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Geometry of the Kepler system in coherent states approach

by

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ABSTRACT. – We consider a family of holomorphic embeddings $\mathcal{K}_f : \mathbf{C}_3 \rightarrow \mathbb{CP}(\mathcal{M}_f)$ of nondegenerate quadric $\mathbf{C}_3$ in $\mathbb{C}^4$ into complex projective Hilbert space $\mathbb{CP}(\mathcal{M}_f)$ parametrized by functions $f \in F_3 \subset C^\infty(\mathbb{R}_+, \mathbb{R})$. We show that there is a natural correspondence between this and the Kepler system. Taking this into account we give a complete classical as well as quantum description of the system in terms of the map $\mathcal{K}_f$. In this way we illustrate in the paper a nontrivial application of the theory developed in our earlier publications.

RÉSUMÉ. – Nous considérons une famille de plongements holomorphes $\mathcal{K}_f : \mathbf{C}_3 \rightarrow \mathbb{CP}(\mathcal{M}_f)$ d'une quadrique non dégénérée $\mathbf{C}_3$ de $\mathbb{C}^4$ dans l'espace projectif complexe $\mathbb{CP}(\mathcal{M}_f)$ paramétrisée par des fonctions $f \in F_3 \subset C^\infty(\mathbb{R}_+, \mathbb{R})$. Nous prouvons qu'il existe une correspondance naturelle avec le système de Kepler. Comme conséquence nous donnons une description complète classique et quantique du problème de Kepler en terme de l'application $\mathcal{K}_f$. Ceci illustre une application non triviale de nos résultats antérieurs.

0. INTRODUCTION

There is an important connection between the classical and quantum descriptions of the harmonic oscilator. It is given by a symplectic embedding of the classical phase space $\mathbb{C}^n$ into the quantum phase space $\mathbb{CP}(\mathcal{M})$ of the oscilator, $\mathcal{M}$ being a complex Hilbert space. This embedding is equivariant with respect to the one-parameter group of symplectomorphisms of $\mathbb{C}^n$ corresponding to the Hamiltonian vector field and the flow on $\mathbb{CP}(\mathcal{M})$ determined by the Schrödinger equation. The quantum states which are the images of the classical states under this embedding, called coherent states, minimize the uncertainty principle and have some other properties which allow us to treat them as being closest to the classical states. Having discovered this kind of states, E. Schrödinger concluded the paper [Sch] with a conjecture that a similar situation should take place for the Kepler system. He also pointed out that this case might turn out to be more complicated from the mathematical point of view. Schrödinger's supposition has been confirmed, in a sense, in [R].

This paper is on the one hand a generalization of [R] to the case of an arbitrary Kähler potential, while on the other hand it provides an example of an application of the formalism developed in [O1], [O2]. According to this formalism, any physical system is described by the triple: a symplectic manifold M, a complex Hilbert space $\mathcal{M}$ and a symplectic embedding $\mathcal{K}: \mathrm{M} \to \mathbb{CP}(\mathcal{M})$. Assuming that the system is in an equilibrium state, we can express the action functional, the transition amplitude between coherent states, the Schrödinger equation propagator and other characteristics of the system in terms of the mapping $\mathcal{K}$ (*see* [O1], [O2]). If the physical system is understood in this sense the classical and quantum descriptions are complementary to each other. The problem of computing the path integral is equivalent to finding $\mathcal{K}: \mathrm{M} \to \mathbb{CP}(\mathcal{M})$ for a given action functional.

In this paper we consider the following triple of objects:

(*a*) A nondegenerate quadric $\mathbf{C}_3 \subset \mathbb{C}^4 \setminus \{0\}$ which is the classical phase space of the regularized Kepler problem (*see* [Ku], [M]).

(*b*) A complex Hilbert space $\mathcal{M}_f$ which is realized as the space of square integrable (with respect to the Liouville measure) holomorphic sections of a line bundle $\mathbb{E}^{0,0}$ over $\mathbf{C}_3$.

(*c*) An embedding $\mathcal{K}_f: \mathbf{C}_3 \to \mathbb{CP}(\mathcal{M}_f)$ which is determined by evaluation functionals on $\mathcal{M}_f$.

This gives rise to a family of physical systems parametrized by a function $f: \mathbb{R}_+ \to \mathbb{R}$ in a subset F_3 of $C^\infty(\mathbb{R}_+, \mathbb{R})$. It is the study of these systems which is the purpose of the paper.

Section 1 contains some facts which are needed to understand the next two sections. We describe SU (2) × SU (2) Hermitian line bundles over the quadric $\mathbf{C}_3$. We also construct an orthonormal basis of the Hilbert space $\mathcal{M}_f$ and find an explicit dependence of the reproducing kernel on the potential $f \in F_3$ [formula (1.16)]. In the special case $f(x)=x^a$, $a>0$, we find an expression for the reproducing kernel $\mathrm{K}_f(\bar{z}, w)$ in terms of the Mittag-Leffler type functions [formula (1.27)]. This generalizes the result of [R] corresponding to the case $f=\mathrm{id}$. Moreover, we formulate conditions on f which guarantee that the space $\mathcal{M}_f$ is ample and the mapping $\mathcal{K}_f$ is symplectic.

In Section 2 we define a family of equilibrium states, that is, mixtures of coherent states $\mathcal{K}_f(z)$, $z \in \mathbf{C}_3$, with an SU(2) × SU(2) invariant weight function $\rho(x)=\phi(\mathrm{H}(x))$, where $\phi:\mathbb{R}_+ \to \mathbb{R}_+$, $\mathrm{H}:\mathbf{C}_3 \to \mathbb{R}$ and $x=(2z\cdot\bar{z})^{1/2}$. Next, using the formalism of [O2] and introducing appropriate coordinates, we show that, for $\mathrm{H}(x)=\frac{1}{2}x(\log \mathfrak{k}_f)'(x)$, (where $\mathfrak{k}_f(x):=\mathrm{K}_f(\bar{z}, z)$), the system $(\mathbf{C}_3, \mathcal{M}_f, \mathcal{K}_f:\mathbf{C}_3 \to \mathbb{CP}(\mathcal{M}_f))$ coincides with the regularized Kepler system.

In Section 3 we compare the quantization based on Ehrenfest's theorem (*see* [E], [O1]) with the Kostant-Souriau quantization. The Ehrenfest quantization is characterized by the property that computing the mean value of a quantized classical observable (function) on coherent states reproduces this function (*see* [O1]). The generators of the group SU(2) × SU(2), constant functions and the Hamiltonian H span the space of functions which are quantizable in the sense of Ehrenfest. Together with the results of Section 2 this shows that the Kepler system is distinguished by Ehrenfest's quantization condition.

The requirement that the Ehrenfest quantization be equivalent to the Kostant-Souriau quantization imposes a condition on f which has the form of an integro-differential equation (3.12). We do not discuss in the paper the question of solvability of this equation. We show, however, that the function $f(x)=x^a$, $a>0$, satisfies this equation asymptotically as $x \to \infty$. This implies that, in the high energy region, the probability density corresponding to the coherent state $\mathcal{K}_f(z)$, $z \in \mathbf{C}_3$, is localized around the point z. Since the embedding $\mathcal{K}_f$ is equivariant with respect to the dynamics of the system, we have obtained those of the states anticipated by Schrödinger the experimental realization of which has been proposed in [Y-S] and [P-S].

1. GEOMETRIC PRELIMINARIES

In this section we describe SU(2) × SU(2) Hermitian line bundles over a nondegenerate quadric in $\mathbb{C}^4$ and the spaces of holomorphic sections of these bundles. These facts will be used in the following sections.

We start by describing some properties of the quadric

$$\mathbf{C}_3 := \{ z \in \mathbb{C}^4 : z \cdot z = z_0^2 + z_1^2 + z_2^2 + z_3^2 = 0,\ z \neq 0 \}. \tag{1.1}$$

It was observed by Souriau [S] that there is a natural isomorphism of manifolds $\mathbf{C}_3 \cong \mathrm{T}_0^* \mathrm{S}^3$, where $\mathrm{T}_0^* \mathrm{S}^3$ is the cotangent bundle to the three-dimensional sphere with the zero section deleted. It is given by

$$z = |u| e + iu, \tag{1.2}$$

where $e = (e_0, \vec{e})$, $u = (u_0, \vec{u}) \in \mathbb{R}^4$ satisfy the conditions

$$e \cdot e = 1, \qquad e \cdot u = 0, \qquad u \neq 0 \tag{1.3}$$

and $|u|^2 = u \cdot u$. Since $\mathbf{C}_3$ is the zero level set of a nondegenerate quadratic form in $\mathbb{C}^4 \setminus \{0\}$, its image under the natural projection onto $\mathbb{CP}(3)$ is isomorphic to $\mathbb{CP}(1) \times \mathbb{CP}(1)$ (*see* e. g. [G-H]). It follows that $\mathbf{C}_3$ is a $\mathbb{C}^*$ principal bundle over $\mathbb{CP}(1) \times \mathbb{CP}(1)$. Let $\mathbb{F}^{k,l} := \mathrm{pr}_1^* (\overset{k}{\otimes} \mathbb{E}) \otimes \mathrm{pr}_2^* (\overset{l}{\otimes} \mathbb{E})$, where pr_i denotes the projection of $\mathbb{CP}(1) \times \mathbb{CP}(1)$ onto the i-th factor, $\mathbb{E} \to \mathbb{CP}(1)$ is the universal bundle and $k, l \in \mathbb{Z}$. The quadric $\mathbf{C}_3$ can be obtained as the pull-back of the universal bundle over $\mathbb{CP}(3)$ with zero section deleted by the Segre imbedding $\mathrm{S} : \mathbb{CP}(1) \times \mathbb{CP}(1) \to \mathbb{CP}(3)$. Comparing the transition functions for this pull-back bundle and $\mathbb{F}^{1,1}$, we get the isomorphism of bundles

$$\mathbf{C}_3 \cong \mathbb{F}^{1,1} \setminus \{ \text{zero section} \}. \tag{1.4}$$

Let $\mathbb{E}^{k,l} := \pi^* (\mathbb{F}^{k,l})$, where $\pi : \mathbf{C}_3 \to \mathbb{CP}(1) \times \mathbb{CP}(1)$.

