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Limiting absorption principle for the Dirac operator

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ABSTRACT. — By using the conjugate operator method for hamiltonians defined as quadratic forms, we prove the limiting absorption principle, the absence of singularly continuous spectrum, the existence and completeness of wave operators for Dirac Hamiltonians with singular, non-local potentials; coulomb-like singularities are allowed.

RÉSUMÉ. — En utilisant la méthode des opérateurs conjugués pour des hamiltoniens définis par des formes quadratiques, nous prouvons un principe d'absorption limite, l'absence de spectre singulier continu, l'existence et la complétude des opérateurs d'onde pour des hamiltoniens de Dirac avec des potentiels singuliers non locaux. Le cas des singularités coulombiennes est aussi couvert.

1. INTRODUCTION

In this paper we study the spectrum of the Dirac Hamiltonian for a quite large class of potentials, including long-range potentials, coulombian singularities, non-local potentials and perturbations by pseudodifferential operators of order one. The main tool that we use is a general result obtained by Anne Boutet de Monvel and Vladimir Georgescu ([4], [5]) concerning the conjugate operator method for Hamiltonians defined as quadratic forms. Their result improves on our previous paper [7], giving a better control on the boundary value of the resolvent. Here we construct a conjugate operator for the free Dirac Hamiltonian and prove that the results in [5] can be applied. As a consequence we show that the spectrum of the total Dirac Hamiltonian in the complement of $\{\pm m\}$ consists only of absolutely continuous spectrum and finitely degenerated eigenvalues having no finite accumulation point with maybe the exception of $\pm m$. We also obtain a limiting absorption principle that implies the one given in [2].

Let us first recall the main facts concerning the free Dirac Hamiltonian. We denote by E a complex four dimensional Hilbert space and we consider the Hilbert space

$$\mathscr{H} := L^2(\mathbb{R}^3)\otimes E \cong L^2(\mathbb{R}^3; E).$$

On E we consider the algebra $L(E)$ of linear operators, $L_H(E)$ the real subspace of hermitian operators and four hermitian operators α_1, α_2, α_3, β satisfying the anticommutation relations:

$$\alpha_j\alpha_k+\alpha_k\alpha_j=2\delta_{jk};\qquad \alpha_j\beta+\beta\alpha_j=0;\qquad \beta^2=1.$$

We denote by the same letters the operators in $\mathscr{H}$ obtained by tensor multiplication with the identity on $L^2(\mathbb{R}^3)$. On $L^2(\mathbb{R}^3)$ we define $P_j := -i\dfrac{\partial}{\partial x_j}$ and the multiplication operator by the variable x_j denoted by Q_j and we observe that they are essentially self-adjoint on $\mathscr{S}(\mathbb{R}^3)$. Let $H^s(\mathbb{R}^3)$ (for $s\in\mathbb{R}$), be the usual Sobolev space on $\mathbb{R}^3$:

$$H^s(\mathbb{R}^3) := \{ f\in\mathscr{S}'(\mathbb{R}^3) \mid (1-\Delta)^{s/2} f\in L^2(\mathbb{R}^3)\}$$

and let us denote $\mathscr{H}^s = H^s(\mathbb{R}^3)\otimes E$ so that $\mathscr{H}^0=\mathscr{H}$. We shall constantly use the notation

$$\langle t\rangle := (1+|t|^2)^{1/2}$$

for t an element in $\mathbb{R}$ or $\mathbb{R}^3$, and by functional calculus for self-adjoint operators. We denote by $\mathscr{F}$ the Fourier transform on $L^2(\mathbb{R}^3)$, multiplied by the identity on E. Let us define now the free Dirac Hamiltonian

$$H_0 := \sum_{j=1}^{3}\alpha_j P_j + m\beta \equiv \alpha\cdot P + m\beta \tag{1.1}$$

where $m>0$. It is then evident that H_0 is self-adjoint on $\mathscr{H}^1$ and we have

$$\sigma(H_0)=\sigma_{ess}(H_0)=(-\infty, -m]\cup[m, +\infty)$$

and

$$\mathscr{F} H_0 \mathscr{F}^{-1}=\alpha\cdot Q+m\beta.$$

Thus H_0 is unitarily equivalent to multiplication by the following $L_H(E)$ valued function on $\mathbb{R}^3$:

$$\tilde{H}_0(k):=\mu(k)(\Pi_+(k)-\Pi_-(k))$$

where

$$\mu(k):=(k^2+m^2)^{1/2} \tag{1.2}$$

$$\Pi_\pm(k):=\frac{1}{2}\pm\frac{1}{2\mu(k)}(\alpha\cdot k+m\beta). \tag{1.3}$$

Then we can define the following operators:

$$\mu(P):=\mathscr{F}^{-1}\mu\mathscr{F};\qquad \Pi_\pm(P):=\mathscr{F}^{-1}\Pi_\pm\mathscr{F}$$

so that $\mu(P)$ is a pseudodifferential operator on $L^2(\mathbb{R}^3)$ with symbol μ given by (1.2) and domain $H^1(\mathbb{R}^3)$, and $\Pi_\pm(P)$ are orthogonal projections in $\mathscr{H}$ defining the orthogonal decomposition:

$$\mathscr{H}:=\mathscr{H}_+\oplus\mathscr{H}_-;\qquad \mathscr{H}_\pm:=\Pi_\pm\mathscr{H}.$$

We can consider the algebraic tensor product $\mathscr{S}(\mathbb{R}^3)\mathring{\otimes}E\equiv\mathscr{E}$ as a subspace of $\mathscr{H}$ and it is straightforward to see that

$$\Pi_\pm(P)\mathscr{E}\subset\mathscr{E}.$$

Also, because $\Pi_\pm(P)$ commute with $\langle P\rangle^s$, one can see that

$$\Pi_\pm(P)\mathscr{H}^s\subseteqq\mathscr{H}^s.$$

From the above relations it is evident that $\Pi_\pm(P)\mathscr{E}$ and $\Pi_\pm(P)\mathscr{H}^s$ are dense in $\mathscr{H}_\pm$. One can define the scale of spaces associated to H_0 [1], and observe that it coincides with the scale of Sobolev spaces $\mathscr{H}^s$. Thus the form domain of H_0 ([9], [17], [19]), will be $\mathscr{H}^{1/2}$.

