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# A generating function for fatgraphs

by

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ABSTRACT. — We study a generating function for the sum over fatgraphs with specified valences of vertices and faces, inversely weighted by the order of their symmetry group. A compact expression is found for general (*i. e.* nor necessarily connected) fatgraphs. This expression admits a matrix integral representation which enables to perform semi-classical computations, leading in particular to a closed formula corresponding to (genus zero, connected) trees.

RÉSUMÉ. — Nous étudions une fonction génératrice pour la somme des graphes épais avec des valences spécifiées aux faces et aux vertex et un poids inverse de l'ordre du groupe de symétrie. Nous trouvons une expression compacte pour les graphes épais généraux (*i.e.* pas nécessairement connexes). Cette expression admet une représentation comme intégrale de matrices, ce qui permet de faire un calcul semi-classique qui conduit en particulier à une formule fermée pour les arbres (genre zéro, connexes).

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**1.** The relation between fatgraphs describing finite cellular decompositions of compact orientable surfaces and arithmetic curves originates in a theorem of Belyi [1] and is described in detail in recent reviews ([2], [3], [4]). It raises the difficult but interesting problem of understanding the action of the Galois group of $\bar{\mathbb{Q}}$ over $\mathbb{Q}$ on fatgraphs. The fatgraphs arise naturally in the Feynman diagrammatic expansion of random matrix models. They consist of ribbons bordered by parallel double lines, joining at vertices and defining faces on a compact orientable surface of given genus. The action of the group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ preserves the numbers $S_v$, $F_v$ of vertices and faces of valency $v$ for each $v \geqq 1$, satisfying

$$\sum v\, S_v = \sum v\, F_v = 2\,A \tag{1}$$

with A the number of edges. For convenience of notation we write $\underline{S}$, $\underline{F}$ for sequences $\{S_v\}$, $\{F_v\}$ satisfying the above relation. The fatgraphs can be built as in a lego game with basic element a half edge (there are 2 A of them). In the following, we reinterpret the sequences $\underline{S}$ and $\underline{F}$ as conjugacy classes of the group of permutations of the 2 A half edges, $\Sigma_{2A}$. Each fat graph (connected or not) admits an automorphism group H or order $h$ ($h$ is a divisor of 2 A in the connected case) as well as a cartographic group (of order a multiple of 2 A in the connected case). The latter is generated by three permutations $\sigma_0$, $\sigma_1$, $\sigma_2$ in $\Sigma_{2A}$, satisfying

$$\sigma_2\,\sigma_1\,\sigma_0 = \mathrm{id}$$

and belonging respectively to the classes $\underline{S}$, $[2^A]$ and $\underline{F}$. The element $\sigma_0$ permutes circularly the half edges around each vertex, $\sigma_1$ interchanges half edges and $\sigma_2$ permutes circularly half edges around faces. Up to overall conjugacy in $\Sigma_{2A}$, the automorphism group H of a fatgraph is the commutant of $\{\sigma_0, \sigma_1, \sigma_2\}$ in $\Sigma_{2A}$.

Let us look for a generating function for the quantities

$$z(\underline{S}, \underline{F}) = \sum_{\mathcal{G}_{\underline{S}, \underline{F}}} \frac{1}{h(\mathcal{G}_{\underline{S}, \underline{F}})} \tag{2}$$

where the sum on the r.h.s. runs over all fatgraphs with given assignments $\underline{S}$ and $\underline{F}$ of vertices and faces subject to the constraints (1), and $h$ is the order of the automorphism group of the fatgraph. If the sum is taken only over *connected* fatgraphs we define a similar and more relevant quantity

$$f(\underline{S}, \underline{F}) = \sum_{\mathcal{G}_{\underline{S}, \underline{F}}\ \text{connected}} \frac{1}{h(\mathcal{G}_{\underline{S}, \underline{F}})} \tag{3}$$

This definition makes clear the following criterion. Given a connected fatgraph or equivalently a finite cellular decomposition of an orientable compact connected surface with specified $\underline{S}$, $\underline{F}$ and order $h$ of its automorphism group, the equality

$$hf(\underline{S}, \underline{F}) = 1$$

ensures that the corresponding curve is defined on $\mathbb{Q}$. This criterion is sufficient but by no means necessary. More generally the number of conjugates of the curve (under $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$) is bounded by $hf(\underline{S}, \underline{F})$.

**2.** To compute the quantity $z(\underline{S}, \underline{F})$ defined in (2), we consider the following problem: count the number $N_{\underline{S}, \underline{F}}$ of triplets of elements $\sigma_0$, $\sigma_1$, $\sigma_2$ in $\Sigma_{2A}$, such that

(i) $\sigma_2 \sigma_1 \sigma_0 = \mathrm{id}$

(ii) $\sigma_0 \in \underline{S}$, $\sigma_1 \in [2^A]$, $\sigma_2 \in \underline{F}$.

This is a particular case of the following situation. Let $C_i$ be the conjugacy classes in a finite group $\Gamma$, $r$ the irreducible representations of $\Gamma$ over $\mathbb{C}$ forming the dual $\tilde{\Gamma}$ of the group, $\chi_r(C)$ the value of the irreducible character in the representation $r$ over the class C and finally $\dim_r \equiv \chi_r(\mathrm{id})$ the dimension of $r$. Set

$$N_{C_2, C_1, C_0} = \begin{cases} \text{number of triplets } (g_2, g_1, g_0) \text{ such that} \\ \qquad g_i \in C_i \quad \text{and} \quad g_2 g_1 g_0 = 1. \end{cases}$$

LEMMA 1 (Frobenius):

$$N_{C_2, C_1, C_0} = \frac{|C_2|\,|C_1|\,|C_0|}{|\Gamma|} \sum_{r \in \tilde{\Gamma}} \frac{\chi_r(C_2)\,\chi_r(C_1)\,\chi_r(C_0)}{\dim_r}$$

*Remark.* – These non negative integers are directly related to the structure constants of the (commutative) algebra of classes

$$C_2 C_1 = \sum_{C_0} N_{C_2, C_1, \bar{C}_0} C_0$$

where $\bar{C}_0 \equiv \{ g \in \Gamma \,|\, g^{-1} \in C_0 \}$.

If $|\underline{S}|$, $|[2^A]|$ and $|\underline{F}|$ stand respectively for the number of elements in the classes $\underline{S}$, $[2^A]$ and $\underline{F}$, we have

$$N_{\underline{S}, \underline{F}} \equiv N_{\underline{S}, [2^A], \underline{F}} = \frac{|\underline{S}|\,|[2^A]|\,|\underline{F}|}{(2A)!} \sum_{r \in \tilde{\Sigma}_{2A}} \frac{\chi_r(\underline{S})\,\chi_r([2^A])\,\chi_r(\underline{F})}{\dim_r}$$

Each fatgraph $\mathscr{G}$ characterized by $\underline{S}$, $\underline{F}$ corresponds to a triplet $\sigma_2$, $\sigma_1$, $\sigma_0$ as above, up to simultaneous conjugacy in $\Sigma_{2A}$ and the number of *distinct* conjugates is $(2\,A)!/h(\mathcal{G})$ as follows from the definition of the automorphism group H of $\mathcal{G}$, of order $h(\mathcal{G})$. Therefore we have

LEMMA 2:

$$z(\underline{S}, \underline{F}) = \sum_{\mathcal{G}_{\underline{S}, \underline{F}}} \frac{1}{h(\mathcal{G}_{\underline{S}, \underline{F}})} = \frac{N_{\underline{S}, \underline{F}}}{(2\,A)!} = \frac{(2\,A-1)!!}{\prod_v v^{S_v+F_v} S_v!\, F_v!} \sum_{r \in \tilde{\Sigma}_{2\,A}} \frac{\chi_r(\underline{S})\, \chi_r([2^A])\, \chi_r(\underline{F})}{\dim_r}$$

where we have used the fact that the number of elements in a class $\underline{K} \equiv [\prod v^{k_v}]$ in $\Sigma_{2\,A}$ is equal to

$$|\underline{K}| = \frac{(2\,A)!}{\prod_v v^{k_v} k_v!}$$

We know of no such explicit formula for the function $f(\underline{S}, \underline{F})$ of eq. (3), *i.e.* when the sum is restricted to connected graphs. Note that $z(\underline{S}, \underline{F})$ is invariant under the interchange $\underline{S} \leftrightarrow \underline{F}$.