PROPOSITION 1.1. — (i) $\forall k, l, m, n \in \mathbb{Z}\ \mathbb{E}^{k,l} \otimes \mathbb{E}^{m,n} \cong \mathbb{E}^{k+m, l+n}$,

(ii) $\forall k \in \mathbb{Z}\ \mathbb{E}^{k,k} \cong \mathbf{C}_3 \times \mathbb{C}$,

(iii) $\mathrm{H}^2 (\mathbf{C}_3, \mathbb{Z}) \cong \mathbb{Z}$,

(iv) *For each complex line bundle* $\mathbb{L}$ *over* $\mathbf{C}_3$, *there is a* $k \in \mathbb{Z}$ *such that* $\mathbb{L} \cong \mathbb{E}^{0,k}$.

Proof. — The isomorphisms (i) and (ii) are obvious.

(iii) We have the following isomorphisms

$$\mathrm{H}^2 (\mathbf{C}_3, \mathbb{Z}) \cong \mathrm{H}^2 (\mathrm{T}_0^* \mathrm{S}^3, \mathbb{Z}) \cong \mathrm{H}^2 (\mathrm{S}^3 \times \mathrm{S}^2, \mathbb{Z}) \cong \mathbb{Z} \tag{1.5}$$

The second one follows from the homotopy equivalence of $\mathrm{T}_0^* \mathrm{S}^3 \cong \mathrm{S}^3 \times (\mathbb{R}^3 \setminus \{0\})$ and $\mathrm{S}^3 \times \mathrm{S}^2$ and the last one is obtained by applying the Künneth formula to the product of spheres $\mathrm{S}^3 \times \mathrm{S}^2$.

(iv) This follows from the fact that the Chern class of $\mathbb{E}^{0,1}$ is the generator of the group $\mathrm{H}^2 (\mathbf{C}_3, \mathbb{Z})$. □

The bundle $\mathbb{E}^{k,l}$ inherits from $\mathbb{E} \to \mathbb{CP}(1)$ the structure of Hermitian line bundle.

The action of the group $\mathrm{SU}(2)$ on $\mathbb{CP}(1)$ induces an action of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on the manifolds $\mathbb{F}^{k,l}$ and $\mathbb{E}^{k,l}$; in particular we obtain an action of $SU(2)\times SU(2)$ on $\mathbf{C}_3$. This action preserves the Hermitian line bundle structure of $\mathbb{E}^{k,l}$.

PROPOSITION 1.2. – (i) $\mathbf{C}_3$ *splits into five-dimensional orbits of the group* $SU(2)\times SU(2)$ *labelled by a positive parameter* $x=(2z\cdot\bar{z})^{1/2}$.

(ii) *The bundles* $\mathbb{E}^{k,k}$, $k\in\mathbb{Z}$, *are isomorphic as* $SU(2)\times SU(2)$ *bundles.* □

We now describe the spaces of global holomorphic sections of the bundles $\mathbb{E}^{k,l}$ which will be denoted, as usual, by $H^0(\mathbf{C}_3, \mathcal{O}(\mathbb{E}^{k,l}))$. To this end, we introduce coordinates on $\mathbf{C}_3$ which are compatible with the fibration of $\mathbf{C}_3$ over $\mathbb{CP}(1)\times\mathbb{CP}(1)$. Let

$$\{V_1:=\pi^{-1}(\mathcal{O}_1\times\mathcal{O}'_1),\ V_2:=\pi^{-1}(\mathcal{O}_1\times\mathcal{O}'_2),\ V_3:=\pi^{-1}(\mathcal{O}_2\times\mathcal{O}'_1),\ V_4:=\pi^{-1}(\mathcal{O}_2\times\mathcal{O}'_2)\}$$

be a covering of $\mathbf{C}_3$, where $\mathcal{O}_i=\{[(\zeta_1,\zeta_2)]\in\mathbb{CP}(1):\zeta_i\neq 0\}$, $i=1,2$, cover the first factor of $\mathbb{CP}(1)\times\mathbb{CP}(1)$ and $\mathcal{O}'_i$, $i=1,2$, which are defined in a similar way, cover the second factor. Let $\mathcal{O}_1\ni[(\zeta_1,\zeta_2\}]\mapsto s_1:=\frac{\zeta_2}{\zeta_1}\in\mathbb{C}$ and $\mathcal{O}_2\ni[(\zeta_1,\zeta_2\}]\mapsto s_2:=\frac{\zeta_1}{\zeta_2}\in\mathbb{C}$ be the coordinates on the first factor of $\mathbb{CP}(1)\times\mathbb{CP}(1)$; the coordinates on the second factor will be denoted by t_1 and t_2. We can now define an atlas of $\mathbf{C}_3$

$$\left.\begin{aligned}\chi_1:&\quad V_1\ni z\mapsto(s_1,t_1,\lambda_1)\in\mathbb{C}\times\mathbb{C}\times\mathbb{C}^*,\\ \chi_2:&\quad V_2\ni z\mapsto(s_1,t_2,\lambda_2)\in\mathbb{C}\times\mathbb{C}\times\mathbb{C}^*,\\ \chi_3:&\quad V_3\ni z\mapsto(s_2,t_1,\lambda_3)\in\mathbb{C}\times\mathbb{C}\times\mathbb{C}^*,\\ \chi_4:&\quad V_4\ni z\mapsto(s_2,t_2,\lambda_4)\in\mathbb{C}\times\mathbb{C}\times\mathbb{C}^*.\end{aligned}\right\}\tag{1.6}$$

Transition functions for this atlas are given by $s_1=\frac{1}{s_2}$, $t_1=\frac{1}{t_2}$, $\lambda_4=t_1\lambda_3=s_1\lambda_2=s_1t_1\lambda_1$. To simplify the notation, we shall write all expression in the chart (V_1,χ_1) and we shall drop the subscript 1. We shall denote the elements of $\mathbf{C}_3$ by z, w and so on, and we shall write $z=(s,t,\lambda)$ and $w=(s',t',\lambda')$. We have the following relations

$$\left.\begin{aligned}z_0&=\frac{1}{2}\lambda(1+st), & z_1&=\frac{1}{2}\lambda(s-t),\\ z_2&=\frac{i}{2}\lambda(s+t), & z_3&=\frac{i}{2}\lambda(1-st),\end{aligned}\right\}\tag{1.7}$$

which imply

$$\left.\begin{aligned}\bar{z}&=(\bar{s},\bar{t},\bar{\lambda}),\\ 2z\cdot w&=\lambda\lambda'(1+ss')(1+tt').\end{aligned}\right\}\tag{1.8}$$

The bundle $\pi_{k,l}: \mathbb{E}^{k,l} \to \mathbf{C}_3$ trivializes over V_α, $\alpha = 1, 2, 3, 4: \pi_{k,l}^{-1}(V_\alpha) \cong V_\alpha \times \mathbb{C}$, transition functions $g_{\alpha\beta}^{k,l}: V_\alpha \times V_\beta \to \mathbb{C}^*$ being

$$\left.\begin{array}{lll} g_{1,2}^{k,l}(z) = t^{-l}, & g_{1,3}^{k,l}(z) = s^{-k}, & g_{1,4}^{k,l}(z) = s^{-k} t^{-l}, \\ g_{2,3}^{k,l}(z) = s^{-k} t^{-l}, & g_{2,4}^{k,l}(z) = s^{-k}, & g_{3,4}^{k,l}(z) = t^{-l}. \end{array}\right\} \quad (1.9)$$

We choose nonvanishing local sections $\sigma_{0,\alpha}: V_\alpha \to V_\alpha \times \mathbb{C}$ corresponding to this trivialization given by $\sigma_{0,\alpha}(z) := (z, 1)$. According to our convention, we write σ_0 rather than $\sigma_{0,1}$.