We want to consider Hamiltonians of the form $H=H_0+V$, associated to quadratic forms with domain $\mathscr{H}^{1/2}$ [9]. For that we shall suppose that V is a symmetric operator $\mathscr{H}^{1/2}\to\mathscr{H}^{-1/2}$ and that the operator in $\mathscr{H}$ associated to the sum $H_0+V:\mathscr{H}^{1/2}\to\mathscr{H}^{-1/2}$ is self-adjoint. The operator V need not be associated to an operator in $\mathscr{H}$ (e. g. it could be a measure, or a distribution of order >0).

In order to state our results in a precise and optimal form, it is convenient to introduce a class of weighted Sobolev spaces. Let $\theta\in C_0^\infty(\mathbb{R}^3)$ be such that $\theta(x)>0$ if $2^{-1}<|x|<2$ and $\theta(x)=0$ otherwise. Choose one more function $\eta\in C_0^\infty(\mathbb{R}^3)$ such that $\eta(x)>0$ if $|x|<1$. Then for any s, $t\in\mathbb{R}$ and $1\leqq p\leqq\infty$ let $\mathscr{H}^s_{t,p}$ be the space of distributions u

which locally belong to $\mathcal{H}^s$ and such that

$$\| \eta(Q) u \|_{\mathcal{H}^s} + \left[\int_1^\infty \{ \| r^t \theta(r^{-1} Q) u \|_{\mathcal{H}^s}^p \} \frac{dr}{r} \right]^{1/p} < \infty.$$

If $p=\infty$, the second term here has to be interpreted as $\sup_{r \geq 1} \| r^t \theta(r^{-1} Q) u \|_{\mathcal{H}^s}$. The left-hand side above is a norm on $\mathcal{H}^s_{t,p}$ which provides this space with a Banach space structure. If one changes the functions η, θ, then an equivalent norm is obtained. We denote $\mathcal{H}^s_t \equiv \mathcal{H}^s_{t,2}$ which are the usual weighted Sobolev spaces defined by the norms $\| \langle P \rangle^s \langle Q \rangle^t u \|$. In section 5 of [5] the real interpolation theory of these spaces is described. We would like to mention the following results. If $t_1 < t < t_2$, $t = (1-\lambda) t_1 + \lambda t_2$ and $p, p_1, p_2 \in [1, \infty]$, $s \in \mathbb{R}$ then:

$$\mathcal{H}^s_{t,p} = (\mathcal{H}^s_{t_1, p_1}, \mathcal{H}^s_{t_2, p_2})_{\lambda, p}$$

(*see* [3] for real interpolation). Moreover, if $1 \leq p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then $(\mathcal{H}^s_{t,p})^* = \mathcal{H}^{-s}_{-t,p'}$.

Our main result is the following theorem (*see* section 4 for the proof and for a slightly different version).

THEOREM. – *Let* V *be a symmetric operator* $\mathcal{H}^{1/2} \to \mathcal{H}^{-1/2}$ *such that the operator* H *in* $\mathcal{H}$ *associated to the sesquilinear form* $H_0 + V : \mathcal{H}^{1/2} \to \mathcal{H}^{-1/2}$ *is self-adjoint and* $D(|H|^{1/2}) = \mathcal{H}^{1/2}$. *Assume that* $V = V_S + V_L$ *where* V_S, V_L *are symmetric operators* $\mathcal{H}^{1/2} \to \mathcal{H}^{-1/2}$ *satisfying the following decay conditions at infinity: there is* $\xi \in C^\infty(\mathbb{R}^3)$ *with* $\xi(x) = 0$ *near zero and* $\xi(x) = 1$ *near infinity such that* $\lim_{r \to \infty} \| \xi(r^{-1} Q) V \|_{\mathfrak{X}} = 0$ (*with* $\| \cdot \|_{\mathfrak{X}}$ *the norm in* $B(\mathcal{H}^{1/2}, \mathcal{H}^{-1/2})$) *and:*

$$\int_1^\infty \| \xi(r^{-1} Q) V_S \|_{\mathfrak{X}} dr + \sum_{j=1}^3 \int_1^\infty \{ \| \xi(r^{-1} Q) [Q_j, V_L] \|_{\mathfrak{X}} + \| \xi(r^{-1} Q) \langle Q \rangle [P_j, V_L] \|_{\mathfrak{X}} + \| \xi(r^{-1} Q) [\alpha_j \beta, V_L] \|_{\mathfrak{X}} \} \frac{dr}{r} < \infty.$$

Then the eigenvalues of H *which are not equal to* $\pm m$ *are of finite multiplicity and can accumulate only at* $+m$ *or* $-m$. H *has no singularly continuous spectrum and the following strong form of the limiting absorption principle holds. Let* $\mathbb{R}_H$ *be the set of* $\lambda \in \mathbb{R}$ *such that* $\lambda \neq \pm m$ *and* λ *is not an eigenvalue of* H *and* $\mathbb{C}_\pm$ *the set of* $z \in \mathbb{C}$ *with* $\pm \operatorname{Im} z > 0$. *Then the holomorphic function*

$$\mathbb{C}_\pm \ni z \mapsto (H - z)^{-1} \in B(\mathcal{H}^{-1/2}_{1/2, 1}; \mathcal{H}^{1/2}_{-1/2, \infty})$$

extends to a weak-continuous function on* $\mathbb{C}_\pm \cup \mathbb{R}_H$.