**3.** To define a generating function for the quantities $z$ and $f$, we introduce two infinite sequences of variables

$$t \equiv \{t_1, t_2, \ldots\} \qquad t' \equiv \{t'_1, t'_2, \ldots\}$$

For short we write $t^{\underline{S}}$ for the monomials $t_1^{S_1} t_2^{S_2} \ldots$ and similarly for $t'^{\underline{F}}$. The generating ("partition") function $Z(t, t')$ is the formal series

$$Z(t, t') = \sum_{\underline{S}, \underline{F}} t^{\underline{S}} t'^{\underline{F}} z(\underline{S}, \underline{F})$$

where the sum is restricted by the condition $\sum v\, S_v = \sum v\, F_v \equiv 0 \bmod 2$. A glance at lemma 2 suggests however a better choice of variables

$$\theta \equiv \left\{ \theta_k = \frac{t_k}{k} \right\} \qquad \theta' \equiv \left\{ \theta'_k = \frac{t'_k}{k} \right\}$$

Without changing the notation, we will consider Z as a function of $\theta$, $\theta'$ and write abusively $Z(\theta, \theta')$. The representations of $\Sigma_{2\,A}$ are indexed by Young tableaux with 2 A boxes so that we can trade the index $r$ appearing in lemma 2 for a Young tableau Y. The same tableau pertains to a representation of the linear group GL(N) for N large enough and one defines the corresponding characters as generalized Schur functions through the Frobenius reciprocity formula

$$\mathrm{ch}_Y(\theta) = \sum_{\text{classes } \underline{v} \in \Sigma_{|Y|}} \chi_Y(\underline{v}) \frac{\theta_1^{v_1}}{v_1!} \frac{\theta_2^{v_2}}{v_2!} \cdots \frac{\theta_{|Y|}^{v_{|Y|}}}{v_{|Y|}!} \tag{4}$$

This leads to

$$\left.\begin{aligned} &Z(\theta, \theta') = \sum_{A \geqq 0} Z_A(\theta, \theta') \\ &Z_A(\theta, \theta') = \sum_{|Y| = 2A} \mathrm{ch}_Y(\theta) \frac{(2A-1)!!\,\chi_Y([2^A])}{\chi_Y([1^{2A}])} \mathrm{ch}_Y(\theta') \end{aligned}\right\} \quad (5)$$

with $Z_0 = 1$, and the convention $(-1)!! = 1$. Recall that the Schur polynomials $p_n(\theta)$ are defined through

$$e^{\sum_{i=1}^{\infty} z^i \theta_i} = \sum_{n=0}^{\infty} z^n p_n(\theta)$$

and that the generalized Schur functions for a Young tableau Y with $f_i$ boxes in the $i$-th line, $f_1 \geqq f_2 \geqq \ldots \geqq f_{2A} \geqq 0$, $\sum f_i = 2A$, read

$$\mathrm{ch}_Y(\theta) = \det [p_{j-i+f_i}(\theta)]_{1 \leqq i, j \leqq 2A} \quad (6)$$

The above expressions enable one to easily compute the quantity under brackets in $Z_A$. Namely attach to Y the strictly decreasing sequence of non negative integers

$$l_i = f_i + 2A - i \qquad 1 \leqq i \leqq 2A \quad (7)$$

For lack of a better name we shall say that a Young tableau is "even" if the number of odd $l$'s equals the number of even ones in the list (7). We have

LEMMA 3:

$$\varphi_Y \equiv \frac{(2A-1)!!\,\chi_Y([2^A])}{\chi_Y([1^{2A}])}$$

$$= (-1)^{A(A-1)/2} \frac{\prod_{l \text{ odd}} l!! \prod_{l' \text{ even}} (l'-1)!!}{\prod_{\substack{l \text{ odd} \\ l' \text{ even}}} (l-l')} \qquad \text{if Y is even}$$

$$= 0 \qquad \text{otherwise.}$$

*Remark.* – When it is not zero, the result of lemma 3 is always a relative odd integer. The proof of lemma 3 uses the reciprocity formula (4) and the determinant form (6) of the generalized Schur function. For instance the denominator $\chi_Y([1^{2A}])$ in the expression computed in lemma 3, is the coefficient of $\theta_1^{2A}/(2A)!$ in the expansion (4). From (6) it has a simple determinant form, which immediately gives

$$\chi_Y([1^{2A}]) = \frac{(2A)!}{l_1!\,l_2! \ldots l_{2A}!} \prod_{1 \leqq i < j \leqq 2A} (l_i - l_j)$$

The numerator of the expression of lemma 3 is proportional to the coefficient of $\theta_2^A/A!$ in the expansion (4). Collecting both terms yields the desired result.

The reader will easily convince him (her) self that the following property holds. Let $Y \equiv \{f_1 \geqq f_2 \geqq \ldots \geqq f_{2A} \geqq 0\}$ be an even Young tableau. Define $l_i$ as in (7). Complete if necessary the $f$ sequence with zeroes or delete some zero $f$'s so that the same sequence now reads

$$f_1 \geqq f_2 \geqq \ldots \geqq f_{2N} \geqq 0$$

for some N. Define now $\{L_i = f_i + 2N - i,\ 1 \leqq i \leqq 2N\}$ as in (7), for this new but equivalent sequence (both pertain to the same tableau Y). In particular the evenness of Y does not depend on N, it is an intrinsic property of the representation. Moreover we have

LEMMA 4:

$$\varphi_Y = (-1)^{A(A-1)/2} \frac{\prod_{l \text{ odd}} l!! \prod_{l' \text{ even}} (l'-1)!!}{\prod_{\substack{l \text{ odd} \\ l' \text{ even}}} (l-l')}$$

$$= (-1)^{N(N-1)/2} \frac{\prod_{L \text{ odd}} L!! \prod_{L' \text{ even}} (L'-1)!!}{\prod_{\substack{L \text{ odd} \\ L' \text{ even}}} (L-L')}$$

This lemma simplifies calculations. For instance, if Y is a single row with 2 A boxes, it is even, we can take $N=1$, $L_1 = 2A+1$, $L_2 = 0$, and the above quantity reduces to

$$\varphi_Y = \frac{(2A+1)!!}{(2A+1)} = (2A-1)!!$$

A further simplification comes from the following remark. Let $\tilde{Y}$ denote the Young tableau conjugate to Y. The corresponding conjugate representation of the symmetric group has same dimension as the representation encoded in Y. It is easy to see that the conjugate of an even tableau is also even. Moreover, the Schur function $ch_{\tilde{Y}}(\theta)$ is obtained from the Schur function $ch_Y(\theta)$ by letting the odd $\theta_{2i+1}$ invariant and changing the even $\theta_{2i} \to -\theta_{2i}$. As a consequence, we have $\chi_{\tilde{Y}}([2^A]) = (-1)^A \chi_Y([2^A])$ and for any even tableau Y

$$\varphi_{\tilde{Y}} = (-1)^{|Y|/2} \varphi_Y$$

where $\varphi_Y$ is defined in lemma 4.

Collecting this information in the generating function Z, with the proviso of lemma 4 and the above remark at hand in order to simplify the expression, we get

PROPOSITION 1. –

$$Z(\theta, \theta') = \sum_{A \geq 0} (-1)^{A(A-1)/2} \times \sum_{\substack{Y \equiv l_1 > \ldots > l_{2A} \geq 0 \\ Y \text{ even}, |Y| = 2A}} \frac{\prod_{l \text{ odd}} l!! \prod_{l' \text{ even}} (l'-1)!!}{\prod_{\substack{l \text{ odd}, \\ l' \text{ even}}} (l-l')} \operatorname{ch}_Y(\theta) \operatorname{ch}_Y(\theta')$$

**4.** It is to be expected that $Z(\theta, \theta')$ can be expressed as the large N limit of some matrix integral. We shall write the integral in two forms. The first one, which we owe to a discussion with I. Kostov, does not exhibit the symmetry between $\theta$ and $\theta'$ (duality between vertices and faces). The second one will be explicitly symmetric. Let X and X′ be diagonal $N \times N$ matrices with non vanishing diagonal elements (they are therefore invertible) and M a generic hermitian $N \times N$ matrix, $dM$ the Lebesgue measure on the $N^2$-dimensional real vector space spanned by M. Set

$$t_k \equiv t_k(X) = \operatorname{tr}(X^k); \qquad \theta_k \equiv \theta_k(X) = \frac{\operatorname{tr}(X^k)}{k}$$

$$t'_k \equiv t_k(X') = \operatorname{tr}(X'^k); \qquad \theta'_k \equiv \theta_k(X') = \frac{\operatorname{tr}(X'^k)}{k}$$

Then in the sense of asymptotic (Feynman fatgraphs) expansion in the variables $\theta_k(X)$, $\theta_k(X')$, as $N \to \infty$, we have

PROPOSITION 2. –

$$Z(\theta(X), \theta(X')) = \frac{\int dM \exp - (1/2) \operatorname{tr}(MX'^{-1} MX'^{-1}) + \sum_{k=1}^{\infty} \theta_k(X) \operatorname{tr}(M^k)}{\int dM \exp - (1/2) \operatorname{tr}(MX'^{-1} MX'^{-1})}$$

Each term of the asymptotic expansion is well behaved if for each pair of diagonal elements $x'_i$, $x'_j$ (where $i$ can be equal to $j$), we have $\operatorname{Re}(x'_i x'_j)^{-1} > 0$. This is readily achieved for X′ positive definite. To make the overall integral well defined, we could require that X be pure imaginary, but this will not cure the divergence of the asymptotic series (*see* below). Since we are only dealing with formal series these subtleties do not affect the algebraic conclusions.

