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Isomorphism of de Rham cohomology and relative Hochschild cohomology of differential operators

by

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ABSTRACT. — We show that one may extend the domain of definition of a differential p-form α on a manifold M, from p-tuples of vector fields on M to p-tuples of differential operators of any order on M. The extended p-form α_S is not unique. It depends on a choice of what we call an *allocation*, S, which is a rule for "filling in" any $(p+1)$-tuple of sufficiently near points of M with a p-simplex having these vertices. For any S, α_S is a $C^\infty(M, \mathbb{R})$-relative Hochschild cochain on the algebra $\mathscr{D}$ of differential operators on M; α_S takes values in $\mathscr{D}$.
The map $\alpha \mapsto \alpha_S$ satisfies the condition

$$(d\alpha)_S = \delta(\alpha_S)$$

where d is the de Rham and δ is the Hochschild differential. If S_1 and S_2 are two allocations, then $\alpha_{S_1} \cong \alpha_{S_2}$ in Hochschild cohomology. The map $\alpha \mapsto \alpha_S$ induces an isomorphism

$$H^p_{DR}(M, \mathbb{R}) \cong H^p(\mathscr{D}, C^\infty(M, \mathbb{R}); \mathscr{D}).$$

Here the latter group is the relative Hochschild cohomology group of $\mathscr{D}$ relative to $C^\infty(M, \mathbb{R})$ with coefficients in $\mathscr{D}$. We indicate some of the applications in the introduction.

Key words : Differential operators, de Rham cohomology, Hochschild cohomology.

RÉSUMÉ. — Nous montrons qu'on peut étendre le domaine de définition d'une p-forme différentielle α sur une variété M, des p-uples de champs de vecteurs sur M aux p-uples d'opérateurs différentiels d'ordre arbitraire sur M. La p-forme α_S ainsi étendue n'est pas unique. Elle dépend d'un

choix de ce que nous appelons une *allocation*, S, qui est une règle pour « remplir » un $(p+1)$-uple quelconque de points de M suffisamment proches par un p-simplexe ayant ces points pour sommets. Pour chaque S, α_S est une cochaîne de Hochschild $C^\infty(M, \mathbb{R})$-relative définie sur l'algèbre $\mathscr{D}$ des opérateurs différentiels sur M à valeurs dans $\mathscr{D}$.
L'application $\alpha \mapsto \alpha_S$ satisfait la condition

$$(d\alpha)_S = \delta(\alpha_S)$$

où d est la différentielle de De Rham et δ est la différentielle de Hochschild. Si S_1 et S_2 sont deux allocations, on a $\alpha_{S_1} \cong \alpha_{S_2}$ en cohomologie de Hochschild. L'application $\alpha \mapsto \alpha_S$ induit un isomorphisme

$$H^p_{DR}(M, \mathbb{R}) \cong H^p(\mathscr{D}, C^\infty(M, \mathbb{R}); \mathscr{D}).$$

Le dernier groupe est le groupe de cohomologie de Hochschild de $\mathscr{D}$ relatif à $C^\infty(M, \mathbb{R})$ à coefficients dans $\mathscr{D}$.

Nous indiquons dans l'introduction quelques applications.

1. INTRODUCTION

Let X_j, $j=1, \ldots, p$ be vector fields on a differentiable manifold M, let $\mathscr{F} = C^\infty(M, \mathbb{R})$, and let α be a p-form on M. One may construct the function $\alpha(X_1, \ldots, X_p)$ which is $\mathscr{F}$-linear in the X_j. Vector fields may be regarded as first order differential operators on $\mathscr{F}$. Let $\mathscr{D}$ denote the associative $\mathbb{R}$-algebra of differential operators on $\mathscr{F}$ and let $H_j \in \mathscr{D}$, $j=1, \ldots, p$.

We shall extend the action of α from p-tuples of vector fields to p-tuples of differential operators of any order. That is, we shall define

$$\alpha_S(H_1, \ldots, H_p) \in \mathscr{D},$$

satisfying the consistency condition

$$\alpha_S(X_1, \ldots, X_p) = \alpha(X_1, \ldots, X_p)/p!$$

with the aid of an additional structure S which we call an allocation on M. Many different allocations S exist, and they give different maps α_S. A connection ∇ on TM is enough to provide an allocation. For an easy-to-grasp description of allocations see the start of Section 2. We may regard α_S as a Hochschild p-cochain on $\mathscr{D}$. We prove that, as $(p+1)$-cochains,

$$(d\alpha)_S = \delta(\alpha_S)$$

where δ is the Hochschild coboundary. The main result of the paper (proved in Section 6) is the following.

THEOREM 1. — *Independent of the choice of* S, *the map* $D_S : \alpha \mapsto \alpha_S$ *induces an isomorphism*

$$\check{D}: \quad H^p_{DR}(M, \mathbb{R}) \cong H^p(\mathcal{D}, \mathcal{F}; \mathcal{D})$$

of the de Rham cohomology of differential forms on M *and the relative Hochschild cohomology of* $\mathcal{D}$*-valued* $\mathcal{F}$*-relative cochains on* $\mathcal{D}$.

The cochain α_S has the following properties.

(*a*) If each H_j is a differential operator of order $h_j > 0$, then

$$\operatorname{ord}(\alpha_S(H_1, \ldots, H_p)) = \sum_{j=1}^{p} (h_j - 1).$$

(*b*) If any H_j is a function, (order zero), then $\alpha_S(H_1, \ldots, H_p) = 0$.

(*c*) For $f \in \mathcal{F}$,

$$\alpha_S(f H_1, H_2, \ldots, H_p) = f \alpha_S(H_1, \ldots, H_p);$$
$$\alpha_S(\ldots, H_j f, H_{j+1}, \ldots) = \alpha_S(\ldots, H_j, f H_{j+1}, \ldots);$$
$$\alpha_S(H_1, \ldots, H_p f) = \alpha_S(H_1, \ldots, H_p) f.$$

Thus α_S is a normalised cochain on $\mathcal{D}/\mathcal{F} \otimes_{\mathcal{F}} \ldots \otimes_{\mathcal{F}} \mathcal{D}/\mathcal{F}$. This product is part of the $(\mathcal{D}, \mathcal{F})$-projective resolution of $\mathcal{D}$, [1]. Its cochains are termed $\mathcal{F}$*-relative* and they form a complex whose cohomology is denoted $H^p(\mathcal{D}, \mathcal{F}; \mathcal{D})$.

The map $\check{D}$ has several uses.

(i) It extends our grasp of the relation between the geometric cohomology of M and the algebraic cohomology of $\mathcal{D}$. In this direction it relates to the work of [2], [3] and [4].

The referee has pointed out that an argument following closely that of [3], but using the spectral sequence for relative p-cochains graded by p and filtered by order, also leads to the isomorphism of Theorem 1. The construction presented here may be helpful in giving a more direct and geometric realisation of the isomorphism.

(ii) For a given allocation, each closed one-form on M gives a derivation on $\mathcal{D}$ and each closed two-form gives a deformation of $\mathcal{D}$. A change of allocation alters the resulting derivation or deformation to an equivalent one.

(iii) Just as we may regard $\alpha_S(H)$ as the operator-valued result of letting a 1-form α act upon H, so we may regard $\alpha_S(H)$ as the result of letting H act on α. H may thus be viewed as an operator-valued de Rham *current*. In quantum mechanics iH is in this view the probability current density operator. Other conserved currents are defined similarly, [5, 6].