The method of proof of theorem 7.4 from [5] immediately gives the following criterion of existence and completeness of wave operators:

COROLLARY. — *Let* V_1, V_2 *be two operators with the same properties as* V *from the theorem. Assume that* $V_1 - V_2 : \mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$ *has image contained in* $\mathscr{H}^{-1/2}_{1/2,1}$ *and extends to a continuous operator from the closure of* $\mathscr{H}^{1/2}$ *into* $\mathscr{H}^{1/2}_{-1/2,\infty}$ *into* $\mathscr{H}^{-1/2}_{1/2,1}$. *Denote* $H_j = H_0 + V_j$ *and* E^c_j *the projection on the subspace of continuity of* H_j. *Then the wave operators:*

$$\underset{t \to \pm\infty}{s\text{-lim}}\, e^{iH_2 t} e^{-iH_1 t} E^c_1$$

exist and have $E^c_2 \mathscr{H}$ *as image.*

We shall make now some comments in relation with the assumptions made on the potential V. Remark first that it is a *non-local* operator in general. As usual, there are two types of conditions on V: a restriction on the local singularities and a decay assumption at infinity. The only local condition comes from the fact that $H_0 + V$ is required to have $\mathscr{H}^{1/2}$ as form domain. This covers the case of finite number of Coulomb centers if the charge of each center is smaller than $\sqrt{3}/2$. More precisely, assume V is the operator of multiplication by $\sum_{j=1}^{n} z_j |x - a_j|^{-1}$ with $a_j \neq a_k$. If $|z_j| < \frac{1}{2}\sqrt{3}$ for each j, then H is self-adjoint on $\mathscr{H}^1$ (*see* lemma 4.2 in [12]). For a much more general statement of the same nature, *see* [21] and also [15], [22]. We do not insist on this point since the problem of finding under which conditions a sesquilinear form $H_0 + V$ on $\mathscr{H}^{1/2}$ is the form of a self-adjoint operator H in $\mathscr{H}$ is outside the scope of this paper. In fact we would like to stress the fact that the conjugate operator method allows us to prove absence of singularly continuous spectrum, existence and completeness of wave operators without any information about the local behaviour of V, besides the fact that H has $\mathscr{H}^{1/2}$ as form-domain.

Let us consider the decay assumptions. Here V is considered to be the sum of a short-range part V_S and a long-range part V_L. From the integral condition it follows that $\|\xi(r^{-1}Q)V_S\|_{\mathfrak{X}} \to 0$ as $r \to \infty$ (remark that the assumptions of the theorem are in fact independent of the choice of ξ), so the condition $\|\xi(r^{-1}Q)V\|_{\mathfrak{X}} \to 0$ is a restriction on V_L only (since only commutators appear in the integral condition on V_L, we need to require explicitly that it vanishes at infinity). The integral assumption on V_S is of Enss-type, so that it may be considered as optimal. If, for example, V_S is the operator of multiplication by the matrix-valued function $V_S(x)$ and $|V_S(x)| \leqq f(|x|)$ with f decreasing, then it is sufficient to have $\int_1^\infty f(r)\,dr < \infty$. The assumptions on the long-range part are much more general than usual (V_L need *not* be a function, it could be a *non-local* operator). In order to see what kind of decay we require for the long-range part, assume V_L is multiplication by a matrix-valued function $V_L(x)$ which is derivable and such that

$$|x||\operatorname{grad} V_L(x)| + \sum_{j=1}^{3} |[\alpha_j \beta, V_L(x)]| \leqq g(|x|)$$

with g decreasing. Then it is sufficient to have $\int_1^\infty g(r)\frac{dr}{r} < \infty$.

Let us mention that in the paper [20], Thaller and Enss describe some interesting propagation properties of the Dirac Hamiltonian. They assume that V is multiplication by a matrix-valued function with a short-range V_S and a long-range V_L part, the local conditions on V_L being stronger than ours. Their local condition on V_S and their decay assumptions, however are weaker than ours. But they are not able to prove absence of singularly continuous spectrum (as it is explicitly stated on p. 153 of their paper) and they do not have any form of the limiting absorption principle. Moreover, in the last section of their paper (devoted to asymptotic completeness) their local assumption on V_S is much stronger, the long-range part V_L being of a very special form.

Finally, let us comment on the corollary of the theorem. The idea is that one starts with a Hamiltonian $H_1 = H_0 + V_1$ of the same form as in the theorem and one adds to it a perturbation $V_2 - V_1 \equiv W$ of short-range type. Then the relative wave operators exist and are complete. One can show that, if W is the operator of multiplication by a matrix-valued function, then the supplementary condition we put on $V_1 - V_2$ in the corollary is equivalent to the short-range assumption $\int_1^\infty \|\xi(r^{-1}Q)W\|_{\mathfrak{X}} < \infty$ (*see* the remark after theorem 8.6 in [5]). This gives an *optimal criterion* for the existence and completeness of the wave operators.

We have recently received a copy of the type-written version of a monograph by B. Thaller devoted to Dirac operator which, besides other qualities, contains an extensive list of references (488 articles). However, only few of them are concerned with the limiting absorption principle (in particular absence of singular continuous spectrum). We have selected the paper [23] by V. Vogelsang which is the most recent and which seems to contain the strongest results. We would like to thank Bernard Thaller for sending us the manuscript of his book before publication.

