prove that the analytic functions $\lambda \mapsto Z_{N, \lambda; \Lambda}(f)$ are of class C^∞ at $\lambda = 0$, and that their Taylor series are Borel-summable of level 1 in any direction, except that of the negative real numbers.

This is an easy consequence of the

PROPOSITION. – *The functions* $\rho \mapsto Z_{N, (\rho); \Lambda}(f)$, *analytic in*

$$\{\rho \in \mathbf{C}^*; |\operatorname{Arg} \rho| < \frac{3\pi}{4}\},$$

are of class C^∞ *at* $\rho = 0$ (14), *and, for any* $\eta > 0$, $R > 0$, *there exists a constant* $C'_{R,\eta}$ (15) *such that, if* $0 \leqq |\rho| \leqq R$, *and* $|\operatorname{Arg} \rho| \leqq \frac{3\pi}{4} - \eta$, *one*

(14) More precisely, their restriction to any sector $\{\rho \in C; |\operatorname{Arg} \rho| < \frac{3\pi}{4} - \eta\}$ is of class C^∞ at $\rho = 0$.

(15) The constants does not depend on $\boldsymbol{n}$, Λ, f, N bounded, and m bounded away from 0; this will be true throughout this chapter, if not otherwise stated.

has for all $n \in \mathbf{N}$

$$\left| \frac{d^n}{d\rho^n} Z_{N,(\rho);\Lambda}(f) \right| \leqq \mathcal{N}(f) e^{K|\Lambda|} |\Lambda|^{n/2} C'^{\,n}_{R,\eta} (n!)^{3/2}. \qquad (3.1.1)$$

3.2. In order to prove this proposition, one wants to show that the successive derivatives of the functions $\rho \mapsto Z_{N,(\rho);\underline{t}}(f)$, defined by (2.0.6), satisfy similar inequalities and converge uniformly for $0 \leqq |\rho| \leqq R$, $|\mathrm{Arg}\,\rho| \leqq \frac{3\pi}{4} - \eta$.

So one wants to give two estimates of $\left(\prod_{r \geqq 1} \prod_{\Delta \in \underline{\Lambda}^r_+} (\tilde{D}^r_\Delta)^{k^r_\Delta}\right) \frac{d^n}{d\rho^n} Z_{N,(\rho);\underline{t}}(f)$, which are obtained by derivation with respect to ρ in the formulae (2.1.9) and (2.9.8), respectively. From (2.1.2) and (2.1.8), one has

$$\frac{d}{d\rho} \mathbf{R}_{\underline{t}}(\sigma) = \frac{-i\alpha}{\sqrt{\mathrm{Re}\,.\,\alpha^2}} \sum_{\Delta \in \underline{\Lambda}(t)} \langle \sigma, 1_\Delta \rangle \mathbf{R}_{\underline{t}}(\sigma) \times [t_\Delta \psi_\Delta \times \psi_\Delta + (1 - t_\Delta) \psi_{\hat{\Delta}} \times \psi_{\hat{\Delta}}] \mathbf{R}_{\underline{t}}(\sigma), \qquad (3.2.1)$$

therefore, from (2.1.4), with $g_\Delta = \{g, \psi_\Delta, \psi_\Delta\}$, as in (2.9),

$$\frac{d}{d\rho} (S_g(\underline{t}; \sigma)\ \mathrm{Det.}\ \mathbf{R}_{\underline{t}}(\sigma)^{1/2}) = \frac{-i\alpha}{2\sqrt{\mathrm{Re}\,.\,\alpha^2}} \sum_{\Delta \in \underline{\Lambda}(t)} \langle \sigma, 1_\Delta \rangle (t_\Delta S_{g_\Delta}(\underline{t}; \sigma) + (1 - t_\Delta) S_{g_{\hat{\Delta}}}(\underline{t}; \sigma))\ \mathrm{Det.}\ \mathbf{R}_{\underline{t}}(\sigma)^{1/2}. \qquad (3.2.2)$$

Given $\underline{t} \in \mathcal{T}_\Lambda$, $k \in \mathbf{N}$ one sets $\mathbf{I}^k_{\underline{t}} = \{\underline{k} = \{k_\Delta \in \mathbf{N}\}_{\Delta \in \underline{\Lambda}(t)};\ |\underline{k}| = k\}$, (with $|\underline{k}| := \sum_{\Delta \in \underline{\Lambda}(t)} k_\Delta$); and for $\underline{k} = \{k_\Delta \in \mathbf{N}\}_{\Delta \in \underline{\Lambda}}$ such that $k_\Delta = 0$ if Δ includes stricly an element of $\underline{\Lambda}(\underline{t})$, one denotes by $H^{\underline{k}}_{\underline{t}}$ the set of the $\underline{h} = \{h_\Delta \in \mathbf{N}\}_{\Delta \in \underline{\Lambda}}$, of which the only non necessarily vanishing components are, for each $\Delta \in \underline{\Lambda}(\underline{t})$, h_Δ and $h_{\hat{\Delta}}$ which verify

$$0 \leqq h_\Delta \leqq \sum_{\Delta_1 \in \underline{\Lambda};\, \Delta_1 \subset \Delta} k_{\Delta_1} \quad \text{and} \quad h_{\hat{\Delta}} = \sum_{\Delta_2 \in \underline{\Lambda};\, \Delta_2 \subset \Delta} k_{\Delta_2} - \sum_{\Delta';\, \hat{\Delta}' = \hat{\Delta}} h_{\Delta'}.$$

Then, from (3.2.2),

$$\frac{d^n}{d\rho^n} (S_g(\underline{t}; \sigma)\ \mathrm{Det.}\ \mathbf{R}_{\underline{t}}(\sigma)^{1/2}) = \left(\frac{-i}{2}\right)^n \sum_{\underline{n} \in \mathbf{I}^n_{\underline{t}}} \frac{n!}{\prod_{\Delta \in \underline{\Lambda}(t)} n_\Delta!} \prod_{\Delta \in \underline{\Lambda}_+} \left(\frac{\alpha}{\sqrt{\mathrm{Re}\,.\,\alpha^2}} \langle \sigma, 1_\Delta \rangle\right)^{n_\Delta} \times \left(\sum_{\underline{h} \in H^{\underline{n}}_{\underline{t}}} \left(\prod_{\Delta \in \underline{\Lambda}(t)} \binom{n_\Delta}{h_\Delta} \right) c_{\underline{t}}(\underline{h})\, S_{g_{\underline{h}}}(\underline{t}; \sigma) \right) \mathrm{Det.}\ \mathbf{R}_{\underline{t}}(\sigma)^{1/2}, \qquad (3.2.3)$$

where, as (2.9), g_h is the family containing g and, for each $\Delta \in \underline{\Lambda}$, $2h_\Delta$ functions equal to $\bar{\psi}_\Delta$; and $0 \leq c_t(\underline{h}) \leq 1$.

Therefore, from (2.1.9), one has ([16])

$$\Big(\prod_{\Delta \in \underline{\Lambda}+} D_\Delta^{(q_\Delta)}\Big) \frac{d^n}{d\rho^n} Z_{N,(\rho);\underline{t}}(f)$$

$$= \sum_{k=0}^{n} [(n-k)!] \sum_{\underline{k} \in I_{\underline{t}}^k} \frac{k!}{\prod_{\Delta \in \underline{\Lambda}(t)} k_\Delta!}$$

$$\times \sum_{\{\underline{q}_j\}_{1 \leq j \leq N};\, \sum_j \underline{q}_j = \underline{q}} \quad \sum_{\{\underline{k}_j\}_{1 \leq j \leq N};\, \sum_j \underline{k}_j = \underline{k}} \quad \sum_{\{\underline{h}_j \in H_{\underline{t}}^{\underline{k}_j}\}_{1 \leq j \leq N}}$$

$$\times \Bigg[\frac{(-i)^{|\underline{q}|+k} \rho^{|\underline{q}|-n+k}}{2^k} \binom{n}{k} \binom{|\underline{q}|}{n-k}$$

$$\times \prod_{\Delta \in \underline{\Lambda}+} \frac{q_\Delta!}{\prod_{j=1}^{N} q_{j\Delta}!} \prod_{\Delta \in \underline{\Lambda}(t)} \frac{k_\Delta!}{\prod_{j=1}^{N} k_{j\Delta}!} \prod_{j=1}^{N} \Big(c_{\underline{t}}(\underline{h}_j) \prod_{\Delta \in \underline{\Lambda}(t)} \binom{k_{j\Delta}}{h_{j\Delta}}\Big)$$

$$\times \Big(\frac{\alpha}{\sqrt{\mathrm{Re}\,.\,\alpha^2}}\Big)^{|\underline{\Lambda}(t)|} \int_{\mathscr{S}'} \prod_{\Delta \in \underline{\Lambda}+} \Big(\Big\langle \sigma, \Big(1_\Delta - \frac{|\Delta|}{|\tilde{\Delta}|} 1_{\tilde{\Delta}}\Big)\Big\rangle$$

$$+ \frac{\alpha}{\sqrt{\mathrm{Re}\,.\,\alpha^2}} \frac{|\Delta|}{|\tilde{\Delta}|} \langle \sigma, {}_{\tilde{\Delta}} \rangle\Big)^{q_\Delta}$$

$$\times \prod_{\Delta' \in \underline{\Lambda}(t)} \Big(\frac{\alpha}{\sqrt{\mathrm{Re}\,.\,\alpha^2}} \langle \sigma, 1_{\Delta'} \rangle\Big)^{k_{\Delta'}} \prod_{j=1}^{N} S_{\mathcal{T}_j \underline{h}_j}^{\underline{q}_j}(\underline{t};\sigma)$$

$$\times \mathrm{Det}\,.\,\mathbf{R}_{\underline{t}}(\sigma)^{N/2}\,.\,e^{-(i/2)(\mathrm{Im}\,.\,\alpha^2/\mathrm{Re}\,.\,\alpha^2) \sum_{\Delta \in \underline{\Lambda}(t)} |\Delta|^{-1} \langle \sigma, 1_\Delta \rangle^2}\,.\,\nu(d\sigma)\Bigg]. \tag{3.2.4}$$

and from (2.9.8)

$$\Big(\prod_{\Delta \in \underline{\Lambda}+} D_\Delta^{(q_\Delta)}\Big) \frac{d^n}{d\rho^n} Z_{N,(\rho);\underline{t}}(f) = \sum_{k=0}^{n} [(n-k)!] \sum_{\underline{k} \in I_{\underline{t}}^k} \frac{k!}{\prod_{\Delta \in \underline{\Lambda}(t)} k_\Delta!}$$

$$\times \sum_{\{\underline{q}_j\}_{1 \leq j \leq N};\, \sum_j \underline{q}_j = \underline{q}} \quad \sum_{X \subset E_{\underline{q}};\, |X| \in 2\mathrm{N}} \quad \sum_{\{Y_j\}_{1 \leq j \leq N};\, \bigcup_j Y_j = E_{\underline{q}} \setminus X,\, Y_i \cap Y_j = \varnothing}$$

$$\times \sum_{\{\underline{k}_j\}_{1 \leq j \leq N};\, \sum_j \underline{k}_j = \underline{k}} \quad \sum_{\{\underline{h}'_j \in H_{\underline{t}}^{\underline{q}_j^{Y_j}}\}_{1 \leq j \leq N}} \quad \sum_{\{\underline{h}''_j \in H_{\underline{t}}^{\underline{k}_j}\}_{1 \leq j \leq N}}$$

([16]) The lines 2, 3 and 4 of the right hand side in this formula, and also lines 2, 3, 4, 5 and 6 in (3.2.5), contribute to the estimate only through harmless constants to some power.

$$\times\left[\frac{(-i)^{2|\underline{q}|-|X|+k}\rho^{2|\underline{q}|-|X|-n+k}}{2^{|\underline{q}|-|X|+k}}\binom{n}{k}\binom{2|\underline{q}|-|X|}{n-k}\right.$$

$$\times\prod_{\Delta\in\underline{\Lambda}+}\frac{q_\Delta!}{\prod_{j=1}^{N}q_{j\Delta}!}\prod_{\Delta\in\underline{\Lambda}(\underline{t})}\frac{k_\Delta!}{\prod_{j=1}^{N}k_{j\Delta}!}$$

$$\times\prod_{j=1}^{N}\left(c_{\underline{t}}(\underline{h}'_j)c_{\underline{t}}(\underline{h}''_j)\prod_{\Delta\in\underline{\Lambda}(\underline{t})}\binom{q_j^{Y_j}{}_\Delta}{h'_{j\Delta}}\binom{k_{j\Delta}}{h''_{j\Delta}}\right)$$

$$\times\left(\sum_{P\in\mathscr{P}_2(X)}\prod_{p\in P}|\Delta_{p_-}\cap\Delta_{p_+}|\right)\prod_{y\in E_{\underline{q}}\setminus X}|\Delta_y|$$

$$\times\left(\frac{\alpha}{\sqrt{\text{Re}\,.\,\alpha^2}}\right)^{|\underline{\Lambda}(\underline{t})|}\int_{\mathscr{S}'}\prod_{\Delta\in\underline{\Lambda}(\underline{t})}\left(\frac{\alpha}{\sqrt{\text{Re}\,.\,\alpha^2}}\langle\sigma,1_\Delta\rangle\right)^{k_\Delta}$$

$$\times\prod_{j=1}^{N}S_{\mathcal{T}^{q_j}_{j\underline{h}'_j+\underline{h}''_j}}(\underline{t};\sigma)\,.\,\text{Det.}\,\mathbf{R}_{\underline{t}}(\sigma)^{N/2}$$

$$\left.\times\, e^{-(i/2)(\text{Im}\,.\,\alpha^2/\text{Re}\,.\,\alpha^2)\sum_{\Delta\in\underline{\Lambda}(\underline{t})}|\Delta|^{-1}\langle\sigma,1_\Delta\rangle^2}\,.\,\nu(d\sigma)\right].\qquad(3.2.5)$$

The estimation of the right hand side of (3.2.4) and (3.2.5) uses the inequalities (2.5.1), (2.7.1), (2.9.9) and

$$\Big\|\prod_{\Delta\in\underline{\Lambda}(\underline{t})}\langle\cdot,1_\Delta\rangle^{k_\Delta}\Big\|_{L^p}\leqq C_p^{|\underline{k}|}\prod_{\Delta\in\underline{\Lambda}(\underline{t})}\left((k_\Delta!)^{1/2}|\Delta|^{k_\Delta/2}\right),\qquad(3.2.6)$$

and also the

LEMMA. — *One has*

$$\sum_{\underline{k}\in I_t^k}\frac{k!}{\prod_{\Delta\in\underline{\Lambda}(\underline{t})}k_\Delta!}\prod_{\Delta\in\underline{\Lambda}(\underline{t})}\left((k_\Delta!)^{1/2}|\Delta|^{k_\Delta/2}\right)\leqq C^k e^{|\underline{\Lambda}(\underline{t})|}|\Lambda|^{k/2}(k!)^{1/2}.\qquad(3.2.7)$$

Proof. — One decomposes the sum in the left hand side of (3.2.7) as the sum of the terms in each of which the set of values taken by the k_Δ's is fixed, (there are at most 2^{k-1} such terms); let T be one of these terms, for which the non vanishing values taken by the k_Δ's are p numbers $(0\leqq p\leqq\frac{k}{2})$, $k_1,\ldots,k_p\geqq 2$, with sum $\sum_{i=1}^{p}k_i=\boldsymbol{k}_1\leqq k$, and $\boldsymbol{k}_2=k-\boldsymbol{k}_1$ times the number 1, one has

$$T\leqq\left[\prod_{i=1}^{p}(k_i!)^{1/2}\sum_{\Delta\in\underline{\Lambda}(\underline{t})}|\Delta|^{k_i/2)}\right]\left[\sum_{\underline{k}'\in I_{\underline{t}}^{\boldsymbol{k}_2}}\frac{\boldsymbol{k}_2!}{\prod_{\Delta\in\underline{\Lambda}(\underline{t})}k'_\Delta!}\prod_{\Delta\in\underline{\Lambda}(\underline{t})}|\Delta|^{k'_\Delta/2}\right],$$

because T is the sub-sum of "non coinciding points" of the right hand side, but

$$\prod_{i=1}^{p}\Big((k_i!)^{1/2}\sum_{\Delta\in\underline{\Lambda}(t)}|\Delta|^{k_i/2}\Big)$$
$$\leqq\prod_{i=1}^{p}\Big((k_i!)^{1/2}|\Delta_0|^{k_i/2-1}\sum_{\Delta\in\underline{\Lambda}(t)}|\Delta|\Big)\leqq C^{k_1}|\Lambda|^{k_1/2}(k_1!)^{1/2},$$

since $|\Delta|\leqq|\Delta_0|$ and $\sum_{\Delta\in\underline{\Lambda}(t)}|\Delta|=|\Lambda|$, and on the other and

$$\sum_{\underline{k}'\in \mathrm{I}_{\underline{t}}^{k_2}}\frac{k_2!}{\prod_{\Delta\in\underline{\Lambda}(t)}k'_\Delta!}\prod_{\Delta\in\underline{\Lambda}(t)}|\Delta|^{k'_\Delta/2}$$
$$\leqq^{(i)}\Big(\sum_{\underline{k}'\in \mathrm{I}_{\underline{t}}^{k_2}}\frac{k_2!}{\prod_{\Delta\in\underline{\Lambda}(t)}k'_\Delta!}\Big)^{1/2}\Big(\sum_{\underline{k}''\in \mathrm{I}_{\underline{t}}^{k_2}}\frac{k_2!}{\prod_{\Delta\in\underline{\Lambda}(t)}k''_\Delta!}\prod_{\Delta\in\underline{\Lambda}(t)}|\Delta|^{k''_\Delta}\Big)^{1/2}$$
$$=^{(ii)}|\underline{\Lambda}(\underline{t})|^{k_2/2}|\Lambda|^{k_2/2}\leqq e^{|\underline{\Lambda}(t)|}|\Lambda|^{k_2/2}(k_2!)^{1/2}.$$

[$^{(i)}$ from Schawarz inequality, $^{(ii)}$ because $|\mathrm{I}_{\underline{t}}^{k_2}|=|\underline{\Lambda}(\underline{t})|^{k_2}$ and $\sum_{\Delta\in\underline{\Lambda}(t)}|\Delta|=|\Lambda|$]. ■