(iv) The map D_S gives another construction of the Poincaré-Cartan form Θ for a class of Lagrangians in the multivariate higher order calculus of variations. The dependence of D_S upon choice of allocation and the independence of $\check{D}$ relate to known properties of Θ, [5].

Section 2 of this paper is a development of the notion of an allocation and some of its properties. In Section 3 we define α_S for a given allocation and in Section 4 prove the fundamental property that $(d\alpha)_S = \delta(\alpha_S)$ where δ is the Hochschild codifferential. This property ensures the existence of the map $\check{D}$ from de Rham to Hochschild cohomology. We also show that $\check{D}$ does not depend on the choice of allocation.

In Section 5 we obtain the $\mathscr{F}$-relative properties (*a*)-(*c*) listed earlier and show that, when restricted to vector fields, α_S reduces to $(\alpha/p!)$. Finally Section 6 contains the proof of Theorem 1.

2. ALLOCATIONS

We give a description of an allocation first, and then its formal definition.

Let M have dimension n and let $(x_0, x_1, \ldots, x_n)$ be a $(n+1)$-tuple of points of M, all "near" x_0, in a sense to be made precise shortly.

An allocation S maps $(x_0, \ldots, x_n)$ into an oriented simplex which we shall denote $S(x_0, \ldots, x_n)$ having vertices $x_0, \ldots, x_n$. It joins pairs of vertices by edges, fills in the triples of edges with 2-faces, and so on. Each p-face is thus prescribed by its vertices. The face does not change if vertices of the simplex which do not belong to this face are moved. Nor does it change, except in orientation, if we permute the vertices of the face to a different order.

If $M = \mathbb{R}^n$, with Euclidean metric, we can choose $S(x_0, \ldots, x_n)$ to be the simplex with straight edges and hyperplane faces. For other M we can choose a positive definite metric, take geodesic edges, and minimal surface faces. Readers may wish to skip the technical remainder of this section and go straight to Section 3.

Consider the (almost) standard simplex Δ_n in $\mathbb{R}^n$ having vertices

$$E_0 = (0, \ldots, 0), \qquad E_1 = (1, 0, \ldots),$$
$$E_2 = (1, 1, 0, \ldots, 0), \ldots, \qquad E_n = (1, 1, 1, \ldots, 1).$$

A point $\mathbf{t} = (t_1, \ldots, t_n) \in \Delta_n$ has $0 \leqq t_n \leqq t_{n-1} \leqq \ldots \leqq t_1 \leqq 1$.

Consider some coordinate chart (U, y^a) in M which contains x_0 and let $x_1, \ldots, x_n$ also lie in U. Then for this set of points the allocation is a map $\Delta_n \to U$, which in coordinates may be written

$$(t_1, \ldots, t_n) \mapsto y^a(x_0, \ldots, x_n; t_1, \ldots, t_n).$$

Here for each j the vertex E_j of Δ_n is mapped to the point x_j of U, *i.e.*

$$y^a(x_0, \ldots, x_n; 1, 1, \ldots, 1, 0, \ldots, 0) = x_j^a,$$

where there are j 1's after the semicolon. We suppose that the points x_j are close enough together that $S(\Delta_n) \subset U$. Now for the formal definition of S.

DEFINITION 1. – *Let* $\mathscr{V} \subset M \times M \times \ldots \times M$, $n+1$ *times, be some neighbourhood of the diagonal subset* $M_D = \{(x, x, \ldots, x); x \in M\}$, *which can be retracted onto* M_D *by a strong deformation retract.*

An allocation *is a smooth map*

$$S: \quad \mathscr{V} \times \Delta_n \to M,$$

satisfying the following conditions.

(i) *If* $\sigma \in S_{n+1}$ *is a permutation of* $(0, \ldots, n)$, *then the n-simplices* $S(x_{\sigma_0}, \ldots, x_{\sigma_n}; \Delta_n)$ *and* $S(x_0, \ldots, x_n; \Delta_n)$ *are identical subsets of* M; *their orientations are related by the factor* $\operatorname{sgn}(\sigma)$.

(ii) $S(x_0, \ldots, x_n; E_p) = x_p$, $0 \leqq p \leqq n$.

(iii) *Let* $[E_{j_0}, \ldots, E_{j_p}]$ *be the face of* Δ_n *having vertices* $E_{j_0}, \ldots, E_{j_p}$, $j_0 < \ldots < j_p$. *Then the image of this face,*

$$S(x_0, \ldots, x_n; [E_{j_0}, \ldots, E_{j_p}]) \equiv S(x_{j_0}, \ldots, x_{j_p}) \tag{1}$$

only depends on the locations of its own vertices $x_{j_0}, \ldots, x_{j_p}$ *and does not depend on the other vertices* x_j, $j \notin \{j_0, \ldots, j_p\}$. *That is, the image of the face is fixed by its own vertices.* □

Note that if we define the boundary of the chain $(x_0, \ldots, x_n)$ to be

$$\partial(x_0, \ldots, x_n) = \sum_{j=0}^{n} (-1)^j (x_0, \ldots, \hat{x}_j, \ldots, x_n), \tag{2}$$

then S and ∂ commute. For example

$$\partial S(x_0, x_1, x_2) = S(x_1, x_2) - S(x_0, x_2) + S(x_0, x_1)$$
$$= S((x_1, x_2) - (x_0, x_2) + (x_0, x_1)) = S\,\partial(x_0, x_1, x_2).$$

Let us develop property (iii) using coordinates.

For $0 \leqq p \leqq n$, let Δ_p be the p-face of Δ_n having vertices $E_0, \ldots, E_p$. Define the associated functions

$$y^a(x_0, \ldots, x_p; t_1, \ldots, t_p) = y^a(x_0, \ldots, x_n; t_1, \ldots, t_p, 0, \ldots, 0)$$
$$= y^a(x_0, \ldots, x_{p-1}, x_p = x_{p+1} = \ldots = x_n; t_1, \ldots, t_p, t_{p+1}, \ldots, t_n) \tag{3}$$

where the second expression does not in fact depend upon $t_{p+1} \ldots t_n$.

These are the coordinates of points in the image of Δ_p in M.

If two vertices x_p and x_q coincide, $p < q$, the n-simplex

$$S(x_0, \ldots, x_p, \ldots, x_q = x_p, x_{q+1} \ldots x_n)$$

collapses to the $(n-1)$-simplex

$$S(x_0, \ldots \hat{x}_q \ldots x_n) \text{ or to } S(x_0, \ldots \hat{x}_p \ldots x_n).$$

These are the same set of points but may differ in orientation. This ambiguity will not bother us as we only confront the unambiguous case $q=p+1$.