In the next section we shall present the conjugate operator method in a version suitable for our purposes. In section 3 we define a conjugate operator A for H_0 and we shall discuss some of its properties. The last section contains the main results of the paper, describing the spectral properties of H by verifying that A is also conjugate to H and using results and methods of [5].

2. CONJUGATE OPERATOR METHOD: THE FORM VERSION

Let us present a short review concerning the conjugate operator method in order to fix the results that we want to use in the sequel. This method initiated in the papers of Mourre ([13], [14]), and has been developped in [1], [5], [6], [7], [16]. We shall briefly recall its main points in the setting of ([4], [5]) that will be used by us.

Let H be a self-adjoint operator in $\mathscr{H}$ and let $\mathscr{G}$ be its form domain. We define on $\mathscr{G}$ the norm

$$\|u\|_{\mathscr{G}} := [(u \mid u) + (u \mid |H| \, u)]^{1/2} \qquad \text{for} \quad u \in \mathscr{G}$$

and denote $\mathscr{G}^*$ its dual. By the Riesz isomorphism we identify $\mathscr{H}$ with $\mathscr{H}^*$ so that we get the following continuous and dense inclusions:

$$\mathscr{G} \subset \mathscr{H} \subset \mathscr{G}^*.$$

Let E be the spectral measure associated to H and let A be a self-adjoint operator on $\mathscr{H}$ and $W_t = \exp\{itA\}$ the unitary group that it generates on $\mathscr{H}$. We denote by $B_\infty(\mathscr{H}_1, \mathscr{H}_2)$ the subspace of compact operators in $B(\mathscr{H}^1, \mathscr{H}^2)$ and $\mathscr{X} := B(\mathscr{G}, \mathscr{G}^*)$.

DEFINITION 2.1. – Let A, H be self-adjoint operators on the Hilbert space $\mathscr{H}$ and let I be an open interval on $\mathbb{R}$. We say that A *is conjugate to* H *in form sense on* I if the following three conditions are satisfied:

1. $W_t \mathscr{G} \subset \mathscr{G}$ for any $t \in \mathbb{R}$.
2. For any $u \in \mathscr{G}$, the application $\mathbb{R} \ni t \mapsto (W_t u \mid HW_t u) \in \mathbb{R}$ is differentiable at $t=0$. Let us denote by B the sesquilinear form defined by the derivative

$$(u \mid Bu) := \left.\frac{d}{dt}\right|_{t=0} (W_t u \mid HW_t u) \qquad \text{for} \quad u \in \mathscr{G}.$$

3. There is a constant $a>0$, and there is an operator K in $B_\infty(\mathscr{G}, \mathscr{G}^*)$ such that for any $u \in \mathscr{G}$ with $E(I)u = u$, we have:

$$(u \mid Bu) \geqq a \|u\|^2 + (u \mid Ku).$$

If H has a conjugate operator A in form sense on I and if H is "*regular*" with respect to the group W_t generated by A, then the abstract theory developed in [4], [5], [6] implies that on I the operator H has no eigenvalues of infinite multiplicity, no accumulation point for the eigenvalues and no singular continuous spectrum. In these conditions one can also prove a limiting absorption principle for H. We make now precise what we mean by "regular" with respect to a group and we recall the exact statement of theorem 7.2 in [5].

DEFINITION 2.2. – Let H and A be self-adjoint on $\mathscr{H}$. We say that an operator $T \in \mathscr{X}$ *is of class* $\mathscr{C}^1(A; \mathscr{G}, \mathscr{G}^*)$ if the following condition holds:

$$\int_0^1 \| W_t T W_{-t} + W_{-t} T W_t - 2T \|_{\mathscr{X}} \frac{dt}{t^2} < \infty. \tag{2.1}$$

In [5] a general method is elaborated in order to prove the regularity of a Halmitonian. In the particular case which is of interest to us, one starts with a free Hamiltonian H_0 which trivially verifies (2.1) [in fact the function $t \mapsto (W_t u \mid H_0 W_t u)$ is of class C^∞ for all $u \in \mathscr{G}$] and then adds a perturbation V which is also of class $\mathscr{C}^1(A; \mathscr{G}, \mathscr{G}^*)$ but for a much more subtle reason.

In order to make an optimal balance between regularity and decay at infinity, two classes of perturbations are considered. The "short-range" perturbations are permited a more singular behaviour but are required to have a faster decay at infinity, while the "long-range" perturbations, for which conditions are imposed on their commutator with A, may have a weaker decay but are supposed to be more regular. This description is due to the fact that the conjugate operator that one usually considers behaves roughly like $\langle Q \rangle$.

We shall now state a theorem which gives a method of proving that the perturbation is of class $\mathscr{C}^1(A; \mathscr{G}, \mathscr{G}^*)$. We do not give details of the proof, since the result is a rather straightforward consequence of theorems 6.2 and 6.3 of [5].