Consider a graph in the above perturbative expansion. A vertex of valance $v$ will contribute a factor $t_v(X) = \operatorname{tr}(X^v)$. Each double line (propagator) corresponding to the pair of indices $i$, $j$ is weighted by a factor $x'_i x'_j$. A "face" of valence $v$ (*i.e.* bounded by $v$ edges) and carrying a given index $i$ will be weighted by a factor $x'^v_i$. When we sum over all the matrix indices running along the lines of the fatgraph, a face of valence $v$ contributes a factor $\sum_i x_i'^v = t_v(X')$. The remaining symmetry factor will contribute $1/h(\mathcal{G})$, where $h(\mathcal{G})$ is the order of the automorphism group of the graph $\mathcal{G}$. Comparing with the definition (5) of Z, this completes the proof of Proposition 2.

As compared to usual one matrix integrals previously considered in various models, we note two specificities.

(*a*) The "potential" term is completely arbitrary. This causes no difficulty as we could require that $\theta_k(X)$ vanish for large enough $k$. Indeed at this stage the introduction of the matrix X can be considered as a trick and we could think of the $\theta_k$'s as arbitrary coupling constants. The above interpretation is however convenient in order to give some flesh to the expression $ch_Y(\theta) \equiv ch_Y(\theta(X))$ in the expansion of proposition 1.

(*b*) What is rather unusual is the form of the quadratic term in the matrix model potential. It involves twice the matrix X′ and is reminiscent of Kontsevich integrals (particularly of the case $p=3$, corresponding to the Boussinesq hierarchy) although not exactly the same. Indeed we are at loss to perform any "angular" average over the unitary group U(N), which would reduce the $N^2$-dimensional integral to an N-dimensional one over the eigenvalues of the argument M. In this respect it is quite fortunate that the combinatorial treatment of the fatgraphs enabled us to obtain directly the expansion of Z in the proposition 1 of the previous section.

At this point the reader might wonder: what is the integral representation of proposition 2 good for? The answer is twofold. As we shall see in section 9, it enables one to obtain an interesting series for the logarithm of Z, the quantity of interest. On a more low-brow level the integral representation and its general properties confirm that the sum over *connected* fatgraphs is indeed obtained by taking the logarithm of Z

$$F(\theta, \theta') = \log Z(\theta, \theta') = \sum_{A \geq 1} F_A(\theta, \theta')$$

with

$$F_A(\theta, \theta') = \sum_{\Sigma v S_v = \Sigma v F_v = 2A} t^{\underline{S}} t'^{\underline{F}} f(\underline{S}, \underline{F})$$

where $f$ is defined in (3). This could of course be also obtained through an algebraic treatment by studying the behaviour of $\mathcal{G}$ under permutations of identical connected parts. To calculate $f$'s explicitly, we use the following

LEMMA 5. –

$$F_A(\theta, \theta') = \frac{1}{A} \sum_{0 \leqq p \leqq A-1} (-1)^p \times \begin{vmatrix} Z_{A-p} & Z_{A-p+1} & . & \dots & Z_{A-1} & Z_A \\ 1 & Z_1 & Z_2 & \dots & Z_{p-1} & Z_p \\ 0 & 1 & Z_1 & \dots & Z_{p-2} & Z_{p-1} \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \dots & 1 & Z_1 \end{vmatrix}$$

This performs directly the inversion of Schur polynomials as

$$\exp\left(\sum_{A \geqq 0} u^A F_A\right) = \sum_{A \geqq 0} u^A Z_A = \sum_{A \geqq 0} u^A p_A(F).$$

The proof of the lemma relies on the inverse of Frobenius reciprocity formula (4), a direct consequence of the orthogonality relations for $\Sigma_n$ characters

$$(\theta_1)^{\nu_1}(2\theta_2)^{\nu_2}\dots(n\theta_n)^{\nu_n} = \sum_{Y : |Y| = \sum_i \nu_i} \chi_Y(\underline{\nu})\, \mathrm{ch}_Y(\theta) \tag{8}$$

To use this let us write formally $F_A = \mathrm{tr}(F^A)/A = t_A(F)$, and express the monomial $A\,t_A(F) = AF_A$ in terms of Schur functions $\mathrm{ch}_Y(F)$, which, thanks to eq. (6), are just some determinants of the $p_k(F) = Z_k$. This means that we apply the formula (8) to the cycle $\underline{\nu} = [A^1]$, $n = A$. The corresponding character $\chi_Y([A^1])$ is known to vanish unless the tableau $Y = Y_{p,q}$, where $Y_{p,q}$ has one row of say $q+1$ boxes and $p \geqq 0$ rows of one box, $p+q+1 = A$ and $\chi_{Y_{p,q}}([A^1]) = (-1)^p$. This leads exactly to an alternating sum over determinants with size ranging from $1 \times 1$ to $A \times A$ and completes the proof of lemma 5.

In spite of signs occuring everywhere, the coefficient of any monomial $t^{\underline{S}} t'^{\underline{E}}$ in F should be a non negative rational number by construction. From the expression of $Z_A$ obtained in section 3 we can then claim that we have an "explicit" formula for the generating function F. However if only in terms of the time (perhaps computer time...) it takes to extract any $f(\underline{S}, \underline{F})$ from these equations for a graph with a reasonable size, we realize at once that our task is not complete. We should rather look for constraints on F which generate faster the required quantities.

A last comment is in order. From the explicit series for $Z(\theta, \theta')$ and Cauchy's determinantal identity, it follows that if Y is an even Young tableau of size $|Y|=2A$, described as before by a sequence of $l$'s (7)

$$\langle \operatorname{ch}_Y(M) \rangle \equiv \frac{\int dM \operatorname{ch}_Y(M) \exp - (1/2) \operatorname{tr}(MX'^{-1} MX'^{-1})}{\int dM \exp - (1/2) \operatorname{tr}(MX'^{-1} MX'^{-1})}$$

$$= (-1)^{A(A-1)/2} \frac{\prod_{l \text{ odd}} l!! \prod_{l' \text{ even}} (l'-1)!!}{\prod_{\substack{l \text{ odd} \\ l' \text{ even}}} (l-l')} \operatorname{ch}_Y(X')$$

and the average is zero if Y is not even. It follows that the Schur functions $\operatorname{ch}_Y$ diagonalize the integral operator, the eigenvalues being zero or the odd integer $\varphi_Y$ computed in lemmas 3.4. A special case of the above reads

$$\langle p_{2A}(M) \rangle = (2A-1)!!\, p_{2A}(X')$$

It is obtained by taking a $1 \times 1$ matrix $X = x$, hence a potential $\sum_{k \geq 1} \frac{x^k}{k} \operatorname{tr}(M^k)$. This also shows the divergent structure of the asymptotic series when X is one-dimensional.