A special case of (3) is

$$\left[\frac{\partial y^a}{\partial t_p}(x_0, \ldots, x_p; t_1, \ldots, t_p)\right]_{x_p=x_{p-1}}=0. \tag{4}$$

This property will be needed in Section 5, as will the following property:

$$\left[\frac{\partial y^a}{\partial x_1^b}(x_0, x_1; t)\right]_{x_1=x_0}=\delta_b^a t f(x_0; t) \tag{5}$$

where f is a function satisfying

$$f(x_0, 1)=1, \quad \text{for all } x_0. \tag{6}$$

To drive (5), note that the conditions

$$y^a(x_0, x_1; 0)=x_0^a$$
$$y^a(x_0, x_0; t)=x_0^a$$

imply that $y^a - x_0^a$ has the form

$$y^a(x_0, x_1; t)-x_0^a=t(x_1^b-x_0^b)f_b^a(x_0, x_1; t) \tag{7}$$

for some functions f_b^a and the condition

$$y^a(x_0, x_1; 1)=x_1^a$$

implies that

$$f_b^a(x_0, x_1; 1)=\delta_b^a.$$

Now the curve from x_0 to x_1, for x_1 near x_0, is "almost linear" in the sense that we may refine (7) to

$$y^a(x_0, x_1; t)=x_0^a+tf(x_0, t)(x_1^a-x_0^a)+(x_1^b-x_0^b)g_b^a(x_0, x_1; t) \tag{8}$$

where the functions g_b^a satisfy $g_b^a \to 0$ as $x_1 \to x_0$. Equation (5) then follows from (8).

3. DIFFERENTIAL FORMS AS HOCHSCHILD COCHAINS ON $\mathscr{D}$

Using an allocation S, we construct for each p-form α, an $\mathbb{R}$-linear map α_s from p-chains $(H_1, \ldots, H_p)$ in $\otimes_{\mathbb{R}}^p \mathscr{D}$ to $\mathscr{D}$,

$$\alpha_S(H_1, \ldots, H_p) \in \mathscr{D}.$$

In the case $p=0$ we show that $(d\alpha)_S=\delta(\alpha_S)$ where δ is the Hochschild codifferential. This is in fact true for all $p>0$ as we show in Section 4.

The following notation occurs repeatedly. Suppose for example

$$H(y)=A^{ab}(y)\frac{\partial}{\partial y^a}\frac{\partial}{\partial y^b}+A(y)$$

is a differential operator in the variables y^a.

Then we write

$$[H(y)f(x,y)]_{y=x}=\left[A^{ab}(x)\frac{\partial^2 f}{\partial y^a\,\partial y^b}(x,y)\right]_{y=x}+A(x)f(x,x).$$

DEFINITION 2. – *Let* $H_j\in\mathcal{D}$, $j=1,\ldots,p$, *let* $\psi\in\mathcal{F}$ *and let* α *be a* p-*form on* M. *Then* α_S *is defined by*

$$\begin{aligned}(\alpha_S(H_1,\ldots,H_p)\psi)(x_0)\\ =\Bigg[H_1(x_1)\Bigg[H_2(x_2)\Bigg[\ldots\Bigg[H_p(x_p)\Bigg[\left(\int_{S(x_0\ldots x_p)}\alpha\right)\\ \times\psi(x_p)\Bigg]\Bigg]_{x_p=x_{p-1}}\Bigg]_{x_{p-1}=x_{p-2}}\ldots\Bigg]_{x_1=x_0}\end{aligned}\qquad\square\quad(9)$$

Example 1. – ($p=0$) If α is a 0-form, it is a (zero-order) differential operator and so is already a 0-cochain on $\mathcal{D}$. The Hochschild 0-cochains on $\mathcal{D}$ are the elements of $\mathcal{D}$. So $\alpha_S=\alpha$ when $\deg\alpha=0$, independent of S. □

Example 2. – ($p=1$) If α is a 1-form, then

$$(\alpha_S(H)\psi)(x_0)=\left[H(x_1)\left[\left(\int_{S(x_0,\,x_1)}\alpha\right).\psi(x_1)\right]\right]_{x_1=x_0}.$$

If $\alpha=df$ the integral of α does not depend on the path allocated by S, only on its ends, and we get

$$\begin{aligned}((df)_S(H)\psi)(x_0)&=[H(x_1)(f(x_1)-f(x_0))\psi(x_1)]_{x_1=x_0}\\ &=(Hf\psi-fH\psi)(x_0)=([H,f]\psi)(x_0).\end{aligned}\qquad\square\quad(10)$$

Recall in Hochschild cohomology that the codifferential $\delta\alpha_S$ of a p-cochain $\alpha_S\in C^p(\mathcal{D},\mathcal{D})$ is the $(p+1)$-cochain given by

$$\begin{aligned}\delta\alpha_S(H_1,\ldots,H_{p+1})&=H_1\,\alpha_S(H_2,\ldots,H_{p+1})-\alpha_S(H_1H_2,H_3,\ldots,H_{p+1})\\ &+\ldots+(-1)^p\alpha_S(H_1,\ldots,H_pH_{p+1})+(-1)^{p+1}\alpha_S(H_1,\ldots,H_p)H_{p+1}.\end{aligned}$$

For example, when $p=1$,

$$\delta\alpha_S(H_1,H_2)=H_1\,\alpha_S(H_2)-\alpha_S(H_1H_2)+\alpha_S(H_1)H_2$$

and for $p=0$, (a 0-cochain is just an element $H_1\in\mathcal{D}$)

$$\delta H_1(H)=[H,H_1].$$

Thus regarding f as a 0-form, and f_S as the corresponding zero-order differential operator $f \in \mathscr{D}$, equation (10) says that for any $H \in \mathscr{D}$

$$(df)_S(H) = [H, f] = \delta(f_S)(H),$$

i.e. $(df)_S = \delta(f_S)$.

Remark 1. — If $S(x_0, x_1)$ is the geodesic through x_0, x_1 for some connection ∇ on TM, then for H given by $H\psi = a^{i_1 \ldots i_k} \psi_{;i_1 \ldots i_k}$, where a is a symmetric contravariant tensor of order k, we write $H = a_k . \nabla^k$, and can obtain

$$\alpha_S(H) = \sum_{j=1}^{k} a_k . (\nabla^{k-j} \alpha \nabla^{j-1}),$$

that is

$$\alpha_S(H)\psi = \sum_{j=1}^{k} a^{i_1 \ldots i_k} (\alpha_{i_1} \psi_{;i_2 \ldots i_j})_{;i_{j+1} \ldots i_k}.$$

So $\alpha_S(H)$ is like a Fréchet derivative,

$$\alpha_S(H) = \left(\frac{\partial H}{\partial \nabla}, \alpha \right) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} (H(\nabla + \varepsilon\alpha) - H(\nabla)).$$

This is related to a definition of the probability current in quantum mechanics, [6]. □

4. THE INDUCED MAP BETWEEN COHOMOLOGY GROUPS

In this section we prove

LEMMA 1. — *For any allocation* S *and p-form* α ($p > 0$)

$$(d\alpha)_S = \delta(\alpha_S).$$

The map $D_S : \alpha \mapsto \alpha_S$ thus passes to cohomology classes giving $\check{D} : [\alpha] \mapsto [\alpha_S]$. We prove that $\check{D}$ does not depend on S, by taking $\check{D}_1$ and $\check{D}_2$ to correspond to different allocations and constructing a homotopy between them.

We showed in Section 3 that Lemma 1 holds for $p = 0$. Before the proof for general p, let us show it for $p = 1$, for clarity.