THEOREM 2.3. – *Let* Λ *be a positive self-adjoint operator in* $\mathscr{H}$ *such that* $(\Lambda + r)^{-1} \mathscr{G} \subset \mathscr{G}$ *for each* $r > 0$ *and* $\| (\Lambda + r)^{-1} \|_{B(\mathscr{G})} \leq \text{Const.} \langle r \rangle^{-1}$. *Assume that* $\Lambda^{-2} \mathscr{H} \subset D(A^2)$ *and that* $A\Lambda^{-1}$, $A^2 \Lambda^{-2}$ *extend to bounded operators in* $\mathscr{G}^*$. *Consider a symmetric operator* $T : \mathscr{G} \to \mathscr{G}^*$. *Then* T *is of class* $\mathscr{C}^1(A; \mathscr{G}, \mathscr{G}^*)$ *if one of the following two conditions is satisfied:*

(S) Short-range perturbations.

$$\int_1^\infty [\| \Lambda^2 (\Lambda + r)^{-2} T \|_{\mathscr{X}} + \| \Lambda (\Lambda + r)^{-1} T \Lambda (\Lambda + r)^{-1} \|_{\mathscr{X}}] \, dr < \infty.$$

(L) Long-range perturbations. – *The derivative* $S = \frac{d}{dt} W_t T W_{-t} |_{t=0}$ *exists weakly in* $\mathscr{X}$ *and:*

$$\int_1^\infty \| \Lambda (\Lambda + r)^{-1} S \|_{\mathscr{X}} \frac{dr}{r} < \infty.$$

Let us mention that the conditions imposed on Λ imply that $(\Lambda + r)^{-1}$ extends to a bounded operator in $\mathscr{G}^*$, so all the terms in the integrals are well defined. In using this theorem, we have $\mathscr{G}$ and $\mathscr{G}^*$ Sobolev spaces, and we can suppose $\Lambda = \langle Q \rangle$, so that we can use some estimates and results from [5]. Our first problem will be that our Sobolev spaces consist of vector-valued functions, and some conditions should be imposed on the matrix part of the commutators. Secondly, we want to include *Coulomb type singularities* which are not relatively compact in form sense with respect to the free Hamiltonian. The conjugate operator we will need is slightly different from the one in [5], [7], and we cannot describe explicitly the unitary group that it generates.

Let us recall now the main theorem in [5] that we shall use. One can show that A, when considered as operator in $\mathscr{G}^*$, is closable; we shall denote $D(A; \mathscr{G}^*)$ the domain of its closure.

DEFINITION 2.4. – Let $\mathscr{K} := (\mathscr{G}^*, D(A; \mathscr{G}^*))_{1/2, 1}$ be the Banach space obtained by *real interpolation* between $D(A; \mathscr{G}^*)$ and $\mathscr{G}^*$, and let $\mathscr{K}^*$ be its dual.

See [5] and [3] for the construction of this space.

THEOREM 2.5. – *Let* H *and* A *be self-adjoint operators on* $\mathscr{H}$ *with* A *conjugate to* H *in form sense on an open bounded interval* I *in* $\mathbb{R}$. *Suppose also that* H *is of class* $\mathscr{C}^1(A; \mathscr{G}, \mathscr{G}^*)$. *Then the spectrum of* H *in* I *has at most a finite number of eigenvalues with finite multiplicities and no singular continuous part. If* $\tilde{I} := I \setminus \sigma_{pp}(H)$ *and* $\tilde{I}_\pm := \{ z \in \mathbb{C} \mid \operatorname{Re} z \in \tilde{I}; \pm \operatorname{Im} z \geqq 0 \}$, *then the function defined by*

$$\mathbb{C} \setminus \sigma(H) \ni z \mapsto (H - z)^{-1} \in B(\mathscr{K}, \mathscr{K}^*)$$

extends to a weak-continuous function on* $\tilde{I}_\pm$.

3. THE CONJUGATE OPERATOR

In this section we shall define a conjugate operator for H_0. Let I be an open bounded interval with its closure contained in $(m, +\infty)$ and J an open neighbourhood of I in (m, ∞). Let $\theta \in C_0^\infty(\mathbb{R})$ have support in J and be equal to 1 on I. Then for $j = 1, 2, 3$ let us define three functions $F_j : \mathbb{R}^3 \to \mathbb{R}$ by the formula

$$F_j(k) := \mu(k)(\theta \circ \mu)(k) \frac{\partial_j \mu(k)}{|\nabla \mu(k)|^2} = \mu^2(k)(\theta \circ \mu)(k) \frac{k_j}{|k|^2} \qquad (3.1)$$

The fact that $m \notin \operatorname{supp} \theta$ implies that $0 \notin \operatorname{supp} F_j$ and thus we have that F_j is of class $C_0^\infty(\mathbb{R}^3)$. Following [5] we define now the operator:

$$\tilde{A} := \frac{1}{2} \sum_{j=1}^{3} \{ Q_j F_j(P) + F_j(P) Q_j \} \equiv \frac{1}{2} \{ Q \cdot F(P) + F(P) \cdot Q \}$$

acting on $\mathscr{E}$. It is a symmetric operator leaving $\mathscr{E}$ invariant and satisfying the relation:

$$\tilde{A} = Q \cdot F(P) - \frac{i}{2} f(P) = F(P) \cdot Q + \frac{i}{2} f(P) \qquad (3.2)$$

where $f : \mathbb{R}^3 \to \mathbb{R}$ is a function of class $C_0^\infty(\mathbb{R}^3)$ defined by:

$$f(k) := (\operatorname{div} F)(k) = 2(\theta \circ \mu)(k) + \mu(k)(\theta' \circ \mu)(k) + \mu^2(k)(\theta \circ \mu)(k) \frac{1}{|k|^2}.$$

(θ' denotes the derivative of θ). In [7] we gave an explicit description for the unitary group generated by $\tilde{A}$ on $L^2(\mathbb{R}^3)$, and using the Nelson lemma we proved that it is essentially self-adjoint on $\mathscr{S}(\mathbb{R}^3)$. Hence it will also be essentially self-adjoint on $\mathscr{E}$ when viewed as acting in $\mathscr{H}$. We have the following commutation relations on $\mathscr{E}$:

$$[\mu(P), \tilde{A}] = -i\mu(P)(\theta \circ \mu)(P)$$
$$[[\mu(P), \tilde{A}], \tilde{A}] = -\mu(P)(\theta \circ \mu)(P)\{(\theta \circ \mu)(P) + \mu(P)(\theta' \circ \mu)(P)\}$$

both commutators being bounded functions of P.