As suggested above there exists a more symmetric form of the matrix integral. To obtain it we observe that

$$\exp \sum_{k \geq 1} \theta_k(X) \operatorname{tr}(M^k) = \prod_{i=1}^{N} \exp \sum_{k \geq 1} x_i^k \frac{\operatorname{tr}(M^k)}{k} = \prod_{i=1}^{N} \frac{1}{\det(1 - x_i M)}$$

Parenthetically we see that if all $x_i$ have a bounded positive imaginary part for instance, this quantity is well defined and is in fact bounded as M runs over hermitian matrices, justifying a remark made before. The inverse determinants can be represented by Gaussian integral over (complex) N-dimensional vectors $v_i$ ($i$ indexes the vectors, not their components)

$$\exp \sum_{k \geq 1} \theta_k(X) \operatorname{tr}(M^k) = \frac{\int \prod_{i=1}^{N} dv_i\, d\bar{v}_i \exp - \sum_{i=1}^{N} \bar{v}_i (1 - x_i M) v_i}{\int \prod_{i=1}^{N} dv_i\, d\bar{v}_i \exp - \sum_{i=1}^{N} \bar{v}_i v_i}$$

Inserting this representation into the integral over M we turn the latter into a Gaussian integral, which is easily performed. Indeed defining an $N \times N$ *complex* matrix V whose $i$-th column is the vector $v_i$ in components, and integrating over M, the partition function Z takes the form

PROPOSITION 3. —

$$Z(\theta, \theta') = \frac{\int dV\, dV^{\dagger}\, e^{-\operatorname{tr}(VV^{\dagger}) + (1/2)\operatorname{tr}(VXV^{\dagger}X')^2}}{\int dV\, dV^{\dagger}\, e^{-\operatorname{tr}(VV^{\dagger})}}$$

$$= \frac{\int dW\, dW^{\dagger}\, e^{-\operatorname{tr}(WX^{-1}W^{\dagger}X'^{-1}) + (1/2)\operatorname{tr}(WW^{\dagger})^2}}{\int dW\, dW^{\dagger}\, e^{-\operatorname{tr}(WX^{-1}W^{\dagger}X'^{-1})}}$$

To make sense of these expressions globally we can take here X and X′ hermitian positive definite while the coefficient $\frac{1}{2}$ could at first be replaced by a parameter $\lambda$ purely imaginary or even better with a negative real part. In both cases one of the terms in the exponential weight is invariant under the transformation $V \to g_1 V g_2$ or $W \to g_1 W g_2$, with $(g_1, g_2) \in U(N) \times U(N)$, revealing the true symmetry of the problem. We observe that in the first dissymetric representation of theorem 2, we were left with a single U(N) invariance. In other words we deal with two hermitian matrices X and X′, the final result being conjugation invariant for both. To integrate over this action on one of them, we had to break the explicit duality between X and X′.

**6.** In spite of the pessimistic remarks made above the formulas obtained so far enable to compute the first few $F_A$'s in a straightforward way. On table I, we display a list of the first few F's up to $F_4$ together with the genus of the corresponding curves in the right column (for each monomial, it is given by Euler's formula $2-2g=\sum(S_v+F_v)-A$). To illustrate the use of this table and our criterion of rationality consider the fatgraph corresponding to the $D_4$ diagram of genus zero depicted on Figure 1. It has 3 vertices of valence 1, 1 vertex of valence 3, 3 edges and a single face of valence 6. It is encoded in $F_3$ in the term

$$f([1^3\,3^1],[6^1])\, t_1^3\, t_3\, t_6' = 18\, f([1^3\,3^1],[6^1])\, \theta_1^3\, \theta_3\, \theta_6'$$

and we read

$$f([1^3\,3^1],[6^1]) = \frac{1}{3}$$

On the other hand the automorphism group of the graph is $\mathbb{Z}/3\,\mathbb{Z}$, hence $h([1^3\,3^1],[6^1])=3$ and

$$h([1^3\,3^1],[6^1])\, f([1^3\,3^1],[6^1]) = 1$$

TABLE I

| | genus |
|---|---|
| $F_1=\theta_1^2\theta'_2+\theta_2\theta_1'^2$ | 0 |
| $F_2=4\theta_4\theta'_4$ | 1 |
| $+4\theta_1^2\theta_2\theta'_4+4\theta_4\theta_1'^2\theta'_2+4\theta_2^2\theta_2'^2+9\theta_1\theta_3\theta'_1\theta'_3$ | 0 |
| $F_3=30\theta_1\theta_5\theta'_6+30\theta_6\theta'_1\theta'_5+24\theta_2\theta_4\theta'_6+24\theta_6\theta'_2\theta'_4+9\theta_3^2\theta'_6+9\theta_6\theta_3'^2$ | 1 |
| $+12\theta_1^2\theta_2^2\theta'_6+12\theta_6\theta_1'^2\theta_2'^2+20\theta_1^2\theta_4\theta'_1\theta'_5+20\theta_1\theta_5\theta_1'^2\theta'_4+30\theta_1\theta_2\theta_3\theta'_1\theta'_5$ | 0 |
| $+30\theta_1\theta_5\theta'_1\theta'_2\theta'_3+48\theta_1\theta_2\theta_3\theta'_2\theta'_4+48\theta_2\theta_4\theta'_1\theta'_2\theta'_3+18\theta_3^2\theta_1'^2\theta'_4+18\theta_1^2\theta_4\theta_3'^2$ | 0 |
| $+12\theta_2^3\theta_3'^2+12\theta_3^2\theta_2'^3+6\theta_1^3\theta_3\theta'_6+6\theta_6\theta_1'^3\theta'_3$ | 0 |
| $F_4=168\theta_8\theta'_8$ | 2 |
| $+120\theta_1^2\theta_6\theta'_8+120\theta_8\theta_1'^2\theta'_6+240\theta_1\theta_2\theta_5\theta'_8+240\theta_8\theta'_1\theta'_2\theta'_5+245\theta_1\theta_7\theta'_1\theta'_7$ | 1 |
| $+192\theta_1\theta_3\theta_4\theta'_8+192\theta_8\theta'_1\theta'_3\theta'_4+96\theta_2^2\theta_4\theta'_8+96\theta_8\theta_2'^2\theta'_4+72\theta_2\theta_3^2\theta'_8+72\theta_8\theta'_2\theta'_3 2$ | 1 |
| $+168\theta_2\theta_6\theta'_1\theta'_7+168\theta_1\theta_7\theta'_2\theta'_6+210\theta_3\theta_5\theta'_1\theta'_7+210\theta_1\theta_7\theta'_3\theta'_5+216\theta_2\theta_6\theta'_2\theta'_6$ | 1 |
| $+112\theta_4^2\theta'_1\theta'_7+112\theta_1\theta_7\theta_4'^2+180\theta_3\theta_5\theta'_2\theta'_6+180\theta_2\theta_6\theta'_3\theta'_5$ | 1 |
| $+96\theta_4^2\theta'_2\theta'_6+96\theta_2\theta_6\theta_4'^2+96\theta_4^2\theta_4'^2+225\theta_3\theta_5\theta'_3\theta'_5$ | 1 |
| $+48\theta_1^3\theta_2\theta'_8+48\theta_8\theta_1'^3\theta'_2\theta'_3+32\theta_1^2\theta_2^3\theta'_8+32\theta_8\theta_1'^2\theta_2'^3+54\theta_1^2\theta_6\theta_1'^2\theta'_6$ | 0 |
| $+35\theta_1^3\theta_5\theta'_1\theta'_7+35\theta_1\theta_7\theta_1'^3\theta'_5+112\theta_1^2\theta_2\theta_4\theta'_1\theta'_7+112\theta_1\theta_7\theta_1'^2\theta'_2\theta'_4+162\theta_2\theta_3^2\theta'_2\theta_3'^2$ | 0 |
| $+63\theta_1^2\theta_3^2\theta'_1\theta'_7+63\theta_1\theta_7\theta_1'^2\theta_3'^2+84\theta_1\theta_2^2\theta_3\theta'_1\theta'_7+84\theta_1\theta_7\theta'_1\theta_2'^2\theta'_3+300\theta_1\theta_2\theta_5\theta'_1\theta'_2\theta'_5$ | 0 |
| $+96\theta_1^2\theta_2\theta_4\theta'_2\theta'_6+96\theta_2\theta_6\theta_1'^2\theta'_2+54\theta_1^2\theta_3^2\theta'_2\theta'_6+54\theta_2\theta_6\theta_1'^2\theta_3'^2+32\theta_2^4\theta_4'^2+32\theta_4^2\theta_2'^4$ | 0 |
| $+144\theta_1\theta_2^2\theta_3\theta'_2\theta'_6+144\theta_2\theta_6\theta'_1\theta_2'^2\theta'_3+144\theta_1\theta_3\theta_4\theta_1'^2\theta'_6+144\theta_1^2\theta_6\theta'_1\theta'_3\theta'_4$ | 0 |
| $+60\theta_1\theta_2\theta_5\theta_1'^2\theta'_6+60\theta_1^2\theta_6\theta'_1\theta'_2\theta'_5+54\theta_2\theta_3^2\theta_1'^2\theta'_6+54\theta_1^2\theta_6\theta'_2\theta_3'^2+288\theta_1\theta_3\theta_4\theta'_1\theta'_3\theta'_4$ | 0 |
| $+75\theta_1^3\theta_5\theta'_3\theta'_5+75\theta_3\theta_5\theta_1'^3\theta'_5+120\theta_1^2\theta_2\theta_4\theta'_3\theta'_5+120\theta_3\theta_5\theta_1'^2\theta'_2\theta'_4+128\theta_2^2\theta_4\theta_2'^2\theta'_4$ | 0 |
| $+180\theta_1\theta_2^2\theta_3\theta'_3\theta'_5+180\theta_3\theta_5\theta'_1\theta_2'^2\theta'_3+180\theta_2\theta_3^2\theta'_1\theta'_2\theta'_5+180\theta_1\theta_2\theta_5\theta'_2\theta_3'^2$ | 0 |
| $+120\theta_1\theta_3\theta_4\theta'_1\theta'_2\theta'_5+120\theta_1\theta_2\theta_5\theta'_1\theta'_3\theta'_4+192\theta_1\theta_3\theta_4\theta_2'^2\theta'_4+192\theta_2^2\theta_4\theta'_1\theta'_3\theta'_4$ | 0 |
| $+64\theta_1^2\theta_2\theta_4\theta_4'^2+64\theta_4^2\theta_1'^2\theta'_2\theta'_4+72\theta_1^2\theta_3^2\theta_4'^2+72\theta_4^2\theta_1'^2\theta_3'^2+8\theta_1^4\theta_4\theta'_8+8\theta_8\theta_1'^4\theta'_4$ | 0 |