Then, from (3.2.4), one has

$$\Big|\Big(\prod_{\Delta\in\underline{\Lambda}+}\mathrm{D}_{\underline{\Delta}}^{q_\Delta}\Big)\frac{d^n}{d\rho^n}\mathrm{Z}_{\mathrm{N},(\rho);\underline{t}}(f)\Big|\leqq\mathcal{N}(f)\,\mathrm{C}_\varepsilon^{|\Lambda|+\underline{\Lambda}(t)|+|\underline{q}|+n}|\Lambda|^{n/2}(n!)^{3/2}$$
$$\times\prod_{\Delta_0\in\underline{\Lambda}_0}(|\underline{q}_{\Gamma\Delta_0}|!)^{1/2}\prod_{\Delta\in\underline{\Lambda}+}\Big((q_\Delta!)|\Delta|^{(1-\varepsilon)q_\Delta}\Big),\quad(3.2.8)$$

and, from (3.2.5),

$$\Big|\Big(\prod_{\Delta\in\underline{\Lambda}+}\mathrm{D}_{\underline{\Delta}}^{q_\Delta}\Big)\frac{d^n}{d\rho^n}\mathrm{Z}_{\mathrm{N},(\rho);\underline{t}}(f)\Big|$$
$$\leqq\mathcal{N}(f)\,\mathrm{C}_\varepsilon^{|\Lambda|+|\underline{\Lambda}(t)|+|\underline{q}|+n}|\Lambda|^{n/2}(n!)^{3/2}$$
$$\times\prod_{\Delta_0\in\underline{\Lambda}_0}(|\underline{q}_{\Gamma\Delta_0}|!)^{1/2}\prod_{\Delta\in\underline{\Lambda}+}\Big((q_\Delta!)^{1/2}|\Delta|^{(1/2-\varepsilon)q_\Delta}\Big).$$
$$\times\sum_{\mathrm{X}\subset\mathrm{E}_{\underline{q}};\ |\mathrm{X}|\in 2\mathbf{N}}\Big[\prod_{\Delta_0\in\underline{\Lambda}_0}(|\underline{q}_{\Gamma\Delta_0}^{\mathrm{X}'}|!)$$
$$\times\Big(\sum_{\mathrm{P}\in\mathcal{P}_2(\mathrm{X})}\prod_{p\in\mathrm{P}}|\Delta_{p_-}\cap\Delta_{p_+}|\Big)\prod_{y\in\mathrm{E}_{\underline{q}}\setminus\mathrm{X}}|\Delta_y|\Big].\quad(3.2.9)$$

One achieves the proof by the substitution of the inequalities (3.2.8), (3.2.9) to the inequalities (2.8.1), (2.9.4) respectively, in the proof of theorem (2.9). ■

3.3. As an immediate consequence of proposition (3.1), one has

THEOREM. — *The function* $\lambda \mapsto Z_{N,\lambda;\Lambda}(f)$, *analytic in*

$$\{\lambda \in \tilde{\mathbf{C}}^*;\ |\operatorname{Arg}.\lambda| < \frac{3\pi}{2}\},$$

are of class C^∞ *at* $\lambda = 0$ *in any sector*

$$\{\lambda \in \tilde{\mathbf{C}};\ |\operatorname{Arg}.\lambda| < \frac{3\pi}{2} - \eta\};$$

their Taylor series at $\lambda = 0$ *are Borel-summable of level* 1 *in all directions except that of negative real numbers* ([17]), *each function being of course the Borel-sum of its series; more precisely, for any* $\eta > 0$, $R > 0$, *there exists a constant* $C_{R,\eta}$ *such that, if* $0 \leqq |\lambda| \leqq R$ *and* $|\operatorname{Arg}.\lambda| \leqq \frac{3\pi}{2} - \eta$, *for any* $n \in \mathbf{N}$, *one has*

$$\left| Z_{N,\lambda;\Lambda}(f) - \sum_{k=0}^{n-1} \left(\frac{d^k}{d\lambda^k} Z_{N,0;\Lambda}(f) \right) \frac{\lambda^k}{k!} \right| \leqq \mathcal{N}(f)\, e^{K|\Lambda|} |\Lambda|^n C_{R,\eta}{}^n n!\, |\lambda|^n. \quad (3.3.1)$$

Proof. — To deduce (3.3.1) from (3.1.1), it suffices to remark that the derivatives of odd order of the function $\rho \mapsto Z_{N,(\rho);\Lambda}(f)$ vanish at $\rho = 0$, (indeed one can change ρ into $-\rho$ in (1.3.1), since the measure ν is even); for if $\psi^{(2n+1)}(0) = 0$, $\Psi(z) = \psi(\sqrt{z})$, one has, recursively on n,

$$\Psi^{(n)}(z) = 2^{-(2n-1)} \int_0^1 \frac{(1-u^2)^{n-1}}{(n-1)!} \psi^{(2n)}(u\sqrt{z})\, du, \qquad (n \geqq 1),$$

from which one deduces

$$|\Psi^{(n)}(z)| \leqq \frac{n!}{(2n)!} \sup_{0 \leqq u \leqq 1} |\psi^{(2n)}(u\sqrt{z})|,$$

and particularly

$$|\Psi^{(n)}| \leqq a(4^\gamma c^2)^n n!)^{1+2\gamma}, \qquad \text{if } |\psi^{(n)}| \leqq ac^n (n!)^{1+\gamma}. \quad \blacksquare$$

4. CONTINUATION OF THE SCHWINGER FUNCTION $S_{N,\lambda}(f)$

4.0. In this chapter, one shows the existence of the limits, as $\Lambda \to \mathbf{R}$, for the functions $S_{N,\lambda;\Lambda}(f)$ defined by (1.1.6), now valid for

([17]) Their Borel-transform of level 1 is therefore analytic and exponentially bounded of order 1 in the cut plane.

$|\mathrm{Arg}\,\lambda|<\frac{3\pi}{2}$, when the denominator does not vanish. For this purpose, one uses the general method of "cluster expansion" of Glimm-Jaffe-Spencer ([18]), the main features of which are recalled briefly in appendix B ([19]). The first step is the construction, for each function $(\Lambda\mapsto Z_{N,\lambda;\Lambda}(f))\in C^{\mathscr{X}_0}$, of an extension $((\Lambda, s)\mapsto Z_{N,\lambda;\Lambda;s}(f))\in C^{\mathscr{X}_0\times S_0}$, such that $Z_{N,\lambda;\Lambda;1_{\mathscr{B}}}(f)=Z_{N,\lambda;\Lambda}(f)$, and to which the theorems (B.3) and (B.4) can be applied.

These functions themselves are obtained as limits, as $r\to\infty$, of the functions $Z_{N,\lambda;\Lambda,r;s}(f)$ which one introduces now.

First, if for β, $\delta_i\in\mathscr{D}_0$ and $s\in S_0$, $H(s\,|\,\beta;\,\delta_1,\delta_2)$ is the coefficient denoted $H^{(2)}(s\,|\,\beta;\,\delta_1,\delta_2)$ in appendix C, one has

$$0\leqq H(s\,|\,\beta;\,\delta_1,\delta_2)\leqq 1 \tag{4.0.1}$$

$$H(s\,|\,\beta;\,\delta_1,\delta_2)=H(s\,|\,\beta;\,\delta_2,\delta_1). \tag{4.0.2}$$

Next, there exists a constant ([20]) C, such that, if $\beta\in\mathscr{D}_0$ and $X\in\mathscr{X}$, one has

$$\|1_X\psi_\Delta\|_{L^2}\leqq C\,e^{-md[\beta,X]}\qquad(\Delta\in\mathscr{D},\ \Delta\subset\beta), \tag{4.0.3}$$

where d is the distance over $\mathbf{R}$, indeed, from (2.0.2), with $\chi_\Delta=\delta_{O_\Delta}$, one has $\psi_\Delta(x)=\frac{1}{\pi}K_0(m(x-O_\Delta))$, but there exists ([21]) a constant c such that $K_0(u)\leqq ce^{-|u|}$, if $|u|\geqq 1$, therefore

$$\int_X \psi_\Delta(x)^2\,dx\leqq\frac{c^2}{\pi^2}\,e^{-2md[\beta,X]}\int_{\mathbf{R}} e^{-m|x|}\,dx:=C^2\,e^{-2md[\beta,X]}.$$

Last, if $M_\delta\in L(L^2(\mathbf{R}))$ is the operator of multiplication by 1_δ, $(\delta\in\mathscr{D}_0)$, for $\underline{t}\in\mathscr{T}_\Lambda$ and $s\in S_0$, one defines $A_{\underline{t};s}(\sigma)\in L(L^2(\mathbf{R}))$ by

$$A_{\underline{t};s}(\sigma):=\sum_{\beta\in\underline{\Lambda}_0}\ \sum_{\delta_1,\delta_2\in\mathscr{D}_0} H(s\,|\,\beta;\,\delta_1,\delta_2)\times\sum_{\Delta\in\underline{\Lambda}} t_\Delta\langle\sigma,1_\beta 1_\Delta\rangle M_{\delta_1}[\psi'_\Delta\times\tilde{\psi}_\Delta+\tilde{\psi}_\Delta\times\psi'_\Delta]M_{\delta_2}, \tag{4.0.4}$$

so that, from (C.1.7), $A_{\underline{t}}(\sigma)=A_{\underline{t};1_{\mathscr{B}}}(\sigma)$; the series is convergent from (4.0.1), (4.0.3) and the definitions (2.0.3), (2.0.4); the

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([18]) One of the aims of this paper being to prepare an analogous analysis in higher dimension, one does not try to use some method too much particular to the dimension d = 1.

([19]) One uses the appendix B with the identification $\mathscr{V}=\mathscr{D}_0$, each element of $\mathscr{B}$ being the common end of some adjacent segments of $\mathscr{D}_0$; one refers to appendix B for the notations.

([20]) This constant is independent of m bounded from below; one chooses l_0 so that $ml_0\geqq 1$.

([21]) K_0 is the modified Bessel function of the second kind; the inequality is classical, it is also an easy consequence of the representation (4.2.3) below.

operator $\mathbf{A}_{\underline{t};s}(\sigma)$ is real and self-adjoint, due to (4.0.2). Then one supposes that *the family* $f=\{f_{j;k_j}\in\mathcal{H}^{-1};\ 1\leqq j\leqq N,\ 1\leqq k_j\leqq l_j\}$ *is such that each function* $\Sigma_m^{-1/2} f_{j;k_j}\in \tilde{f}$ *has its support included is some segment of* $\mathcal{D}_0$ ([22]), one sets $f_\Lambda=\{f\in f;\ \text{supp.}\ \Sigma_m^{-1/2}f\subset\Lambda\}$, and one defines $Z_{N,(\rho);\underline{t};s}(f)\equiv Z_{N,\rho^2/8;\underline{t};s}(f)$ by the substitution of $\mathbf{A}_{\underline{t};s}(\sigma)$ to $\mathbf{A}_{\underline{t}}(\sigma)$ and of f_Λ to $\tilde{f}$ ([23]) in (2.0.6), the analog of (2.1.6) is

$$Z_{N,(\rho);\underline{t};s}(f): =\left(\frac{\alpha}{\sqrt{\text{Re}.\alpha^2}}\right)^{|\underline{\Lambda}_r|}\int_{\mathcal{S}'}\prod_{j=1}^{N}\left(S\mathcal{T}_{\Lambda_j}(\underline{t};s;\sigma).\text{Det}.\ \mathbf{R}_{\underline{t};s}(\sigma)^{1/2}\right) \times e^{-(i/2)(\text{Im}.\alpha^2/\text{Re}.\alpha^2)\sum_{\Delta\in\underline{\Lambda}_r}|\Delta|^{-1}\langle\sigma,1_\Delta\rangle^2}\,\nu(d\sigma), \quad (4.0.5)$$

where $\mathbf{R}_{\underline{t};s}(\sigma)$ and $S\mathcal{T}_{\Lambda_j}(\underline{t};s;\sigma)$ are defined by substitution in (2.1.2) and (2.1.4).

4.1. The existence of the limits

$$Z_{N,(\rho);\Lambda;s}(f):=\lim_{r\to\infty} Z_{N,(\rho);\Lambda,r;s}(f), \quad (4.1.1)$$

[with, of course, $Z_{N,(\rho);\Lambda,r;s}(f):=Z_{N,(\rho);1_{\Lambda^r_+};s}(f)$]; and the inequality ([24])

$$|Z_{N,(\rho);\Lambda;s}(f)|\leqq\mathcal{N}(f_\Lambda)\,e^{K_1|\Lambda|}, \quad (4.1.2)$$

are proved by the following adaptation of the argument of chapter 2.

First, from (4.0.4), and owing to (4.0.2), for $\Delta\in\underline{\Lambda}_+$, the analogue of formula (2.1.5) is

$$D_\Delta(S_g(\underline{t};s;\sigma)\ \text{Det}.\ \mathbf{R}_{\underline{t};s}(\sigma)^{1/2}) =\frac{-i\rho\alpha}{\sqrt{\text{Re}.\alpha^2}}\langle\sigma,1_\Delta\rangle \times\sum_{\delta',\tilde{\delta}\in\mathcal{D}_0} H(s|\Delta^0;\delta',\tilde{\delta})\, S_{g^{(\Delta;\delta',\tilde{\delta})}}(\underline{t};s;\sigma)\,\mathbf{R}_{\underline{t};s}(\sigma)^{1/2}, \quad (4.1.3)$$

where $\Delta^0\in\underline{\Lambda}_0$ is determined by $\Delta^0\supset\Delta$, and $g^{(\Delta;\delta',\tilde{\delta})}=\{g,\ M_{\delta'}\psi'_\Delta,\ M_{\tilde{\delta}}\tilde{\psi}_\Delta\}$; therefore if, given $\underline{q}\in Q_\Lambda$ and $\underline{\delta}',\ \underline{\tilde{\delta}}\in\mathcal{D}_0^{E_{\underline{q}}}$ ([25]), one denotes by $g^{[\underline{q};\underline{\delta}',\underline{\tilde{\delta}}]}$ *the*

([22]) This condition allows to verify the condition L.(i) of the definition (B.4); see remark ([1]).

([23]) This convention is introduced to verify the condition L.(iii) of (B.4), with, for $\Sigma_{Z(f)|Z(\varphi)}$, the set Σ_f of segments of $\mathcal{D}_0$ which contain the support of at least one function of $\tilde{f}$; it does not change the value of the limit as $\Lambda\to\mathbf{R}$.

([24]) Where now, according to the hypothesis done on $\mathbf{f}$,

$$\mathcal{N}(f)=C^{|f|}\prod_{\Delta_0\in\mathcal{D}_0}[|f_{\Delta_0}|!]^{1/2}\prod_{f\in\mathbf{f}}\|f\|_{\mathcal{H}^{-1}},$$

with $f_{\Delta_0}:=\{f\in f;\ \text{supp}\,\Sigma_m^{-1/2}f\subset\Delta_0\}$.

([25]) As in (2.9), $E_{\underline{q}}$ is the set containing, for each $\Delta\in\mathcal{D}$, q_Δ elements equal to Δ.

family of L^2*-functions containing* g *and, for each* $\Delta \in E_{\underline{q}}$, *the functions* $M_{\delta'_\Delta} \psi'_\Delta$ *and* $M_{\tilde{\delta}_\Delta} \tilde{\psi}_\Delta$, the formula analogous to (2.1.9) becomes

$$\Big(\prod_{\Delta \in \underline{\Delta}+} D_\Delta^{q_\Delta}\Big) Z_{N,(\rho);\underline{t};s}(f) = \sum_{\{\underline{q}_j\}_{1\le j\le N};\sum_j \underline{q}_j = \underline{q}} \prod_{\Delta\in\underline{\Delta}+} \frac{q_\Delta !}{\prod_{j=1}^N q_{j\Delta}!} \cdot$$

$$\times \left(\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\right)^{|\underline{\Delta}(t)|} \int_{\mathscr{S}'} \prod_{\Delta\in\underline{\Delta}+} \left(-i\rho\left[\left\langle \sigma, \left(1_\Delta - \frac{|\Delta|}{|\tilde{\Delta}|} 1_{\tilde{\Delta}}\right)\right\rangle + \frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}} \frac{|\Delta|}{|\tilde{\Delta}|} \langle \sigma, 1_{\tilde{\Delta}} \rangle \right]\right)^{q_\Delta}$$

$$\times \prod_{j=1}^N \Big(\sum_{\underline{\delta}'_j, \underline{\tilde{\delta}}_j \in \mathscr{D}_0^{E_{\underline{q}_j}}} \{ (\prod_{\Delta\in E_{\underline{q}_j}} H(s\,|\,\Delta^0;\, \delta'_{j\Delta}, \tilde{\delta}_{j\Delta}))\, S_{\tilde{f}_{j\Delta}}^{[\underline{q}_j;\underline{\delta}'_j,\underline{\tilde{\delta}}_j]}(\underline{t};\, s;\, \sigma)\}\Big)$$
$$\times \mathrm{Det}.\, \mathbf{R}_{\underline{t};s}(\sigma)^{N/2} . e^{-(i/2)(\mathrm{Im}.\alpha^2/\mathrm{Re}.\alpha^2)\sum_{\Delta\in\underline{\Delta}(t)} |\Delta|^{-1}\langle\sigma, 1_\Delta\rangle^2} . \nu(d\sigma) \quad . \quad (4.1.4)$$