In defining the conjugate operator we would like it to have a positive commutator with H_0. We define:

$$A := \Pi_+(P)\tilde{A}\Pi_+(P) + \Pi_-(P)\tilde{A}\Pi_-(P). \qquad (3.3)$$

Because $\Pi_\pm(P)$ and A are defined on $\mathscr{E}$ and leave it invariant, the same things will hold true for A also.

PROPOSITION 3.1. — 1. *If we denote by* E_0 *the spectral measure of* H_0, *if* I *is a bounded, open interval with its closure contained in* $(m, +\infty)$ *and* A *is as above, then*

$$E_0(I)\, i[H_0, A]\, E_0(I) \geqq a E_0(I)$$

with $a > 0$.

2. *The following relation holds on* $\mathscr{E}$:

$$[[H_0, A], A] = -(\theta \circ \mu)\{(\theta \circ \mu) + \mu(\theta' \circ \mu)\} H_0.$$

Proof. — 1. We begin by computing the commutator $[H_0, A]$ on $\mathscr{E}$ where it is well defined:

$$\begin{aligned}[H_0, A] &= [\mu(\Pi_+ - \Pi_-), (\Pi_+ \tilde{A} \Pi_+ + \Pi_- \tilde{A} \Pi_-)] \\ &= [\mu\Pi_+, \Pi_+ \tilde{A} \Pi_+] - [\mu\Pi_-, \Pi_- \tilde{A} \Pi_-] = \Pi_+[\mu, \tilde{A}]\Pi_+ - \Pi_-[\mu, \tilde{A}]\Pi_- \\ &= -i(\theta \circ \mu)(P)(\Pi_+ \mu \Pi_+ - \Pi_- \mu \Pi_-) = -i H_0 (\theta \circ \mu)(P)\end{aligned}$$

where we used the fact that $\mu(P)$ commutes with $\Pi_\pm(P)$ being both functions of P only and μ being a scalar function. Thus $[H_0, A]$ defines a bounded operator on $\mathscr{H}$ and one has

$$E_0(I)\, i[H_0, A]\, E_0(I) = H_0(\theta \circ \mu)(P) E_0(I) = H_0 E_0(I) > a E_0(I)$$

where $a > m > 0$, taking into account that $I \subset (m, +\infty)$.

2. The relation is obtained by a straightforward calculation. ■

It is evident that to study the negative part of the spectrum one only has to take $-A$ as conjugate operator.

LEMMA 3.2. — *The following relation holds on* $\mathscr{E}$:

$$A=\tilde{A}+\frac{1}{2}\{(\Pi_+-\Pi_-)[\tilde{A},\Pi_+]+[\Pi_+,\tilde{A}](\Pi_+-\Pi_-)\}$$
$$=\tilde{A}+i m \mu^{-2}(\theta\circ\mu)(\alpha\cdot P)\beta$$

so that A *is essentially self-adjoint on* $\mathscr{E}$ *and its closure has the same domain as* $\tilde{A}$.

Proof.

$$A=\Pi_+\tilde{A}\Pi_++\Pi_-\tilde{A}\Pi_-$$
$$=\frac{1}{2}\{(\Pi_++\Pi_-)\tilde{A}+\tilde{A}(\Pi_++\Pi_-)\}+\frac{1}{2}\{(\Pi_+[\tilde{A},\Pi_+]+[\Pi_+,\tilde{A}]\Pi_+\}$$
$$+\frac{1}{2}\{(\Pi_-[\tilde{A},\Pi_-]+[\Pi_-,\tilde{A}]\Pi_-\}$$
$$=\tilde{A}+\frac{1}{2}\{(\Pi_+-\Pi_-)[\tilde{A},\Pi_+]+[\Pi_+,\tilde{A}](\Pi_+-\Pi_-)\}$$

where we have used the fact that $\Pi_++\Pi_-=1$ so that $[\Pi_+,\tilde{A}]=-[\Pi_-,\tilde{A}]$. Using the formulas (1.3), (3.1) and (3.2) we obtain

$$(\Pi_+-\Pi_-)=\mu^{-1}H_0;\qquad [\tilde{A},\Pi_+]=i m\mu^{-1}\beta(\theta\circ\mu)$$
$$(\Pi_+-\Pi_-)[\tilde{A},\Pi_+]+[\Pi_+,\tilde{A}](\Pi_+-\Pi_-)=2 i m\mu^{-2}(\theta\circ\mu)(\alpha\cdot P)\beta.$$

From the relations above one can see that $\tilde{A}-A\in B(\mathscr{H})$ and a standard perturbation argument implies that A will be essentially self-adjoint on $\mathscr{E}$ and its closure will have the same domain as $\tilde{A}$. ■

PROPOSITION 3.3. — 1. *The unitary group* W_t *generated by* A *on* $\mathscr{H}$ *leaves* $\mathscr{H}^s$ *invariant for any* $t, s\in\mathbb{R}$.

