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Floquet operators without absolutely continuous spectrum

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ABSTRACT. — We extend some recent results of Howland [1], concerning the absence of absolutely continuous spectrum of Floquet operators for time periodic perturbations of discrete hamiltonians with increasing gaps. Our result cover the case of n coupled pulsed rotors (the case $n=1$ has been considered by Bellissard [2]):

$$H(t) = -\frac{d^2}{d\theta^2}\mathbf{1}_n + V(\theta, t)$$

on $0<\theta<2\pi$ with periodic boundary conditions, and $V_{i,j}(\theta, t) \in \mathscr{C}^2$; $i, j = 1, \ldots, n$.

RÉSUMÉ. — Nous faisons une extension de quelques résultats récents de Howland, [1] concernant l'absence du spectre absolument continu des opérateurs de Floquet pour les perturbations périodiques d'hamiltoniens discrets avec des discontinuités croissantes. Notre résultat couvre le cas de n roteurs pulsatoires couplés (le cas $n=1$ avait été considéré par Bellissard [2]);

$$H(t) = -\frac{d^2}{d\theta^2}\mathbf{1}_n + V(\theta, t)$$

avec conditions périodiques sur $0<\theta<2\pi$ et $V_{i,j}(\theta, t) \in \mathscr{C}^2$; $i, j = 1, \ldots, n$.

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1. THE RESULT

For the origin of the problem and previous results the reader is referred to [1], [2].

Let H_0 be a positive self-adjoint operator in $\mathscr{H}$ with discrete spectrum. Suppose that

$$\left.\begin{array}{ll} \sigma(H_0)=\bigcup_{n=1}^{\infty}\sigma_n, & \operatorname{mult}\sigma_n\leqq m, \\ \max\limits_{\lambda,\,\mu\in\sigma_n}|\lambda-\mu|\leqq d<\infty, & \operatorname{dist}(\sigma_{n+1},\sigma_n)\geqq cn^{\alpha} \end{array}\right\} \tag{1}$$

where m, d, c, α are independent of n. Let $V(t)$ be a strongly continuous family of uniformly bounded self-adjoint operators satisfying

$$V(t+2\pi)=V(t).$$

Consider the evolution given by

$$i\frac{d}{dt}U(t)=(H_0+V(t))\,U(t),\qquad U(0)=1$$

and let M be the corresponding monodromy matrix *i.e.*

$$M=U(2\pi).$$

THEOREM 1. — *If* $\alpha>1/2$ *and* $V(t)$ *is norm* $\mathscr{C}^2$ *then* M *has no absolutely continuous spectrum.*

Remarks. — 1. Actually Howland considered, instead of the monodromy matrix M, the Floquet hamiltonian corresponding to $H_0+V(t)$ *i.e.*

$$K=-i\frac{d}{dt}+H_0+V(t)$$

defined in $L^2[0,2\pi]\otimes\mathscr{H}$ with periodic boundary conditions. Since the spectral properties of M and K are related (*see* e.g. [3], [4]) it is irrelevant which one of the operators is considered.

2. Our result removes the condition, imposed by Howland, that σ_n consists of a nondegenerate eigenvalue. Actually the fact that this condition can be removed has been conjectured in [1].

3. The proof is based on the adiabatic expansion machinery as developed in [5], [6] and references therein. At the expense of imposing more differentiability on $V(t)$ one can lower the value of α in Theorem 1 but the proofs are more involved.

2. PROOF

A finite number of constants will appear during the proof; for simplicity we shall denote all of them by the same letter c. Some technical points will be stated as lemmas and their proof will be given at the end. We shall use the standard notations: $\frac{d^k}{dt^k} f(t) = f^{(k)}(t)$ and $AB - BA = [A, B]$.

Consider the hamiltonian

$$H(t) = H_0 + V(t).$$

Since $V(t)$ is uniformly bounded, without restricting the generality (with an eventual relabeling of the spectrum) one can assume that the spectrum, $\sigma(t)$, of $H(t)$ satisfies (1) uniformly in t. Let Γ_n be a circle enclosing σ_n, of radius cn^α, and satisfying

$$\operatorname{dist}(\Gamma_n, \sigma(t)) \geqq cn^\alpha$$

and $P_n(t)$ be the spectral projection of $H(t)$ corresponding to $\sigma_n(t)$ *i.e.*

$$P_n(t) = \frac{1}{2\pi i} \int_{\Gamma_n} (H(t) - z)^{-1} dz.$$

Obviously, from

$$P_n^{(1)}(t) = -\frac{1}{2\pi i} \int_{\Gamma_n} (H(t) - z)^{-1} V^{(1)}(t) (H(t) - z)^{-1} dz$$