We have

$$\int_{S(x_0, x_1, x_2)} d\alpha = \int_{\partial S(x_0, x_1, x_2)} \alpha$$

and

$$\partial S(x_0, x_1, x_2) = S(x_1, x_2) - S(x_0, x_2) + S(x_0, x_1).$$

Hence, from Definition 2

$$
\begin{aligned}
(d\alpha)_S(\mathrm{H}_1, \mathrm{H}_2)\psi)(x_0) &= \left[\mathrm{H}_1(x_1)\left[\mathrm{H}_2(x_2)\left[\left(\int_{S(x_0, x_1, x_2)} d\alpha\right)\right.\right.\right. \\
&\quad \times \psi(x_2)\Big]\Big]_{x_2=x_1}\Big]_{x_1=x_0} = \left[\mathrm{H}_1(x_1)\left[\mathrm{H}_2(x_2)\left[\left(\int_{S(x_1, x_2)} \alpha\right).\psi(x_2)\right]\right]_{x_2=x_1}\right]_{x_1=x_0} \\
&\quad - \left[\mathrm{H}_1(x_1)\left[\mathrm{H}_2(x_2)\left[\left(\int_{S(x_0, x_2)} \alpha\right).\psi(x_2)\right]\right]_{x_2=x_1}\right]_{x_1=x_0} \\
&\quad + \left[\mathrm{H}_1(x_1)\left[\mathrm{H}_2(x_2)\left[\left(\int_{S(x_0, x_1)} \alpha\right).\psi(x_2)\right]\right]_{x_2=x_1}\right]_{x_1=x_0}
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
&= \mathrm{H}_1(x_0)\left[\mathrm{H}_2(x_2)\left[\left(\int_{S(x_0, x_2)} \alpha\right).\psi(x_2)\right]\right]_{x_2=x_0} \\
&\quad - \left[\mathrm{H}_1(x_1)\mathrm{H}_2(x_1)\left[\left(\int_{S(x_0, x_1)} \alpha\right).\psi(x_1)\right]\right]_{x_1=x_0} \\
&\quad + \left[\mathrm{H}_1(x_1)\left[\left(\int_{S(x_0, x_1)} \alpha\right).\mathrm{H}_2(x_2)\psi(x_2)\right]\right]_{x_2=x_1}\Big]_{x_1=x_0} \\
&= ((\mathrm{H}_1\,\alpha_S(\mathrm{H}_2) - \alpha_S(\mathrm{H}_1\,\mathrm{H}_2) + \alpha_S(\mathrm{H}_1)\,\mathrm{H}_2)\psi)(x_0) = (\delta\alpha_S(\mathrm{H}_1, \mathrm{H}_2)\psi)(x_0).
\end{aligned}
\tag{12}
$$

The expressions (11) and (12) agree term by term. In the first term of (11), x_1 may be set equal to x_0 at once, in the second term, x_2 may be set equal to x_1 in the bracket $[\ \ ]_{x_2=x_1}$ simply by writing x_1 instead of x_2, and in the third term the α-integral is independent of x_2 so that $\mathrm{H}_2(x_2)$ acts only on $\psi(x_2)$.

The proof of Lemma 1 for general p follows the same pattern but with more middle terms.

$$
\begin{aligned}
&((d\alpha)_S(\mathrm{H}_1, \ldots, \mathrm{H}_{p+1})\psi)(x_0) \\
&\qquad = \sum_{j=0}^{p+1} (-1)^j \left[\mathrm{H}_1(x_1)\left[\ldots\left[\mathrm{H}_{p+1}(x_{p+1})\right.\right.\right. \\
&\qquad\quad \times \left[\left(\int_{S(x_0, \ldots, \hat{x}_j, \ldots, x_{p+1})} \alpha\right).\psi(x_{p+1})\right]\Big]_{x_{p+1}=x_p}\Big]\ldots\Big]_{x_1=x_0}
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
&= \mathrm{H}_1(x_0)\left[\mathrm{H}_2(x_2)\left[\ldots\left[\left(\int_{S(x_0, x_2 \ldots, x_{p+1})} \alpha\right).\psi(x_{p+1})\right]\right]\ldots\right]_{x_2=x_0} \\
&\quad + \sum_{j=1}^{p} (-1)^j \left[\mathrm{H}_1(x_1)\ldots\left[\mathrm{H}_j(x_j)\mathrm{H}_{j+1}(x_j)\left[\mathrm{H}_{j+2}(x_{j+2})\left[\ldots\right.\right.\right.\right.
\end{aligned}
$$

$$\times\left[\left(\int_{S(x_0,\ldots,\hat{x}_{j+1},\ldots,x_{p+1})}\alpha\right).\psi(x_{p+1})\right]\Bigg]\ldots\Bigg]_{x_{j+2}=x_j}\ldots\Bigg]_{x_1=x_0}$$

$$+(-1)^{p+1}\left[H_1(x_1)\left[\ldots H_p(x_p)\left[\left(\int_{S(x_0,\ldots,x_p)}\alpha\right)\right.\right.\right.$$

$$\left.\left.\left.\times H_{p+1}(x_p)\psi(x_p)\right]\right]_{x_p=x_{p-1}}\ldots\right]_{x_1=x_0}$$

$$=\left(\left(H_1\alpha_S(H_2,\ldots,H_{p+1})+\sum_{j=1}^{p}(-1)^j\alpha_S(H_1,\ldots,H_jH_{j+1},\ldots,H_{p+1})\right.\right.$$

$$\left.\left.+(-1)^{p+1}\alpha_S(H_1,\ldots,H_p)H_{p+1}\right)\psi\right)(x_0)$$

$$=((\delta\alpha_S)(H_1,\ldots,H_{p+1})\psi)(x_0). \quad (14)$$

Hence if $\alpha\sim\alpha'$ in de Rham cohomology so that $\alpha-\alpha'=d\beta$ for some de Rham $(p-1)$-form β, then

$$\alpha_S-\alpha'_S=(\alpha-\alpha')_S=(d\beta)_S=\delta(\beta_S).$$

This means that $\alpha_S\sim\alpha'_S$ in Hochschild cohomology so that $D_S:\alpha\mapsto\alpha_S$ induces a map

$$\check{D}:\quad H^p_{DR}(M,\mathbb{R})\to H^p(\mathcal{D},\mathcal{D}).$$

In Section 6 we prove that $\check{D}$ is an isomorphism of H^p_{DR} with the *relative* Hochschild cohomology group [1] $H^p(\mathcal{D},\mathcal{F};\mathcal{D})$.

Note that (13) may be written using the notation (2) as

$$((d\alpha)_S(H_1,\ldots,H_{p+1})\psi)(x_0)$$

$$=\left[H_1(x_1)\left[\ldots\left[H_{p+1}(x_{p+1})\left[\left(\int_{S\partial(x_0,\ldots,x_{p+1})}\alpha\right)\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\times\psi(x_{p+1})\right]\right]_{x_{p+1}=x_p}\ldots\right]_{x_1=x_0}$$

$$\equiv(\alpha_{S\partial}(H_1,\ldots,H_{p+1})\psi)(x_0)$$

where the middle expression defines $\alpha_{S\partial}$. Equations (13) and (14) show that

$$(d\alpha)_S=\alpha_{S\partial}=\delta(\alpha_S). \quad (15)$$

We end this section with a two-step proof that $\check{D}$ does not depend on the choice of allocation. We show that any two allocations are homotopic, and that this implies that they give the same map $\check{D}$.

For $\lambda\in[0,1]$ let

$$J_\lambda:\quad \mathcal{V}\times\Delta_n\to\mathcal{V}\times\Delta_n\times[0,1]$$

be the inclusion

$$(x_0, \ldots, x_n; \mathbf{t}) \mapsto (x_0, \ldots, x_n; \mathbf{t}, \lambda)$$

where $\mathscr{V}$ is given in Definition 1.