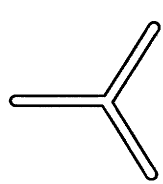

FIG. 1. – The genus 0 graph "$D_4$", with $A=3$, $\underline{S}=[1^3\,3^1]$ and $\underline{F}=[6^1]$.

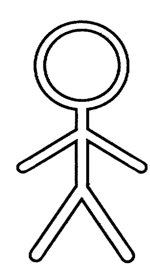

FIG. 2. – The genus 0 "dessin d'enfant", with $A=7$, $\underline{S}=[1^4\,3^2\,4^1]$ and $\underline{F}=[1^1\,13^1]$.

This shows that the corresponding curve is defined over $\mathbb{Q}$ as expected.

One can also compute easily a number of other contributions to F. For instance all terms which only involve $\theta_1$ and $\theta_2$ are obtained from a Gaussian integral and are all of genus zero. They read

$$\sum_{A \geqq 1} A\, 2^{A-1} (\theta_1^2 \theta_2^{A-1} \theta'_{2A} + \theta_2^A \theta'^2_A) \tag{9}$$

where the expression has to be symmetrized in $\theta$, $\theta'$, except for the completely symmetric term $\theta_2^2 \theta_2'^2$ which appears only once.

Another sequence easily computed corresponds to graphs with only one face (or only one vertex), which are necessarily connected. It contributes to $F_A$ through

$$\Phi_A(\theta)\, \theta'_{2A} + (\theta \leftrightarrow \theta')$$

where we omit the symmetrization for the completely symmetric term $\theta_{2A} \theta'_{2A}$ and

$$\Phi_A(\theta) = \sum_{p=0}^{A-1} (-1)^p (2p-1)!!\,(2A-2p-3)!!$$

$$\times [2(A-2p-1) \sum_{r=0}^{2p} p_r(-\theta) p_{2A-r}(\theta) - (2p+1) p_{2p+1}(-\theta) p_{2A-2p-1}(\theta)]$$

From this we extract the term, non-zero iff A is even, of genus $\frac{A}{2}$ corresponding to one 2 A-legged vertex surrounded by one face of valence 2 A

$$2 \sum_{p=0}^{A-1} (-1)^p (2p-1)!!\,(2A-2p-3)!!\,(A-2p-1)\, \theta_{2A} \theta'_{2A}$$

$$= \frac{1+(-1)^A}{2} \frac{2A}{A+1} (2A-1)!!\, \theta_{2A} \theta'_{2A}$$

The coefficient is an even integer, in agreement with table I (we find 4 for A = 2, genus 1 and 168 for A = 4, genus 2).

Pursuing we have a term, non zero iff A is odd, of genus $\frac{A-1}{2}$ with a vertex of valence one and a vertex of valence 2 A − 1, and only one face of valence 2 A

$$\frac{1-(-1)^A}{2} 2(2A-1)!!\, \theta_{2A-1} \theta_1 \theta'_{2A} \qquad \text{for} \quad A > 1$$

For A = 1 omit the factor 2, *i.e.* we get $\theta_1^2 \theta'_2$ (as already noted in $F_1$ on table I).

The next term, non zero for A even, of genus $\frac{A-2}{2}$ reads

$$\frac{1+(-1)^{A}}{2} 2A(2A-3)!!\,\theta_1^2\,\theta_{2A-2}\,\theta'_{2A} \qquad \text{for} \quad A>2$$

and so on...

All the results collected so far suggest the following

CONJECTURE 1. — In the expansion of F as an asymptotic series of $\theta$, $\theta'$, the coefficient of any monomial is a non-negative *integer*

Equivalently

$$\prod_v v^{S_v+F_v} \sum_{\mathscr{G}_{\underline{S},\underline{F}}} \frac{1}{h(\mathscr{G}_{\underline{S},\underline{F}})} \in \mathbb{N}.$$

Since as mentioned in section 1, for a connected fatgraph $h(\mathscr{G}_{\underline{S},\underline{F}})$ divides 2 A, when it is not zero [4], the sum

$$f(\underline{S},\underline{F}) = \sum_{\mathscr{G}_{\underline{S},\underline{F}}} \frac{1}{h(\mathscr{G}_{\underline{S},\underline{F}})}$$

is a positive rational with denominator a divisor of 2 A. The conjecture would imply that it can be reduced to the form

$$f(\underline{S},\underline{F}) = \frac{p}{q}$$

where $p$ is an integer and $q$ is the greatest common divisor of 2 A and $\prod_v v^{S_v+F_v}$. For instance take the term $f([1^1\,3^1],[1^1\,3^1])\,t_1\,t_3\,t'_1\,t'_3$ with $2A=4$ and $\prod v^{s_v+F_v}=9$ relatively prime. The conjecture (true in this case) implies that $f$ be an integer (1 here).

The above results motivate another conjecture on the form of the genus 0 part of F, but expanding it on monomials of the form

$$\frac{t^{\underline{S}}}{\underline{S}!}\,\frac{t'^{\underline{F}}}{\underline{F}!} \equiv \prod_v \frac{t_v^{S_v}}{S_v!}\,\frac{t'\,v^{F_v}}{F_v!}$$

CONJECTURE 2. — As an asymptotic expansion on monomials of the form $\frac{t^{\underline{S}}}{\underline{S}!}\,\frac{t'^{\underline{F}}}{\underline{F}!}$ the genus zero contribution to F has non negative integer coefficients

Equivalently

$$\prod_v S_v!\,F_v! \sum_{\substack{\mathscr{G}_{\underline{S},\underline{F}} \\ \text{of genus } 0}} \frac{1}{h(\mathscr{G}_{\underline{S},\underline{F}})} \in \mathbb{N}.$$

TABLE II

$$F_1^{(0)} = \frac{t_1^2}{2!} t'_2 + t_2 \frac{t_1'^2}{2!}$$

$$F_2^{(0)} = \frac{t_1^2}{2!} t_2 t'_4 + t_4 \frac{t_1'^2}{2!} t'_2 + \frac{t_2^2}{2!} \frac{t_2'^2}{2!} + t_1 t_3 t'_1 t'_3$$

$$\begin{aligned} F_3^{(0)} = {} & t_1 t_2 t_3 t'_1 t'_5 + t_1 t_5 t'_1 t'_2 t'_3 + t_1 t_2 t_3 t'_2 t'_4 + t_2 t_4 t'_1 t'_2 t'_3 + 2 \frac{t_1^3}{3!} t_3 t'_6 \\ & + 2 t_6 \frac{t_1'^3}{3!} t'_3 + 2 \frac{t_1^2}{2!} \frac{t_2^2}{2!} t'_6 + 2 t_6 \frac{t_1'^2}{2!} \frac{t_2'^2}{2!} + 2 \frac{t_1^2}{2!} t_4 t'_1 t'_5 + 2 t_1 t_5 \frac{t_1'^2}{2!} t'_4 \\ & + 2 \frac{t_3^2}{2!} \frac{t_1'^2}{2!} t'_4 + 2 t_4 \frac{t_1^2}{2!} \frac{t_3'^2}{2!} + 2 \frac{t_2^3}{3!} \frac{t_3'^2}{2!} + 2 \frac{t_3^2}{2!} \frac{t_2'^3}{3!} \end{aligned}$$