one obtains the estimate

$$\| P_n^{(1)}(t) \| \leqq cn^{-\alpha}. \tag{2}$$

Consider the operator

$$B(t) = i \sum_{n=1}^{\infty} P_n(t) P_n^{(1)}(t).$$

Due to (2), $B(t)$ is everywhere defined and

$$\| B(t) \| \leqq c \left(\sum_{n=1}^{\infty} n^{-2\alpha} \right)^{1/2}$$

LEMMA 1. — *$B(t)$ is self-adjoint and:*

$$iP_n^{(1)}(t) = [P_n(t), B(t)]$$

Consider now the operator

$$H_1(t) = H(t) + B(t).$$

Again one can suppose that $\sigma(H_1(t))$ satisfies (1) and then

$$P_{1,n}(t) = \frac{1}{2\pi i} \int_{\Gamma_n} (H_1(t) - z)^{-1} dz$$

can be defined. By direct verification one can see that $B(t)$ is norm differentiable and

$$\| B^{(1)}(t) \| \leqq c$$

uniformly in t. Let

$$B_1(t) = \sum_{n=1}^{\infty} P_{1,n}(t) \{ i P_{1,n}^{(1)}(t) - [H(t), P_{1,n}(t)] \}$$

LEMMA 2. — $B_1(t)$ *is trace-class and self-adjoint,*

$$\| B_1(t) \|_1 \leqq c.$$

Consider $U_A(t)$ given by

$$i \dot{U}_A(t) = (H(t) - B_1(t)) U_A(t), \qquad U_A(0) = 1.$$

LEMMA 3:

$$P_{1,n}(t) = U_A(t) P_{1,n}(0) U_A(t)^*. \tag{3}$$

Consider now the "Möller" operator corresponding to the pair $H(t)$, $H_1(t)$ *i. e.*

$$\Omega(t) = U_A(t)^* U(t).$$

By the usual computation

$$\Omega(t) - 1 = i \int_0^t U_A(u)^* B_1(u) U_A(u) \Omega(u) \, du. \tag{4}$$

From (4) and Lemma 2 it follows that $\Omega(t) - 1$ is trace class

$$\| \Omega(t) - 1 \|_1 \leqq c.$$

The monodromy matrix M can be written as

$$M = U(2\pi) = U_A(2\pi) + U_A(2\pi)(\Omega(2\pi) - 1). \tag{5}$$

Since $P_{1,n}(2\pi) = P_{1,n}(0)$, from (3) it follows that $[U_A(2\pi), P_{1,n}(2\pi)] = 0$ and since $P_{1,n}(2\pi)$ are finite dimensional, this implies that $U_A(2\pi)$ has pure point spectrum. This together with the fact that the second term in the r. h. s. of (5) is trace class, finishes the proof in view of the fact that by Theorem 1 in [7] the absolutely continuous spectrum is invariant under trace class perturbations.

Proof of Lemma 1. — Let Q_N be the spectral projection of $H(t)$ corresponding to $\bigcup_{N+1}^{\infty} \sigma_n$. Notice that (the variable t is omitted)

$$Q_N^{(1)} = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} (H - ix - y_N)^{-1} V^{(1)} (H - ix - y_N)^{-1} \, dx \tag{6}$$

where y_N is a point between σ_N and σ_{N+1} such that

$$\operatorname{dist}(y_N, \sigma(H)) \geqq c n^{\alpha}. \tag{7}$$

From (6) and (7) it follows

$$\lim_{N \to \infty} \| Q_N^{(1)} \| = 0. \tag{8}$$

Consider now

$$B_N = i \sum_{n=1}^{N} P_n P_n^{(1)} + i Q_N Q_N^{(1)}.$$

From (8) and

$$\left\| B - i \sum_{n=1}^{N} P_n P_n^{(1)} \right\| \leqq c \left(\sum_{n=N+1}^{\infty} n^{-2\alpha} \right)$$

it follows that

$$\lim_{N \to \infty} \| B - B_N \| = 0. \tag{9}$$

Using the identities

$$P_j P_n = P_j \delta_{j,n}$$

$$P_n Q_N = 0 \qquad \text{for} \quad n \leqq N$$

$$\sum_{n=1}^{N} P_n + Q_N = 1$$

one can see that

$$B_N = B_N^*$$

and for $n \leqq N$

$$[P_n, B_N] = i P_n^{(1)}$$

which together with (9) finishes the proof.

Proof of Lemma 2. – It is sufficient to prove that

$$\| P_{1,n} \{ i P_{1,n}^{(1)} - [H, P_{1,n}] \} \| \leqq c n^{-2\alpha}.$$

Using the identities

$$P_{1,n} = P_n - \frac{1}{2\pi i} \int_{\Gamma_n} (H - z)^{-1} B (H - z)^{-1} dz$$
$$+ \frac{1}{2\pi i} \int_{\Gamma_n} (H - z)^{-1} B (H_1 - z)^{-1} B (H - z)^{-1} dz \tag{10}$$

$$H (H - z)^{-1} = 1 + z (H - z)^{-1}$$

and Lemma 1 one obtains

$$iP_{1,n}^{(1)}-[H, P_{1,n}]=-\frac{1}{2\pi}\left(\int_{\Gamma_n}(H-z)^{-1}B(H-z)^{-1}dz\right)^{(1)}$$
$$+\frac{1}{2\pi}\left(\int_{\Gamma_n}(H-z)^{-1}B(H_1-z)^{-1}B(H-z)^{-1}dz\right)^{(1)}$$
$$-\left[B, \frac{1}{2\pi i}\int_{\Gamma_n}(H_1-z)^{-1}B(H-z)^{-1}dz\right] \quad (11)$$