Expanding an arbitrary $\sigma \in \mathrm{H}^0(\mathbf{C}_3, \mathcal{O}(\mathbb{E}^{k,l}))$ into a power series and using the compatibility conditions implied by (1.9), we obtain

PROPOSITION 1.3. — *Any* $\sigma \in \mathrm{H}^0(\mathbf{C}_3, \mathcal{O}(\mathbb{E}^{k,l}))$, *when restricted to* V_1, *is of the form* $\psi\sigma_0$, *where* $\psi \in \mathcal{O}(V_1)$ *has the power series expansion*

$$\psi(s, t, \lambda) = \sum_{c=\min\{k,l\}}^{\infty} \sum_{a=0}^{c-k} \sum_{b=0}^{c-l} \mathrm{A}_{abc} s^a t^b \lambda^c, \qquad \mathrm{A}_{abc} \in \mathbb{C}. \quad \square \quad (1.10)$$

The natural action Σ of the group

$$\mathrm{SU}(2) \times \mathrm{SU}(2) = \left\{ (g, h) = \left(\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \begin{pmatrix} \gamma & \delta \\ -\bar{\delta} & \bar{\gamma} \end{pmatrix} \right) \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \times \mathrm{Mat}_{2\times 2}(\mathbb{C}) : |\alpha|^2 + |\beta|^2 = 1 = |\gamma|^2 + |\delta|^2 \right\} \text{ on } \mathbf{C}_3$$

is given in the coordinates (s, t, λ) by

$$\Sigma(g, h)(s, t, \lambda) = ((-\bar{\beta} + \bar{\alpha} s)/(\alpha + \beta s), (-\bar{\delta} + \bar{\gamma} t)/(\gamma + \delta t), \lambda(\alpha + \beta s)(\gamma + \delta t)). \quad (1.11)$$

The corresponding action T on the sections of $\mathbb{E}^{k,l}$ is

$$(\mathrm{T}(g, h)\psi\sigma_0)(s, t, \lambda) = \left(\alpha + \beta \frac{\bar{\beta} + \alpha s}{\bar{\alpha} - \beta s}\right)^k \left(\gamma + \delta \frac{\bar{\delta} + \gamma t}{\bar{\gamma} - \delta t}\right)^l \times \psi(\Sigma(g^{-1}, h^{-1})(s, t, \lambda))\, \sigma_0(s, t, \lambda). \quad (1.12)$$

This implies

PROPOSITION 1.4. — *The subspaces* $\mathrm{V}_n \subset \mathrm{H}^0(\mathbf{C}_3, \mathcal{O}(\mathbb{E}^{k,l}))$ *spanned by sections of the form*

$$\tau(s, t, \lambda) = \left(\sum_{a=0}^{c-k} \sum_{b=0}^{c-l} \mathrm{A}_{abc} s^a t^b \lambda^c \right) \sigma_0(s, t, \lambda), \quad (1.13)$$

where $c = n - 1$, *are* $\mathrm{T}(\mathrm{SU}(2) \times \mathrm{SU}(2))$ *invariant and*

$$\dim_{\mathbb{C}} \mathrm{V}_n = (n + |k|)(n + |l|). \quad \square \quad (1.14)$$

From now on we restrict our considerations to the bundle $\mathbb{E}^{0,0}$.

To describe SU(2) × SU(2) invariant Hermitian structures on $\mathbb{E}^{0,0}$, let us note that the section σ_0 extends to a unique (up to a constant coefficient) SU(2) × SU(2) invariant nonvanishing global section of $\mathbb{E}^{0,0}$. This follows from T(SU(2) × SU(2)) invariance of V_1 and from formula (1.12). An SU(2) × SU(2) invariant Hermitian structure H_f on $\mathbb{E}^{0,0}$ is defined by setting

$$H_f(\sigma_0(z), \sigma_0(z)) := e^{-f(x)}, \tag{1.15}$$

where $x^2 = 2z.\bar{z}$ and $f \in C^\infty(\mathbb{R}_+, \mathbb{R})$. Thus, the mapping $f \mapsto H_f$ establishes a 1−1 correspondence between $C^\infty(\mathbb{R}_+, \mathbb{R})$ and the space of SU(2) × SU(2) invariant smooth Hermitian structures on $\mathbb{E}^{0,0}$. The Hermitian structure H_f determines a metric connection ∇^f on $\mathbb{E}^{0,0}$ (*see* [We]) such that

$$\nabla^f \sigma_0 = (\partial f) \otimes \sigma_0, \tag{1.16}$$

where $\sigma_0 : C_3 \to \mathbb{E}^{0,0}$ is the global section defined above. The curvature (1, 1)-form of this connection will be denoted by $-i\omega_f$, *i.e.*

$$\omega_f := i \operatorname{curv} \nabla^f = i\partial\bar{\partial} f. \tag{1.17}$$

In what follows we shall be assuming that the function f satisfies the following conditions

$$\forall x \in \mathbb{R}_+, \quad f'(x) \neq 0 \quad \text{and} \quad f'(x) + xf''(x) \neq 0, \tag{1.18}$$

$$\forall c \in \mathbb{N}, \quad I_f(c) := \int_0^\infty y^{2c+2}\left(h(y) + \frac{y}{3}h'(y)\right)e^{-f(y)}\,dy < \infty, \tag{1.19}$$

where $h(y) = [f'(y)]^3$.

Condition (1.18) is equivalent to nondegeneracy of the curvature form ω_f. Condition (1.19) is equivalent to the fact that, for each $n \in \mathbb{N}$, the spaces V_n are contained in the space $\mathcal{M}_f$ of square-integrable holomorphic sections

$$\mathcal{M}_f := \left\{ \sigma \in H^0(C_3, \mathcal{O}(\mathbb{E}^{k,l})) : \langle \sigma | \sigma \rangle_f := \int_{C_3} H_f(\sigma, \sigma)\, d\mu_f < \infty \right\} \tag{1.20}$$

where $d\mu_f := \omega_f \wedge \omega_f \wedge \omega_f$ is the Liouville form corresponding to the symplectic form ω_f. It is a standard fact that $\mathcal{M}_f$ is complete (*see* [W]).

Let $\sigma_{a,b,c} = \psi_{a,b,c}\sigma_0$, where

$$\psi_{a,b,c}(s, t, \lambda) := s^a t^b \lambda^c \quad c = 0, 1, \ldots, a, b = 0, 1, \ldots, c. \tag{1.21}$$

These sections form an orthogonal basis of $\mathcal{M}_f$, their norms being

$$\begin{aligned} \|\sigma_{a,b,c}\|_f^2 &= \langle \sigma_{a,b,c} | \sigma_{a,b,c} \rangle_f \\ &= A I_f(c)\, B(a+1, c-a+1)\, B(b+1, c-b+1), \end{aligned} \tag{1.22}$$

where A is a constant independent of f and $B(u, v)$ is the Euler beta function.

Writing $\sigma=\psi\sigma_0$ in the frame σ_0, we can identify $\mathcal{M}_f$ with the space of holomorphic functions on $\mathbf{C}_3$ which are square-integrable with respect to the measure $e^{-f}\,d\mu_f$. Let

$$e_z(\sigma):=\psi(z) \qquad \text{for} \quad \sigma\in\mathcal{M}_f \quad \text{and} \quad z\in\mathbf{C}_3. \tag{1.23}$$

be the evaluation functional. One knows [W] that this is a continuous functional on $\mathcal{M}_f$. It follows that, for each $z\in\mathbf{C}_3$, there is a $\mathrm{K}_f(z)\in\mathcal{M}_f$ such that

$$\forall\,\sigma=\psi\sigma_0\in\mathcal{M}_f, \qquad \psi(z)=\langle \mathrm{K}_f(z)\,|\,\sigma\rangle_f. \tag{1.24}$$

Expressing $\mathrm{K}_f(z)=\mathrm{K}_f(\bar{z},\,\cdot)\,\sigma_0$ in the frame σ_0, we get the reproducing property

$$\mathrm{K}_f(\bar{z},\,w)=\langle \mathrm{K}_f(z)\,|\,\mathrm{K}_f(w)\rangle_f. \tag{1.25}$$

Expressing $\mathrm{K}_f(z)$ in the basis $\|\sigma_{a,b,c}\|_f^{-1}\sigma_{a,b,c}$ we obtain the following expression for the reproducing kernel

$$\mathrm{K}_f(\bar{z},\,w)=\frac{1}{\mathrm{A}}\sum_{c=0}^{\infty}\sum_{a=0}^{c}\sum_{b=0}^{c}\frac{(\bar{s}s')^a(\bar{t}t')^b(\bar{\lambda}\lambda')^c}{\mathrm{I}_f(c)\,\mathrm{B}(a+1,\,c-a+1)\,\mathrm{B}(b+1,\,c-b+1)}$$
$$=\mathrm{M}\sum_{c=0}^{\infty}\frac{(c+1)^2}{\mathrm{I}_f(c)}(2\,w\cdot\bar{z})^c. \tag{1.26}$$

From this formula we see that the diagonal $\mathrm{K}_f(\bar{z},\,z)$ is an even function of $x=(2\,z\cdot\bar{z})^{1/2}$. It is convenient to introduce the notation $\mathfrak{k}_f(x):=\mathrm{K}_f(\bar{z},\,z)$.

Example. – Let $f(x)=x^a$, $a>0$. Then

$$\mathrm{K}_f(\bar{z},\,w)=\mathrm{M}\frac{d}{dv}v\frac{d}{dv}v\,\mathrm{E}_{a/2}(v;\,3)\big|_{v=2\,\bar{z}\cdot w}$$
$$=\frac{\mathrm{M}\,a^2}{4}\left(\mathrm{E}_{a/2}(v;\,1)+\left(\frac{4}{a}-3\right)\mathrm{E}_{a/2}(v;\,2)\right.$$
$$\left.+\left(2-\frac{2}{a}\right)^2\mathrm{E}_{a/2}(v;\,3)\right)\Bigg|_{v=2\,\bar{z}\cdot w}, \tag{1.27}$$

where $\mathrm{E}_\rho(v;\,\beta)$ is a function of Mittag-Leffler type (*see* [D] and Section 3). In particular, for $f=\mathrm{id}$ we get the formula

$$\mathrm{K}_{\mathrm{id}}(\bar{z},\,w)=\mathrm{M}\left(\frac{\operatorname{sh} v}{v}+\operatorname{ch} v\right)\Bigg|_{v=(2\,\bar{z}\cdot w)^{1/2}},$$

which was obtained in [R].