$$\begin{aligned} F_4^{(0)} = {} & \frac{t_1^2}{2!} t_2 t_4 t'_1 t'_7 + t_1 t_7 \frac{t_1'^2}{2!} t'_2 t'_4 + t_1 t_3 t_4 t'_1 t'_2 t'_5 + t_1 t_2 t_5 t'_1 t'_3 t'_4 + t_1 t_3 t_4 \frac{t_1'^2}{2!} t'_6 \\ & + \frac{t_1^2}{2!} t_6 t'_1 t'_3 t'_4 + 2 t_1 \frac{t_2^2}{2!} t_3 t'_1 t'_7 + 2 t_1 t_7 t'_1 \frac{t_2'^2}{2!} t'_3 + 2 \frac{t_1^2}{2!} t_2 t_4 t'_2 t'_6 + 2 t_2 t_6 \frac{t_1'^2}{2!} t'_2 t'_4 \\ & + 2 \frac{t_1^2}{2!} \frac{t_3^2}{2!} t'_2 t'_6 + 2 t_2 t_6 \frac{t_1'^2}{2!} \frac{t_3'^2}{2!} + 2 t_1 \frac{t_2^2}{2!} t_3 t'_2 t'_6 + 2 t_2 t_6 t'_1 \frac{t_2'^2}{2!} t'_3 + 2 t_1 t_2 t_5 \frac{t_1'^2}{2!} t'_6 \\ & + 2 \frac{t_1^2}{2!} t_6 t'_1 t'_2 t'_5 + 2 \frac{t_1^2}{2!} t_2 t_4 t'_3 t'_5 + 2 t_3 t_5 \frac{t_1'^2}{2!} t'_2 t'_4 + 2 t_1 \frac{t_2^2}{2!} t_3 t'_3 t'_5 + 2 t_3 t_5 t'_1 \frac{t_2'^2}{2!} t'_3 \\ & + 2 t_2 \frac{t_3^2}{2!} t'_1 t'_2 t'_5 + 2 t_1 t_2 t_5 t'_2 \frac{t_3'^2}{2!} + 2 \frac{t_1^2}{2!} t_2 t_4 \frac{t_4'^2}{2!} + 2 \frac{t_4^2}{2!} \frac{t_1'^2}{2!} t'_2 t'_4 + 2 t_1 t_3 t_4 t'_1 t'_3 t'_4 \\ & + 2 t_2 \frac{t_3^2}{2!} \frac{t_1'^2}{2!} t'_6 + 2 \frac{t_1^2}{2!} t_6 t'_2 \frac{t_3'^2}{2!} + 2 \frac{t_2^2}{2!} t_4 t'_1 t'_3 t'_4 + 2 t_1 t_3 t_4 \frac{t_2'^2}{2!} t'_4 + 2 \frac{t_2^2}{2!} t_4 \frac{t_2'^2}{2!} t'_4 \\ & + 2 t_2 \frac{t_3^2}{2!} t'_2 \frac{t_3'^2}{2!} + 3 t_1 t_2 t_5 t'_1 t'_2 t'_5 + 4 \frac{t_1^2}{2!} \frac{t_3^2}{2!} t'_1 t'_7 + 4 t_1 t_7 \frac{t_1'^2}{2!} \frac{t_3'^2}{2!} + 4 \frac{t_1^2}{2!} \frac{t_3^2}{2!} \frac{t_4'^2}{2!} \\ & + 4 \frac{t_4^2}{2!} \frac{t_1'^2}{2!} \frac{t_3'^2}{2!} + 6 \frac{t_1^4}{4!} t_4 t'_8 + 6 t_8 \frac{t_1'^4}{4!} t'_4 + 6 \frac{t_1^3}{3!} t_2 t_3 t'_8 + 6 t_8 \frac{t_1'^3}{3!} t'_2 t'_3 \\ & + 6 \frac{t_1^2}{2!} \frac{t_2^3}{3!} t'_8 + 6 t_8 \frac{t_1'^2}{2!} \frac{t_2'^3}{3!} + 6 \frac{t_1^3}{3!} t_5 t'_1 t'_7 + 6 t_1 t_7 \frac{t_1'^3}{3!} t'_5 + 6 \frac{t_1^2}{2!} t_6 \frac{t_1'^2}{2!} t'_6 \\ & + 6 \frac{t_1^3}{3!} t_5 t'_3 t'_5 + 6 t_3 t_5 \frac{t_1'^3}{3!} t'_5 + 6 \frac{t_2^4}{4!} \frac{t_4'^2}{2!} + 6 \frac{t_4^2}{2!} \frac{t_2'^4}{4!} \end{aligned}$$

We display these integers on table II up to the genus 0 contribution to $F_4$. They are computed from the data of table I. These integers are much smaller than those occuring in table I, suggesting that the second conjecture is sharper than the first one in genus 0. In higher genus, the above numbers are no longer integers, but still have relatively small denominators.

7. The combination of Frobenius formula (4) and the expression of characters of the linear group as determinants of Schur polynomials (6) enables one to perform rather efficiently some non trivial computations. Take the example shown on Figure 2 of a graph considered by Osterlé (following Grothendieck, some authors use the french name "dessin d'enfant" for what we call here fatgraphs, this applies particularly well to this

example). This genus 0 fatgraph corresponds to the monomial $\theta_1^4 \theta_3^2 \theta_4 \theta'_1 \theta'_{13}$ in $F_7$, with the same coefficient in $Z_7$. This is because a graph with two faces, one of which is of valence 1, is necessarily connected. Moreover the only characters $ch_Y(\theta')$ which have a non vanishing coefficient of $\theta'_1 \theta'_{13}$ corresponds to Young tableaux with a row of $p$ boxes and $14-p$ rows of one box or one row with $p \geqq 2$ boxes, one row with 2 boxes and $12-p$ rows of one box. In fact the only tableaux actually contributing are $[14^1]$ $(p=14)$ and $[1^{14}]$ $(p=0)$ in the first sequence, while only even $p$'s contribute in the second sequence (they are the only even ones). Finally we use the (anti)symmetry of the prefactor $\varphi_Y$ defined in lemma 3 under conjugation of the tableau [here $(-1)^A = -1$], which reduces the computation to four cases, with one vanishing contribution. This yields the coefficient of $t_1^4 t_3^2 t_4 t'_1 t'_{13}$

$$f([1^4 3^2 4^1], [1^1 13^1]) = \frac{2}{3^2.4.13}\left(\frac{13!!}{48} - \frac{13.9!!}{24} + \frac{13.5!!3!!}{16}\right) = 10.$$

The graph of Figure 2 has an automorphism group reduced to unity. It follows from section 1 that the number of conjugates of the corresponding rational curve is at most ten. As Osterlé and collaborators have shown, this number is effectively 10 in the present case.

**8**. The generating function Z defined in (5) admits a number of specializations. For instance when the matrix X′ is equal to the unit matrix, and $\theta_k = -x^{k-2}/k$ for $k \geqq 3$ while $\theta_1 = \theta_2 = 0$, the integral representation of proposition 2 reduces to an integral considered by Penner [5] in the computation of the virtual Euler characteristic of the mapping class group of punctured Riemann surfaces (*see* also [6]). More precisely, under the specialization

$$\theta_1 = \theta_2 = 0 \qquad \theta_k = -\frac{x^{k-2}}{k} \qquad k \geqq 3$$

$$\theta'_k = -\frac{y}{k} \qquad k \geqq 1$$

we obtain for $F = \log Z$ the following series

$$F(x, y) = \sum_{n>0,\, 2g-2+n>0} x^{2(2g-2+n)} y^n \frac{B_{2g}}{2g(2g-2+n)} \binom{2g-2+n}{n} \tag{10}$$

with $B_{2g}$ the Bernoulli numbers defined through

$$\frac{1}{2} + \frac{t}{e^t - 1} = \sum_{g \geqq 0} B_{2g} \frac{t^{2g}}{(2g)!}.$$

In the sum (10) the integers $n$ and $g$ stand respectively for the number of faces (there is a factor $y$ for each face) and the genus of the corresponding fatgraphs. When $g=0$ the coefficient is understood as a limiting value

$$\frac{B_{2g}}{2g(2g-2+n)}\binom{2g-2+n}{n}\Big|_{g=0}=-\frac{(n-3)!}{n!}.$$

For instance when $n=3$, $g=0$ we get from table I the terms

$$4\theta_4\theta_1'^2\theta_2'+18\theta_3^2\theta_1'^2\theta_4'+12\theta_3^2\theta_2'^3$$

Under the above specialization this yields

$$4\left(-\frac{x^2}{4}\right)(-y)^2\left(-\frac{y}{2}\right)+18\left(-\frac{x}{3}\right)^2(-y)^2\left(-\frac{y}{4}\right)$$
$$+12\left(-\frac{x}{3}\right)^2\left(-\frac{y}{2}\right)^3=-\frac{1}{6}x^2y^3$$

in agreement with (10). Similarly for $n=1$, $g=1$, the contributions from table I are

$$4\theta_4\theta_4'+9\theta_3^2\theta_6'$$

which under the specialization yield

$$4\left(-\frac{x^2}{4}\right)\left(-\frac{y}{4}\right)+9\left(-\frac{x}{3}\right)\left(-\frac{y}{6}\right)=\frac{1}{12}x^2y$$

again in agreement with (10) by using $B_2=\frac{1}{6}$.