DEFINITION 3. — *Two allocations* S_0 *and* S_1 *are* homotopic *if there is a smooth map*

$$\chi: \quad \mathscr{V} \times \Delta_n \times [0, 1] \to M$$

such that the following conditions are satisfied:

(i) $\chi \circ J_\lambda$ *is an allocation for each fixed* λ

(ii) $\chi \circ J_\lambda = S_\lambda$ *at* $\lambda = 0$ *and* $\lambda = 1$. □

Evidently *any* two allocations S_0 and S_1 are homotopic as we may define the homotopy χ to be

$$\chi(x_0, \ldots, x_n; \mathbf{t}, \lambda) = S_0(S_0(x_0, \ldots, x_n; \mathbf{t}), S_1(x_0, \ldots, x_n; \mathbf{t}); \lambda).$$

As in (1), for $(x_0, \ldots, x_p) \in \mathscr{V}$ we denote $(\chi \circ J_\lambda)(x_0, \ldots, x_p)$ to be the face of the $\chi \circ J_\lambda$-simplex having vertices $x_0, \ldots, x_p$, and introduce the $p+1$-dimensional subset

$$\chi(x_0, \ldots, x_p) = \bigcup_{0 \leqq \lambda \leqq 1} (\chi \circ J_\lambda)(x_0, \ldots, x_p).$$

Then

$$\partial \chi(x_0, \ldots, x_p) = S_1(x_0, \ldots, x_p) - S_0(x_0, \ldots, x_p) - \chi(\partial(x_0, \ldots, x_p))$$

using the notation (2).

We wish to show that if α is a closed de Rham p-form, then $\alpha_{S_1} - \alpha_{S_0}$ is an exact Hochschild p-cochain.

In Definition 2, for $\alpha_{S_1} - \alpha_{S_0}$, the α-integral is over the subset

$$S_1(x_0, \ldots, x_p) - S_0(x_0, \ldots, x_p) = \partial \chi(x_0, \ldots, x_p) + \chi \partial(x_0, \ldots, x_p).$$

Now

$$\int_{\partial \chi} \alpha = \int_{\chi} d\alpha = 0,$$

so by (15),

$$\alpha_{S_1} - \alpha_{S_0} = \alpha_{\chi \partial} = \delta \alpha_\chi,$$

where α_χ is the Hochschild $(p-1)$-cochain given by

$$(\alpha_\chi(H_1, \ldots, H_{p-1})\psi)(x_0) = \left[H_1(x_1)\left[\ldots\left[H_{p-1}(x_{p-1})\left[\left(\int_{\chi(x_0 \ldots x_{p-1})} \alpha\right) \times \psi(x_{p-1})\right]\right]_{x_{p-1}=x_{p-2}} \ldots\right]\right]_{x_1=x_0}.$$

5. CONSISTENCY

This section comprises a proof of the following result:

LEMMA 2. — *If* $X_1, \ldots, X_p$ $(p \geqq 0)$ *are vector fields on* M, *then*

$$\alpha_S(X_1, \ldots, X_p) = \frac{1}{p!}\alpha(X_1, \ldots, X_p). \tag{16}$$

We begin by proving the $\mathscr{F}$-linearity properties (*a*)-(*c*) of α_S listed in the introduction. Let $f \in \mathscr{F} \subset \mathscr{D}$; then the operator $H \circ f : \psi \mapsto Hf\psi$. It follows from Definition 2 that

$$\alpha_S(H_1, \ldots, H_j, fH_{j+1}, \ldots) = \alpha_S(H_1, \ldots, H_j f, H_{j+1}, \ldots),$$

because

$$H_j(x_j)\,[f(x_{j+1})\,H_{j+1}(x_{j+1})\,\Psi(x_j, x_{j+1})]_{x_{j+1}=x_j} = (H_j f)(x_j)\,[H_{j+1}(x_{j+1})\,\Psi(x_j, x_{j+1})]_{x_{j+1}=x_j}.$$

Thus α_S defined on $\mathscr{D} \otimes_R \ldots \otimes_R \mathscr{D}$ factors through $\mathscr{D} \otimes_{\mathscr{F}} \ldots \otimes_{\mathscr{F}} \mathscr{D}$.

It is also clear from Definition 2 that

$$\alpha_S(fH_1, H_2, \ldots) = f\alpha_S(H_1, H_2, \ldots)$$
$$\alpha_S(\ldots, H_p f) = \alpha_S(\ldots, H_p) \circ f.$$

In consequence,

$$[\alpha_S(H_1, \ldots, H_p), f] = \sum_{j=1}^{p} \alpha_S(H_1, \ldots, [H_j, f], \ldots, H_p). \tag{17}$$

Suppose ord $H_j = 0$, *i.e.*: $H_j \in \mathscr{F}$ for some j. Then

$$\left[H_j(x_j)\left[H_{j+1}(x_{j+1})\ldots\left[H_p(x_p)\left(\int_{S(x_0\ldots x_p)}\alpha\right)\times\psi(x_p)\right]_{x_p=x_{p-1}}\ldots\right]_{x_{j+1}=x_j}\right]_{x_j=x_{j-1}}$$
$$= H_j(x_{j-1})\left[H_{j+1}(x_{j+1})\left[H_{j+2}(x_{j+2})\ldots\left[H_p(x_p)\left[\left(\int_{S(x_0,\ldots,x_{j-1},x_{j-1},x_{j+1}\ldots x_p)}\alpha\right).\psi(x_p)\right]_{x_p=x_{p-1}}\ldots\right]_{x_{j+1}=x_{j-1}}\right.\right.\right. = 0$$

since the α-integral is identically zero for all $x_0, \ldots, x_{j-1}, x_{j-1}, \ldots, x_p$. So if any H_j is just a function then

$$\alpha_S(H_1, \ldots, H_p) = 0.$$

Hence α_S is a $\mathscr{F}$-normalised cochain, [1, 7], factoring through

$$\mathscr{D}/\mathscr{F} \otimes_{\mathscr{F}} \ldots \otimes_{\mathscr{F}} \mathscr{D}/\mathscr{F}.$$

The function $\alpha_S(H_1, \ldots, H_p)\psi$, when the differentiations which lie in the H_j's are carried out, consists of several terms. In any non-zero term *every* operator H_j must differentiate the factor $\int_{S(x_0 \ldots x_p)} \alpha$ at least once. The maximum number of times that the other factor, ψ, can be differentiated is thus

$$\operatorname{ord}(\alpha_S(H_1, \ldots, H_p)) = \sum_{j=1}^{p} (\operatorname{ord} H_j - 1).$$

It follows that if $X_1, \ldots, X_p$ are vector fields then

$$\operatorname{ord}(\alpha_S(X_1, \ldots, X_p)) = 0 \tag{18}$$

and

$$\alpha_S(X_1, \ldots, fX_j, \ldots, X_p) = f\alpha_S(X_1, \ldots, X_j, \ldots, X_p) \tag{19}$$

since each commutator $[X_k, f]$, (*see* (17)) is a function.