2. THE REGULARIZED KEPLER SYSTEM

In [O2] it was shown that any mechanical system can be described by a triple $(\mathrm{M},\,\mathcal{M},\,\mathcal{K}:\mathrm{M}\to\mathbb{CP}(\mathcal{M}))$, where M is a differentiable manifold, $\mathcal{M}$

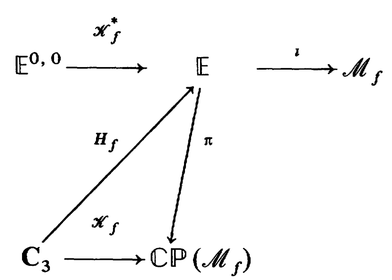
is a complex Hilbert space and $\mathscr{K}$ is a symplectic embedding in the sense that the pull-back $\mathscr{K}^* \omega_{FS}$ of the Fubini-Study form ω_{FS} on $\mathbb{CP}(\mathscr{M})$ is nondegenerate. According to [O2], the symplectic manifold $(M, \mathscr{K}^* \omega_{FS})$ is interpreted as the classical phase space of the system. The symplectic manifold $(\mathbb{CP}(\mathscr{M}), \omega_{FS})$ is the space of pure quantum states of the system and the states in $\mathscr{K}(M)$ are interpreted as coherent states. Let $P(m)$ denote the orthogonal projection onto $\mathscr{K}(m)$, $m \in M$, and let $\rho \in L^1(M, d\mu_L)$, where $d\mu_L := \overset{n}{\wedge} \mathscr{K}^* \omega_{FS}$ is the Liouville form. Then the operator

$$P(\rho) := \int_M \rho(m) P(m) \, d\mu_L(m) \tag{2.1}$$

corresponds to a mixed state of the system. Among the mixed states one distinguishes a family of equilibrium states (*see* [O2]) which can be characterized by means of a mapping $H: M \to \mathbb{R}$. The mapping H depends functionally on $\mathscr{K}: M \to \mathbb{CP}(\mathscr{M})$ and is interpreted as the energy of the system. It was shown in [O2], by means of the notion of formal path integral, how the action functional depends on $\mathscr{K}$ and H. In this approach the mapping $\mathscr{K}: M \to \mathbb{CP}(\mathscr{M})$ is a primary object, the action functional as well as other characteristics of the system being dependent on it.

In this section, basing on Chapter II of [O2], we shall give a physical interpretation of the triple $(\mathbf{C}_3, \mathscr{M}_f, \mathscr{K}_f: \mathbf{C}_3 \to \mathbb{CP}(\mathscr{M}_f))$, where $\mathscr{K}_f$ is defined by $\mathscr{K}_f(z) := [K_f(z)]$, $z \in \mathbf{C}_3$ [*see* (1.24)]. In order to make the paper self contained, we shall repeat step by step some considerations of [O2] restricting them to the model being studied here.

First we show that $\mathscr{K}_f$ is a symplectic embedding. Let us consider the diagram



(2.2)

where $\mathbb{E} := \{(l, v) \in \mathbb{CP}(\mathscr{M}_f) \times \mathscr{M}_f : v \in l\}$ is the universal complex line bundle over $\mathbb{CP}(\mathscr{M}_f)$, π and ι denote the projections of $\mathbb{E}$ onto the first and second factor of the product $\mathbb{CP}(\mathscr{M}_f) \times \mathscr{M}_f$ and $H_f(z) := (\mathscr{K}_f(z), K_f(z))$. The bundle $\mathbb{E}^{0,0}$ being trivial, we obtain a holomorphic morphism of bundles $\mathscr{K}_f^*: \mathbb{E}^{0,0} \to \mathbb{E}$ given by $\mathscr{K}_f^*(\sigma_0(z)) := H_f(z)$. With this definition of $\mathscr{K}_f^*$, the pull-back $\mathscr{K}_f^* H_{FS}$ of the canonical Hermitian structure of $\mathbb{E}$ to $\mathbb{E}^{0,0}$ takes the form

$$\mathcal{K}_f^* \mathrm{H}_{\mathrm{FS}}(\sigma_0(z), \sigma_0(z)) = \mathrm{K}_f(\bar{z}, z), \tag{2.3}$$

where $\mathrm{K}_f(\bar{z}, z)$ is the diagonal of the reproducing kernel (1.26). The (1, 1)-form

$$\mathcal{K}_f^* \omega_{\mathrm{FS}} = i\,\partial\bar{\partial} \log \mathrm{K}_f(\bar{z}, z) \tag{2.4}$$

is nondegenerate what follows from the proposition:

PROPOSITION 2.1. — *If f satisfies* (1.18) *and* (1.19) *then these conditions are also satisfied by* $\log \mathfrak{k}_f(x)$.

Proof. — Condition (1.18) holds for any even analytic function $g(x) = \sum_{n=0}^{\infty} a_n x^{2n}$ with positive coefficients ($a_n > 0$) and the radius of convergence $\mathrm{R} = \infty$. It follows from (1.26) that $\mathfrak{k}_f$ is such a function.

From (1.26) it follows that $1/\mathfrak{k}_f$ is a positive Schwartz function on $\mathbb{R}_+ \cup \{0\}$. The inequality $\mathfrak{k}_f(x) \geq a_{2n} x^{4n}$ for all $n \in \mathbb{N} \cup \{0\}$ implies that $u_f := 1/\mathfrak{k}_f^{1/4}$ is a Schwartz function. Now, for all $c \in \mathbb{N} \cup \{0\}$, the condition (1.19) takes the following form:

$$\mathrm{I}_{\log \mathfrak{k}_f}(c) = \int_0^{\infty} y^{2c+2} [u_f'(y)]^2 [u_f(y) u_f'(y) + y u_f(y) u_f''(y) - y (u_f'(y))^2]\, dy < \infty,$$

and it is fulfilled because the derivative of a Schwartz function is a Schwartz function and the product of Schwartz functions is a Schwartz function. □

Because of $\mathrm{V}_n \subset \mathcal{M}_f$, $n \in \mathbb{N} \cup \{0\}$, the functions from $\mathcal{M}_f$ separate points of $\mathbf{C}_3$, which implies injectivity of $\mathcal{K}_f$. We have thus shown that $\mathcal{K}_f$ is a symplectic embedding of $(\mathbf{C}_3, \mathcal{K}_f^* \omega_{\mathrm{FS}})$ into $(\mathbb{CP}(\mathcal{M}_f), \omega_{\mathrm{FS}})$.

Given the mapping $\mathrm{K}_f : \mathbf{C}_3 \to \mathcal{M}_f$, we can write down an explicit form of the transition amplitude

$$a_f(z, w) = \frac{\langle \mathrm{K}_f(z) | \mathrm{K}_f(w) \rangle_f}{\langle \mathrm{K}_f(z) | \mathrm{K}_f(z) \rangle_f^{1/2} \langle \mathrm{K}_f(w) | \mathrm{K}_f(w) \rangle_f^{1/2}} = \frac{\mathrm{K}_f(\bar{z}, w)}{\mathrm{K}_f(\bar{z}, z)^{1/2} \mathrm{K}_f(\bar{w}, w)^{1/2}} \tag{2.5}$$

between the coherent states $\mathcal{K}_f(z)$ and $\mathcal{K}_f(w)$, which will be identified in what follows with the classical states z and w respectively. Suppose that the system is in a mixed state $\mathrm{P}_f(\rho)$ defined by a normalized $\left(\int_{\mathbf{C}_3} \rho\, d\mu_{\mathrm{L},f} = 1\right)$ weight function $\rho \in \mathrm{L}^1(\mathbf{C}_3, d\mu_{\mathrm{L},f})$, where $d\mu_{\mathrm{L},f} = \mathcal{K}_f^* \omega_{\mathrm{FS}} \wedge \mathcal{K}_f^* \omega_{\mathrm{FS}} \wedge \mathcal{K}_f^* \omega_{\mathrm{FS}}$. Then the probability of finding the system in a coherent state $z \in \mathbf{C}_3$ is given by

$$\langle \mathrm{P}_f(\rho) \rangle (z) := \mathrm{Tr}(\mathrm{P}_f(\rho)\,\mathrm{P}_f(z)) = \int_{\mathbf{C}_3} |a_f(z, w)|^2 \rho(w)\, d\mu_{\mathrm{L}, f}(w), \quad (2.6)$$

where $\mathrm{P}_f(z)$ is the orthogonal projection in $\mathscr{M}_f$ onto the state $\mathscr{K}_f(z)$. Interpreting $\langle \mathrm{P}_f(\rho) \rangle : \mathbf{C}_3 \to \mathbb{R}_+ \cup \{0\}$ as the weight function of the mixed state $\mathrm{P}_f(\langle \mathrm{P}_f(\rho) \rangle)$, we introduce (following [O2]) the condition of stability of mixed states with respect to the interaction of the system with a classical device which measures the probability of finding the system in a state z, provided that it is localized in $\mathscr{K}_f(\mathbf{C}_3)$ with probability density given by ρ. Namely, we require that the level sets of the function $\langle \mathrm{P}_f(\rho) \rangle$ are the same as those of ρ, which is equivalent to the following condition

$$\left.\begin{array}{c} \exists\, \mathrm{H} : \mathbf{C}_3 \to \mathbb{R}_+ \ \&\ \text{bijections } \phi, \tilde{\phi} : \mathbb{R}_+ \to \mathbb{R}_+ \text{ such that} \\ \rho = \phi \circ \mathrm{H} \ \&\ \langle \mathrm{P}_f(\rho) \rangle = \tilde{\phi} \circ \mathrm{H}. \end{array}\right\} \quad (2.7)$$

The mixed states $\mathrm{P}_f(\rho)$ that satisfy (2.7) will be called equilibrium states of the system and H will be interpreted as the energy function of the system.