In the same way, the analogous (2.9.6) can be written

$$\left(\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\right)^{-1} \partial_{1_\Delta}(S_g(\underline{t};\, s;\, \sigma)\, \mathrm{Det}.\, \mathbf{R}_{\underline{t};s}(\sigma)^{1/2}) = \frac{-i\rho}{2} |\Delta|$$
$$\times \sum_{\delta^-, \delta^+ \in \mathscr{D}_0} H(s\,|\,\Delta^0;\, \delta^-, \delta^+)[t_{\tilde{\Delta}} S_{g(\tilde{\Delta};\delta^-,\delta^+)}(\underline{t};\, s;\, \sigma)$$
$$+ (1-t_{\tilde{\Delta}}) S_{g(\hat{\tilde{\Delta}};\delta^-,\delta^+)}(\underline{t};\, s;\, \sigma)]\, \mathrm{Det}.\, \mathbf{R}_{\underline{t}}(\sigma)^{1/2}, \quad (4.1.5)$$

then, for $\underline{h} = \{h_\Delta\}_{\Delta\in\underline{\Delta}}$ and $\underline{\delta}^-, \underline{\delta}^+ \in \mathscr{D}_0^{E_{\underline{h}}}$, one denotes by $g_{[\underline{h};\underline{\delta}^-,\underline{\delta}^+]}$ *the family of* L^2*-functions containing* g *and, for each* $\Delta\in E_{\underline{h}}$, *the functions* $M_{\delta^-_\Delta}\psi_\Delta$ *and* $M_{\delta^+_\Delta}\psi_\Delta$; one obtains for the formula analogous to (2.9.8),

$$\Big(\prod_{\Delta\in\underline{\Delta}+} D_\Delta^{q_\Delta}\Big) Z_{N,(\rho);\underline{t};s}(f)$$
$$= \sum_{\{\underline{q}_j\}_{1\le j\le N};\sum_j \underline{q}_j=\underline{q}}\ \sum_{X\subset E_{\underline{q}};\,|X|\in 2\mathbb{N}}\ \sum_{\{Y_j\}_{1\le j\le N};\,\bigcup_j Y_j = E_{\underline{q}}\setminus X,\, Y_i\cap Y_j=\varnothing}\ \sum_{\{\underline{h}_j\in H_{\underline{t}}^{\underline{q}_j^{Y_j}}\}_{1\le j\le N}}$$
$$\times\left[\frac{(-i\rho)^{2|\underline{q}|-|X|}}{2^{|\underline{q}|-|X|}} \prod_{\Delta\in\underline{\Delta}+} \frac{q_\Delta!}{\prod_{j=1}^N q_{j\Delta}!} \prod_{j=1}^N \left(c_{\underline{t}}(\underline{h}_j) \prod_{\Delta\in\underline{\Delta}(t)} \binom{q_j^{Y_j}{}_\Delta}{h_{j\Delta}}\right)\right.$$
$$\times \Big(\sum_{P\in\mathscr{P}_2(X)} \prod_{p\in P} |\Delta_{p_-}\cap\Delta_{p_+}|\Big) \prod_{y\in E_{\underline{q}}\setminus X} |\Delta_y|$$
$$\times \left(\frac{\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\right)^{|\underline{\Delta}(t)|} \int_{\mathscr{S}'} \prod_{j=1}^N \Big(\sum_{\underline{\delta}'_j, \underline{\tilde{\delta}}_j \in \mathscr{D}_0^{E_{\underline{q}_j}}}\ \sum_{\underline{\delta}_j^-, \underline{\delta}_j^+ \in \mathscr{D}_0^{E_{\underline{h}_j}}}$$
$$\times \{ (\prod_{\Delta\in E_{\underline{q}_j}} H(s\,|\,\Delta^0;\, \delta'_{j\Delta}, \tilde{\delta}_{j\Delta}))$$

$$\left(\prod_{\Delta' \in E_{\underline{h}_j}} H(s \mid \Delta'^0; \delta^-_{j\Delta'}, \delta^+_{j\Delta'}) \right) S_{\tilde{f}\, {}_{j\Lambda[\underline{h}_j; \underline{\delta}^-_j, \underline{\delta}^+_j]}^{[\underline{q}_j; \underline{\delta}'_j, \underline{\tilde{\delta}}_j]}} (\underline{t}; s; \sigma) \})$$

$$\left. \times \operatorname{Det.} \mathbf{R}_{\underline{t}; s}(\sigma)^{N/2} . e^{-(i/2)(\operatorname{Im}. \alpha^2/\operatorname{Re}. \alpha^2) \sum_{\Delta \in \underline{\Lambda}(\underline{t})} |\Delta|^{-1} \langle \sigma, 1_\Delta \rangle^2} . \nu(d\sigma) \right]. \quad (4.1.6)$$

The proof of (4.1.1), (4.1.2) [26], rests mainly on the following inequality, analogous to (2.4.1) and (2.9.9), to be proved in paragraph (4.2),

LEMMA. — *If, as in lemma* (2.4), $\underline{q} \in Q_\Lambda$ *and* $\underline{t} \in \mathcal{T}_\Lambda$ *verify* (2.1.1), *one has*

$$\begin{aligned} | S_{g[\underline{h}; \underline{\delta}^-, \underline{\delta}^+]}^{[\underline{q}; \underline{\delta}', \underline{\tilde{\delta}}]} (\underline{t}; s; \sigma) | \leq \mathcal{N}_2(g) C_{\varepsilon, \zeta}^{|\underline{q}| + |\underline{h}|} & \\ \times \prod_{\Delta_1 \in E_{\underline{q}}} & m^{-1/2} e^{-(1-\zeta) m (d[\Delta^0_1, \delta'_{\Delta_1}] + d[\Delta^0_1, \tilde{\delta}_{\Delta_1}])} \\ \times \prod_{\Delta_2 \in E_{\underline{h}}} & m^{-1} e^{-(1-\zeta) m (d[\Delta^0_2, \delta^-_{\Delta_2}] + d[\Delta^0_2, \delta^+_{\Delta_2}])} \\ \times \prod_{\Delta_0 \in \underline{\Delta}_0} & ([\underline{q}_{\Gamma\Delta_0} |!]^{1/2} [|\underline{h}_{\Gamma\Delta_0}|!]) \prod_{\Delta \in \underline{\Lambda}_+} ((q_\Delta !)^{1/2} |\Delta|^{(1/2-\varepsilon) q_{j\Delta}}). \quad (4.1.7) \end{aligned}$$

One also needs

$$\| \operatorname{Det.} \mathbf{R}_{\underline{t}; s} \|_{L^p} \leq C^{|\Lambda|}, \quad (4.1.8)$$

analogous to (2.5.1), with the same condition, and easily deduced from (4.0.3).

4.2. The proof of (4.1.7) mimics that of lemma (2.4), according to the inequalities (4.2.1) and (4.2.8) below. First, the analog of lemma (2.3) is

LEMMA. — *Given* $\underline{k} = \{ k_\Delta \}_{\Delta \in \mathcal{D}}$, $\underline{\delta} \in \mathcal{D}_{0-}^{E_{\underline{k}}}$, *and* $g = \{ g_j \in L^2 \}_{1 \leq j \leq l}$ *a family such that each function* g_j *has its support included in some segment of* $\mathcal{D}_0$, *one has*

$$\begin{aligned} \left\| \left(\mathop{\vee}_{j=1}^{l} g_j \right) \vee \left(\mathop{\vee}_{\Delta \in E_{\underline{k}}} M_{\delta_\Delta} \psi_\Delta \right) \right\|_{L^{2 \vee (|\underline{k}|+1)}} & \\ \leq \mathcal{N}_2(g) C_\zeta^{|\underline{k}|} & \prod_{\Delta_0 \in \mathcal{D}_0} ([|\underline{k}_{\top\Delta_0}|!]^{1/2}) \\ & \times \prod_{\Delta \in E_{\underline{k}}} m^{-1/2} e^{-(1-\zeta) md[\Delta^0, \delta_\Delta]}, \quad (4.2.1) \end{aligned}$$

[26] One does not give more details, since this proof is a particular case of that of paragraphs (4.3), (4.4).

where

$$\mathcal{N}_2(g) = C^l \prod_{\Delta_0 \in \mathcal{D}_0} ([|g_{\upharpoonright \Delta_0}|!]^{1/2}) \prod_{j=1}^{l} \|g_j\|_{L^2},$$
$$g_{\upharpoonright \Delta_0} = \{g \in g;\ \text{supp}\ g \subset \Delta_0\}.$$

Proof. – As $M_{\gamma_1} M_{\gamma_2} = 0$ if $\gamma_1 \neq \gamma_2$, one has

$$\| \vee_{\Delta \in E_{\underline{k}}} M_{\delta_\Delta} \psi_\Delta \| = \prod_{\gamma \in \mathcal{D}_0} \| \vee_{\Delta \in \underline{\delta}^{-1}(\gamma)} M_\gamma \psi_\Delta \|,$$

then, from (2.4.2) and (4.0.3),

$$\| \vee_{\Delta \in \underline{\delta}^{-1}(\gamma)} M_\gamma \psi_\Delta \| \leqq C^{|\underline{\delta}^{-1}(\gamma)|} (|\underline{\delta}^{-1}(\gamma)|!)^{1/2} \times \prod_{\Delta \in \underline{\delta}^{-1}(\gamma)} m^{-1/2} e^{-(1-\zeta_1) md [\Delta^0, \gamma]},$$

but, $\sum_{\Delta_0 \in \mathcal{D}_0} e^{-2\zeta_2 md [\Delta_0, \gamma]} \leqq C'_{\zeta_2}$, therefore if

$$\underline{\delta}^{-1}(\gamma)_{\upharpoonright \Delta_0} := \{\Delta \in \underline{\delta}^{-1}(\gamma);\ \Delta \subset \Delta_0\},$$
$$(|\underline{\delta}^{-1}(\gamma)|!) . e^{-2\zeta_2 |\underline{\delta}^{-1}(\gamma)_{\upharpoonright \Delta_0}| md[\Delta_0, \gamma]} \leqq C'^{|\underline{\delta}^{-1}(\gamma)|}_{\zeta_2} \prod_{\Delta_0 \in \underline{\Delta}_0} (|\underline{\delta}^{-1}(\gamma)_{\upharpoonright \Delta_0}|!);$$

from what one deduces (4.2.1) since, on the one hand

$$\sum_{\gamma \in \mathcal{D}_0} |\underline{\delta}^{-1}(\gamma)_{\upharpoonright \Delta_0}| = |\underline{k}_{\upharpoonright \Delta_0}| \quad \text{so that} \quad \prod_{\gamma \in \mathcal{D}_0} (|\underline{\delta}^{-1}(\gamma)_{\upharpoonright \Delta_0}|!) \leqq |\underline{k}_{\upharpoonright \Delta_0}|!$$

and, on the other hand,

$$\left\| (\vee_{j=1}^{l} g_j) \vee (\vee_{\Delta \in E_{\underline{k}}} M_{\delta_\Delta} \psi_\Delta) \right\| \leqq C^{1+|\underline{k}|} \left\| \vee_{j=1}^{l} g_j \right\| \| \vee_{\Delta \in E_{\underline{k}}} M_{\delta_\Delta} \psi_\Delta \|,$$

and

$$\| \vee_{j=1}^{l} g_j \| = \prod_{\Delta_0 \in \mathcal{D}_0} \| \vee_{g \in g \upharpoonright \Delta_0} g \| \leqq \prod_{\Delta_0 \in \mathcal{D}_0} (|g_{\upharpoonright \Delta_0}|!)^{1/2} \prod_{j=1}^{l} \|g_j\|. \qquad \blacksquare$$

Next

LEMMA. – *Let* β, δ, Δ_0 *be segments of* $\mathcal{D}_0$, *and* $\Delta_1 \in \mathcal{D}_+$, $\Delta_1 \subset \beta$, *then for any* $\varepsilon > 0$, *there exists a constant* C_ε *such that, for any* $r \geqq 1$,

$$\sum_{\Delta \in \mathcal{D}_r;\ \Delta \subset \Delta_0} |(M_\delta \psi'_{\Delta_1}, M_\delta \psi'_\Delta)| \leqq C_\varepsilon |\Delta_1|^{(1-\varepsilon)} e^{-m(d[\beta, \delta] + d[\Delta_0, \delta])}. \quad (4.2.2)$$

Proof. – One has $\psi'_\Delta(x) = \frac{1}{\pi}\Big(K_0(m(x-O_\Delta)) - K_0(m(x-O_{\hat{\Delta}}))\Big)$, and, one deduces from [7], (3.53), the representation

$$\frac{1}{\pi} K_0(mx) = \frac{1}{\pi}\int_0^\infty K^{-1/2}\,\frac{1}{2\sqrt{m^2+K}}\,e^{-\sqrt{m^2+K}\,|x|}\,dK, \qquad (4.2.3)$$

indeed, $(-D^2+m^2)^{-1/2}$ is the operator of convolution by $\frac{1}{\pi} K_0(m.)$, and $(-D^2+m^2)^{-1}$ the operator of convolution by $\frac{1}{2m}e^{-m|\cdot|}$. Therefore, one has first

$$\|M_\delta \psi'_\Delta\|_{L^2} \leqq C|\Delta|e^{-md[\Delta^0,\delta]}, \qquad \text{if} \quad d[\Delta^0,\delta] > 0, \qquad (4.2.4)$$

indeed, for $x \in \delta$,

$$\psi'_\Delta(x) \leqq \frac{|\Delta|}{2} \sup_{y \in [O_\Delta, O_{\hat{\Delta}}]+x} \Big(\frac{m}{\pi}|K'_0(my)|\Big),$$

but if $d[\Delta^0,\delta] > 0$, one has $|my| \geqq ml_0 \geqq 1$, and one deduces easily from (4.2.3) that $|K'_0(u)| \leqq ce^{-|u|}$, if $|u| \geqq 1$, thus

$$\int_\delta |\psi'(x)|^2\,dx \leqq \frac{c^2 m^2}{4\pi^2}|\Delta|^2 e^{-2md[\Delta^0,\delta]}\int_{\mathbb{R}} e^{-2m|x|}\,dx.$$

Next

$$\|M_\delta \psi'_\Delta\|_{L^p} \leqq \|\psi'_\Delta\|_{L^p} \leqq \begin{cases} C_p|\Delta|^{1/p}, & 1<p<\infty, \\ C_1|\Delta|\log|\Delta|, & p=1, \end{cases} \qquad (4.2.5)$$

indeed,

$$\frac{1}{2M}|e^{-M|x|} - e^{-M|x-y|}| \leqq \frac{1-e^{-M|y|}}{2M}\max\{e^{-M|x|}, e^{-M|x-y|}\}$$
$$\leqq \frac{1-e^{-M|y|}}{2M}(e^{-M|x|} + e^{-M|x-y|}),$$

therefore

$$\|\psi'_\Delta\|_{L^p} \leqq \frac{1}{\pi}\|F_{|\Delta|/2}\|_{L^p}$$

with

$$F_a(x) = \int_0^\infty K^{-1/2}\,\frac{1-e^{-a\sqrt{m^2+K}}}{\sqrt{m^2+K}}\,e^{-\sqrt{m^2+K}\,|x|}\,dK,$$

then, one deduces (4.2.5) from the estimate

$$\|F_a\|_{L^p}\leqq\int_0^\infty K^{-1/2}\,\frac{1-e^{-a\sqrt{m^2+K}}}{\sqrt{m^2+K}}\,\|e^{-\sqrt{m^2+K}\,|\,.\,|}\|_{L^p}\,dK$$
$$\leqq\left(\int_{\mathbf{R}}e^{-p|y|}\,dy\right)^{1/p}\int_0^\infty K^{-1/2}\,\frac{1-e^{-a\sqrt{m^2+K}}}{\sqrt{m^2+K}}\,(m^2+K)^{-1/(2p)}\,dK$$
$$\leqq C'_p\left(a\int_0^{a^{-2}}K^{-1/2}(m^2+K)^{-1/(2p)}\,dK+\int_{a^{-2}}^{\infty}K^{-1/2}(m^2+K)^{-1/2\,(1+1/p)}\,dK\right).$$

Now, with the hypothesis of the lemma,

(i) if $d[\beta,\delta]>0$ and $d[\Delta_0,\delta]>0$, according to (4.2.4),

$$\sum_{\Delta\in\mathscr{D}_r;\,\Delta\subset\Delta_0}|(M_\delta\psi'_{\Delta_1},M_\delta\psi'_\Delta)|\leqq\sum_{\Delta\in\mathscr{D}_r;\,\Delta\subset\Delta_0}\|M_\delta\psi'_{\Delta_1}\|_{L^2}\|M_\delta\psi'_\Delta\|_{L^2}$$
$$\leqq C|\Delta_1|e^{-md[\beta,\delta]}\sum_{\Delta\in\mathscr{D}_r;\,\Delta\subset\Delta_0}|\Delta|e^{-md[\Delta_0,\delta]}$$
$$=(C|\Delta_0|)|\Delta_1|e^{-m(d[\beta,\delta]+d[\Delta_0,\delta])},$$

(ii) if $d[\beta,\delta]>0$ and $d[\Delta_0,\delta]=0$, with an obvious adaptation of (2.4.5),

$$\sum_{\Delta\in\mathscr{D}_r;\,\Delta\subset\Delta_0}|(M_\delta\psi'_{\Delta_1},M_\delta\psi'_\Delta)|\leqq\mathrm{var.}_{\Delta_0}[\Sigma_m^{-1/2}M_\delta\psi'_{\Delta_1}]$$
$$=\|1_{\Delta_0}(\Sigma_m^{-1/2}M_\delta\psi'_{\Delta_1})'\|_{L^1}\leqq|\Delta_0|^{1/2}\|(\Sigma_m^{-1/2}M_\delta\psi'_{\Delta_1})'\|_{L^2}$$
$$\leqq|\Delta_0|^{1/2}\|M_\delta\psi'_{\Delta_1}\|_{L^2}\leqq(C|\Delta_0|^{1/2})|\Delta_1|e^{-md[\beta,\delta]},$$

(iii) if $d[\beta,\delta]=0$ and $d[\Delta_0,\delta]>0$,

$$\sum_{\Delta\in\mathscr{D}_r;\,\Delta\subset\Delta_0}|(M_\delta\psi'_{\Delta_1},M_\delta\psi'_\Delta)|\leqq\|1_{\Delta_0}(\Sigma_m^{-1/2}M_\delta\psi'_{\Delta_1})'\|_{L^1}$$
$$=\left\|1_{\Delta_0}\left(\frac{1}{\pi}K_0(m.)*M_\delta\psi'_{\Delta_1}\right)'\right\|_{L^1}\leqq\left\|1_{\Delta_0}\left(\frac{m}{\pi}K'_0(m.)*M_\delta\psi'_{\Delta_1}\right)\right\|_{L^1}$$
$$\leqq\sup_{y\in\delta}\int_{\Delta_0}\frac{m}{\pi}|K'_0(m(x-y))|\,dx\,\|M_\delta\psi'_{\Delta_1}\|_{L^1}\leqq C_\varepsilon|\Delta_1|^{(1-\varepsilon)}e^{-md[\Delta_0,\delta]},$$

the last inequality comes from (2.4.5) and the inequality

$$|K'_0(u)|\leqq ce^{-|u|},\quad\text{if }|u|\geqq 1,$$

(iv) if $d[\beta,\delta]=0$ and $d[\Delta_0,\delta]=0$, due to the equivalence of the norms

$$(\|\varphi\|_{L^p}+\|\varphi'\|_{L^p})\qquad\text{and}\qquad\|(-D^2+1)^{1/2}\varphi\|_{L^p}$$

over the space $\mathscr{H}_p^1$, $(1<p<\infty)$ [27],

$$\sum_{\Delta\in\mathscr{D}_r;\,\Delta\subset\Delta_0} |(M_\delta\psi'_{\Delta_1}, M_\delta\psi'_\Delta)| \leqq \| 1_{\Delta_0}(\Sigma_m^{-1/2} M_\delta\psi'_{\Delta_1})' \|_{L^1}$$
$$\leqq |\Delta_0|^{(1-1/p)} \| (\Sigma_m^{-1/2} M_\delta\psi'_{\Delta_1})' \|_{L^p}$$
$$\leqq (C_p^1 |\Delta_0|^{(1-1/p)}) \| M_\delta\psi'_{\Delta_1} \|_{L^p} \leqq C_p |\Delta_1|^{1/p},$$

the last inequality is deduced from (2.4.5), and it suffices to choose p such that $\frac{1}{p}\geqq(1-\varepsilon)$. ■