Yet another interesting specialization is when only finitely many θ's are non zero and X′ is again the unit matrix. The integral representation of proposition 2 reduces to the standard one matrix model which can be dealt with, using orthogonal polynomial techniques.

**9.** Until now no significant use of the integral representation of proposition 2 was made. Let us now show how it can be of some help through a saddle point method. In general in matrix models the latter is applicable only after performing the angular integration. The present case is an exception as we will soon see. Let us look in proposition 2 for stationary points of the integral over M in the numerator. We find the equation

$$X'^{-1}MX'^{-1}=\sum_{k\geq 1}k\theta_k(X)M^{k-1} \tag{11}$$

For short write $\theta_k(X)\equiv\theta_k$. We are going to solve this equation through the Lagrange inversion method, starting from the term corresponding to $k=1$ on the r.h.s., namely $\theta_1$ id. The solution $M_0$ thus obtained is readily seen to commute with X′. Let us recall the Lagrange original problem. For a given analytic function $\varphi(x, p)$ of $x$, $p$ a parameter, let us consider the equation

$$m=p+\varphi(m, p)$$

and suppose that it has a unique solution $m(p)$ for $p$ small enough. Then the function

$$\partial_x \log[x-p-\varphi(x, p)]$$

is analytic except at $x=m(p)$, where it has a single pole with residue 1. We use the Cauchy formula on a contour C surrounding this pole to rewrite

$$\begin{aligned}\psi(m(p), p)&=\int_{C}\frac{dx}{2i\pi}\psi(x, p)\,\partial_x\left[\log(x-p)+\log\left(1-\frac{\varphi(x, p)}{(x-p)}\right)\right]\\ &=\psi(p, p)+\sum_{k\geqq 1}\frac{1}{k!}\partial_x^{k-1}[\partial_x\psi(x, p)\,\varphi(x, p)^k]|_{x=p}\end{aligned}$$

Returning to our problem, equation (11) is the stationarity condition for the function

$$\begin{aligned}&F(M)=\mathrm{Tr}(G(M))\\ &G(M)=-\frac{1}{2}(MX'^{-1})^2+\theta_1 M+V(M)\\ &V(M)=\sum_{k\geqq 2}\theta_k M^k\end{aligned}$$

Let $P=\theta_1 X'^2$, hence $M_0=P\left(1+\frac{V'(M_0)}{\theta_1}\right)$, as $M_0$ and $X'$ commute (we can treat $M_0$ and P as commuting scalars). Applying the Lagrange method above to the functions

$$\begin{aligned}\psi(M, P)=G&=-\frac{1}{2}M^2P^{-1}\theta_1+M\theta_1+V(M)\\ &=\frac{\theta_1 P}{2}-\frac{\theta_1(M-P)^2P^{-1}}{2}+V(M)\\ \varphi(M, P)&=\frac{P}{\theta_1}V'(M),\end{aligned}$$

we finally get

$$G(M_0, P)=\frac{\theta_1 P}{2}+V(P)$$

$$+\sum_{n=1}^{\infty}\frac{P^n}{n!\,\theta_1^n}\partial_x^{n-1}[(V'(x))^{n+1}-\theta_1 P^{-1}(x-P)(V'(x))^n]_{x=P}$$

$$=\frac{\theta_1 P}{2}+V(P)+\sum_{n=1}^{\infty}\frac{P^n}{(n+1)!\,\theta_1^n}\partial_P^{n-1}V'(P)^{n+1}.$$

To get F, we still have to take the trace of the above expression, *i.e.* expand it in powers of P and use $\mathrm{Tr}(P^k)=\theta_1^k\,\mathrm{Tr}(X'^{2k})=2k\,\theta'_{2k}\,\theta_1^k$. This suggests to rewrite the result symbolically, by setting $P=\theta_1\,\theta'^2$, acting on G with the operator $\theta'\partial_{\theta'}\equiv 2P\partial_P$, and finally substituting $\theta'^{2k}\to\theta'_{2k}$. We have

$$F^{[0]}(\theta, \theta')=\theta_1 P+2\sum_{n=0}^{\infty}P\partial_P\frac{P^n}{(n+1)!\,\theta_1^n}\partial_P^{n-1}(V'(P))^{n+1}\Bigg|_{P^k=\theta_1^k\theta'_{2k}},$$

where the $n=0$ term in the sum is to be understood as $2PV'(P)$. Substituting $V(P)=\sum_{k\geqq 2}\theta_k P^k$, we get

$$F^{[0]}(\theta, \theta')=\sum_{\substack{r_1, r_2, \ldots \geqq 0\\ 2+\sum(i-2)r_i=0}}\frac{2}{\sum r_i}\frac{(\sum r_i)!}{\prod r_i!}\prod(i\theta_i)^{r_i}\theta'_{2(r_1+r_2+\ldots-1)}$$

A few comments are in order.

(i) This is the so-called tree approximation to the logarithm of the integral of proposition 2. A tree here means a fatgraph of genus zero with only one face (only one $\theta'$ appears in each monomial of the expansion of $F^{[0]}$, and the genus is given by the Euler formula $2-2g=1+\sum r_i-(\sum r_i-1)=2$, with $S=\sum r_i$ vertices, $F=1$ face and $A=\sum r_i-1=\sum ir_i/2$ edges). Therefore $F^{[0]}$ is only the tree piece of the genus zero energy $F^{(0)}$ of table II.

(ii) The coefficients appearing in $F^{[0]}$ are integers generalizing the Catalan numbers, in agreement with conjecture 1. To prove it, write in various ways

$$\frac{2}{\sum r_i}\frac{(\sum r_i)!}{\prod r_i!}=\frac{2}{r_k}\frac{(\sum r_i-1)!}{(r_k-1)!\prod_{i\neq k}r_i!}$$

therefore for all nonvanishing $r_k$'s,

$$\frac{2}{\sum r_i}\times(\text{integer})=\frac{2}{r_k}\times(\text{integer}) \tag{12}$$

and the $r$'s are subject to the constraint $2+\sum(i-2)r_i=0$. The l.h.s. of (12) is a rational with a denominator dividing all $r_k$'s, hence a divisor of 2 due to the constraint. If one of the $r_k$'s at least is odd, the denominator is one, and the number (12) is an integer. If all the $r_k$'s are even, the denominator of (12) has to divide all $r_k/2$, hence is one, again due to the constraint. In all cases we proved that (12) is an integer. It is a generalization of the Catalan numbers $\binom{2n}{n}/(n+1)$. In terms of these numbers the connected function for genus zero tree-fatgraphs defined in (3) reads

$$f([\prod i^{r_i}],[(2(\sum r_i-1))^1])=\frac{1}{2\mathrm{A}}\frac{2}{\sum r_i}\frac{(\sum r_i)!}{\prod r_i!}$$

with the constraint $2+\sum(i-2)r_i=0$, and $2\mathrm{A}=\sum ir_i=2(\sum r_i-1)$.

(iii) The expression of $\mathrm{F}^{[0]}$ agrees also with conjecture 2. To see why, notice that $\sum r_i \geqq 2$ and rewrite

$$\frac{2}{\sum r_i}\frac{(\sum r_i)!}{\prod r_i!}(\prod(i\theta_i)^{r_i})\,\theta'_{2(\Sigma r_i-1)}=(\sum r_i-2)!\left(\prod\frac{t_i^{r_i}}{r_i!}\right)t'_{2(\Sigma r_i-1)}.$$

(iv) The expression of $\mathrm{F}^{[0]}$ simplifies in the case of tree fatgraphs with only one- and $k$-valent vertices

$$\mathrm{F}^{[0]}(\theta_1,0,\ldots,0,\theta_k,0,\ldots;\theta')$$
$$=\sum_{n\geqq 0}\frac{2}{2+n(k-1)}\binom{2+n(k-1)}{n}\theta_1^{2+n(k-2)}(k\theta_k)^n\,\theta'_{2+2n(k-1)}$$

In the particular case of the star fatgraph (A vertices of valence 1 and 1 vertex of valence A, corresponding to the term $n=1$ in the above sum) we find

$$f([1^{\mathrm{A}}\mathrm{A}^1],[2\mathrm{A}^1])=\frac{1}{\mathrm{A}}$$

which exhibits the cyclic group $\mathbb{Z}_{\mathrm{A}}$ as automorphism group.