In what follows, we shall restrict our considerations to the family of equilibrium states defined by $\mathrm{SU}(2) \times \mathrm{SU}(2)$ invariant weight functions, *i.e.* $\rho = \rho(x) = \phi(\mathrm{H}(x))$, where $x = (2 z \cdot \bar{z})^{1/2}$ labels the orbits of $\mathrm{SU}(2) \times \mathrm{SU}(2)$ on $\mathbf{C}_3$. Due to $\mathrm{SU}(2) \times \mathrm{SU}(2)$ equivariance of $\mathscr{K}_f : \mathbf{C}_3 \to \mathbb{CP}(\mathscr{M}_f)$, the space of $\mathrm{SU}(2) \times SU(2)$ invariant equilibrium states is parametrized by the mappings of $\mathbf{C}_3$ to $\mathbb{R}$ which are constant on the orbits of $\mathrm{SU}(2) \times \mathrm{SU}(2)$.

In view of the equality

$$d\mu_{\mathrm{L}, f}(v) = \rho_0^{-1}(x)\, \mathfrak{k}_f(x)\, e^{-f(x)}\, d\mu_f(v) \quad (2.8)$$

where

$$\left.\begin{array}{c} \rho_0(x) = \dfrac{h(x) + (x/3)\, h'(x)}{k(x) + (x/3)\, k'(x)}\, \mathfrak{k}_f(x)\, e^{-f(x)}, \\ h(x) = [f'(x)]^3, \qquad k(x) = [\mathfrak{k}'_f(x)]^3, \qquad x^2 = 2\, \bar{v} \cdot v, \end{array}\right\} \quad (2.9)$$

the state $\mathrm{P}_f(\rho_0)$ is an invariant equilibrium state and it is representated by the identity operator

$$1 = \mathrm{P}_f(\rho_0) = \int_{\mathbf{C}_3} \frac{|\mathrm{K}_f(v) \rangle \langle \mathrm{K}_f(v)|}{\langle \mathrm{K}_f(v) | \mathrm{K}_f(v) \rangle_f}\, \rho_0(x)\, d\mu_{\mathrm{L}, f}(v). \quad (2.10)$$

Raising (2.10) to the N-th power and taking the matrix element between the vectors $\frac{\langle K_f(z)|}{\langle K_f(z)|K_f(z)\rangle_f^{1/2}}$ and $\frac{|K_f(w)\rangle}{\langle K_f(w)|K_f(w)\rangle_f^{1/2}}$ we get

$$a_f(z, w) = \lim_{N\to\infty} \int_{C_3} \rho_0(x_2)\, d\mu_{L,f}(v_2) \ldots \int_{C_3} \rho_0(x_{N-1})\, d\mu_{L,f}(v_{N-1}) \times a_f(z, v_2)\, a_f(v_2, v_3) \ldots a_f(v_{N-1}, w). \quad (2.11)$$

Since the left-hand side of the N-th power of (2.10) does not depend on N, passing to the limit as $N \to \infty$ does not affect the amplitude $a_f(z, w)$. According to the law of multiplication of amplitudes, the expression

$$a_f(z, v_2)\, a_f(v_2, v_3) \ldots a_f(v_{N-1}, w)$$

is the transition amplitude of the system from the state $z = v_1$ to the state $w = v_N$ through the intermediate states $v_2, \ldots, v_{N-1}$, *i.e.* the probability amplitude of the process $z = v_1, v_2, \ldots, v_N = w$. If we approximate this process by a piecewise smooth curve $\gamma : [\tau_a, \tau_b] \to \mathbf{C}_3$, that is, if we put $v_k = \gamma(\tau_k)$, where $\tau_k = \frac{k}{N}(\tau_b - \tau_a) + \tau_a$, we obtain the following expression

$$a_f(z; \gamma; w) = \lim_{N\to\infty} a_f(z, v_2)\, a_f(v_2, v_3) \ldots a_f(v_{N-1}, w) = \exp\left(i \int_\gamma \operatorname{Im} \frac{\langle K_f | dK_f \rangle_f}{\langle K_f | K_f \rangle_f} \right) \quad (2.12)$$

for the transition amplitude from z to w along the path γ. If we introduce the symbol of formal integration over all piecewise smooth paths joining the stats z and w

$$\int \prod_{\tau\in[\tau_a, \tau_b]} d_{\mathscr{K}_f}[\gamma(\tau)] := \lim_{N\to\infty} \int_{C_3} \rho_0(x_2)\, d\mu_{L,f}(v_2) \ldots \int_{C_3} \rho_0(x_{N-1})\, d\mu_{L,f}(v_{N-1}) \quad (2.13)$$

we can express the transition amplitude as a formal path integral

$$a_f(z, w) = \int \prod_{\tau\in[\tau_a, \tau_b]} d_{\mathscr{K}_f}[\gamma(\tau)] \times \exp\left\{ i \int_{\tau_a}^{\tau_b} \left(\operatorname{Im} \frac{\langle K_f | dK_f \rangle_f}{\langle K_f | K_f \rangle_f} \lrcorner \frac{d\gamma}{d\tau} \right) d\tau \right\}, \quad (2.14)$$

Thus, according to the superposition law for amplitudes, $a_f(z, w)$ is the "sum" of the transition amplitudes $a_f(z; \gamma; w)$. If we allow only those processes that do not change the energy of the system, then the measures $\rho_0(x_k)\, d\mu_{L,f}(v_k)$ in the right-hand side of (2.13) should be replaced by the measures

$$\delta(\mathrm{H}(\gamma(\tau_k))-\mathrm{E})\,\rho_0(x_k)\,d\mu_{\mathrm{L},f}(v_k)$$
$$=\int_{-\infty}^{+\infty}\exp\left\{-i(\mathrm{H}(\gamma(\tau_k))-\mathrm{E})\,\Lambda(\tau_k)\right\}d\Lambda(\tau_k)\,\rho_0(x_k)\,d\mu_{\mathrm{L},f}(v_k) \qquad (2.15)$$

which contain a δ-factor corresponding to the constraint $\mathrm{H}(\gamma(\tau))=\mathrm{E}=\mathrm{Const.}$ As a result we obtain an expression for the transition amplitude between z and w

$$a_f(z,w;\,\mathrm{H}=\mathrm{E}=\mathrm{Const.})=\int\prod_{\tau\in[\tau_a,\tau_b]}d_{\mathscr{K}_f}[\gamma(\tau)]\,d\Lambda(\tau)$$
$$\times\exp\left\{i\int_{\tau_a}^{\tau_b}\left(\mathrm{Im}\frac{\langle \mathrm{K}_f\,|\,d\mathrm{K}_f\rangle_f}{\langle \mathrm{K}_f\,|\,\mathrm{K}_f\rangle_f}\,\lrcorner\,\frac{d\gamma}{d\tau}-(\mathrm{H}(\gamma(\tau))-\mathrm{E})\,\Lambda(\tau)\right)d\tau\right\}, \qquad (2.16)$$

which is compatible with the energy conservation law. Integration over the Lagrange multipliers $\prod_{\tau\in[\tau_a,\tau_b]} d\Lambda(\tau)$ takes into account the contribution to the integral due to the freedom of choice of the parametrization of a path. Fixing such a parametrization Λ_0 can be understood as a choice of a classical instrument (a clock) that measures time. Formally, this corresponds to replacing the measure $\prod_{\tau\in[\tau_a,\tau_b]} d\Lambda(\tau)$ in (2.16) by $\prod_{\tau\in[\tau_a,\tau_b]}\delta(\Lambda(\tau)-\Lambda_0(\tau))\,d\Lambda(\tau)$. Finally we get

$$a_f(z,w;\,\mathrm{H}=\mathrm{E}=\mathrm{Const.};\,\Lambda_0)=\exp[-i(\tau_b-\tau_a)\,\mathrm{E}]\int\prod_{\tau\in[\tau_a,\tau_b]}d_{\mathscr{K}_f}[\gamma(\tau)]$$
$$\times\exp\left\{i\int_{\tau_a}^{\tau_b}\left(\mathrm{Im}\frac{\langle \mathrm{K}_f\,|\,d\mathrm{K}_f\rangle_f}{\langle \mathrm{K}_f\,|\,\mathrm{K}_f\rangle_f}\,\lrcorner\,\frac{d\gamma}{d\tau}-\mathrm{H}(\gamma(\tau))\right)d\tau\right\}, \qquad (2.17)$$