One deduces elementarily from (4.2.2) the inequality analogous to (2.4.8), that is

COROLLARY. — *If now, for* $\Delta\in\mathscr{D}_+$, $\delta\in\mathscr{D}_0$ *one sets*

$$u_{\Delta,\delta} = |\Delta|^{-(1/2-\varepsilon)} e^{(1-\zeta)\,md\,[\Delta^0,\delta]} M_\delta\psi'_\Delta, \tag{4.2.6}$$

then, for $\xi_{\Delta,\delta}=|\Delta|^{1/2}e^{-md\,[\Delta^0,\delta]}$, *one has*

$$\sup_{\Delta_1\in\mathscr{D}_+,\,\delta_1\in\mathscr{D}_0}\left(\frac{1}{\xi_{\Delta_1,\delta_1}}\sum_{\Delta_2\in\mathscr{D}_+,\,\delta_2\in\mathscr{D}_0}\xi_{\Delta_2,\delta_2}|(u_{\Delta_1,\delta_1}, u_{\Delta_2,\delta_2})|\right)\leqq C_{\varepsilon,\zeta}. \tag{4.2.7}$$

4.3. Now, with the notations of appendix B, let, for $b\in\mathscr{B}$, $\partial^{(b)}=\frac{\partial}{\partial s_b}$, and for $\Gamma\in\mathscr{C}_0$, $\partial^\Gamma=\prod_{b\in\Gamma}\partial^{(b)}$ if $\Gamma\neq\varnothing$, $(\partial^\varnothing=I)$; then, if $\underline{t}\in\mathscr{T}_\Lambda$ verifies (2.1.1), one deduces from (4.0.4), taking (4.0.2) into account

$$\partial^{(b)}(S_g(\underline{t};s;\sigma)\,\mathrm{Det.}\,\mathbf{R}_{\underline{t};s}(\sigma)^{1/2})$$
$$=\frac{-i\rho\alpha}{2\sqrt{\mathrm{Re}.\alpha^2}}\prod_{\Delta\in\underline{\Lambda}(\underline{t})}\langle\sigma,1_\Delta\rangle\sum_{\delta^1,\delta^2\in\mathscr{D}_0}\partial^{(b)}H(s|\Delta^0;\delta^1,\delta^2)$$
$$\times(t_\Delta S_{g(\Delta;\delta^1,\delta^2)}(\underline{t};s;\sigma)+(1-t_\Delta)S_{g(\hat{\Delta};\delta^1,\delta^2)}(\underline{t};s;\sigma))$$
$$\times\,\mathrm{Det.}\,\mathbf{R}_{\underline{t};s}(\sigma)^{1/2}, \tag{4.3.1}$$

and therefore

$$\partial^\Gamma(S_g(\underline{t};s;\sigma)\,\mathrm{Det.}\,\mathbf{R}_{\underline{t};s}(\sigma)^{1/2})$$
$$=\sum_{P\in\mathscr{P}(\Gamma)}\sum_{\Delta_*\in\underline{\Lambda}(\underline{t})^P}\sum_{\underline{\delta}^1,\underline{\delta}^2\in\mathscr{D}_0^P}\prod_{C\in P}\frac{-i\rho\alpha}{\sqrt{\mathrm{Re}.\alpha^2}}\langle\sigma,1_{\Delta_c}\rangle\partial^C H(s|\Delta_C^0;\delta_C^1,\delta_C^2)\Big)$$
$$\times\sum_{\hat{\cdot}^*\in\{\cdot,\hat{\cdot}\}^P}c_{\underline{t}}(\hat{\Delta}_*^*)S_{g[\hat{\Delta}_*^*;\underline{\delta}^1,\underline{\delta}^2]}(\underline{t};s;\sigma)\,\mathrm{Det.}\,\mathbf{R}_{\underline{t};s}(\sigma)^{1/2}, \tag{4.3.2}$$

where $\mathscr{P}(\Gamma)$ is the set of partitions of Γ; $0\leqq c_{\underline{t}}(\hat{\Delta}_*^*)\leqq 1$; and $g_{[\hat{\Delta}_*^*;\underline{\delta}^1,\underline{\delta}^2]}$ *contains* g, *and, for each* $C\in P$, $M_{\delta_C^1}\psi_{\hat{\Delta}_C^*}$ *and* $M_{\delta_C^2}\psi_{\hat{\Delta}_C^*}$. Then one has

[27] See for example [16], theorem 2.5.6.

LEMMA. – *For any* $K>0$, *one has, provided m is sufficiently large,* (*with* C *not depending on* K),

$$\|\partial^{\Gamma_j} S_{\tilde{f}_{j\Delta}{}^{[\underline{q}_j;\,\underline{\delta}'_j,\,\underline{\tilde{\delta}}_j]}_{[\underline{h}_j;\,\underline{\delta}_j^-,\,\underline{\delta}_j^+]}}(\underline{t};\,s;\,.)\,\mathrm{Det.}\,\mathbf{R}_{\underline{t};\,s}(.)^{1/2}\|_{L^p}$$
$$\leqq \mathcal{N}(f_{j\Delta})\,C^{|\underline{q}|+|\underline{h}|+|\Lambda|+|\underline{\Lambda}(\underline{t})|}\,e^{-K|\Gamma_j|}$$
$$\times\prod_{\Delta_1\in E_{\underline{q}}} m^{-1/2}\,e^{-(1-\zeta)\,m\,(d[\Delta_1^\circ,\,\delta'_{\Delta_1}]+d[\Delta_1^\circ,\,\tilde{\delta}_{\Delta_1}])}$$
$$\times\prod_{\Delta_2\in E_{\underline{h}_j}} m^{-1}\,e^{-(1-\zeta)\,m\,(d[\Delta_2^\circ,\,\delta_{\Delta_2}^-]+d[\Delta_2^\circ,\,\delta_{\Delta_2}^+])}$$
$$\times\prod_{\Delta_0\in\underline{\Lambda}_0}([\underline{q}_j\lceil_{\Delta_0}|!]^{1/2}[|\underline{h}_j\lceil_{\Delta_0}|!])\prod_{\Delta\in\underline{\Lambda}_+}((q_{j\Delta}!)^{1/2}|\Delta|^{(1/2)-\varepsilon)\,q_{j\Delta}}). \qquad (4.3.3)$$

Proof. – The bound on $S_{g[\hat{\Delta}_*^*;\,\underline{\delta}^1,\,\underline{\delta}^2]}$, with $g=\tilde{f}_{j\Delta}{}^{[\underline{q}_j;\,\underline{\delta}'_j,\,\underline{\tilde{\delta}}_j]}_{[\underline{h}_j;\,\underline{\delta}_j^-,\,\underline{\delta}_j^+]}$, given by (4.1.7), depends on $\hat{\Delta}_*^*$ only through $\Delta_*^0\in\underline{\Lambda}_0^P$, and therefore can be factorized; so, given $\underline{\beta}\in\underline{\Lambda}_0^P$. one has

$$\|\sum_{\Delta_*\in\underline{\Lambda}(\underline{t})^P;\,\Delta^0{}_*=\underline{\beta}}\;\prod_{C\in P}\langle\,.,\,1_{\Delta_C}\rangle\|_{L^p}$$
$$\leqq^{(i)} C_1^{|P|}\prod_{\Delta_0\in\underline{\Lambda}_0}\left(\sum_{\vartheta\in(\underline{\Lambda}(\underline{t})\lceil_{\Delta_0})^{\underline{\beta}^{-1}(\Delta_0)}}\frac{|\underline{\beta}^{-1}(\Delta_0)|!}{\prod_{\Delta\in\underline{\Lambda}(\underline{t})\lceil_{\Delta_0}}|\vartheta^{-1}(\Delta)|!}\right.$$
$$\left.\times\prod_{\Delta\in(\underline{\Lambda}(\underline{t})\lceil_{\Delta_0}}(|\vartheta^{-1}(\Delta)|!)^{1/2}|\Delta|^{|\vartheta^{-1}(\Delta)|/2}\right)$$
$$\leqq^{(ii)} C_1^{|P|}\prod_{\Delta_0\in\underline{\Lambda}_0}C_2^{|\underline{\beta}^{-1}(\Delta_0)|}\,e^{|\underline{\Lambda}(\underline{t})\lceil_{\Delta_0}|}|\Delta_0|^{|\underline{\beta}^{-1}(\Delta_0)|}(|\underline{\beta}^{-1}(\Delta_0)|!)^{1/2}$$
$$\leqq C^{|\Gamma|}\,e^{|\underline{\Lambda}(\underline{t})|}\prod_{\Delta_0\in\underline{\Lambda}_0}(|\underline{\beta}^{-1}(\Delta_0)|!)^{1/2},$$

[$^{(i)}$ from (3.2.6); $^{(ii)}$ from (3.2.7), with $\underline{\Lambda}(\underline{t})\lceil_{\Delta_0}:=\{\Delta\in\underline{\Lambda}(\underline{t});\ \Delta\subset\Delta_0\}$ in place of $\underline{\Lambda}(\underline{t})$]. To achieve the proof, one bounds $\partial^C H(s\,|\,\Delta_C^0;\,\delta_C^1,\,\delta_C^2)$ according to (C.2.5); one notes that, for β, $\delta\in\mathscr{D}_0$, $\beta\neq\delta$, one has $d[\beta,\,\delta]=l_0(\boldsymbol{d}(\beta,\,\delta)-1)$, so that, for any given τ, if m is sufficiently large, one has [28]

$$\prod_{\delta\in\mathscr{D}_0} m^{-1/4}\,e^{-(1-\zeta)\,md\,[\beta,\,\delta]}\,e^{\tau\boldsymbol{d}(\beta,\,\delta)}\leqq 1, \qquad (4.3.4)$$

therefore, if one choses τ, and accordingly m, sufficiently large, one can conclude using (C.3.1), (C.3.3) with $\nu=3/2$, and taking into account (4.1.8). ■

4.4. To compute the derivatives $(\prod_{\Delta\in\underline{\Lambda}_+}D_\Delta^{q_\Delta})\,\partial^\Gamma Z_{N,\,(\rho);\,\underline{t};\,s}(f)$, one applies ∂^Γ to the expressions (4.1.4), and (4.1.6). One obtains

---

([28]) The factor $m^{-1/4}$ is used to control the terms where $\boldsymbol{d}(\beta,\,\delta)=1$.

LEMMA. — *For any* $K_2>0$, *one has* ([29]), *if m is sufficiently large,*

$$
\begin{aligned}
&\Big|\Big(\prod_{\Delta\in\underline{\Delta}_+} D_\Delta^{q_\Delta}\Big)\partial^\Gamma Z_{N,(\rho);\underline{t};s}(f)\Big| \\
&\qquad\leqq \mathcal{N}(f_\Lambda)\, C_\varepsilon^{|\Lambda|+|\underline{\Delta}(t)|+|\underline{q}|}\, e^{-K_2|\Gamma|}\prod_{\Delta_0\in\underline{\Delta}_0}(|\underline{q}\restriction_{\Delta_0}|!)^{1/2} \\
&\qquad\qquad\times\prod_{\Delta\in\underline{\Delta}_+}\Big((q_\Delta!)\,|\Delta|^{(1-\varepsilon)q_\Delta}\Big),
\end{aligned}\tag{4.4.1}
$$

and

$$
\begin{aligned}
&\Big|\Big(\prod_{\Delta\in\underline{\Delta}_+} D_\Delta^{q_\Delta}\Big)\partial^\Gamma Z_{N,(\rho);\underline{t};s}(f)\Big| \\
&\qquad\leqq \mathcal{N}(f_\Lambda)\, C_\varepsilon^{|\Lambda|+|\underline{\Delta}(t)|+|\underline{q}|}\, e^{-K_2|\Gamma|}\prod_{\Delta_0\in\underline{\Delta}_0}(|\underline{q}\restriction_{\Delta_0}|!)^{1/2} \\
&\qquad\qquad\times\prod_{\Delta\in\underline{\Delta}_+}\Big((q_\Delta!)^{1/2}\,|\Delta|^{((1/2)-\varepsilon)q_\Delta}\Big) \\
&\qquad\qquad\times\sum_{X\subset E_{\underline{q}};\,|X|\in 2\mathbb{N}}\Big[\prod_{\Delta_0\in\underline{\Delta}_0}(|\underline{q}^{X'}\restriction_{\Delta_0}|!) \\
&\qquad\qquad\times\Big(\sum_{P\in\mathscr{P}_2(X)}\prod_{p\in P}|\Delta_{p_-}\cap\Delta_{p_+}|\Big)\prod_{y\in E_{\underline{q}}\setminus X}|\Delta_y|\Big].
\end{aligned}\tag{4.4.2}
$$

Proof. — One applies ∂^Γ under the integral sign in the formulas (4.1.4) and (4.1.6); one estimates the derivatives with respect to s of the different factors by means of (4.3.3) and (C.3.4), with τ chosen sufficiently large and m such that (4.3.3) is satisfied; one concludes, taking (2.7.1) into account. ■

One completes the argument as in chapter 2, with the inequalities (4.4.1) and (4.4.2) in place of (2.8.1) and (2.9.4) respectively, one obtains from the theorem (A.0)

PROPOSITION. — *The function* $Z_{N,(\rho);\Lambda,r;s}(f)$ *and their derivatives* $\partial^\Gamma Z_{N,(\rho);\Lambda,r;s}(f)$ *converge, uniformly for* $s\in S_0$, *as* $r\to\infty$, *to functions* $Z_{N,(\rho);\Lambda;s}(f)$ *and their derivatives, which satisfy*

$$|\partial^\Gamma Z_{N,(\rho);\Lambda;s}(f)|\leqq\mathcal{N}(f_\Lambda)\,e^{K_1|\Lambda|-K_2|\Gamma|},\tag{4.4.4}$$

where K_2 *can be supposed arbitrarily large, provided m is large enough.*

4.5. The application of the theorems (B.3) to $(\Lambda;s)\mapsto Z_{N,(\rho);\Lambda;s}(\varnothing)$ and (B.4) to $(\Lambda;s)\mapsto Z_{N,(\rho);\Lambda;s}(f)$ requires to check that these functions verify P-i to P-vi, and L-i to L-iv, respectively.

For sufficiently large m, P-v and L-iv come simply from the inequality (4.4.4); next $\rho\mapsto Z_{N,(\rho);\Delta_0;0}(\varnothing)$, which does not depend on $\Delta_0\in\mathscr{D}_0$, is

([29]) With the hypothesis of lemmas (2.8) and (2.9).

continuous in $\{\rho \in \mathbf{C};\ |\text{Arg}\ \rho| \leqq \frac{3\pi}{4} - \eta\}$, for any $\eta > 0$, and equas 1 for $\rho = 0$, therefore given $K_0 > 0$, for each $\eta > 0$, there exists $R'_\eta > 0$ [30], such that P-iv is satisfied whenever $|\text{Arg}\ \rho| \leqq \frac{3\pi}{4} - \eta$ and $|\rho| \leqq R'_\eta$; the other conditions are easily checked on the function $(\Lambda;\ s) \mapsto Z_{N,\,(\rho);\,\Lambda,\,r;\,s}(f)$, and then transferred to the functions of interest, due to the uniformity of the convergence as $r \to \infty$; in particular, the decoupling at $s = 0$, (conditions P-i and L-i), comes from the fact that, under the hypothesis of the definition (B.2), the corresponding resolvent factorizes:

$$\mathbf{R}_{\Lambda,\,r;\,1_\Gamma s}(\sigma) = \prod_{i=1} \mathbf{R}_{\Lambda \cap X_i,\,r;\,1_{\Gamma \cap X_i} - s}(\sigma),$$

and the variables $\langle \sigma, 1_\Delta \rangle$'s which appear in different factors are mutually independent; and the regularity at infinity, (conditions P-ii and L-ii), is deduced from the lemma (C.2) by means of Lebesgue's dominated convergence theorem.

Therefore, one has, (with $\lambda = \rho^2/8$),

THEOREM. – *Given $m > 0$, for any $\eta > 0$, there exists $R_\eta > 0$ such that, if $|\text{Arg}\ \lambda| \leqq \frac{3\pi}{2} - \eta$, $|\lambda| \leqq R_\eta$, (and if l_0 is sufficiently large), the functions $\lambda \mapsto S_{N,\,\lambda;\,\Lambda}(f) := Z_{N,\,\lambda;\,\Lambda}(f)/Z_{N,\,\lambda;\,\Lambda}(\varnothing)$ are well defined* [31], *and converge, as $\Lambda \to \mathbf{R}$, to functions $S_{N,\,\lambda}(f)$, analytic in λ and continuous at $\lambda = 0$, which continue the functions defined by* (1.1.7)*; moreover* [32]

$$|S_{N,\,\lambda}(f)| \leqq \mathcal{N}(f). \tag{4.5.1}$$

Indeed, for m sufficiently large, this statement is only a paraphrase of the previous analysis; one suppresses the restriction over m by a classical scaling argument [4], that is, one remarks that, for $\xi > 0$,

$$S_{N,\,m,\,\lambda)}(x) = \xi^{1/2}\, S_{N,\,(\xi m,\,\xi^3 \lambda)}(\xi^{-1} x), \qquad (x \in \mathbf{R}^l). \qquad \blacksquare$$

---

[30] This constant is independent of m bounded from below.

[31] That means that the denominator does not vanish.

[32] With a possible change, [depending uniformly on (m, λ) in any compact set], of the constant which appears in the definition of $\mathcal{N}(f)$.

One can also derive from (B.5.3) cluster properties for the "truncated" Schwinger functions.

5. BOREL SUMMABILITY FOR THE SCHWINGER FUNCTIONS $S_{N,\lambda}(f)$

5.0. The functions $(\Lambda; s) \mapsto \frac{d^n}{d\rho^n} Z_{N,(\rho);\Lambda;s}(f)$ do not decouple at $s=0$ [33], first one decomposes each of them as a sum of functions which decouple.