We are now sufficiently prepared to prove Lemma 2. Again we start with the case $p=1$ to indicate the pattern. Let α be a 1-form. We wish to prove that

$$\alpha_S\left(\frac{\partial}{\partial x^a}\right)(x_0) = \alpha_a(x_0).$$

The simplex $S(x_0, x_1)$ is the curve $t \mapsto y(x_0, x_1; t)$, $0 \leqq t \leqq 1$. So

$$\alpha_S\left(\frac{\partial}{\partial x^a}\right)(x_0) = \left[\frac{\partial}{\partial x_1^a}\left[\int_0^1 dt \frac{\partial y^b}{\partial t}(x_0, x_1; t)\,\alpha_b(y(x_0, x_1; t))\right]\right]_{x_1 = x_0} \tag{20}$$

$$= \int_0^1 dt.\left[\frac{\partial}{\partial t}\frac{\partial y^b}{\partial x_1^a}(x_0, x_1; t)\right]_{x_1 = x_0} .\,\alpha_b(x_0) \tag{21}$$

$$= \alpha_a(x_0) \int_0^1 dt \frac{d}{dt}(tf(x_0, t)) = \alpha_a(x_0). \tag{22}$$

In (20),

$$\frac{\partial y^b}{\partial t}(x_0, x_0; t) = 0,$$

a special case of (4), so the $\partial\alpha/\partial x$ term disappears. The passage from (21) to (22) uses (5) and (6).

We now prove the general result.

Proof of Lemma 2. — Let $\deg \alpha = p \geqq 0$. Equations (18) and (19) show that α_S is $\mathscr{F}$-valued and $\mathscr{F}$-multilinear on vector fields. So to prove

$\alpha_S(X_1, \ldots, X_p)=\frac{1}{p!}\alpha(X_1, \ldots, X_p)$ it is enough to prove

$$\alpha_S\left(\frac{\partial}{\partial x^{a_1}}, \ldots, \frac{\partial}{\partial x^{a_p}}\right)(x_0)=\frac{1}{p!}\alpha_{a_1\ldots a_p}(x_0).$$

From Definition 2 we have

$$\begin{aligned}\alpha_S\left(\frac{\partial}{\partial x^{a_1}}, \ldots, \frac{\partial}{\partial x^{a_p}}\right)(x_0) &= \left[\frac{\partial}{\partial x_1^{a_1}}\left[\ldots\left[\frac{\partial}{\partial x_p^{a_p}}\int_0^1 dt_1 . \int_0^{t_1} dt_2 \ldots \int_0^{t_{p-1}} dt_p \right.\right.\right.\\ &\quad\left.\left.\left.\times \frac{\partial y^{b_1}}{\partial t_1}\ldots\frac{\partial y^{b_p}}{\partial t_p}\alpha_{b_1\ldots b_p}(y(x_0\ldots x_p;\, t_1\ldots t_p))\right]_{x_p=x_{p-1}}\ldots\right]\right]_{x_1=x_0}\\ &= \int_0^1 dt_1\ldots\int_0^{t_{p-1}} dt_p\left[\frac{\partial}{\partial x_1^{a_1}}\left[\ldots\left[\frac{\partial}{\partial x_p^{a_p}}\left[\frac{\partial y^{b_1}}{\partial t_1}\ldots\frac{\partial y^{b_p}}{\partial t_p}\right.\right.\right.\right.\\ &\quad\left.\left.\left.\left.\times\alpha_{b_1\ldots b_p}(y(x_0\ldots x_p;\, t_1\ldots t_p))\right]\right]_{x_p=x_{p-1}}\ldots\right]\right]_{x_1=x_0}. \qquad (23)\end{aligned}$$

upon taking the $\partial/\partial x$ operators inside the t-integrals, these variables being independent.

Consider the innermost bracket. The action of $\frac{\partial}{\partial x_p^{a_p}}$ on the $(p+1)$ factors to its right produces $(p+1)$ terms, but all except one vanish on setting $x_p=x_{p-1}$ as a result of equation (4). The survivor is

$$\begin{aligned}A=&\left(\frac{\partial y^{b_1}}{\partial t_1}\ldots\frac{\partial y^{b_{p-1}}}{\partial t_{p-1}}\right)(x_0, \ldots, x_{p-1};\, t_1\ldots t_{p-1})\\ &\times\left[\frac{\partial}{\partial x_p^{a_p}}\frac{\partial}{\partial t_p}y^{b_p}(x_0, \ldots, x_p;\, t_1\ldots t_p)\right]_{x_p=x_{p-1}}\\ &\times\alpha_{b_1\ldots b_p}(y(x_0\ldots x_{p-1};\, t_1\ldots t_{p-1}))\end{aligned}$$

We are using the notation of (3),

$$y(x_0, \ldots, x_{p-1};\, t_1, \ldots, t_{p-1})=y(x_0, \ldots, x_p;\, t_1, \ldots, t_p)\big|_{x_p=x_{p-1}}.$$

Now consider the next bracket out, which is

$$\left[\frac{\partial}{\partial x_{p-1}^{a_{p-1}}}A\right]_{x_{p-1}=x_{p-2}}$$

The action of $\partial/\partial x_{p-1}^{a_{p-1}}$ again produces $p+1$ terms, but since

$$\left[\frac{\partial y^{b_{p-1}}}{\partial t_{p-1}}(x_0, \ldots, x_{p-1};\, t_1, \ldots, t_{p-1})\right]_{x_{p-1}=x_{p-2}}=0,$$

the only survivor is the term

$$\left(\frac{\partial y^{b_1}}{\partial t_1}\cdots\frac{\partial y^{b_{p-2}}}{\partial t_{p-2}}\right)(x_0\ldots x_{p-2};\, t_1\ldots t_{p-2})$$
$$\times\left[\frac{\partial}{\partial x_{p-1}^{a_{p-1}}}\,\frac{\partial}{\partial t_{p-1}}\,y^{b_{p-1}}(x_0\ldots x_{p-1};\, t_1\ldots t_{p-1})\right]_{x_{p-1}=x_{p-2}}$$
$$\times\left[\frac{\partial}{\partial x_p^{a_p}}\,\frac{\partial}{\partial t_p}\,y^{b_p}(x_0\ldots x_{p-2},\, x_p;\, t_1\ldots t_{p-2},\, t_p)\right]_{x_p=x_{p-2}}$$
$$\times\,\alpha_{b_1\ldots b_p}(y(x_0\ldots x_{p-2};\, t_1\ldots t_{p-2})).$$

Here we use the fact (*cf.* (3)),

$$y^{b_p}(x_0,\,\ldots,\,x_{p-2},\,x_{p-1},\,x_p;\,t_1,\,\ldots,\,t_p)\big|_{x_{p-1}=x_{p-2}}$$
$$=y^{b_p}(x_0,\,\ldots,\,x_{p-2},\,x_p;\,t_1,\,\ldots,\,t_{p-2},\,t_p).$$

Working outwards through all the square brackets, we find that the expression (23) becomes

$$\alpha_{b_1\ldots b_p}(x_0)\int_0^1 dt_1\ldots\int_0^{t_{p-1}} dt_p\prod_{j=1}^{p}\left[\frac{\partial}{\partial t_j}\,\frac{\partial}{\partial x_j^{a_j}}\,y^{b_j}(x_0,\,x_j;\,t_j)\right]_{x_j=x_0}$$
$$=\alpha_{a_1\ldots a_p}(x_0)\int_0^1 dt_1\ldots\int_0^{t_{p-1}} dt_p\prod_{j=1}^{p}\frac{\partial}{\partial t_j}(t_j\,f(x_0;\,t_j))=\frac{1}{p!}\,\alpha_{a_1\ldots a_p}(x_0).\qquad\square$$

6. PROOF OF THEOREM 1

Here we prove that $\check{D}$ is both injective and surjective by introducing a map $\check{a}$ in the reverse direction. Let $C^p_H\,(=C^p_H(\mathscr{D},\,\mathscr{F};\,\mathscr{D}))$ and Z^p_H be respectively the space of $\mathscr{F}$-relative Hochschild p-cochains and the subspace of closed p-cochains. Let $C^p_{DR}(M,\,\mathbb{R})$ be the space of de Rham p-forms.