where the parameter $\tau=\int_{\tau_a}^{\tau}\Lambda_0(s)\,ds$ can be interpreted as time. According to Feynman's approach to the path integral, we obtain from (2.17) the following expression

$$\mathrm{S}_{f,\mathrm{H}}[\gamma]=\int_{\tau_a}^{\tau_b}\left(\mathrm{Im}\frac{\langle \mathrm{K}_f\,|\,d\mathrm{K}_f\rangle_f}{\langle \mathrm{K}_f\,|\,\mathrm{K}_f\rangle_f}\,\lrcorner\,\frac{d\gamma}{d\tau}-\mathrm{H}(\gamma(\tau))\right)d\tau \qquad (2.18)$$

for the action functional of the system $(\mathbf{C}_3,\mathscr{M}_f,\mathscr{K}_f:\mathbf{C}_3\to\mathbb{CP}(\mathscr{M}_f))$ in the equilibrium state $\mathrm{P}_f(\phi\circ\mathrm{H})$, where we have assumed that the energy function H is $\mathrm{SU}(2)\times\mathrm{SU}(2)$ invariant. The whole theory depends on two functional parameter $f:\mathbb{R}_+\to\mathbb{R}$ and $\phi\circ\mathrm{H}:\mathbf{C}_3\to\mathbb{R}_+$, which will be fixed in the next section, where we shall analyze quantum aspects of the problem. As far as the classical mechanics is concerned, the extremals of (the equation) $\frac{\delta S_{f,H}[\gamma]}{\delta\gamma}=0$ are described by Hamilton's equation

$$\mathcal{K}_f^* \omega_{FS} \lrcorner \frac{d\gamma}{d\tau} = dH, \tag{2.19}$$

which in our case has an explicit solution

$$\left.\begin{aligned} & s(\tau)=s(0), \quad t(\tau)=t(0), \quad |\lambda|(\tau)=|\lambda(0)|, \\ & (\arg\lambda)(\tau)=\frac{H'(x_0)f^2(x_0)}{f(x_0)[f'(x_0)+x_0 f''(x_0)]-x_0 f'^2(x_0)}\tau+\arg\lambda(0), \end{aligned}\right\} \tag{2.20}$$

where $z(0)=(s(0), t(0), \lambda(0))$ is the state of the system at the moment $\tau=0$ and $x_0^2=2z(0)\cdot\bar{z}(0)$.

In particular, we see that the choice of f and H affects only parametrization of trajectories.

In order to give a physical interpretation of the Hamiltonian system $(\mathbf{C}_3, \mathcal{K}_f^* \omega_{FS}, H(x))$, let us consider the following imbedding of symplectic manifolds

$$\Phi: \quad T_0^* \mathbb{R}^3 \overset{\Psi^{-1}}{\to} T_0^* S^3 \rightrightarrows \mathbf{C}_3, \tag{2.21}$$

where the second arrow stands for the diffeomorphism (1.2), $T_0^* \mathbb{R}^3$ is the cotangent bundle to $\mathbb{R}^3$ with zero section deleted and Ψ^{-1} is the inverse of the mapping

$$\Psi(e, u)=\begin{bmatrix}\vec{y}\\ \vec{\eta}\end{bmatrix}:=\begin{bmatrix}\frac{1}{1-e_0}\vec{e} \\ -2|u|(\log \mathfrak{k}_f)'(2|u|)[(1-e_0)\vec{u}+u_0\vec{e}]\end{bmatrix}, \tag{2.22}$$

which, up to the factor $-2|u|(1-e_0)^{-2}(\log \mathfrak{k}_f)'(2|u|)$, is the derivative of the stereographic projection $S^3\setminus\{\infty\}\to\mathbb{R}^3$. The domain of Ψ is $T_0^*(S^3\setminus\{\infty\})\cong T_0(S^3\setminus\{\infty\})$. After a calculation (2.21) becomes

$$\begin{pmatrix} z_0 \\ \vec{z}\end{pmatrix}=\Phi(\vec{y}, \vec{\eta})=-(\log \mathfrak{k}_f)'\left(\varphi_f\left(\frac{1}{2}|\vec{\eta}|(1+|\vec{y}|^2)\right)\right)$$
$$\times\left[|\vec{\eta}|\begin{pmatrix}\frac{1}{2}(|\vec{y}|^2-1)\\ \vec{y}\end{pmatrix}+i\begin{pmatrix}\vec{y}.\vec{\eta}\\ \frac{1}{2}(1+|\vec{y}|^2)\vec{\eta}-(\vec{\eta}.\vec{y})\vec{y}\end{pmatrix}\right] \tag{2.23}$$

where $(\vec{y}, \vec{\eta})\in\mathbb{R}^3\times(\mathbb{R}^3\setminus\{0\})\cong T_0^*\mathbb{R}^3$, $z=\Phi(\vec{y}, \vec{\eta})\in\Phi(T_0^*\mathbb{R}^3)$ and φ_f is the inverse function of $\frac{1}{2}x(\log \mathfrak{k}_f)'(x)$.

Pulling $\mathcal{K}_f^* \omega_{FS}$ and H back on $T_0^* \mathbb{R}^3$ we obtain the following Hamiltonian system

$$\left(T_0^* \mathbb{R}^3,\ \omega_f = d(\vec{\eta} \cdot d\vec{y}),\ H \circ \varphi_f\left(\frac{1}{2}|\vec{\eta}|(1+|\vec{y}|^2)\right)\right), \tag{2.24}$$

which, for $H(x) = \frac{1}{2} x (\log \mathfrak{k}_f)'(x)$, coincides with Moser's regularization of the Kepler problem (*see* [Ku])

$$\left(T_0^* \mathbb{R}^3,\ \omega_f = d(\vec{\eta} \cdot d\vec{y}),\ \frac{1}{2}|\vec{\eta}|(1+|\vec{y}|^2)\right). \tag{2.25}$$

To see this, let us relate the variables $(\vec{y}, \vec{\eta})$ to the momentum $\vec{p}$ and the position $\vec{q}$ in the following way

$$\left.\begin{aligned} \vec{y} &= (-2E(\vec{q}, \vec{p}))^{-1/2} \vec{p}, \\ \vec{\eta} &= (-2E(\vec{q}, \vec{p}))^{1/2} \vec{q}, \end{aligned}\right\} \tag{2.26}$$

where $E(\vec{q}, \vec{p}) = \frac{1}{2}|\vec{p}|^2 - |\vec{q}|^{-1}$ is the energy function of the reduced Kepler problem. We assume that the range of $(\vec{q}, \vec{p})$ is such that $E(\vec{q}, \vec{p}) < 0$, that is, we consider only bound states. We have

$$\frac{1}{2}|\vec{\eta}|(1+|\vec{y}|^2) = (-2E(\vec{q}, \vec{p}))^{-1/2}, \tag{2.27}$$

which implies that the trajectories of the system (2.24) coincide with those of the reduced Kepler problem, *i.e.* (2.20) gives the Hamiltonian flow for the Kepler problem.

Concluding, we see that the Hamiltonian system $\left(C_3,\ \mathcal{K}_f^* \omega_{FS},\ \frac{1}{2} x (\log \mathfrak{k}_f)'(x)\right)$ is Moser's regularization of the Kepler problem.

3. THE QUANTUM ASPECTS

In this section we discuss the problem of quantizing classical observables (functions) for the regularized Kepler problem following the procedure given in [O2]. The procedure is based on the classical Ehrenfest theorem (*see* [E]). If $H = H^+ \in B(\mathcal{M}_f)$ is an Hermitian operator, the one-parameter group $U_H(t) = \exp(itH)$, $t \in \mathbb{R}$, generated by H uniquely determines a flow $U'_H(t)$, $t \in \mathbb{R}$, on the universal bundle $\mathbb{E} \to \mathbb{CP}(\mathcal{M}_f)$ which preserves both the metric and holomorphic structures of the bundle. The Poisson algebra $(C^\infty(C_3, \mathbb{R}), \{\cdot, \cdot\}_f)$ defined by the symplectic form $\mathcal{K}_f^* \omega_{FS}$ is isomorphic to an algebra of vector fields on $\mathbb{E}^{0,0}$ which are infinitesimal automorphisms of the quantum bundle $(\mathcal{K}_f^* \mathbb{E} \to \mathbf{C}_3, \mathcal{K}_f^* \nabla^{FS}, \mathcal{K}_f^* H_{FS})$. This isomorphism holds in general case; its explicit form is given in [K] (*see* theorem 4.2.1). Let Y_φ be the vector field on $\mathbb{E}^{0,0}$ corresponding to a function $\varphi \in C^\infty(\mathbf{C}_3, \mathbb{R})$. We say that the pair (H, φ) satisfies Ehrenfest's condition if

(*a*) the flow $[U_H(t)]$ preserves $\mathcal{K}_f(\mathbf{C}_3) \subset \mathbb{CP}(\mathcal{M}_f)$;

(*b*) the vector field Y_φ is the infinitesimal generator of the flow

$$\sigma'_\varphi(t) := \mathcal{K}_f^{*-1} \circ U'_H(t) \circ \mathcal{K}_f^*.$$

This condition means that the Hamiltonian dynamics of the system coincides with the Schrödinger dynamics, which corresponds to the assertion of Ehrenfest's theorem. In [O2] it was shown that the space C_E of functions φ for which there is only one $H \in B(\mathcal{M}_f)$ such that (H, φ) satisfies Ehrenfest's conditions is a Lie subalgebra of the Poisson algebra $(C^\infty(\mathbf{C}_3, \mathbb{R}), \{\cdot, \cdot\}_f)$ and the assignement $Q_E : \varphi \mapsto iH$ is a monomorphism of this subalgebra into the Lie algebra $(B(\mathcal{M}_f), [\cdot, \cdot])$. This monomorphism will be called the Ehrenfest's quantization. Conditions (*a*) and (*b*) imply that the vector field Y_φ is complete and its flow $\sigma'_\varphi(t)$ consists of automorphisms of the quantum bundle. Moreover, $\sigma'_\varphi(t)$ is a holomorphic flow because $\mathcal{K}_f$ is an antiholomorphic embedding.