For $\underline{u} \in \mathbf{R}^{\mathscr{D}_0}$, one sets

$$A_{\underline{t};s;\underline{u}}(\sigma) := \sum_{\beta \in \underline{\Lambda}_0} u_\beta \sum_{\delta_1, \delta_2 \in \mathscr{D}_0} H(s \mid \beta; \delta_1, \delta_2) \times \sum_{\Delta \in \underline{\Lambda}} t_\Delta \langle \sigma, 1_\beta 1_\Delta \rangle M_{\delta_1} [\psi'_\Delta \times \bar{\psi}_\Delta + \bar{\psi}_\Delta \times \psi'_\Delta] M_{\delta_2}, \quad (5.0.1)$$

so that, if $\underline{t} \in \mathscr{T}_\Lambda$, $A_{\underline{t};s}(\sigma) = A_{\underline{t};s;1_{\underline{\Lambda}_0}}(\sigma)$; one notes $Z_{N,(\rho);\underline{t};s;\underline{u}}(f)$ the functions obtained by substitution of $A_{\underline{t};s;\underline{u}}$ to $A_{\underline{t};s}$, and for $\underline{\beta} \in \mathscr{D}_0^{e_n}$, ($n \in \mathbf{N}$, $e_n = \{1, \ldots, n\}$), one sets

$$Z^{\underline{\beta}}_{N,(\rho);\underline{t};s}(f) := \frac{1}{\rho^n} \left(\prod_{j=1}^{n} \frac{\partial}{\partial u_{\beta_j}} \right) Z_{N,(\rho);\underline{t};s;1_{\underline{\Lambda}_0}}(f), \quad (5.0.2)$$

then one has [34]

$$\frac{d^n}{d\rho^n} Z_{N,(\rho);\underline{t};s}(f) = \sum_{\underline{\beta} \in \underline{\Lambda}_0^{e_n}} Z^{\underline{\beta}}_{N,(\rho);\underline{t};s}(f). \quad (5.0.3)$$

5.1. One computes the expressions similar to (3.2.4), (3.2.5) [35]

$$\Big(\prod_{\Delta \in \underline{\Lambda}_+} D_\Delta^{q_\Delta} \Big) Z^{\underline{\beta}}_{N,(\rho);\underline{t};s}(f)$$

$$= \sum_{\underline{k} \in \mathbf{N}^{\underline{\Lambda}_0};\, 0 \le k_\gamma \le |\underline{\beta}^{-1}(\gamma)|} \; \sum_{\underline{k} \in \Gamma_{\underline{t}}^{\underline{k}}} \left(\prod_{\gamma \in \underline{\Lambda}_0} ([|\underline{\beta}^{-1}(\gamma)| - k_\gamma]!) \frac{k_\gamma!}{\prod_{\Delta \in \underline{\Lambda}(\underline{t});\, \Delta \subset \gamma} k_\Delta!} \right)$$

$$\times \sum_{\{\underline{q}_j\}_{1 \le j \le N};\, \sum_j \underline{q}_j = \underline{q}} \; \sum_{\{\underline{k}_j\}_{1 \le j \le N};\, \sum_j \underline{k}_j = \underline{k}} \; \sum_{\{\underline{h}_j \in H_{\underline{t}}^{\underline{k}_j}\}_{1 \le j \le N}}$$

$$\times \left[\frac{(-i)^{|\underline{q}| + |\underline{k}|} \rho^{|\underline{q}| - n + |\underline{k}|}}{2^{|\underline{k}|}} \left(\prod_{\gamma \in \underline{\Lambda}_0} \binom{|\underline{\beta}^{-1}(\gamma)|}{k_\gamma} \binom{|q \Gamma_\gamma|}{|\underline{\beta}^{-1}(\gamma)| - k_\gamma} \right) \right.$$

$$\times \prod_{\Delta \in \underline{\Lambda}_+} \frac{q_\Delta!}{\prod_{j=1}^{N} q_{j\Delta}!} \prod_{\Delta \in \underline{\Lambda}(\underline{t})} \frac{k_\Delta!}{\prod_{j=1}^{N} k_{j\Delta}!} \prod_{j=1}^{N} \left(c_{\underline{t}}(\underline{h}_j) \prod_{\Delta \in \underline{\Lambda}(\underline{t})} \binom{k_{j\Delta}}{h_{j\Delta}} \right)$$

---

[33] See the definition (B.2) and the conditions P-i and L-i.

[34] One notes that $Z^{\underline{\beta}}_{N,(\rho);\underline{t};s}(f) = 0$, if $\underline{\beta}(e_n) \not\subset \underline{\Lambda}_0$.

[35] In this formula, the lines 2, 3, 4 of the right hand side – and in the subsequent formula the lines 2, 3, 4, 5, 6, 7 – are harmless.

$$\times\left(\frac{\alpha}{\sqrt{\operatorname{Re}.\alpha^2}}\right)^{|\underline{\Delta}(\underline{t})|}\int_{\mathscr{S}'}\prod_{\Delta\in\underline{\Delta}+}\left(\left\langle\sigma,\left(1_\Delta-\frac{|\Delta|}{|\tilde{\Delta}|}1_{\tilde{\Delta}}\right)\right\rangle + \frac{\alpha}{\sqrt{\operatorname{Re}.\alpha^2}}\frac{|\Delta|}{|\tilde{\Delta}|}\langle\sigma,1_{\tilde{\Delta}}\rangle\right)^{q_\Delta}$$

$$\times\prod_{\Delta'\in\underline{\Delta}(\underline{t})}\left(\frac{\alpha}{\sqrt{\operatorname{Re}.\alpha^2}}\langle\sigma,1_{\Delta'}\rangle\right)^{k_{\Delta'}}\prod_{j=1}^{N}\Bigg(\sum_{\delta_j',\tilde{\delta}_j\in\mathscr{D}_0^{E_{\underline{q}_j}}}\ \sum_{\delta_j^1,\delta_j^2\in\mathscr{D}_0^{E_{\underline{h}_j}}}$$

$$\times\Big\{\Big(\prod_{\Delta'\in E_{\underline{q}_j}}H(s|\Delta'^0;\delta'_{j\Delta'},\tilde{\delta}_{j\Delta'})\Big)\Big(\prod_{\Delta''\in E_{\underline{h}_j}}H(s|\Delta''^0;\delta^1_{j\Delta''},\delta^2_{j\Delta''})\Big)$$

$$S_{\tilde{f}_{j\Lambda}}{}^{[\underline{q}_j;\,\underline{\delta}_j',\,\underline{\tilde{\delta}}_j]}_{[\underline{h}_j;\,\underline{\delta}_j^-,\,\underline{\delta}_j^+]}(\underline{t};s;\sigma)\Big\}\Bigg)$$

$$\times\operatorname{Det}.\mathbf{R}_{\underline{t};s}(\sigma)^{N/2}.e^{-(i/2)(\operatorname{Im}.\alpha^2/\operatorname{Re}.\alpha^2)\sum_{\Delta\in\underline{\Delta}(\underline{t})}|\Delta|^{-1}\langle\sigma,1_\Delta\rangle^2}.\nu(d\sigma)\Bigg],\qquad(5.1.1)$$

where, for $\underline{k}\in\mathbb{N}^{\underline{\Lambda}_0}$, $I_{\underline{t}}^{\underline{k}}:=\{\underline{k}=(k_\Delta\in\mathbb{N})_{\Delta\in\underline{\Delta}(\underline{t})};\ |\underline{k}\lceil_\gamma|=\boldsymbol{k}_\gamma,\ (\gamma\in\Lambda_0)\}$, and

$$\Big(\prod_{\Delta\in\underline{\Delta}+}D_\Delta^{q_\Delta}\Big)Z^{\beta}_{\bar{N},(\rho);\underline{t};s}(f)$$

$$=\sum_{\underline{k}\in\mathbb{N}^{\underline{\Lambda}_0};\,0\leq\boldsymbol{k}_\gamma\leq|\beta^{-1}(\gamma)|}\ \sum_{\underline{k}\in I_{\underline{t}}^{\underline{k}}}\left(\prod_{\gamma\in\underline{\Lambda}_0}([|\underline{\beta}^{-1}(\gamma)|-\boldsymbol{k}_\gamma]!)\frac{\boldsymbol{k}_\gamma!}{\prod_{\Delta\in\underline{\Delta}(\underline{t});\,\Delta\subset\gamma}k_\Delta!}\right)$$

$$\times\sum_{\{\underline{q}_j\}_{1\leq j\leq N};\,\sum_j\underline{q}_j=\underline{q}}\ \sum_{X\subset E_{\underline{q}};\,|X|\in 2N}\ \sum_{\{Y_j\}_{1\leq j\leq N};\,\bigcup_j Y_j=E_{\underline{q}}\setminus X,\,Y_i\cap Y_i=\varnothing}$$

$$\times\sum_{\{\underline{k}_j\}_{1\leq j\leq N};\,\sum_j\underline{k}_j=\underline{k}}\ \sum_{\{\underline{h}_j'\in H_{\underline{t}}^{\underline{q}_j^{Y_j}}\}_{1\leq j\leq N}}\ \sum_{\{\underline{h}_j''\in H_{\underline{t}}^{\underline{k}_j}\}_{1\leq j\leq N}}$$

$$\times\Bigg[\frac{(-i)^{2|\underline{q}|-|X|+|\underline{k}|}\rho^{2|\underline{q}|-|X|-n+|\underline{k}|}}{2^{|\underline{q}|-|X|+|\underline{k}|}}$$

$$\times\left(\prod_{\gamma\in\underline{\Lambda}_0}\binom{|\underline{\beta}^{-1}(\gamma)|}{\boldsymbol{k}_\gamma}\binom{2|\underline{q}\lceil_\gamma|-|X\lceil_\gamma|}{|\underline{\beta}^{-1}(\gamma)|-\boldsymbol{k}_\gamma}\right)$$

$$\times\prod_{\Delta\in\underline{\Delta}+}\frac{q_\Delta!}{\prod_{j=1}^N q_{j\Delta}!}\ \prod_{\Delta\in\underline{\Delta}(\underline{t})}\frac{k_\Delta!}{\prod_{j=1}^N k_{j\Delta}!}$$

$$\prod_{j=1}^{N}\left(c_{\underline{t}}(\underline{h}_j')\,c_{\underline{t}}(\underline{h}_j'')\prod_{\Delta\in\underline{\Delta}(\underline{t})}\binom{q_j^{Y_j}{}_\Delta}{h'_{j\Delta}}\binom{k_{j\Delta}}{h''_{j\Delta}}\right)$$

$$\times\Big(\sum_{P\in\mathscr{P}_2(X)}\prod_{p\in P}|\Delta_{p_-}\cap\Delta_{p_+}|\Big)\prod_{y\in E_{\underline{q}}\setminus X}|\Delta_y|$$

$$\times\left(\frac{\alpha}{\sqrt{\operatorname{Re}.\alpha^2}}\right)^{|\underline{\Delta}(\underline{t})|}\int_{\mathscr{S}'}\prod_{\Delta\in\underline{\Delta}(\underline{t})}\left(\frac{\alpha}{\sqrt{\operatorname{Re}.\alpha^2}}\langle\sigma,1_\Delta\rangle\right)^{k_\Delta}$$

$$\times \prod_{j=1}^{N} \Big(\sum_{\delta_j', \tilde{\delta}_j \in \mathscr{D}_0^{E_{qj}}} \ \sum_{\delta_j^-, \delta_j^+ \in \mathscr{D}_0^{E_{h'j}}} \ \sum_{\delta_j^1, \delta_j^2 \in \mathscr{D}_0^{E_{h''j}}}$$

$$\times \{ (\prod_{\Delta \in E_{qj}} H(s \,|\, \Delta^0; \delta_{j\Delta}, \tilde{\delta}_{j\Delta})) (\prod_{\Delta' \in E_{h'j}} H(s \,|\, \Delta'^0; \delta_{j\Delta'}^-, \delta_{j\Delta'}^+))$$

$$\times (\prod_{\Delta'' \in E_{h''j}} H(s \,|\, \Delta''^0; \delta_{j\Delta''}^1, \delta_{j\Delta''}^2))$$

$$\times S_{\tilde{f}_{j\Lambda}}{}^{[q_j; \delta_j', \tilde{\delta}_j]}_{[h_j; \delta_j^-, \delta_j^+] + [h_j'', \delta_j^1, \delta_j^2]} (\underline{t}; s; \sigma) \} \Big)$$

$$\times \text{Det. } \mathbf{R}_{\underline{t}; s}(\sigma)^{N/2} . e^{-(i/2)(\text{Im} . \alpha^2/\text{Re} . \alpha^2) \sum_{\Delta \in \underline{\Delta}(t)} |\Delta|^{-1} \langle \sigma, 1_\Delta \rangle^2} . \nu(d\sigma) \Big] \quad (5.1.2)$$

5.2. One computes the derivatives $(\prod_{\Delta \in \underline{\Delta}+} D_\Delta^{q_\Delta}) \partial^\Gamma Z^{\underline{\beta}}_{N, (\rho); \underline{t}; s}(f)$, and an estimation similar to that of lemma (4.4), gives, under the same hypothesis,

$$| (\prod_{\Delta \in \underline{\Delta}+} D_\Delta^{q_\Delta}) \partial^\Gamma Z^{\underline{\beta}}_{N, (\rho); \underline{t}; s}(f) |$$

$$\leqq \mathcal{N}(f_\Lambda) C_\varepsilon^{|\Lambda| + |\underline{\Delta}(t)| + |\underline{q}| + n} e^{-K_2 |\Gamma|}$$

$$\times \prod_{\gamma \in \underline{\Delta}_0} (|\underline{\beta}^{-1}(\gamma)|!)^{3/2} \prod_{\Delta_0 \in \underline{\Delta}_0} (|\underline{q} \lceil_{\Delta_0}|!)^{1/2}$$

$$\times \prod_{\Delta \in \underline{\Delta}+} \Big((q_\Delta !) |\Delta|^{(1-\varepsilon) q_\Delta} \Big), \quad (5.2.1)$$

and

$$| (\prod_{\Delta \in \underline{\Delta}+} D_\Delta^{q_\Delta}) \partial^\Gamma Z^{\underline{\beta}}_{N, (\rho); \underline{t}; s}(f) |$$

$$\leqq \mathcal{N}(f_\Lambda) C_\varepsilon^{|\Lambda| + |\underline{\Delta}(t)| + |\underline{q}| + n} e^{-K_2 |\Gamma|}$$

$$\times \prod_{\gamma \in \underline{\Delta}_0} (|\underline{\beta}^{-1}(\gamma)|!)^{3/2} \prod_{\Delta_0 \in \underline{\Delta}_0} (|\underline{q} \lceil_{\Delta_0}|!)^{1/2}$$

$$\times \prod_{\Delta \in \underline{\Delta}+} \Big((q_\Delta !)^{1/2} |\Delta|^{(1/2-\varepsilon) q_\Delta} \Big) . \sum_{X \subset E_{\underline{q}};\ |X| \in 2N} \Big[\prod_{\Delta_0 \in \underline{\Delta}_0} (|\underline{q}_\Gamma^{X'} \Delta_0|!)$$

$$\times (\sum_{P \in \mathscr{P}_2(X)} \prod_{p \in P} |\Delta_{p_-} \cap \Delta_{p_+}|) \prod_{y \in E_{\underline{q}} \setminus X} |\Delta_y| \Big]. \quad (5.2.2)$$

Then, by substitution of (5.2.1), (5.2.2) to (2.8.1), (2.9.4) in the proof of chapter 2, one deduces

PROPOSITION. — *For all* $n \in N$, $\underline{\beta} \in D_0^{e_n}$, *the functions* $Z^{\underline{\beta}}_{N, (\rho); \Lambda, r; s}(f)$ *and their derivatives* $\partial^\Gamma Z^{\underline{\beta}}_{N, (\rho); \Lambda, r; s}(f)$ *converge, uniformly with respect to* $s \in S_0$ *and to* ρ *in any compact set, as* $r \to \infty$ *towards functions* $Z^{\underline{\beta}}_{N, (\rho); \Lambda; s}(f)$ *and* *their derivatives, which verify*

$$|\partial^\Gamma Z^{\underline{\beta}}_{N,(\rho);\Lambda;s}(f)| \leqq \mathcal{N}(f_\Lambda) C^n \prod_{\gamma\in\underline{\Lambda}_0} (|\underline{\beta}^{-1}(\gamma)|!)^{3/2} e^{K_1|\Lambda|-K_2|\Gamma|}, \quad (5.2.3)$$

with K_2 *arbitrarily large if m is large enough; moreover, from* (5.0.3),

$$\frac{d^n}{d\rho^n} Z_{N,(\rho);\Lambda;s}(f) = \sum_{\underline{\beta}\in\underline{\Lambda}^{e_n}_0} Z^{\underline{\beta}}_{N,(\rho);\Lambda;s}(f). \quad (5.2.4)$$

5.3. Now, for $n\in\mathbb{N}$, one sets $e_n^+ = \{0, 1, \ldots, n\}$, and for $v\subset e_n^+$, $\underline{\beta}\in\mathcal{D}_0^{e_n}$, one denotes by $\underline{\beta}\lceil_v$ the restriction of $\underline{\beta}$ to $v\cap e_n$; let

$$F^{\underline{\beta}}_{N,(\rho);\Lambda;s\,|\,v}(f) := \begin{cases} Z^{\underline{\beta}\lceil v}_{N,(\rho);\Lambda;s}(f), & \textit{if } 0\in v \\ Z^{\underline{\beta}\lceil v}_{N,(\rho);\Lambda;s}(\varnothing), & \textit{if } 0\notin v \end{cases} \quad (5.3.1)$$

and, with $\mathcal{P}(e_n^+)$ the set of partitions of e_n^+,

$$S^{\underline{\beta}}_{N,(\rho);\Lambda;s}(f) := \sum_{P\in\mathcal{P}(e_n^+)} (-1)^{|P|-1}.(|P|-1)! \prod_{v\in P} \frac{F^{\underline{\beta}}_{N,(\rho);\Lambda;s|v}(f)}{Z_{N,(\rho);\Lambda;s}(\varnothing)}, \quad (5.3.2)$$

then if $S_{N,(\rho);\Lambda;s}(f) : Z_{N,(\rho);\Lambda;s}(f)/Z_{N,(\rho);\Lambda;s}(\varnothing)$, one deduces from (5.2.4),

$$\frac{d^n}{d\rho^n} S_{N,(\rho);\Lambda;s}(f) = \sum_{\underline{\beta}\in\underline{\Lambda}^{e_n}_0} S^{\underline{\beta}}_{N,(\rho);\Lambda;s}(f). \quad (5.3.3)$$

One now gives two estimates of $S^{\underline{\beta}}_{N,(\rho);\Lambda;s}(f)$:

LEMMA. – *For any* $\eta>0$, *there exists* $R'_\eta>0$ *such that, if* $|\text{Arg}\,\rho| \leqq \frac{3\pi}{4} - \eta$ *and* $|\rho|\leqq R'_\eta$, *and if m is sufficiently large, then for all* $n\in\mathbb{N}$, $\underline{\beta}\in\mathcal{D}_0^{e_n}$, *one has* (36)

$$|S^{\underline{\beta}}_{N,(\rho);\Lambda;s}(f)| \leqq \mathcal{N}(f_\Lambda) C^n n! \prod_{\gamma\in\mathcal{D}_0} (|\underline{\beta}^{-1}(\gamma)|!)^{1/2} \times e^{-\mathbf{D}[\Sigma_f\{\beta_1\},\ldots,\{\beta_n\}]}, \quad (5.3.4)$$

where Σ_f *is the set of segments of* $\mathcal{D}_0$ *which contain the support of at least one function of* $\tilde{f}$, *and* **D** *is defined by* (B.5.1).