2. H_0 *is of class* $\mathscr{C}^1(A;\mathscr{G},\mathscr{G}^*)$.

Proof. — In [7] we proved that the unitary group generated by $\tilde{A}$ on $L^2(\mathbb{R}^3)$ leaves H^s invariant for all $t, s\in\mathbb{R}$ and hence the same will hold true for $\mathscr{H}^s$. Let $\tilde{A}^{(s)}$ be the generator of the group $\tilde{W}_t$ on $\mathscr{H}^s$, that will be a C_0-group but no longer unitary. Evidently on its domain $\tilde{A}^{(s)}$ coincides with $\tilde{A}$ ([4], [5]). Let us define

$$B:=\frac{1}{2}\{(\Pi_+-\Pi_-)[\tilde{A},\Pi_+]+[\Pi_+,\tilde{A}](\Pi_+-\Pi_-)\}\in B(\mathscr{H}).$$

Using now the explicit form of B given by Lemma 3.2, we see that it commutes with $\langle P\rangle^s$ so that it is bounded on any $\mathscr{H}^s$. Thus let us define

$$A^{(s)}:=\tilde{A}^{(s)}+B$$

on the domain of $\tilde{A}^{(s)}$. Then Theorem 3.1 in [8] implies that $A^{(s)}$ generates a C_0-group on $\mathscr{H}^s$ that we shall denote $W_t^{(s)}$. We see that $W_t^{(s)}=W_t$ on the domain of $\tilde{A}^{(s)}$ in $\mathscr{H}^s$, so that $W_t^{(s)}$ is simply the restriction of W_t to $\mathscr{H}^s$ and thus leaves it invariant. The second point of Proposition 2.1 implies that $[[H_0, A], A] \in B(\mathscr{H}) \subset \mathscr{X}$ so that the function $\mathbb{R} \ni t \mapsto W_t H_0 W_{-t} \in \mathscr{X}$ is twice differentiable for the strong-topology and thus H_0 is of class $\mathscr{C}^1(A; \mathscr{G}, \mathscr{G}^*)$. ■

To conclude: *For* H_0 *given by* (1.1) *and* A *given by* (3.3) *and for an open, bounded interval* I *with its closure contained in* $(m, +\infty)$, *we proved that* A *is conjugate in form sense to* H_0 *on* I *and* H_0 *is of class* $\mathscr{C}^1(A; \mathscr{G}, \mathscr{G}^*)$.

LEMMA 3.4. – 1. W_t *has polynomial growth in* t *on* $\mathscr{H}^1 = D(H_0)$.
2. $[H_0, A]$ *and* $[[H_0, A], A]$ *are in* $B(\mathscr{H}^s)$ *for any* $s \in \mathbb{R}$.

Proof. – 1. We shall use a remark in [5] saying that W_t has polynomial growth in t on $D(H_0) = \mathscr{H}^1$ if $[H_0, A] \in B(\mathscr{H}^\theta, \mathscr{H})$ for some $\theta < 1$. More precisely, in this case one has $\|W_t\|_{B(\mathscr{H}^1)} \leq c\langle t \rangle^m$ for $m = (1-\theta)^{-1}$. In our case $[H_0, A] \in B(\mathscr{H})$ so that we can take $\theta = 0$ and thus $m = 1$.

2. This statement is a simple consequence of the fact that the two operators are bounded on $\mathscr{H}$ and commute with $\langle P \rangle^s$ which defines the norm on $\mathscr{H}^s$. ■

Before we close this section let us discuss a second method of defining the conjugate operator for H_0. One could try to define directly the unitary group W_t by an explicit formula, so that its generator be conjugate to H_0; one can expand of course the details in various cases but the following heuristic argument is there only as an illustration. One considers a flow $\tilde{\Phi}_t$ in $\mathbb{R}^3$ given by

$$\frac{d}{dt}\tilde{\Phi}_t = -X \circ \tilde{\Phi}_t; \qquad \tilde{\Phi}_0(k) = k, \quad \forall k \in \mathbb{R}^3$$

where $X : \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field with globally Lipschitz components. Then one shows that

$$(\tilde{W}_t f)(k) := \{\det(\nabla \tilde{\Phi}_t(k))\}^{1/2} f(\tilde{\Phi}_t(k))$$

is the Fourier transform of the wanted unitary group, and its generator is

$$\tilde{A} = \frac{1}{2}[X(P) \cdot Q + Q \cdot X(P)].$$

One can then try the following definition

$$(\hat{W}_t f)(k) := U^{-1}(k)\{\det(\nabla \Phi_t(k))\}^{1/2} U(\Phi_t(k)) f(\Phi_t(k))$$

where

$$U = \exp(-S); \qquad S := \beta \frac{\alpha \cdot k}{2|k|} \operatorname{arctg}\left(\frac{|k|}{m}\right)$$

$$\frac{d}{dt}\Phi_t(k) = F(\Phi_t(k)); \qquad \Phi_0(k) = k, \quad \forall k \in \mathbb{R}^3$$

with the functions $F_j : \mathbb{R}^3 \to \mathbb{R}$ given by the formula (3.1). Then one gets:

$$A = \tilde{A} + F(P) \cdot \nabla S$$

that is also a bounded perturbation of $\tilde{A}$, and

$$[H_0, A] = -i(\theta \circ \mu) H_0.$$

4. SPECTRAL ANALYSIS OF THE DIRAC HAMILTONIAN

We consider in this section a perturbed Dirac Hamiltonian of the form $H_0 + V$ where $V : \mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$ is a symmetric operator such that the operator H in $\mathscr{H}$ associated to the sum $H_0 + V : \mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$ is self-adjoint and $D(|H|^{1/2}) = \mathscr{H}^{1/2}$. We shall also impose a decay condition for the perturbation. We say that a bounded operator $T : \mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$ is *small at infinity* if there is $\xi \in C^\infty(\mathbb{R}^3)$ with $\xi(x) = 0$ near zero and $\xi(x) = 1$ near infinity such that $\lim_{r \to \infty} \| \xi(r^{-1} Q) T \|_{\mathscr{X}} = 0$, where $\mathscr{X} = B(\mathscr{H}^{1/2}, \mathscr{H}^{-1/2})$. It is obvious that this is independent of ξ. Now we can state the decay assumptions on V: one can write $V = V_S + V_L$ with V_S, V_L symmetric operators $\mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$, V_L small at infinity and