We shall not deal here with the problem of an explicit description of the algebra C_E of classical observables which are quantizable in the sense of Ehrenfest. Our further considerations will be restricted to those elements $h \in C_E$ which satisfy

(*a'*) $$\forall \xi \in \mathbb{E}^{0,0}, \ \mathcal{K}_f^*(\sigma_h(t)\xi) = \exp(i\phi(t))\, U'_H(t)\, \mathcal{K}_f^*(\xi).$$

This condition implies condition *a*) and is equivalent to the fact that the section σ_0 which trivializes the bundle $\mathcal{K}_f^* \mathbb{E} \to \mathbf{C}_3$ is invariant [up to phase factor $\exp(i\phi(t))$]. Taking into account that the metric structure $\mathcal{K}_f^* H_{FS}$ is $\sigma'_h(t)$-invariant, we see that (*a'*) is also equivalent to $\sigma_h(t)$-invariance of the diagonal of the reproducing kernel

$$\mathfrak{k}_f(x) = K_f(\bar{z}, z) = \langle K_f(z) \,|\, K_f(z) \rangle_f = \mathcal{K}_f^* H_{FS}(\sigma_0(z), \sigma_0(z)). \quad (3.1)$$

Thus, the subalgebra $C_{E,0} \subset C_E$ of classical observables which satisfy (*a'*) and (*b*) consists of the functions which generate holomorphic flows leaving $x = (2z \cdot \bar{z})^{1/2}$ invariant.

Let $\Gamma(h) = X + \bar{X}$ denote the holomorphic Hamiltonian vector field generated by $h \in C_{E,0}$. Thus $\mathcal{K}_f^* \omega_{FS} \lrcorner \Gamma(h) = dh$ implies that

$$h = iX(\log \mathfrak{k}_f) + \psi, \quad (3.2)$$

where $\psi \in \mathcal{O}(\mathbf{C}_3)$. Since $(X + \bar{X})(\log \mathfrak{k}_f) = 0$, we have $\psi = \text{Const.} = \alpha$ and

$$h = \frac{i}{2}(X - \bar{X})(\log \mathfrak{k}_f) + \psi, \quad (3.3)$$

Each holomorphic Hamiltonian vector field $X+\bar{X}$ leaving $\mathfrak{f}_f$ invariant is easily seen to be of the form $\Gamma(h)$, where h is given by (3.3). Therefore, in order to describe the algebra $C_{E,0}$, it suffices to know the group of biholomorphisms of $\mathbf{C}_3$ preserving $x^2=2z\cdot\bar{z}$. It can be shown that this is the group $U(1)\times SO(4)$ (*see* [R]). It follows that any $h\in C_{E,0}$ is given by (3.3), where the holomorphic vector field X corresponds to an element of the Lie algebra $u(1)\oplus so(4)\cong u(1)\oplus su(2)\oplus su(2)$ under the homomorphism induced by the action of $U(1)\times SU(2)\times SU(2)$ given by (1.11). We thus see that the algebra $C_{E,0}$ is spanned by

$$\left.\begin{aligned}\vec{M}(s,t,\lambda)&=\frac{x}{2}(\log\mathfrak{f}_f)'(x)\,\vec{v}(s),\\ \vec{N}(s,t,\lambda)&=\frac{x}{2}(\log\mathfrak{f}_f)'(x)\,\vec{v}(t),\\ E(s,t,\lambda)&=\frac{x}{2}(\log\mathfrak{f}_f)'(x),\\ J(s,t,\lambda)&\equiv 1,\end{aligned}\right\}\tag{3.4}$$

where $\vec{v}(s):=\left(\frac{\operatorname{Re}s}{1+|s|^2},\frac{\operatorname{Im}s}{1+|s|^2},\frac{1-|s|^2}{1+|s|^2}\right)$.

This basis is compatible with the decomposition $u(1)\oplus su(2)\oplus su(2)$. The constant function $J(s,t,\lambda)\equiv 1$ spans the kernel of the homomorphism $\Gamma: C_{E,0}\to \operatorname{Ham}(\mathbf{C}_3)$. The functions M_k, N_k, $k=1, 2, 3$, form a natural basis of the algebra $su(2)\oplus su(2)$ and satisfy the commutation relations $\{M_1, M_j\}=\varepsilon_{ijk}M_k$, $\{N_i, N_j\}=\varepsilon_{ijk}N_k$, $\{M_i, N_j\}=0$. Functions of the form $H_{AB}:=AE+B$, $A, B\in\mathbb{R}$, span the center of $C_{E,0}$. The flows $\sigma_{M_k}(t)$ and $\sigma_{N_k}(t)$ are given by (1.11). The corresponding one-parameter groups are $T(\sigma_{M_k}(t))\,i\,T(\sigma_{N_k}(t))$, where the representation T is given by (1.12). The function E is exactly the Hamiltonian of the regularized Kepler problem and the flow $\sigma_E(t)$ generated by E is the time evolution flow of the regularized Kepler problem given by (2.10). The flow $\sigma'_J(t)$ generated by the constant function J is the constant phase flow e^{it} of the line bundle $\mathbb{E}^{0,0}$.

Quantizing (3.4) in the sense of Ehrenfest, we get

$$\begin{aligned}Q_E(M_1)&=\frac{i}{2}\left(s\lambda\frac{\partial}{\partial\lambda}-s^2\frac{\partial}{\partial s}+\frac{\partial}{\partial s}\right),\\ Q_E(M_2)&=\frac{1}{2}\left(s\lambda\frac{\partial}{\partial\lambda}-s^2\frac{\partial}{\partial s}-\frac{\partial}{\partial s}\right),\\ Q_E(M_3)&=\frac{i}{2}\lambda\frac{\partial}{\partial\lambda}-is\frac{\partial}{\partial s},\end{aligned}$$

$$\left.\begin{aligned} Q_E(N_1) &= \frac{i}{2}\left(t\lambda\frac{\partial}{\partial\lambda} - t^2\frac{\partial}{\partial t} + \frac{\partial}{\partial t}\right), \\ Q_E(N_2) &= \frac{1}{2}\left(t\lambda\frac{\partial}{\partial\lambda} - t^2\frac{\partial}{\partial t} - \frac{\partial}{\partial t}\right), \\ Q_E(N_3) &= \frac{i}{2}\lambda\frac{\partial}{\partial\lambda} - \mathrm{it}\frac{\partial}{\partial t}, \\ Q_E(E) &= 2\lambda\frac{\partial}{\partial\lambda}, \\ Q_E(J) &= 1. \end{aligned}\right\} \tag{3.5}$$

The operators $Q_E(M_k)$ and $Q_E(N_k)$ are (up to the factor $-2i$) the generators of the representation $T: SU(2)\times SU(2) \to \mathrm{Aut}\,\mathcal{M}_f$ given by (1.12) and the Hamiltonian $\hat{H}_{AB} := Q_E(AE+B)$ satisfies the obvious conditions

$$[\hat{H}_{AB}, Q_E(M_k)] = [\hat{H}_{AB}, Q_E(N_k)] = 0, \qquad \text{for} \quad k=1, 2, 3. \tag{3.6}$$

Moreover, the subspaces V_n, $n \in \mathbb{N}$ [*see* (1.13)], are the eigenspaces of $\hat{H}_{AB}$ corresponding to the eigenvalues

$$E_n = 2A(n-1) + B \tag{3.7}$$

and, for each $\sigma = \psi\sigma_0 \in \mathcal{M}_f$ and each $\tau \in \mathbb{R}$, we have

$$(\exp(i\tau\hat{H}_{AB})\sigma)(s, t, \lambda) = [e^{iB\tau}\psi(s, t, e^{2iA\tau}\lambda)]\,\sigma_0(s, t, \lambda) \tag{3.8}$$

It was shown in [O2] that in the general case evaluating the mean value of the operator which is the Ehrenfest quantization of a classical observable on all possible coherent states gives again this function. More precisely, the operation $\langle . \rangle$ of taking the mean value on coherent states is a Lie algebra isomorphism which is the inverse of Q_E. Thus, for the case of the Kepler problem we get

$$\langle Q_E(h) \rangle (z) = \mathrm{Tr}(Q_E(h)P(z)) = h(z) \qquad \text{for} \quad h \in C_E. \tag{3.9}$$

It follows from (2.20) and (3.8) that the embedding $\mathcal{K}_f: C_3 \to \mathbb{CP}(\mathcal{M}_f)$ is equivariant with respect to the Hamiltonian dynamics on C_3 and the Schrödinger dynamics on $\mathbb{CP}(\mathcal{M}_f)$. After passing to the Heisenberg picture, this together with (3.9) shows that the classical dynamics corresponding to $h \in C_E$ is consistent with the quantum dynamics corresponding to $Q_E(h)$.