Proof. – Due to the proposition (5.2), one can apply the theorem (B.5) to the family $\{F^{\underline{\beta}}_{N,(\rho);\,.\,;\,.\,|\,v}(f)\}_{v\subset e_n^+}$, of functions linked to the partition function $Z_{N,(\rho);\,.\,;\,.}(\varnothing)$, (if $|\rho|$ is small enough so that the condition P-iv is verified); here, the family $\{\Sigma_j\}_{0\leqq j\leqq n}$ of the theorem (B.5) is given by $\Sigma_0 = \Sigma_f$, $\Sigma_j = \{\beta_j\}$, $(1\leqq j\leqq n)$.

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(36) *See* note (32).

One deduces (5.3.4) from (B.5.3) and (5.2.3), by means of the inequality

$$\sum_{P \in \mathscr{P}(e_n^+)} (|P|-1)! \prod_{v \in P} (|v|!) \leqq 3^n n!,$$

proved by recurrence over n. ■

LEMMA. — *With the hypothesis of the preceding lemma,*

$$S^{\underline{\beta}}_{N,(\rho);\Lambda;s}(f) | \leqq \mathcal{N}(f_\Lambda) C^n \prod_{\gamma \in \mathscr{D}_0} (|\underline{\beta}^{-1}(\gamma)|!)^{3/2}. \qquad (5.3.5)$$

Proof. — For $\underline{\delta} \in \mathscr{D}_0^v$ and $\underline{z} \in \mathbf{C}^{\mathscr{D}_0}$, following an idea of Spencer [14], one sets

$$Y^{\underline{\delta}}_{N,(\rho);\Lambda;s}[\underline{Z}] := \sum_{w \subset v} \left(\prod_{\gamma \in \mathscr{D}_0} \left(\frac{|\underline{\delta}^{-1}(\gamma)|}{|\underline{\delta}\lceil_w^{-1}(\gamma)|} \right)^{-1} \times \frac{z_\gamma^{|\underline{\delta}\lceil_w^{-1}(\gamma)|}}{|\underline{\delta}\lceil_w^{-1}(\gamma)|!} \right) Z^{\underline{\delta}\lceil_w}_{N,(\rho);\Lambda;s}(\varnothing), \qquad (5.3.6)$$

so that, from (5.0.2), for $\underline{\delta} \in \mathscr{D}_0^v$, and $w \subset v$,

$$Z^{\underline{\delta}\lceil_w}_{N,(\rho);\Lambda;s}(\varnothing) = \left(\prod_{\gamma \in \mathscr{D}_0} \frac{\partial^{|\underline{\delta}\lceil_w^{-1}(\gamma)|}}{\partial z_\gamma^{|\underline{\delta}\lceil_w^{-1}(\gamma)|}} \right) Y^{\underline{\delta}}_{N,(\rho);\Lambda;s}[\underline{0}], \qquad (5.3.7)$$

therefore, for $n \in \mathbf{N}$, $\underline{\beta} \in \mathscr{D}_0^{e_n}$, one has from (5.3.1), (5.3.2),

$$S^{\underline{\beta}}_{N,(\rho);\Lambda;s}(f) = \sum_{v \subset e_n} \left(\prod_{\gamma \in \mathscr{D}_0} \frac{\partial^{|\underline{\beta}\lceil_v^{-1}(\gamma)|}}{\partial z_\gamma^{|\underline{\beta}\lceil_v^{-1}(\gamma)|}} \right) \frac{Z^{\underline{\beta}\lceil_{e_n \setminus v}}_{N,(\rho);\Lambda;s}(f)}{Y^{\underline{\beta}\lceil_v}_{N,(\rho);\Lambda;s}[\underline{0}]}; \qquad (5.3.8)$$

(the right hand side of (5.3.8) is meaningful, since

$$Y^{\underline{\delta}}_{N,(\rho);\Lambda;s}[\underline{0}] = Z_{N,(\rho);\Lambda;s}(\varnothing) \neq 0,$$

so that $Y^{\underline{\delta}}_{N,(\rho);\Lambda;s}[\underline{z}] \neq 0$, for $\underline{z}$ in some neighbourhood of $\underline{0}$).

Now, if for $\underline{\delta} \in \mathscr{D}_0^v$, $\gamma \in \underline{\delta}(v)$, one sets

$$R^{\underline{\delta}}(\gamma) = \xi |\underline{\delta}^{-1}(\gamma)|^{-1/2}, \qquad (5.3.9)$$

one can choose the constant ξ sufficiently small so that, if $|z_\gamma| \leqq R^{\underline{\delta}}(\gamma)$, $(\gamma \in \underline{\delta}(v))$, the function $Y^{\underline{\delta}}_{N,(\rho);.;.}[\underline{z}]$ is a partition function —definition (B.3)— to which each function $Z^{\underline{\beta}}_{N,(\rho);.;.}(f)$ is linked —definition (B.4)—, provided m is sufficiently large.

Indeed, first, each function $Z^{\underline{\alpha}}_{N,(\rho);.;.}(\varnothing)$, $(\underline{\alpha} \in \mathscr{D}_0^w)$, is linked to $Z_{N,(\rho);.;.}(\varnothing)$, and one has $|\Sigma_{Z^{\underline{\alpha}}|Z}| \leqq |w|$, therefore, from (5.2.3) and (B.4.2),

$$\left| \frac{Z^{\underline{\alpha}}_{N,(\rho);\Lambda;s}(\varnothing)}{Z_{N,(\rho);\Lambda;s}(\varnothing)} \right| \leqq C^{|w|} \prod_{\gamma \in \underline{\Lambda}_0} (|\underline{\alpha}^{-1}(\gamma)|!)^{3/2},$$

thus, from (5.3.6),

$$\left|\frac{Y^{\underline{\delta}}_{N,(\rho);\gamma;0}[z]-Z_{N,(\rho);\gamma;0}(\varnothing)}{Z_{N,(\rho);\gamma;0}(\varnothing)}\right|\leqq\sum_{j=1}^{|\underline{\delta}^{-1}(\gamma)|}|z_\gamma|^j C^j (j!)^{1/2}\leqq\frac{1}{2},$$

the last inequality if $|z_\gamma|\leqq R^{\underline{\delta}}(\gamma)$, (and ξ sufficiently small), then, as $Z_{N,(\rho);\gamma;0}(\varnothing)$ satisfies P-iv, (written for $K'_0>K_0$), one has

$$|Y^{\underline{\delta}}_{N,(\rho);\gamma;0}[z]|\geqq\frac{1}{2}e^{-K'_0},$$

and the function $Y^{\underline{\delta}}_{N,(\rho);.;.}[z]$ verifies P-iv.

The other conditions P and L are easily checked.

Then, one deduces from (5.2.3) and (B.4.2), for $\underline{\beta}\in\mathscr{D}_0^{e_n}$, $v\subset e_n$, if $|z_\gamma|\leqq R^{\underline{\beta}\lceil_v}(\gamma)$,

$$\left|\frac{Z^{\underline{\beta}\lceil e_n\setminus v}_{N,(\rho);\Lambda;s}(f)}{Y^{\underline{\beta}\lceil_v}_{N,(\rho);\Lambda;s}[0]}\right|\leqq\mathscr{N}(f)C^n\prod_{\gamma\in\underline{\Lambda}_0}(|\underline{\beta}\lceil_{e_n\setminus v}{}^{-1}(\gamma)|!)^{3/2},\qquad(5.3.10)$$

(because

$$\Sigma_{Z^{\underline{\beta}\lceil e_n\setminus v}(f)\,|\,Y^{\underline{\beta}\lceil_v}}\subset\Sigma_{Z^{\underline{\beta}\lceil e_n\setminus v}(f)\,|\,Z(\varnothing)}\cup\Sigma_{Y^{\underline{\beta}\lceil_v}\,|\,Z(\varnothing)},$$

so that $|\Sigma_{Z^{\underline{\beta}\lceil e_n\setminus v}(f)\,|\,Y^{\underline{\beta}\lceil_v}}|\leqq|f|+n$), but, from Cauchy's formula,

$$\left(\prod_{\gamma\in\mathscr{D}_0}\frac{\partial^{|\underline{\beta}\lceil_v{}^{-1}(\gamma)|}}{\partial z_\gamma{}^{|\underline{\beta}\lceil_v{}^{-1}(\gamma)|}}\right)\frac{Z^{\underline{\beta}\lceil e_n\setminus v}_{N,(\rho);\Lambda;s}(f)}{Y^{\underline{\beta}\lceil_v}_{N,(\rho);\Lambda;s}[0]}=\left(\prod_{\gamma\in\underline{\beta}(v)}\frac{|\underline{\beta}\lceil_v{}^{-1}(\gamma)|!}{2i\pi}\right)$$
$$\times\int_{|z_\gamma|=R^{\underline{\beta}\lceil_v(\gamma)}}\frac{Z^{\underline{\beta}\lceil e_n\setminus v}_{N,(\rho);\Lambda;s}(f)}{Y^{\underline{\beta}\lceil_v}_{N,(\rho);\Lambda;s}[z]}\prod_{\gamma\in\underline{\beta}(v)}\frac{dz_\gamma}{z_\gamma{}^{(1+|\underline{\beta}\lceil_v{}^{-1}(\gamma)|)}},\qquad(5.3.11)$$

and one deduces (5.3.5) from (5.3.8), (5.3.9), (5.3.10) and (5.3.11). ■

5.4. Taking the geometric mean of (5.3.4) and (5.3.5), one obtains

$$|S^{\underline{\beta}}_{N,(\rho);\Lambda;s}(f)|\leqq\mathscr{N}(f_\Lambda)C^n(n!)^{1/2}\prod_{\gamma\in\mathscr{D}_0}(|\underline{\beta}^{-1}(\gamma)|!)$$
$$\times e^{-(1/2)D[\Sigma_f,\{\beta_1\},\ldots,\{\beta_n\}]},\qquad(5.4.1)$$

but, according to [2], there exists a constant C_{Σ_f} such that

$$\sum_{\underline{\beta}\in\mathscr{D}_0{}^{e_n}}\left(\prod_{\gamma\in\mathscr{D}_0}|\underline{\beta}^{-1}(\gamma)|!\right)e^{-(1/2)D[\Sigma_f,\{\beta_1\},\ldots,\{\beta_n\}]}\leqq C^n_{\Sigma_f}n!,\qquad(5.4.2)$$

therefore, from (5.3.3) and (5.4.1),

$$\left|\frac{d^n}{d\rho^n}S_{N,(\rho);\Lambda;s}(f)\right|\leqq\mathscr{N}(f_\Lambda)C^n(n!)^{3/2}.\qquad(5.4.3)$$

Now, according to the theorem (B.4), each function $\frac{F^{\beta}_{N,(\rho);\Lambda;s|v}(f)}{Z_{N,(\rho);\Lambda;s}(\varnothing)}$ converges, as $\Lambda \to \mathbf{R}$, uniformly for ρ in any compact set, therefore, from (5.3.2) and (5.3.3), $\frac{d^n}{d\rho^n} S_{N,(\rho);\Lambda}(f)$, converges, uniformly for ρ in any compact set, to the derivative $\frac{d^n}{d\rho^n} S_{N,(\rho)}(f)$, and, from (5.4.3),

$$\left|\frac{d^n}{d\rho^n} S_{N,(\rho)}(f)\right| \leqq \mathcal{N}(f)\, C^n (n!)^{3/2}. \tag{5.4.4}$$

Therefore, with a proof similar to that of theorem (3.3), one has

THEOREM. — *The functions* $S_{N,\lambda}(f)$, *analytic in the domain described in the theorem* (4.5), *are of class* C^∞ *at* $\lambda = 0$ *in any sector*

$$|\operatorname{Arg}\lambda| \leqq \frac{3\pi}{2} - \eta, \ (\eta > 0);$$

their Taylor series are Borel-summable of level 1 *in all directions except that of negative real numbers, and, for any* $\eta > 0$, *there exists* $R_\eta > 0$, $C_\eta > 0$, *such that, if* $|\operatorname{Arg}\lambda| \leqq \frac{3\pi}{2} - \eta$, $|\lambda| \leqq R_\eta$, *then*

$$\left| S_{N,\lambda}(f) - \sum_{k=0}^{n-1} \left(\frac{d^k}{d\lambda^k} S_{N,0}(f) \right) \frac{\lambda^k}{k!} \right| \leqq \mathcal{N}(f)\, C_\eta^n\, n!\, |\lambda|^n. \tag{5.4.5}$$

COROLLARY. — *The Borel transform of level* 1 *of* $\lambda \mapsto S_{N,\lambda}(f)$ *is analytic and exponentially bounded of order* 1 *(uniformly in any sector* $|\operatorname{Arg} z| \leqq \pi - \eta$*), in a cut plane* $\mathbf{C} \setminus]-\infty, -a_N]$; *for* $N = 1$,

$$-a_1 = -\frac{1}{3} m^3 = -\left(2 \sup_{h \in \mathcal{H}^1} \left\{ \|h\|_{L^4} / (\|h\|_{L^2}^2 + \|h'\|_{L^2}^2)^{1/2} \right\}\right)^{-4} m^3$$

is the Lipatov singularity, [15].

A. The phase-space expansion

One just summarises, in a "model independent" version, the main result of the so called "phase-space expansion" ([37]).

([37]) This method, introduced in [5], has been used in one or an other form by different authors: *see* for example [9]. Unfortunately there does not exist any written version to which one can refer precisely.

One omits the proofs, which are more or less directly derived from the litterature; (they are available as preprint).

A.0. Let E be an euclidean space of dimension d, one denotes by $\mathscr{D}_0$ the set of cells of a cubic lattice of spacing l_0, and, for $r \geqq 1$, by $\mathscr{D}_r$ the set of cells of the lattice of spacing $\nu^{-r} l_0$ obtained by subdivision of $\mathscr{D}_{r-1}$, (ν is a fixed integer); one sets

$$\mathscr{D} = \bigcup_{r=0}^{\infty} \mathscr{D}_r, \qquad \mathscr{D}_+ = \bigcup_{r=1}^{\infty} \mathscr{D}_r, \qquad \mathscr{D}^r = \bigcup_{s=0}^{r} \mathscr{D}_s, \qquad \mathscr{D}^r_+ = \bigcup_{s=1}^{r} \mathscr{D}_s.$$

For $\Delta \in \mathscr{D}$, $|\Delta|$ stands for the volume of Δ, (so that $|\Delta| = \nu^{-rd} l_0^d$, if $\Delta \in \mathscr{D}_r$), and, if $\Delta \in \mathscr{D}_r$, ($r \geqq 1$), $\hat{\Delta}$ the unique element of $\mathscr{D}_{r-1}$ such that $\Delta \subset \hat{\Delta}$.

If Λ is a finite union of cells of $\mathscr{D}_0$, one sets $\underline{\Lambda}_r := \{ \Delta \in \mathscr{D}_r;\ \Delta \subset \Lambda \}$, ($r \in \mathbf{N}$), and defines $\underline{\Lambda}$, $\underline{\Lambda}_+$, $\underline{\Lambda}^r$, $\underline{\Lambda}^r_+$, in the same way as above.

One sets $\mathscr{T}^r_\Lambda := [0, 1]^{\underline{\Lambda}^r_+}$, ($r \geqq 1$), and $\mathscr{T}_\Lambda := \bigcup_{r \geqq 1} \mathscr{T}^r_\Lambda \subset [0, 1]^{\underline{\Lambda}_+}$ ([38]);

for $\underline{t} = \{ t_\Delta \}_{\Delta \in \underline{\Lambda}_+} \in \mathscr{T}_\Lambda$ one denotes by $\underline{\Lambda}(\underline{t}) \subset \underline{\Lambda}$ the family defined by

$$\Delta \in \underline{\Lambda}(\underline{t}) \quad \textit{iff} \quad \begin{cases} \forall \Delta_1 \subsetneq \Delta: \quad t_{\Delta_1} = 0, \\ \textit{and, (if } \Delta \in \underline{\Lambda}_+), \\ \exists \Delta_2 \subsetneq \hat{\Delta}: \quad t_{\Delta_2} \neq 0, \end{cases} \tag{A.0.1}$$

then $\underline{\Lambda}(\underline{t})$ is a family of pairwise disjoint cells, the union of which covers Λ.

One the other hand, one sets $D_\Delta = \dfrac{\partial}{\partial t_\Delta}$, and, for $r \geqq 1$, $\Delta \in \underline{\Lambda}^r_+$,

$$\tilde{D}^r_\Delta = \sum_{\Delta_r \in \underline{\Lambda}_r;\ \Delta_r \subset \Delta} D_{\Delta_r}, \tag{A.0.2}$$

and, if $\underline{k} = \{ k^r_\Delta \}_{r \geqq 1, \Delta \in \underline{\Lambda}^r_+}$, is a family of integers, one notes $Q(\underline{k})$ the family of the $\underline{q} = \{ q_\Delta \}_{\Delta \in \underline{\Lambda}_+}$ such that $\prod_{r \geqq 1} \prod_{\Delta \in \underline{\Lambda}^r_+} \tilde{D}^{r\, k^r_\Delta}_\Delta = \sum_{\underline{q} \in Q(\underline{k})} \prod_{\Delta \in \underline{\Lambda}_+} D^{q_\Delta}_\Delta$.