Define the map

$$a:\quad C^p_H\to C^p_{DR}$$

by

$$(a\mathrm{A})(X_1,\,\ldots,\,X_p)=\sum_{\sigma\in S_p}\operatorname{sgn}(\sigma)\,\mathrm{A}(X_{\sigma_1},\,\ldots,\,X_{\sigma_p})$$

for $\mathrm{A}\in C^p_H$ and vector fields $X_1,\,\ldots,\,X_p$. The restriction of A to vector fields is $\mathscr{F}$-multilinear in the vector fields, so $a\mathrm{A}$ is indeed a de Rham p-form. It follows from Lemma 2 that

$$a\circ D_S=\mathrm{id}\,(C^p_{DR}).$$

One can verify by direct calculation that $a(\delta A)=d(aA)$ so that a passes to cohomology as a map

$$\check{a}:\quad H^p(\mathscr{D}, \mathscr{F}; \mathscr{D}) \to H^p_{DR}(M, \mathbb{R}),$$

where

$$\check{a}\circ\check{D}=\mathrm{id}\,(H^p_{DR}).$$

Hence $\check{D}$ is injective. To show that $\check{D}$ is also surjective it is enough to prove the following.

LEMMA 3. — *The map $\check{a}$ is injective.*

The proof of Lemma 3 depends upon two other lemmas.

LEMMA 4. — *Let $\mathscr{P}_{k,p}$ $(1\leqq p\leqq k)$ be the vector space over $\mathbb{R}$ whose basis consists of ordered partitions of the set $K=\{1, 2, \ldots, k\}$ into p non-empty subsets. Let $\mathscr{P}_{k,k+1}\subset\mathscr{P}_{k,k}$ be the one-dimensional subspace spanned by the element*

$$\sum_{\sigma\in S_k}(\operatorname{sgn}\sigma)(\sigma_1, \sigma_2, \ldots, \sigma_k). \tag{24}$$

For $2\leqq p\leqq k$ let $\partial_{k,p}:\mathscr{P}_{k,p}\to\mathscr{P}_{k,p-1}$ be given by:

$$\partial_{k,p}(I_1, I_2, \ldots, I_p)=(I_1\cup I_2, I_3, \ldots, I_p)-(I_1, I_2\cup I_3, I_4, \ldots, I_p)+\ldots +(-1)^p(I_1, \ldots, I_{p-1}\cup I_p). \tag{25}$$

and let $\partial_{k,k+1}:\mathscr{P}_{k,k+1}\to\mathscr{P}_{k,k}$ be the inclusion. Then the sequence

$$0\to\mathscr{P}_{k,k+1}\xrightarrow{\partial_{k,k+1}}\mathscr{P}_{k,k}\xrightarrow{\partial_{k,k}}\mathscr{P}_{k,k-1}\xrightarrow{\partial_{k,k-1}}\ldots\xrightarrow{\partial_{k,2}}\mathscr{P}_{k,1}\to 0 \tag{26}$$

is exact.

Proof. — The sequence (26) is the homology sequence for a barycentric triangulation of the sphere S^{k-2} using appropriate labelling ([1]).

Consider k distinct points $x_1, \ldots, x_k$ in $\mathbb{R}^{k-1}$ whose convex hull forms a $(k-1)$-simplex Δ_{k-1}. The boundary $\partial\Delta_{k-1}$ triangulates S^{k-2}. We follow [8], Section 1.4.

Add the barycentres of all p-faces, $p\leqq k-2$ of this triangulation to obtain the set of vertices of a finer triangulation of S^{k-2}. Relabel all the vertices and barycentres by ordered partitions of K into two subsets as follows. The vertex x_j is renamed $(\{j\}, K-\{j\})$. For any subset $J=\{j_1, \ldots, j_{p+1}\}\subset K$, the barycentre $x_{j_1\ldots j_{p+1}}$ of the p-face having vertices $x_{j_1}, \ldots, x_{j_{p+1}}$ is renamed $(J, K-J)$. Thus the ordered 2-partitions

([1]) This delightful argument was given to us by Alastair King and Johan Dupont, to whom we are most grateful.

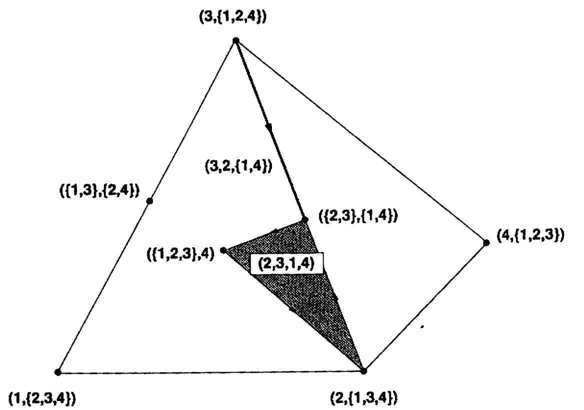
(J, K − J) of K correspond one-to-one with the vertices of the refined triangulation of S^{k-2}.

A p-simplex of the refined triangulation has $p+1$ vertices of the form

$$(J_1, K-J_1), (J_2, K-J_2), \ldots, (J_{p+1}, K-J_{p+1})$$

where $J_1 \subset J_2 \subset \ldots \subset J_{p+1}$. Label this p-simplex, with this orientation, by the $(p+2)$-partition

$$(J_1, J_2-J_1, J_3-J_2, \ldots, J_{p+1}-J_p, K-J_{p+1}). \tag{27}$$



The figure illustrates the labelling of some points, lines and 2-faces in the (123)-plane of the refined triangulation of S^2.

So for $k \geqq p \geqq 2$ the labelling gives an isomorphism from the vector space of $(p-2)$-chains of this barycentric triangulation of S^{k-2} to $\mathscr{P}_{k,p}$. We now show it is a chain map with respect to the simplicial boundary operator ∂ and the operator $\partial_{k,p}$ in (25).

The boundary of the face (27) is the signed sum of the subfaces whose vertices are

$$(J_1, K-J_1), \ldots, (J_j, K-J_j)^{\wedge}, \ldots, (J_{p+1}, K-J_{p+1})$$

where $\hat{}$ denotes omission. The above subface is labelled by the ordered $(p+1)$-partition

$$(J_1, J_2-J_1, \ldots, J_{j-1}-J_{j-2}, J_{j+1}-J_{j-1}, \\ J_{j+2}-J_{j+1}, \ldots, J_{p+1}-J_p, K-J_{p+1}).$$

Noting that

$$J_{j+1}-J_{j-1}=(J_j-J_{j-1}) \cup (J_{j+1}-J_j)$$

and writing $J_0=\varphi$, $J_{p+2}=K$, we see that

$$\partial(J_1, J_2-J_1, \ldots, J_{p+1}-J_p, J_{p+2}-J_{p+1})$$
$$=\sum_{j=1}^{p+1}(-1)^{j+1}(J_1, J_2-J_1, \ldots, (J_j-J_{j-1})\cup(J_{j+1}-J_j), J_{j+2}-J_{j+1}, \ldots, J_{p+2}-J_{p+1})$$

in agreement with equation (25).