So far we only performed the tree approximation to the integral of proposition 2. Let us now proceed and compute the "one loop" corrections by the semi-classical approximation at the above stationary point $\mathrm{M}_0$. We have to study

$$\mathrm{Z}=\frac{\int d\mathrm{M}\,e^{\mathrm{S}(\mathrm{M},\mathrm{X}')}}{\int d\mathrm{M}\,e^{\mathrm{S}_0(\mathrm{M},\mathrm{X}')}}$$

with

$$S(M, X') = -\frac{1}{2}\operatorname{Tr}(MX'^{-1})^2 + \sum_{k \geq 2} \theta_k \operatorname{Tr}(M^k)$$

$$S_0(M, X') = -\frac{1}{2}\operatorname{Tr}(MX'^{-1})^2.$$

Semi-classically, one expands S up to second order around the stationary point $M_0$

$$S(M, X') = F^{[0]} + \frac{1}{2} \sum_{\alpha, \beta, \gamma, \delta} (M - M_0)_{\alpha, \beta} \frac{\partial^2 S}{\partial M_{\alpha, \beta} \partial M_{\gamma, \delta}} (M_0)(M - M_0)_{\gamma, \delta} + \dots$$

and integrates over the Gaussian corrections. Taking into account the contribution from the denominator, we get

$$\log Z = F^{[0]} + F^{[1]} + \dots$$

$$F^{[1]} = -\frac{1}{2} \log\det(QR^{-1}),$$

where Q and R are linear operators acting on the space of hermitian matrices, defined by

$$QM = X'^{-1} M X'^{-1} - \sum_{k \geq 2} k\theta_k \sum_{\substack{r, s \geq 0 \\ r+s=k-2}} M_0^r M M_0^s$$

$$RM = X'^{-1} M X'^{-1}$$

$$R^{-1} M = X' M X'.$$

Set $QR^{-1} = 1 - K$, where

$$KM = \sum_{k \geq 2} t_k \sum_{\substack{r, s \geq 0 \\ r+s=k-2}} M_0^r X' M X' M_0^s. \tag{13}$$

From now on we return to the notation $t_k$ for $k\theta_k = \operatorname{Tr}(X^k)$. The one loop correction reads

$$F^{[1]} = -\frac{1}{2}\log\det(1-K) = \frac{1}{2}\sum_{N \geq 1} \frac{\operatorname{Tr}(K^N)}{N}.$$

Using the definition of the operator K and the fact that $M_0$ and $X'$ commute, we get

$$F^{[1]} = \frac{1}{2}\sum_{N \geq 1} \frac{1}{N} \sum_{r_1, \dots, r_N, s_1, \dots, s_N \geq 0} \left(\prod_{i=1}^{N} t_{r_i + s_i + 2}\right) \times \operatorname{Tr}(X'^N M_0^{r_1 + \dots + r_N}) \operatorname{Tr}(X'^N M_0^{s_1 + \dots + s_N}). \tag{14}$$

To compute $M_0^r$, we use the Lagrange method again, with the functions $\psi(M, P) = M^r$ and $\varphi(M, P) = (P/t_1)\, V'(M) = P \sum_{k \geqq 2} (t_k/t_1) M^{k-1}$, $P = t_1 X'^2$, and we get

$$M_0^r = P^r + \sum_{n=1}^{\infty} \frac{P^n}{n!\, t_1^n} \partial_P^{n-1} [r P^{r-1} V'(P)^n]$$

$$= t_1^t X'^r + r \sum_{n=1}^{\infty} \sum_{k_1, \ldots, k_n \geqq 1} t_1^{k_1 + \ldots + k_n + r - n} t_{k_1+1} \ldots t_{k_n+1}$$

$$\times \frac{X'^{2(k_1 + \ldots + k_n + r)}}{(k_1 + \ldots + k_n + r)} \binom{k_1 + \ldots + k_n + r}{n}.$$

Let us reorganize the summation over $k_i$ into a sum over the $n_j = |\{i, k_i = j\}|$ for $j \geqq 2$ and $n_1 = \sum k_i - n + r$, $n = \sum n_i$, $\sum k_i = \sum i n_i$, and take the trace after multiplication by $X'^N$

$$\mathrm{Tr}(M_0^r X'^N) = r \sum_{\substack{n_1, n_2, \ldots \geqq 0 \\ r + \Sigma (i-2) n_i = 0}} \frac{(\sum n_i - 1)!}{\prod n_i!} (\prod t_i^{n_i})\, t'_{N + 2 \Sigma r_i}$$

where we understand the sum as yielding $t'_N$ when $r = 0$ (this can be summarized in the convention $0 \times (-1)! = 1$). Substituting this into (14), we have

$$F^{[1]} = \sum_{N \geqq 1} \frac{1}{2N} \sum_{\substack{r_a, s_a \geqq 0 \\ a = 1, \ldots, N}} (\sum r_a)(\sum s_a)\, t_{r_1 + s_1 + 2} \ldots t_{r_N + s_N + 2}$$

$$\times \sum_{\substack{n_i \geqq 0 \\ \Sigma r_a + \Sigma (i-2) n_i = 0}} \sum_{\substack{m_i \geqq 0 \\ \Sigma s_a + \Sigma (i-2) m_i = 0}} \frac{(\Sigma n_i - 1)!}{\Pi n_i!} \frac{(\Sigma m_i - 1)!}{\Pi m_i!}$$

$$\times t_1^{n_1 + m_1} t_2^{n_2 + m_2} t_3^{n_3 + m_3} \ldots t'_{N + 2 \Sigma n_i} t'_{N + 2 \Sigma m_i}$$

Here we use again the convention $0 \times (-1)! = 1$ when $\sum n_i = 0$ (resp. $\sum m_i = 0$), which implies $\sum r_a = 0$ (resp. $\sum s_a = 0$).

Of course the contributions in $F^{[0]} + F^{[1]}$ up to $A = 4$ agree with the data in tables I and II. Remarkably the expression involves only genus zero fatgraphs! This is readily seen by applying Euler's formula with $F = 2$ faces (two $t'$), $S = \sum (n_i + m_i) + N$ vertices, and

$$A = [\sum (r_a + s_a) + \sum i (n_i + m_i) + 2N]/2 = S$$

edges, hence $2 - 2g = F - A + S = 2$. We do not fully understand this phenomenon. Perhaps higher genus contributions would correspond to other non trivial saddle points. We could pursue the semi-classical expansion beyond "one loop" order. But the expressions become quite cumbersome.

**10.** In this note we gathered information about the sum of the inverse orders of the automorphism groups of fatgraphs with specified valences of faces and vertices. We gave a compact expression for the generating function for general fatgraphs. The connected case however proved to be more subtle. Although in principle we just had to take the logarithm F of the previous generating function, we were not able to find such a compact and ready-to-use expression in that case.

The introduction of the matrix integral of proposition 2 should shed some light on the problem of calculating directly the generating function F for connected fatgraphs. We were only able to perform a semi-classical expansion around a small stationary point of the matrix model action, yielding apparently only genus zero contributions to F (this remains for us slightly mysterious). But one should try to extract more information from this matrix model, presumably by implementing the equations of motion in an efficient way.

## REFERENCES

[1] G. V. BELYI, On Galois Extensions of a Maximal Cyclotomic Field, *Math. U.S.S.R. Izvestija*, Vol. **14**, 1980, pp. 247-256.

[2] V. A. VOEVODSKY and G. SHABAT, Drawing Curves Over Number Fields, in *A. Grothendieck Festschrift*, Birkhäuser.

[3] J. WOLFART, Mirror-Invariant Triangulations of Riemann Surfaces, Triangle Groups and Grothendieck Dessins: Variations on a Thema of Belyi, *Mathematik Seminar der Universität Frankfurt*, 1992.

[4] M. BAUER and C. ITZYKSON, Triangulations, *Publications de la R.C.P.* 25, Strasbourg (to appear) .

[5] R. C. PENNER, Perturbative Series and the Moduli Space of Riemann Surfaces, *J. Diff. Geom.*, Vol. **27**, 1988, pp. 35-53.

[6] C. ITZYKSON and J.-B. ZUBER, Matrix Integration and Combinatorics of the Modular Groups, *Comm. Math. Phys.*, Vol. **134**, 1990, pp. 197-207.

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