One has

THEOREM. – *Let* $\mathscr{Z} : (\bigcup_{\Lambda \subset \mathscr{D}_0} \mathscr{T}_\Lambda) \to \mathbf{C}$ *a function, the restriction of which to each* $\mathscr{T}^r_\Lambda$ *is of class* C^∞, *one supposes that there exist constants* C_0, C, β, γ *and* $\eta > 0$, *such that, if* $\underline{t} \in \mathscr{T}_\Lambda$ *and* $\underline{q} \in Q(\underline{k})$ *verify* $t_\Delta = 1$ *and* $q_\Delta = 0$, *for any* $\Delta \in \underline{\Lambda}_+$ *which includes strictly some element of* $\underline{\Lambda}(\underline{t})$ ([39]), *one has*

$$\Big|\Big(\prod_{r \geqq 1} \prod_{\Delta \in \underline{\Lambda}^r_+} \tilde{D}^{r\, k^r_\Delta}_\Delta\Big) \mathscr{Z}(\underline{t})\Big| \leqq C_0\, C^{|\underline{\Lambda}(\underline{t})| + |\underline{k}|}$$
$$\times \prod_{\Delta_0 \in \underline{\Lambda}_0} (|\underline{\Lambda}(\underline{t}) \lceil_{\Delta_0}|!)^\beta \Big(\prod_{r \geqq 1} \prod_{\Delta \in \underline{\Lambda}^r_+} [k^r_\Delta !]^\gamma |\Delta_r|^{\eta\, k^r_\Delta}\Big), \tag{A.0.3}$$

([38]) One imbeds naturally $\mathscr{T}^r_\Lambda$ in $[0, 1]^{\Lambda_+}$, ($t_\Delta = 0$ if $\Delta \in \mathscr{D}_s$, $s > r$).

([39]) This condition is a restriction on the hypothesis: it suffices to check (A.0.3), *only* in this case.

where $\underline{\Lambda}(\underline{t})\lceil_{\Delta_0}=\{\Delta\in\underline{\Lambda}(\underline{t});\ \Delta\subset\Delta_0\}$; $|\underline{\Lambda}(\underline{t})|$, [*resp.* $|\underline{\Lambda}(\underline{t})\lceil_{\Delta_0}|$], *is the number of elements of* $\underline{\Lambda}(\underline{t})$, [*resp.* $\underline{\Lambda}(\underline{t})\lceil_{\Delta_0}$]; $|\Delta_r|$ *is the volume of an element of* $\underline{\Lambda}_r$; and $|\underline{k}|=\sum_{r\geq 1}\sum_{\Delta\in\underline{\Lambda}^r_+}k^r_\Delta$; *then*

(i) *there exists a constant* K, *such that, for any* $\Lambda\subset\mathscr{D}_0$, $n\in\mathbf{N}$,

$$|\mathscr{Z}(1_{\underline{\Lambda}^n_+})|\leqq C_0\,e^{K|\Lambda|},\qquad \text{(A.0.4)}$$

where $1^n_{\underline{\Lambda}+}\in\mathscr{T}_\Lambda$ *is the indicatrix function of* $\underline{\Lambda}^n_+$, *and* $|\Lambda|$ *the volume of* Λ;

(ii) *for each* $\Lambda\subset\mathscr{D}_0$, *the sequence* $\{\mathscr{Z}(1^n_{\underline{\Lambda}+})\}_{n\in\mathbf{N}}$ *is convergent; moreover, if a family of functions* $\mathscr{Z}_\alpha$ *satisfies* (A.0.3) *uniformly, then the sequences* $\{\mathscr{Z}_\alpha(1^n_{\underline{\Lambda}+})\}_{n\in\mathbf{N}}$ *are uniformly bounded and converge uniformly.*

B. Summary (40) of the "cluster expansion" of Glimm-Jaffe-Spencer

Let $\mathscr{B}$ and $\mathscr{V}$ be respectively the (countable) set of the lines and the set of the vertices of a connected graph (41) which has the following properties

– each line $b\in\mathscr{B}$ has distincts ends,

– there is at most 1 line between two (distinct) vertices,

– each vertex $\alpha\in\mathscr{V}$ is the end of at most r_0 distinct lines.

B.1. One denotes by $\mathscr{C}$ the set of the parts of $\mathscr{B}$, by $\mathscr{C}_0$ the set of the finite parts, and by $\mathscr{C}_1$ the set of the parts which are either finite or of finite complement.

One says that *a subset* S_* *of* $S=[0,1]^{\mathscr{B}}$ *is stable if* (42).

(i) S_* *contains the constant functions* **0** *and* **1**,

(ii) *if* $s\in S_*$, $t\in S$ *are such that there exists* $\Gamma\in\mathscr{C}_0$, *so that* $1_{\Gamma'}t=1_{\Gamma'}s$, *then* $t\in S_*$, (Γ' is the complement of $\Gamma\subset\mathscr{B}$, and 1_Γ is the indicatrix function of Γ). One notes that, if S_* is stable, and if $s\in S^*$, then $1_\Gamma s$ and $1_{\Gamma'}s$ belong to S_*, for any $\Gamma\in\mathscr{C}_1$.

Let $S_*\subset S$ be some stable subset, for any $b\in\mathscr{B}$, one defines $\delta^{(b)}:\mathbf{C}^{s*}\to\mathbf{C}^{s*}$ by

$$\delta^{(b)}f(s):=f(s)-f(1_{\{b\}'}s),\qquad (f\in\mathbf{C}^{S*},\ s\in S_*),\qquad \text{(B.1.1)}$$

(40) This theory appears first in [4]; the paragraph (B.5) comes from [3]; one can find a "model independent" version (with minor changes, and some oddities), and the proofs collected in [11], appendix A.

(41) In the usual situation – which is also that of this paper – there is given some cubic lattice $\mathscr{D}_0$ in $\mathbf{R}^d$, and $\mathscr{V}=\mathscr{D}_0$ and $\mathscr{B}$ is the set of common faces of two adjacent cubes. Here one choses this more abstract presentation, having in mind situations where, (as in appendix A), there is given a family $\mathscr{D}$ of lattices, with then $\mathscr{V}=\mathscr{D}$, $\mathscr{B}$ containing bounds between adjacent cubes of the same size, but also between Δ and $\hat{\Delta}$.

(42) *See* note (44) below.

[then $(\delta^{(b)})^2=\delta^{(b)}$, $\delta^{(a)}\delta^{(b)}=\delta^{(b)}\delta^{(a)}$, $(a, b\in\mathscr{B})$]; and, for $\Gamma\in\mathscr{C}_0$, $\delta^\Gamma: \mathbf{C}^{S_*}\to\mathbf{C}^{S_*}$ by

$$\delta^\Gamma:=\prod_{b\in\Gamma}\delta^{(b)},\quad \text{if } \Gamma\neq\varnothing;\qquad \delta^{\varnothing}f=f,\quad (f\in\mathbf{C}^{S_*}).\qquad (B.1.2)$$

DEFINITION. – *One says that* $f\in\mathbf{C}^{S_*}$, (*where* S_* *is stable*), *is regular at infinity if*

$$f(s)=\lim_{\Gamma\in\mathscr{C}_0;\,\Gamma\to\mathscr{B}} f(1_\Gamma s),\qquad (s\in S_*),\qquad (B.1.3)$$

B.2. Let $\mathscr{X}$ be the set of the parts of $\mathscr{V}$, and $\mathscr{X}_0$, $(\mathscr{X}_0^{(n)})$, the set of the finite parts, (of the parts with n elements).

For $X\in\mathscr{X}$, one denotes by $X_-\in\mathscr{C}$, $(X_+\in\mathscr{C})$ the set of the lines of $\mathscr{B}$ of which the two ends, (respectively at least one end), belong to X.

If $X\in\mathscr{X}$ and $\Gamma\in\mathscr{C}$ one says that X is Γ-connected if the subgraph $(X, \Gamma\cap X_-)$ is connected, X is said to be connected if it is $\mathscr{B}$-connected, [*i.e.* if (X, X_-) is connected].

DEFINITION. – *One says that* $F\in\mathbf{C}^{\mathscr{X}_0\times S_*}$, (*where* S_* *is stable*), *decouples at* $s=0$ *if, for any* $\Lambda\in\mathscr{X}_0$, $s\in S_*$, $\Gamma\in\mathscr{C}_1$, *and any finite partition* $\{X_i\}_{1\leqq i\leqq n}$ *of* $\mathscr{V}$ *such that each connected component of* $(\mathscr{V}, \Gamma)$ *is included in one of the* (X_i, X_{i-}), *one has*

$$F(\Lambda, 1_\Gamma s)=\prod_{i=1}^{n} F(\Lambda\cap X_i, 1_{\Gamma\cap X_{i-}} s).\qquad (B.2.1)$$

B.3. Let $S_*\subset S$ be some stable subset, one sets

DEFINITION. – *One says that* $Z\in\mathbf{C}^{\mathscr{X}_0\times S_*}$ *is a partition function if*

P-i Z *decouples at* $s=0$,

P-ii *for any* $\Lambda\in\mathscr{X}_0$, $Z(\Lambda, .)\in\mathbf{C}^{S_*}$ *is regular at infinity*,

P-iii *for any* $s\in S_*$, $Z(\varnothing, s)=1$,

P-iv *there exists a constant* $K_0\geqq 0$ *such that* $|Z(\Delta, 0)|\geqq e^{-K_0}$, *for any* $\Delta\in\mathscr{X}_0^{(1)}$,

P-v *there exist constants* $K_1\geqq 0$, $K_2\geqq 2(K_0+K_1+3r_0)$, *such that*

$$|\delta^\Gamma Z(\Lambda, 1_\Gamma s)|\leqq e^{K_1|\Lambda|-K_2|\Gamma|},\qquad (\Lambda\in\mathscr{X}_0, \Gamma\in\mathscr{C}_0, s\in S_*),\quad (B.3.1)$$

P-vi *for any* $\Lambda\in\mathscr{X}_0$, *there exists* $a_\Lambda>0$ *such that* $|Z(\Lambda, s)|\leqq a_\Lambda$, $(s\in S_*)$.

THEOREM. – *Let* $Z\in\mathbf{C}^{\mathscr{X}_0\times S_*}$ *be a partition function, then*

$$|Z(\Lambda, s)|\geqq\frac{1}{4}e^{-(K_0+r_0)|\Lambda|}>0,\qquad (\Lambda\in\mathscr{X}_0, s\in S_*).\qquad (B.3.2)$$

B.4. Given a partition function $Z\in\mathbf{C}^{\mathscr{X}_0\times S_*}$, one sets

DEFINITION. – *One says that a function* $Z\in\mathbf{C}^{\mathscr{X}_0\times S_*}$ *is linked to* Z *if*

L-i F *decouples at* $s=0$,

L-ii *for any* $\Lambda \in \mathscr{X}_0$, $F(\Lambda, .) \in \mathbf{C}^{S_*}$ *is regular at infinity,*

L-iii *there exists* $\Sigma_{F|Z} \in \mathscr{X}_0$ *such that* $F(\Lambda, s) = Z(\Lambda, s)$, $(s \in S_*)$, *if* $\Lambda \cap \Sigma_{F|Z} = \varnothing$,

L-iv *there exist constants* $C > 0$, $K_1 \geqq 0$, $K_2 \geqq 2\,(K_0 + K_1 + 3\,r_0)$, *such that*

$$|\delta^\Gamma F(\Lambda, 1_\Gamma s)| \leqq C e^{K_1|\Lambda| - K_2|\Gamma|}, \qquad (\Lambda \in \mathscr{X}_0, \Gamma \in \mathscr{C}_0, s \in S_*), \quad (B.4.1)$$

and (B.3.1) *are simultaneously verified.*

One has

THEOREM. – *Let Z be a partition function, and F a function linked to Z, then*

(i) *one has*

$$\left|\frac{F(\Lambda, s)}{Z(\Lambda, s)}\right| \leqq C e^{(K_0 + K_1 + 3 r_0)|\Sigma_{F|Z}|}, \qquad (\Lambda \in \mathscr{X}_0, s \in S_*), \quad (B.4.2)$$

(ii) *the family* $\left\{\frac{F(\Lambda, s)}{Z(\Lambda, s)}\right\}_{\Lambda \in \mathscr{X}_0}$ *converges as* $\Lambda \to \mathscr{V}$, *uniformly with respect to* $s \in S_*$.

Moreover, if $\{Z_i\}_{i \in I}$ *is a family of partition functions, and if, for each* $i \in I$, F_i *is a function linked to* Z_i, *such that* P-iv, P-v, L-iv *are verified with constants independent of* $i \in I$, *and that there exists* $X \in \mathscr{X}_0$ *such that* $\Sigma_{F_i|Z_i} \subset X$, *for all* $i \in I$, *then the convergence is uniform with respect to* $i \in I$.

B.5. One supposes that is given a family $\{Y_j \in \mathscr{X}_0\}_{1 \leqq j \leqq n}$, one sets

$$\mathbf{D}[\{Y_j\}_{1 \leqq j \leqq n}] := = \min\left\{\left|X \setminus \left(\bigcup_{j=1}^{n} Y_j\right)\right|; X \in \mathscr{X}_0, \left(\bigcup_{j=1}^{n} Y_j\right) \subset X, \text{ X is "connected mod. } \{Y_j\}_{1 \leqq j \leqq n}\text{"}\right\}, \quad (B.5.1)$$

where X is "connected mod. $\{Y_j\}_{1 \leqq j \leqq n}$" means that, for any decomposition $X = X_1 \cup X_2$ of X into two nonempty unions of connected parts, there exists some $j \in \{1, \ldots, n\}$ such that $Y_j \cap X_1 \neq \varnothing$ and $Y_j \cap X_2 \neq \varnothing$; (if the Y_j's are connected, then X is connected).

On the other hand, let $Z \in \mathbf{C}^{\mathscr{X}_0 \times S_*}$ be a partition function, and, for each $u \subset e_n := \{1, \ldots, n\}$, let F_u be a function linked to Z, such that

T-i *for each* $u \subset e_n$, $\Sigma_{F_u|Z} = Y_u := \bigcup_{j \in u} Y_j$,

T-ii *if* $u \subset v \subset e_n$, *then* $F_v(\Lambda, s) = F_u(\Lambda, s)$, $(s \in S_*)$, *if* $\Lambda \cap Y_{v \setminus u} = \varnothing$.

One sets, for each $u \subset e_n$, $S_u := \frac{F_u}{Z}$, and

$$S^T_{e_n} := \sum_{P \in \mathscr{P}(e_n)} (-1)^{|P|-1} (|P|-1)! \prod_{u \in P} S_u, \qquad (B.5.2)$$

where $\mathscr{P}(e_n)$ is the set of partitions of e_n. Then

THEOREM. – *With the hypothesis above, one has for any* $\Lambda \in \mathscr{X}_0$, $s \in S_*$,

$$|S^T_{e_n}(\Lambda, s)| \leqq \Big(\sum_{P \in \mathscr{P}(e_n)} (|P|-1)! \prod_{u \in P} C_u\Big) \times e^{(K_0+K_1+3r_0)\sum_{j=1}^{n} |Y_j|} e^{-D[\{Y_j\}_{1 \leqq j \leqq n}]}, \quad (B.5.3)$$

[*where* C_u *is the constant which occurs in* (B.4.1), *written down for* F_u].

C. The coefficients $H^{(k)}(s \mid \alpha; \beta_1, \ldots, \beta_k)$

C.1. One now introduces some natural coefficients ([43]) which can be helpfull in concrete applications of the theory summarized in appendix B, (one uses the notations of B).

First, for $\Gamma \in \mathscr{C}$ and $s \in S$, one sets

$$A_\Gamma(s) := \prod_{a \in \Gamma} s_a \prod_{b \in \Gamma'} (1-s_b), \qquad (C.1.1)$$

then $S_0 := \{ s \in S;\ \sum_{\Gamma \in \mathscr{C}} A_\Gamma(s) = 1 \}$, is stable.

Next, for α, $\beta \in \mathscr{V}$ and $\Gamma \in \mathscr{C}$, one denotes by $\mathscr{X}_\Gamma(\alpha, \beta)$ the set of the $X \in \mathscr{X}_0$ which are Γ-connected and contain α and β; then, according to ([4], p. 219), one has $|\{X \in \mathscr{X}_\Gamma(\alpha, \beta);\ |X| = p\}| \leqq r_0^{2(p-1)}$, so that, if $\tau > 2 \log r_0$, the series

$$G_\Gamma(\alpha, \beta) := \sum_{X \in \mathscr{X}_\Gamma(\alpha, \beta)} e^{-\tau(|X|-1)} \qquad (C.1.2)$$

is convergent, (of course $G_\Gamma(\alpha, \beta) = 0$ if $\mathscr{X}_\Gamma(\alpha, \beta) = \varnothing$); one notes $G(\alpha, \beta) := G_{\mathscr{B}}(\alpha, \beta)$.

Then one defines

$$\tilde{H}^{(1)}_\Gamma(\alpha; \beta) := \frac{G_\Gamma(\alpha, \beta)}{G(\alpha, \beta)}, \qquad (\Gamma \in \mathscr{C};\ \alpha,\ \beta \in \mathscr{V}), \qquad (C.1.3)$$

then, for each integer k,

$$\tilde{H}^{(k)}_\Gamma(\alpha; \beta_1, \ldots, \beta_k) := \prod_{j=1}^{k} \tilde{H}^{(1)}_\Gamma(\alpha; \beta_j), \qquad (\Gamma \in \mathscr{C};\ \alpha,\ \beta_j \in \mathscr{V}), \quad (C.1.4)$$

([43]) The useful fact is the existence of coefficients having the suitable properties, not the particular realization.

and last

$$\left.\begin{aligned} H^{(k)}(s \mid \alpha; \beta_1, \ldots, \beta_k) := \sum_{\Gamma \in \mathscr{C}} A_\Gamma(s)\, \tilde{H}_\Gamma^{(k)}(\alpha; \beta_1, \ldots, \beta_k), \\ (s \in S_0;\ \alpha, \beta_j \in \mathscr{V}). \end{aligned}\right\} \quad (C.1.5)$$

One has

$$0 \leqq H^{(k)}(s \mid \alpha; \beta_1, \ldots, \beta_k) \leqq 1, \quad (C.1.6)$$

$$H^{(k)}(1_{\mathscr{B}} \mid \alpha; \beta_1, \ldots, \beta_k) = 1. \quad (C.1.7)$$

C.2. First one has

LEMMA. – *For any* $\alpha, \beta_j \in \mathscr{V}$, *the function* $s \mapsto H^{(k)}(s \mid \alpha; \beta_1, \ldots, \beta_k)$, $(s \in S_0)$, *is regular at infinity* ([44]).

Proof. – From [11], lemma (B.1), if $f(s) = \sum_{\Gamma \in \mathscr{C}} A_\Gamma(s) \tilde{f}_\Gamma$, $(s \in S_0)$, with $\tilde{f}$ some bounded function defined on $\mathscr{C}$, then f is regular at infinity if and only if $\tilde{f}_\Gamma = \lim_{C \in \mathscr{C}_0;\, C \to \mathscr{B}} \tilde{f}_{\Gamma \cap C}$, for all $\Gamma \in \mathscr{C}$.