The vector (24) spanning $\mathscr{P}_{k,k+1}$ is the sphere S^{k-2} itself which generates $H_{k-2}(S^{k-2})=\mathbb{R}$. Since for $2<p<k$, $H_{p-2}(S^{k-2})$ vanishes and since the map $\partial_{k,2}:\mathscr{P}_{k,2}\to\mathscr{P}_{k,1}$ is the augmentation, (26) is an exact sequence as required. □

Lemma 4 implies that there exist homotopies for $1\leqq p\leqq k$,

$$\tau_{k,p}:\quad \mathscr{P}_{k,p}\to\mathscr{P}_{k,p+1}$$

such that

$$\tau_{k,p-1}\circ\partial_{k,p}+\partial_{k,p+1}\circ\tau_{k,p}=\mathrm{id}(\mathscr{P}_{k,p}). \tag{28}$$

Let

$$\operatorname{ord}\mathbf{H}=\sum_{j=1}^{p}\operatorname{ord}H_j$$

be the total order of the p-chain $\mathbf{H}=(H_1, \ldots, H_p)$. This is the standard filtration on $\otimes_{\mathbb{R}}^{*}\mathscr{D}$.

LEMMA 5. — *Let* A *be a cochain in* C_H^p *and let* $k\geqq p$. *If* A *satisfies*
(i)

$$A(\mathbf{H})=0 \qquad \textit{for} \quad \operatorname{ord}\mathbf{H}<k; \tag{29}$$

(ii) $\delta A=0$ (*or* $aA=0$ *in the special case* $k=p$);
then there exists $B\in C_H^{p-1}$ *such that*

$$\delta B(\mathbf{H})=A(\mathbf{H}) \qquad \textit{for} \quad \operatorname{ord}\mathbf{H}\leqq k.$$

Proof. — Consider first any cochain $A\in C_H^p$, not necessarily closed, satisfying (29). Since A is $\mathscr{F}$-relative, for $f\in\mathscr{F}$ and for ord $\mathbf{H}=k$,

$$[A(\mathbf{H}), f]=\sum_{j=1}^{p}A(H_1, \ldots, [H_j, f], \ldots, H_p)=0.$$

So $A(\mathbf{H})\in\mathscr{F}$ when ord $\mathbf{H}=k$. The important fact for us will be that for given vector fields $X_1, \ldots, X_k$ on M, such an A determines a map $A_{k,p}:\mathscr{P}_{k,p}\to\mathscr{F}$ by:

$$A_{k,p}(I_1, \ldots, I_p)=A(\prod_{j\in I_1}X_j, \ldots, \prod_{j\in I_p}X_j) \tag{30}$$

The condition (29) ensures that the right hand side of (30) is not altered by any rearrangement of the factors X_j in a given argument $\prod_{j \in I_r} X_j$. Note that for $k>p$,

$$(\delta A)_{k,\,p+1} = -A_{k,\,p} \circ \partial_{k,\,p+1} \tag{31}$$

since

$$\begin{aligned}
(\delta A)_{k,\,p+1}(I_1, \ldots, I_{p+1}) &= (\delta A)\big(\prod_{j \in I_1} X_j, \ldots, \prod_{j \in I_{p+1}} X_j\big) \\
&= \sum_{r=1}^{p} (-1)^r A\big(\prod_{j \in I_1} X_j, \ldots, \prod_{j \in I_r \cup I_{r+1}} X_j, \ldots, \prod_{j \in I_{p+1}} X_j\big) \\
&= A_{k,\,p}\left(\sum_{r=1}^{p} (-1)^r (I_1, \ldots, I_r \cup I_{r+1}, \ldots, I_{p+1})\right) \\
&= -A_{k,\,p} \circ \partial_{k,\,p+1}(I_1, \ldots, I_{p+1}).
\end{aligned}$$

The special case $k=p$ is identical, with a replacing δ:

$$(a A)_{p,\,p+1} = -A_{p,\,p} \circ \partial_{p,\,p+1}. \tag{32}$$

For ord $\mathbf{H}<k$ define $B(\mathbf{H})=0$. For the $(p-1)$-chain of order k written as

$$\mathbf{H} = \big(\prod_{j \in I_1} X_j, \ldots, \prod_{j \in I_{p-1}} X_j\big), \tag{33}$$

let $B(\mathbf{H}) \in \mathscr{F}$ be given by

$$B(\mathbf{H}) = -A_{k,\,p} \circ \tau_{k,\,p-1}(I_1, \ldots, I_{p-1}) \equiv B_{k,\,p-1}(I_1, \ldots, I_{p-1}).$$

For ord $\mathbf{H}>k$ define $B(\mathbf{H})$ arbitrarily, consistent with the above.
Then for the p-chain $\mathbf{H}$ of order k,

$$\mathbf{H} = \big(\prod_{j \in I_1} X_j, \ldots, \prod_{j \in I_p} X_j\big),$$

$$\begin{aligned}
\delta B(\mathbf{H}) = (\delta B)_{k,\,p}(I_1, \ldots, I_p) &= -B_{k,\,p-1} \circ \partial_{k,\,p}(I_1, \ldots, I_p) \\
&= A_{k,\,p} \circ \tau_{k,\,p-1} \circ \partial_{k,\,p}(I_1, \ldots, I_p),
\end{aligned}$$

which, because $\delta A = 0$ ($a A = 0$ if $k=p$) and from (31) and (32), becomes

$$\begin{aligned}
A_{k,\,p} \circ (\tau_{k,\,p-1} \circ \partial_{k,\,p} + \partial_{k,\,p+1} \circ \tau_{k,\,p})(I_1, \ldots, I_p) \\
= A_{k,\,p}(I_1, \ldots, I_p) \quad \text{from (28)} \\
= A(\mathbf{H}). \quad \square
\end{aligned}$$

We now return to the proof that $\check{a}$ is injective.

Proof of Lemma 3. – Consider $A \in C^p_H$ such that $\delta A = a A = 0$. We show that $A = \delta B$ for some $B \in C^{p-1}_H$ by constructing $B(\mathbf{H})$ inductively for successively increasing $k = \text{ord}\, \mathbf{H}$.

We start with $k=p$. For ord $\mathbf{H}<p$, $\mathrm{A}(\mathbf{H})=0$. Therefore, from Lemma 5, there exists $\mathrm{B}^{(1)}\in \mathrm{C}_{\mathbf{H}}^{p-1}$ such that $\mathrm{A}(\mathbf{H})=\delta\,\mathrm{B}^{(1)}(\mathbf{H})$ for ord $\mathbf{H}\leqq p$. Thus $\mathrm{A}^{(1)}\equiv \mathrm{A}-\delta\,\mathrm{B}^{(1)}$ satisfies $\mathrm{A}^{(1)}(\mathbf{H})=0$ for ord $\mathbf{H}\leqq p$ and moreover $\delta\,\mathrm{A}^{(1)}=0$. So $\mathrm{A}^{(1)}$ satisfies the conditions of Lemma 5 for $k=p+1$. Proceeding in this way we see that $\mathrm{A}=\delta\,\mathrm{B}$ where $\mathrm{B}(\mathbf{H})=\sum_{j=1}^{1+\mathrm{ord}\,\mathbf{H}} \mathrm{B}^{(j)}(\mathbf{H})$. □

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