The second term in the r. h. s. of (11) is bounded in norm by $cn^{-2\alpha}$ by a direct estimate. Consider now the other terms. Notice that by (10)

$$P_{1,n}=P_n+R_n$$

with

$$\|R_n\|\leqq cn^{-\alpha}.$$

Since all the terms in the r. h. s. of (11) are of order $n^{-\alpha}$ we are left with the estimation of

$$P_n\left[B, \frac{1}{2\pi i}\int_{\Gamma_n}(H_1-z)^{-1}B(H-z)^{-1}dz\right] \quad (12)$$

and

$$P_n\left(\int_{\Gamma_n}(H-z)^{-1}B(H-z)^{-1}dz\right)^{(1)}.$$

Now

$$P_n\left(\int_{\Gamma_n}(H-z)^{-1}B(H-z)^{-1}dz\right)^{(1)}=\left(P_n\int_{\Gamma_n}(H-z)^{-1}B(H-z)^{-1}dz\right)^{(1)}$$
$$-P_n^{(1)}\int_{\Gamma_n}(H-z)^{-1}B(H-z)^{-1}dz.$$

The use of (2), $[P_n, (H-z)^{-1}]=0$ and

$$P_n B=iP_n P_n^{(1)} \quad (13)$$

gives the needed estimation. To estimate (12) use the fact that

$$(H_1-z)^{-1}B(H-z)^{-1}=(H-z)^{-1}B(H_1-z)^{-1}$$

and again (13) and (2).

For the self-adjointness of B_1 notice that

$$H-B_1=i\sum_{n=1}^{\infty}P_{1,n}P_{1,n}^{(1)}+\sum_{n=1}^{\infty}P_{1,n}HP_{1,n} \quad (14)$$

and apply the proof of Lemma 1 to the first term in the r. h. s. of (14).

Proof of Lemma 3. – The lemma is a generalisation [6] of the Krein-Kato transformation matrix lemma [8], [9]. Consider for $f \in \mathscr{D}(\mathrm{H})$

$$\mathrm{F}(t)=\mathrm{U_A}(t)^*\,\mathrm{P}_{1,n}(t)\,\mathrm{U_A}(t)f.$$

By direct computation

$$i\,\mathrm{F}^{(1)}=\mathrm{U_A^*}\{i\,\mathrm{P}_{1,n}^{(1)}+[\mathrm{P}_{1,n},\,\mathrm{H}-\mathrm{B}_1]\}\,\mathrm{U_A}f. \tag{15}$$

Then using (14) and Lemma 1 applied to $\mathrm{P}_{1,n}$, the curly braket in the r.h.s. of (15) vanishes. This means that $\mathrm{U_A}(t)^*\,\mathrm{P}_{1,n}(t)\,\mathrm{U_A}(t)$ does not depend on t and since at $t=0$ it coincides with $\mathrm{P}_{1,n}(0)$, the proof is finished.

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REFERENCES

[1] J. S. Howland, Floquet Operators with Singular Spectrum, *Ann. Inst. Henri Poincaré*, Vol. **49**, 1989, pp. 309-323; J. S. Howland, Floquet Operators with Singular Spectrum. II. *Ann. Inst. Poincaré*, Vol. **49**, 1989, pp. 325-334.

[2] J. Bellissard, Stability and Instability in Quantum Mechanics, In *Trends and Developments in the Eighties*, Albeverio and Blanchard Eds., World Scientific, Singapore, 1985.

[3] K. Yajima, Scattering Theory for Schrödinger Equations with Potential Periodic in Time, *J. Math. Soc. Japan*, Vol. **29**, 1977, pp. 729-743.

[4] J. S. Howland, Scattering Theory for Hamiltonians Periodic in Time, *Indiana J. Math.*, Vol. **28**, 1978, pp. 471-494.

[5] G. Nenciu, Asymptotic Invariant Subspaces, Adiabatic Theorems and Block Diagonalisation, in *Recent Developments in Quantum Mechanics*, A. Boutet de Monvel *et al.* Eds., Kluver Academic Publishers, Dordrecht, 1991.

[6] G. Nenciu, Linear Adiabatic Theory. Exponential Estimates, *Commun. Math. Phys.*, Vol. **152**, 1993, pp. 479-496.

[7] M. S. Birman, M. G. Krein, On the Theory of Wave and Scattering Operators, *Dokl. Acad. Nauk S.S.S.R.*, Vol. **144**, 1962, pp. 475-478 (English translation: *Soviet Math.*, Vol. **3**, 1962, pp. 740-744.

[8] S. G. Krein, *Linear Differential Equations in Banach Spaces*, AMS Translations of Mathematical Monographs, Vol. **29**, Providence, 1971.

[9] T. Kato, *Perturbation Theory for Linear Operators*, Springer, Berlin, 1976.

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