Inserting the Hamiltonian function H_{AB} into the formal path integral formula (2.17) we obtain (*see* [O2] for general case) the propagator for the Schrödinger time evolution expressed in terms of coherent states [*see* formula (3.8)]

$$a(z, w; \mathrm{H_{AB}}=\mathrm{E}=\text{Const.}; \Lambda_0)=\frac{\langle \mathrm{K}_f(z)\,|\exp(i(\tau_b-\tau_a)\,\hat{\mathrm{H}}_{\mathrm{AB}})\,|\,\mathrm{K}_f(w)\rangle_f}{\langle \mathrm{K}_f(z)\,|\,\mathrm{K}_f(z)\rangle_f^{1/2}\langle \mathrm{K}_f(w)\,|\,\mathrm{K}_f(w)\rangle_f^{1/2}}$$

$$=\exp(i\,\mathrm{B}(\tau_b-\tau_a))\frac{\mathrm{K}_f(\bar{z}, \exp(2\,i\,\mathrm{A}(\tau_b-\tau_a))\,w)}{(\mathrm{K}_f(\bar{z}, z)\,\mathrm{K}_f(\bar{w}, w))^{1/2}}. \quad (3.10)$$

Let us recall that by formal path integration we mean the limit of N-time integration over phase space given by (2.13).

We now discuss the connection of Ehrenfest's quantization with the Kostant-Souriau quantization. A discussion of this problem for an arbitrary system is given in [O2]. In the case of the Kepler problem the situation is as follows. Applying the geometric quantization procedure to the quantum bundle $(\mathbb{E}^{0,0}\to \mathbf{C}_3, \nabla^f, \mathrm{H}_f)$ with antiholomorphic polarization on the complex manifold $\mathbf{C}_3$, we obtain

$$\mathrm{C_{KS}}=\left\{\frac{x}{2}f'(x)\,\vec{v}(s), \frac{x}{2}f'(x)\,\vec{v}(t), \frac{x}{2}f'(x), 1\right\} \quad (3.11)$$

as the space of quantizable functions and $\mathbb{CP}(\mathcal{M}_f)$ as the space of pure states. The operators $\mathrm{Q_{KS}}(h)$, for $h\in \mathrm{C_{KS}}$, are given by (3.5) as for Ehrenfest's quantization.

We thus see that, in this case, the two quantization procedures differ only by the subalgebra of quantizable functions $\mathrm{C_{E.0}}\neq \mathrm{C_{KS}}$. This implies, in particular, that the operation $\langle\,.\,\rangle$ of taking mean values of the operators $\mathrm{Q_{KS}}(h)$ on the coherent states $\mathcal{K}_f(z)$ does not give functions from $\mathrm{C_{KS}}$. Thus, in the case of the Kostant-Souriau quantization, we do not have the consistency of quantum and classical descriptions which we had in Ehrenfest's quantization.

Let F_3 denote the set of potentials $f:\mathbf{C}_3\to\mathbb{R}_+$ which satisfy (1.18) and (1.19). The functions from F_3 parametrize the quantum bundles $(\mathbb{E}^{0,0}\to \mathbf{C}_3, \nabla^f, \mathrm{H}_f)$, and hence also the mechanical systems $(\mathbf{C}_3, \mathcal{M}_f, \mathcal{K}_f:\mathbf{C}_3\to\mathbb{CP}(\mathcal{M}_f))$, where $\mathcal{K}_f$ is the antiholomorphic embedding described in the preceding section. Taking the pull-back by $\mathcal{K}_f$ of the universal quantum bundle $(\mathbb{E}\to\mathbb{CP}(\mathcal{M}_f), \nabla^{\mathrm{FS}}, \mathrm{H_{FS}})$ we obtain a new quantum bundle

$$(\mathcal{K}_f^*\,\mathbb{E}\to \mathbf{C}_3, \mathcal{K}_f^*\nabla^{\mathrm{FS}}, \mathcal{K}_f^*\,\mathrm{H_{FS}})\cong(\mathbb{E}^{0,0}\to \mathbf{C}_3, \nabla^{\log \mathfrak{k}_f}, \mathrm{H}_{\log \mathfrak{k}_f})$$

described by the potential $\log \mathfrak{k}_f$ which, by Proposition 2.1 also belongs to F_3. Summing up, we obtain a mapping $\mathrm{J}:F_3\to F_3$ given by $f\mapsto\log\mathfrak{k}_f$. Finding a fixed point of J amounts to finding a potential f such that $\mathcal{K}_f^*\,\mathrm{H_{FS}}=\mathrm{H}_f$, which implies that $\mathcal{K}_f^*\,\nabla^{\mathrm{FS}}=\nabla^f$ and $\mathcal{K}_f^*\,\omega_{\mathrm{FS}}=\omega_f$. Thus the quantum bundle which is the pull-back by $\mathcal{K}_f$ of $(\mathbb{E}\to\mathbb{CP}(\mathcal{M}_f), \nabla^{\mathrm{FS}}, \mathrm{H_{FS}})$ coincides with the initial quantum bundle. It follows that, for $\mathrm{J}(f)=f$, Ehrenfest's quantization is equivalent to the Kostant-Souriau quantization.

We have thus obtained a natural from a physical (and geometrical) point of view criterion for choosing the functional parameter of our theory

$$F_3 \ni f \mapsto (\mathbf{C}_3, \mathcal{M}_f, \mathcal{K}_f : \mathbf{C}_3 \to \mathbb{CP}(\mathcal{M}_f)).$$

The condition for J to have a fixed point takes the form of a nonlinear integro-differential equation for the function f:

$$e^{f(x)} = \mathfrak{k}_f(x) = \sum_{c=0}^{\infty} \frac{(c+1)^2}{\mathrm{I}_f(c)} x^{2c}, \tag{3.12}$$

where $\mathrm{I}_f(c)$ is given by (1.13).

An essential difficulty in studying the properties of this equation is due to the fact that each term of the sum in the right-hand side depends functionally on f. In particular it is hard to decide whether there exist solutions in F_3. However, there is a class of functions in F_3 which satisfy the equation asymptotically. Namely, for $f(x) = x^a$, $a > 0$, the reproducing kernel $\mathfrak{k}_f(x)$ can be expressed in terms of functions of Mittag-Leffler type [*see* (1.27)]. We recall that a Mittag-Leffler type function is an entire function of a complex variable u given by the power series

$$\mathrm{E}_\rho(u; \mu) := \sum_{k=0}^{\infty} \frac{u^k}{\Gamma(\mu + k/\rho)}, \tag{3.13}$$

where $\rho > 0$ and $\mu \in \mathbb{C}$. A detailed description of these functions can be found in [D]. We shall need the following two lemmas from [D].

LEMMA 3.1. – *Suppose that* $\rho > 1/2$, μ *is a complex number and* α *is a real number such that*

$$\frac{\pi}{2\rho} < \alpha < \min\{\pi, \pi/\rho\}.$$

Then, for any $p \in \mathbb{N}$ *and for* $|u| \to \infty$ *with* $|\arg u| \leqq \alpha$, *we have the following asymptotic formula*

$$\mathrm{E}_\rho(u; \mu) = \rho\, u^{\rho(1-\mu)} e^{u^\rho} - \sum_{k=1}^{p} \frac{u^{-k}}{\Gamma(\mu - k/\rho)} + \mathcal{O}(|u|^{-1-p}). \quad \square$$

LEMMA 3.2. – *Suppose* $\rho \in (0, 1/2]$, $p \in \mathbb{N}$ *and* $\mu \in \mathbb{C}$. *Then, for* $|u| \to \infty$, *we have the following asymptotic formula*

$$\mathrm{E}_\rho(u; \mu) = \rho \sum_{|\arg u + 2\pi n| \leqq \pi/(2\rho)} (u^\rho e^{i 2\pi\rho n})^{1-\mu} e^{e^{i 2\pi\rho n} u^\rho} - \sum_{k=1}^{p} \frac{u^{-k}}{\Gamma(\mu - k/\rho)} + \mathcal{O}(|u|^{-1-p}),$$

where in the first sum the summation runs over those $n = 0, \pm 1, \ldots$ *which satisfy the inequality under the sum symbol.* $\square$

The lemmas imply that the function $f(x)=x^a$ satisfies (3.12) asymptotically. More precisely, we have

$$\lim_{x\to\infty} \mathfrak{k}_f(x)\,e^{-x^a}=\frac{M}{8}a^3. \tag{3.14}$$

It can also be shown that there is no $a>0$ for which (3.12) holds identically.

The asymptotic symplectic coherent state $\mathscr{K}_f(z)$ described above or any coherent state for which $\mathscr{K}_f$ is a symplectic embedding have the following properties: the square of the modulus of its wave function has a maximum in the point z and the higher is the energy of the system the better it is localized around this point. Thereby these states are the ones which can be obtained in experiments proposed in the papers [P-S] and [Y-S].

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