Therefore the lemma rests on the fact that $\mathscr{X}_\Gamma(\alpha, \beta) = \bigcup_{C \in \mathscr{C}_0} \mathscr{X}_{\Gamma \cap C}(\alpha, \beta)$, since the series (C.1.2) is convergent. ■

Next, for $\Gamma \in \mathscr{C}$ one defines $d_\Gamma : \mathscr{V} \times \mathscr{V} \to \bar{\mathbb{N}}$ by

$$\left.\begin{aligned} d_\Gamma(\alpha, \beta) := \min\{(|X| - 1);\ X \in \mathscr{X}_0,\ \alpha, \beta \in X,\ X \text{ is } \Gamma\text{-connected}\}, \\ (\alpha, \beta \in \mathscr{V}), \end{aligned}\right\} \quad (C.2.1)$$

with $d_\Gamma(\alpha, \beta) := +\infty$, if the set of the right hand side is empty; one notes $d = d_{\mathscr{B}}$; then, for $\alpha \in \mathscr{V}$ and $b \in \mathscr{B}$, one sets

$$D(\alpha, \{b\}) := \min\{|X|;\ X \in \mathscr{X}_0,\ \alpha \in X,\ b \in X_+,\ X \text{ is connected}\}, \quad (C.2.2)$$

and, for $\Gamma \in \mathscr{C}_0$,

$$D(\alpha, \Gamma) := \max_{b \in \Gamma} D(\alpha, \{b\}), \quad \text{if } \Gamma \neq \varnothing; \qquad D(\alpha, \varnothing) := 0; \quad (C.2.3)$$

and on the other hand, for $\Gamma \in \mathscr{C}_0$,

$$L(\Gamma) := \min\{(|X| - 1);\ X \in \mathscr{X}_0,\ X_- \supset \Gamma,\ X \text{ is connected}\}, \text{ if } \Gamma \neq \varnothing;$$
$$L(\varnothing) := 0. \quad (C.2.4)$$

Then, if one sets $\partial^{(b)} = \dfrac{\partial}{\partial s_b}$, $(b \in \mathscr{B})$, and for $\Gamma \in \mathscr{C}_0$, $\partial^\Gamma = \prod_{b \in \mathscr{C}_0} \partial^{(b)}$ if $\Gamma \neq \varnothing$, $(\partial^\varnothing = I)$,

([44]) This property fails if one replaces S_0 by S; here is the main reason to introduce stable subsets of S.

THEOREM. — *There exists a constant* M *such that, if* $\eta', \eta''>0$ *and* $\tau \geqq \tau_0 > \eta' + \eta'' + 2\log r_0$, *one has for any* $C \in \mathscr{C}_0$; $\alpha, \beta_j \in \mathscr{V}$,

$$|\partial^C H^{(k)}(s\,|\,\alpha;\beta_1,\ldots,\beta_k)| \leqq M\,4^{k|C|} e^{-r_0^{-1}(\tau - 2\log r_0 - \eta' - \eta'')|C| - \eta' L(C) - \eta'' D(\alpha, C)} e^{\tau \sum_{j=1}^{k} d(\alpha,\beta_j)}. \quad (C.2.5)$$

Proof. — From [11], lemma (B.2), if $\mathcal{J}^{(j)}$, $1 \leqq j \leqq k$, are k bounded functions defined on $\mathscr{C}$, if $\mathcal{J}_\Gamma = \prod_{j=1}^{k} \mathcal{J}_\Gamma^{(j)}$, and if $f^{(j)}(s) = \sum_{\Gamma\in\mathscr{C}} A_\Gamma(s)\,\mathcal{J}_\Gamma^{(j)}$ and $f(s) = \sum_{\Gamma\in\mathscr{C}} A_\Gamma(s)\,\mathcal{J}_\Gamma$, one has, for $C \in \mathscr{C}_0$,

$$\partial^C f(s) = \sum_{\Gamma\in\mathscr{C};\,\Gamma\supset C} A_\Gamma(s \vee 1_C) \times \left(\sum_{\{C_j\}_{1\leqq j\leqq k};\ \bigcup_{j=1}^{k} C_j = C} \ \prod_{j=1}^{k} \delta^{C_j} f^{(j)}(1_{(\Gamma\setminus C)\cup C_j}) \right), \quad (C.2.6)$$

where the sum over the C_j's contains $(2^k-1)^{|C|}$ terms ([45]).

Now let $f^{(j)}(s) = \sum_{\Gamma\in\mathscr{C}} A_\Gamma(s)\,G_\Gamma(\alpha,\beta_j)$, $C_j \in \mathscr{C}_0$, and $\Gamma\in\mathscr{C}$ such that $\Gamma \supset C_j$, one has

$$\partial^{C_j} f^{(j)}(1_\Gamma) = \sum_{B\subset C_j} (-1)^{|B|} G_{\Gamma\setminus B}(\alpha,\beta_j); \quad (C.2.7)$$

then if $X \in \mathscr{X}_\Gamma(\alpha,\beta_j)$ is such that $C_j \not\subset X_-$, $(C_j \neq \varnothing)$, let $b \in C_j \setminus X_-$, for any $B \subset C_j$ containing b, one has $X \in \mathscr{X}_{\Gamma\setminus B}(\alpha,\beta_j)$ if and only if $X \in \mathscr{X}_{\Gamma\setminus(B\setminus\{b\})}(\alpha,\beta_j)$; therefore the term corresponding to X either does not appear in $G_{\Gamma\setminus B}(\alpha,\beta_j)$ neither in $G_{\Gamma\setminus(B\setminus\{b\})}(\alpha,\beta_j)$, or it appears in both, with opposite signs; in all cases it disappears in $\partial^{C_j} f^{(j)}(1_\Gamma)$ which consequently is a sum of terms of the form $e^{-\tau(|X|-1)}$, where $X_- \supset C_j$.

Let $X_j \in \mathscr{X}_\Gamma(\alpha,\beta_j)$ be such that $X_{j-} \supset C_j$, and $X = \bigcup_{j=1}^{k} X_j$; then $X_- \supset C$, X is connected, since the X_j's are connected and contain α, and $\sum_{j=1}^{k} |X_j| - 1 = \sum_{j=1}^{k} |X_j \setminus \{\alpha\}| = |X \setminus \{\alpha\}| = |X| - 1$; therefore one has

(i) $|X| - 1 \geqq D(\alpha, C)$, since $\alpha \in X$ and $C \subset X_-$,

(ii) $|X| - 1 \geqq L(C)$, since $C \subset X_-$,

(iii) $|X| - 1 \geqq |C|/r_0$, since $|X| \geqq 2|C|/r_0$, and, either $|C|/r_0 \geqq 1$, or $|C|/r_0 < 1$ and $|X| \geqq 2$, (if $|X| = 1$, $X_- = \varnothing$ and $|C| = 0$).

([45]) The C_j's *are not* pairwise disjoint.

Therefore, from (C.2.6), (C.2.7), and due to $\sum_{\Gamma \in \mathscr{C};\, \Gamma \supset C} A_\Gamma(s) \leqq 1$, one obtains

$$|\partial^C f(s)| \leqq M\, e^{-r_0^{-1}(\tau - 2\log r_0 - \eta' - \eta'')\,|C| - \eta' L(C) - \eta'' D(\alpha, C)} \times \sum_{\{C_j\}_{1 \leqq j \leqq k};\ \bigcup_{j=1}^{k} C_j = C} \ \prod_{j=1}^{k} 2^{|C_j|}, \qquad (C.2.8)$$

with $M = \sum_{p=0}^{\infty} e^{-(\tau_0 - 2\log r_0)\, p}$;

then the sum contains less than $2^{k|C|}$ terms, one has $\sum_{j=1}^{k} |C_j| \leqq k\,|C|$, and last, from (C.2.1), $G(\alpha, \beta_j) \geqq e^{-\tau d(\alpha, \beta_j)}$, therefore one deduces (C.2.5) from (C.2.8). ■

C.3. The factor $e^{-\tau r_0^{-1}|C|}$ of (C.2.5) is used to give a "small" factor, the factor $e^{-\eta' L(C)}$ to sum over partitions, and the factor $e^{-\eta'' D(\alpha, C)}$ to dominate factorials, according to the following propositions.

PROPOSITION 1 ([46]). – *There exists a constant* K *such that, for any* $\eta' \geqq 2 r_0$

$$\sum_{P \in \mathscr{P}(\Gamma)} \prod_{C \in P} e^{-\eta' L(C)} \leqq e^{K|\Gamma|}, \qquad (C.3.1)$$

where $\mathscr{P}(\Gamma)$ *is the set of partitions of* Γ.

Proof. – First, $\sum_{P \in \mathscr{P}(\Gamma)} \prod_{C \in P} e^{-\eta' L(C)} \leqq \prod_{C \in P} (1 + e^{-\eta' L(C)}) \leqq e^{\sum_{C \subset \Gamma} e^{-\eta' L(C)}}$.

Next, for each integer p, one has

$$|\{C;\ C \subset \Gamma,\ L(C) = p\}| \leqq 2\,|\Gamma|\, r_0^{2(p+1)}\, 2^{r_0 (p+1)}, \qquad (C.3.2)$$

indeed, let $C \subset \Gamma$ be such that $L(C) = p$, there exists some connected $X \in \mathscr{X}_0$, such that $|X| = p+1$ and $C \subset X_-$ and there exists $\delta \in \mathscr{V}$, one end of some line of Γ, such that $\delta \in X$; but there are at most $2|\Gamma|$ elements of $\mathscr{V}$ which are one end of some line of Γ, given δ one of these points, there are at most $r_0^{2(p+1)}$ connected sets X such that $|X| = p+1$ and $\delta \in X$, and last, for such an X, $|X_-| \leqq r_0 (p+1)$ so that $|\{C \in \mathscr{C}_0;\ C \subset X_-\}| \leqq 2^{r_0(p+1)}$.

Then, from (C.3.2), if η' is sufficiently large,

$$\sum_{C \subset \Gamma} e^{-\eta' L(C)} \leqq 2\,|\Gamma| \sum_{p=1}^{\infty} r_0^{2(p+1)}\, 2^{r_0(p+1)}\, e^{-\eta' p} := K\,|\Gamma|. \qquad ■$$

([46]) This proposition is sketched from [4], proposition (8.2).

PROPOSITION 2 [47]. – *Given* $\nu>0$, *if* $\eta''>(\nu+1)\log r_0$ [48], *there exists a constant* K *such that, for any* $\Gamma\in\mathscr{C}_0$ *and* $P\in\mathscr{P}(\Gamma)$, *one has*

$$\sum_{\underline{\alpha}\in\mathscr{V}^P}\prod_{C\in P}e^{-\eta''D(\alpha_C,C)}\prod_{\beta\in\mathscr{V}}(|\alpha^{-1}(\beta)|!)^{\nu}\leqq e^{K|\Gamma|}. \qquad (C.3.3)$$

Proof. – One has

$$\sum_{\underline{\alpha}\in\mathscr{V}^P}\prod_{C\in P}e^{-\eta''D(\alpha_C,C)}\prod(|\alpha^{-1}(\beta)|!)^{\nu}$$
$$\leqq\Big(\prod_{C\in P}\sum_{\gamma\in\mathscr{V}}e^{-(\eta''/(\nu+1))D(\gamma,C)}\Big)\Big(\sup_{\underline{\alpha}\in\mathscr{V}^P}\prod_{\beta\in\mathscr{V};\,\underline{\alpha}^{-1}(\beta)\neq\varnothing}\phi(P,\underline{\alpha},\beta)\Big),$$

with $\phi(P,\underline{\alpha},\beta)=e^{-(\nu\eta''/(\nu+1))\sum_{C\in\underline{\alpha}^{-1}(\beta)}D(\beta,C)}(|\alpha^{-1}(\beta)|!)^{\nu}$.

Then, first, from (C.2.3), $D(\gamma,C)\geqq D(\gamma,\{b\})$, for all $b\in C$; but for all $b\in\mathscr{B}$, one has $|\{\gamma\in\mathscr{V};\,D(\gamma,\{b\})=p\}|\leqq 2r_0^{(p-1)}$, thus

$$\sup_{b\in\mathscr{B}}\sum_{\gamma\in\mathscr{V}}e^{-(\eta''/(\nu+1))D(\gamma,\{b\})}\leqq\sum_{p=1}^{\infty}2r_0^{(p-1)}e^{-(\eta''/(\nu+1))p}:=e^{K'},$$

and therefore $\prod_{C\in P}\sum_{\gamma\in\mathscr{V}}e^{-(\eta''/(\nu+1))D(\gamma,C)}\leqq e^{K'|P|}\leqq e^{K'|\Gamma|}$.

Next, let $\varepsilon>0$ be such that $(1-\varepsilon)\eta''>(\nu+1)\log r_0$; given $\underline{\alpha}\in\mathscr{V}^P$ and $\beta\in\mathscr{V}$, one can suppose that $\varepsilon|\underline{\alpha}^{-1}(\beta)|\geqq 1$, [otherwise $\phi(P,\underline{\alpha},\beta)\leqq\varepsilon^{-\nu/\varepsilon}:\leqq e^{K''}$], and one denotes by n the largest integer such that $r_0^n\leqq\varepsilon|\underline{\alpha}^{-1}(\beta)|$, so that, on the one hand,

$$n>\frac{\log|\underline{\alpha}^{-1}(\beta)|}{\log r_0}-\Big(\frac{|\log\varepsilon|}{\log r_0}+1\Big),$$

and on the other hand, since $|\{b\in\mathscr{B};\,D(\gamma,\{b\})<n\}|\leqq r_0^n$, one has

$$|\{C\in\underline{\alpha}^{-1}(\beta);\,D(\beta,C)<n\}|\leqq r_0^n\leqq\varepsilon|\underline{\alpha}^{-1}(\beta)|,$$

thus $|\{C\in\underline{\alpha}^{-1}(\beta);\,D(\beta,C)\geqq n\}|\geqq(1-\varepsilon)|\underline{\alpha}^{-1}(\beta)|$, therefore

$$\phi(P,\underline{\alpha},\beta)\leqq e^{-(\nu\eta''/(\nu+1))(1-\varepsilon)|\underline{\alpha}^{-1}(\beta)|n}(|\underline{\alpha}^{-1}(\beta)|!)^{\nu}$$
$$\leqq e^{-\nu((1-\varepsilon)\eta''/(\nu+1)\log r_0)-1)|\underline{\alpha}^{-1}(\beta)|\log|\underline{\alpha}^{-1}(\beta)|}$$
$$\times e^{((1-\varepsilon)\nu\eta''/(\nu+1))((|\log\varepsilon|/\log r_0)+1)|\underline{\alpha}^{-1}(\beta)|}$$
$$\leqq\sup_{u\geqq 1}e^{-\nu((1-\varepsilon)\eta''/(\nu+1)\log r_0-1)u\log u+((1-\varepsilon)\nu\eta''/(\nu+1))((|\log\varepsilon|/\log r_0)+1)u}:\leqq e^{K''};$$

but $|\{\beta\in\mathscr{V};\,|\underline{\alpha}^{-1}(\beta)|\neq 0\}|\leqq|P|$, therefore

$$\sup_{\underline{\alpha}\in\mathscr{V}^P}\prod_{\beta\in\mathscr{V};\,\underline{\alpha}^{-1}(\beta)\neq\varnothing}\phi(P,\underline{\alpha},\beta)\leqq e^{K''|P|}\leqq e^{K''|\Gamma|}. \qquad ■$$

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[47] This proposition is an adaptation of [4], lemma 10.2.

[48] In many usual cases, $|\{b\in\mathscr{B};\,D(\gamma,\{b\})<p\}|$ and $|\{\gamma\in\mathscr{V};\,D(\gamma,\{b\})=p\}|$ are polynomialy bounded with respect to p, (in place of the general exponential bounds used here), then it suffices to suppose $\eta''>0$.

One of the useful consequences is

COROLLARY. — *For any* $K>0$, *one has for sufficiently large* τ (49),

$$\left|\partial^{\Gamma}\left(\prod_{i=1}^{n} H^{(k)}(s\,|\,\alpha^{i};\ \beta_{1}^{i},\ \ldots,\ \beta_{k}^{i})\right)\right| \leqq 2^{n} e^{K|\Gamma|} \prod_{i=1}^{n} e^{\tau \sum_{j=1}^{k} d(\alpha^{i},\, \beta_{j}^{i})}, \quad (C.3.4)$$

for any $\Gamma \in \mathscr{C}_0$, $n \in \mathbf{N}$, *and* α^i, $\beta_j^i \in \mathscr{V}$, $(1 \leqq i \leqq n,\ 1 \leqq j \leqq k)$.

Proof. — One has

$$\partial^{\Gamma}\left(\prod_{i=1}^{n} H^{(k)}(s\,|\,\alpha^{i};\ \beta_{1}^{i},\ \ldots,\ \beta_{k}^{i})\right)$$

$$=\left(\sum_{P \in \mathscr{P}(\Gamma)} \sum_{\varphi \in I(P,\, e_n)} \prod_{C \in P} \partial^{C} H^{(k)}(s\,|\,\alpha^{\varphi(C)};\ \beta_{1}^{\varphi(C)},\ \ldots,\ \beta_{k}^{\varphi(C)})\right)$$

$$\times\left(\prod_{l \in \varphi(P)'} H^{(k)}(s\,|\,\alpha^{l};\ \beta_{1}^{l},\ \ldots,\ \beta_{k}^{l})\right),$$

where $I(P, e_n)$ is the set of the injections of P into $e_n = \{1, \ldots, n\}$; but if, for $\gamma \in \mathscr{V}$, $n(\gamma) := |\{i \in e_n;\ \alpha_i = \gamma\}|$, one has, for any $\underline{\delta} \in \mathscr{V}^{P}$,

$$|\{\varphi \in I(P, e_n);\ \alpha^{\varphi(C)} = \delta^{C},\ (C \in P)\}|$$

$$= \prod_{\gamma \in \mathscr{V}} \binom{n(\gamma)}{|\underline{\delta}^{-1}(\gamma)|} |\underline{\delta}^{-1}(\gamma)|! \leqq \prod_{\gamma \in \mathscr{V}} 2^{n(\gamma)} |\underline{\delta}^{-1}(\gamma)|! = 2^{n} \prod_{\gamma \in \mathscr{V}} |\underline{\delta}^{-1}(\gamma)|!,$$

therefore

$$\sum_{\varphi \in I(P,\, e_n)} \prod_{C \in P} e^{-\eta D(\alpha^{\varphi(C)},\, C)} \leqq 2^{n} \sum_{\underline{\delta} \in \mathscr{V}^{P}} \prod_{C \in P} e^{-\eta D(\delta^{C},\, C)} \prod_{\gamma \in \mathscr{V}} |\underline{\delta}^{-1}(\gamma)|!,$$

and one deduces (C.3.4) from (C.2.5), (C.3.1) and (C.3.3). ■

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