$$\int_1^\infty \left\| \left(\frac{\langle Q \rangle}{\langle Q \rangle + r} \right)^2 V_S \right\|_{\mathscr{X}} dr < \infty. \tag{H.1}$$

$$\sum_{j=1}^3 \int_1^\infty \left\{ \left\| \frac{\langle Q \rangle}{\langle Q \rangle + r} [Q_j, V_L] \right\|_{\mathscr{X}} + \left\| \frac{\langle Q \rangle^2}{\langle Q \rangle + r} [P_j, V_L] \right\|_{\mathscr{X}} + \left\| \frac{\langle Q \rangle}{\langle Q \rangle + r} [\alpha_j \beta, V_L] \right\|_{\mathscr{X}} \right\} \frac{dr}{r} < \infty. \tag{H.2}$$

From (H.1) it follows easily that $V_S : \mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$ is small at infinity, so V has the same property. Assume for the moment only that H is self-adjoint in $\mathscr{H}$ and $V : \mathscr{H}^{1/2} \to \mathscr{H}^{-1/2}$ is small at infinity. For z a complex number in the resolvent set of H and H_0, clearly one has

$$(z - H)^{-1} - (z - H_0)^{-1} = (z - H)^{-1} V (z - H_0)^{-1} \tag{4.1}$$

as operators $\mathscr{H}^{-1/2} \to \mathscr{H}^{1/2}$.

Observe that $(z - H_0)^{-1} \in B(\mathscr{H}, \mathscr{H}^{+1})$ and V is a compact operator $\mathscr{H}^1 \to \mathscr{H}^{-1/2}$. The first assertion is trivial; for the second, since V is the norm limit in $\mathscr{X}$ of $V(1 - \xi(r^{-1}Q))$ as $r \to \infty$, it is enough to assume that $V = V\eta(Q)$ for some $\eta \in C_0^\infty(\mathbb{R}^3)$; but $\eta(Q) : \mathscr{H}^1 \to \mathscr{H}^{1/2}$ is compact. It follows that the right-hand side of (4.1) is a compact operator in

$B(\mathscr{H}, \mathscr{H}^{1/2})$. In conclusion

$$(z-H)^{-1}-(z-H_0)^{-1}\in B_\infty(\mathscr{H}, \mathscr{H}^{1/2}).$$

Using Stone-Weierstrass theorem, one now gets

$$\varphi(H)-\varphi(H_0)\in B_\infty(\mathscr{H})$$

for each $\varphi:\mathbb{R}\to\mathbb{C}$ continuous and convergent to zero at infinity; if $\varphi(\lambda)=o(\lambda^{-1})$, one even gets $\varphi(H)-\varphi(H_0)\in B_\infty(\mathscr{H}^{-1/2+\varepsilon}, \mathscr{H}^{1/2})$ for any $\varepsilon>0$. In particular, H *and* H_0 *have the same essential spectrum.*

From now on we assume that (H.1), (H.2) are fulfilled.

PROPOSITION 4.1. – *If* A *is the operator given by* (3.3), *then* V *is of class* $\mathscr{C}^1(A; \mathscr{G}, \mathscr{G}^*)$.

Proof. – We shall prove that the hypotheses of theorem 2.3 hold.

1. From Lemma 3.2 we get $A=\tilde{A}+i\sum_{j=1}^{3}\alpha_j\beta G_j(P)\equiv\tilde{A}+i\alpha\beta\cdot G(P)$, where $G_j(k)$ are given by:

$$G_j(k):=\frac{mk_j}{\mu^2(k)}(\theta\circ\mu)(k)$$

and are scalar, bounded functions of k, defining thus bounded operators on $\mathscr{H}^s$ for any $s\in\mathbb{R}$, commuting with $\alpha_j\beta$ and with any matrix-valued function of P. Thus we write:

$$A=\sum_{j=1}^{3}F_j(P)Q_j+\frac{i}{2}f(P)+i\sum_{j=1}^{3}\alpha_j\beta G_j(P).$$

We have

$$A^2=\sum_{j,k}F_jF_kQ_jQ_k\pm i\{\sum_{j,k}F_j(\partial_jF_k)Q_k+(f+2\sum_j\alpha_j\beta G_j)\sum_k F_kQ_k\}$$
$$+\left\{i\left(\frac{1}{2}\sum_j F_j\partial_j f+\sum_j\alpha_j\beta G_j\right)-\left(\frac{1}{2}f+\sum_j\alpha_j\beta G_j\right)^2\right\},$$

where all the functions of P are bounded scalar functions, so that evidently $\langle Q\rangle^{-2}\mathscr{H}\subset D(A^2)$ and the operators $A\langle Q\rangle^{-1}$, $A^2\langle Q\rangle^{-2}$ extend to bounded operators in $\mathscr{H}^{-1/2}$. Clearly then we may take $\Lambda=\langle Q\rangle$ in Theorem 2.3. In this particular situation, one can use the argument given in section 8 of [5] and deduce that the second term in the condition (S) of Theorem 2.3 is bounded by the first one. So condition (S) reduces to

$$\int_1^\infty\left\|\left(\frac{\langle Q\rangle}{\langle Q\rangle+r}\right)^2 T\right\|_{\mathscr{X}}dr<\infty. \tag{4.2}$$

Thus for the short-range part of the perturbation we only have to verify condition (4.2) that is evidently true for V_S due to hypothesis (H.1). We conclude that V_S is of class $\mathscr{C}^1(A; \mathscr{G}, \mathscr{G}^*)$.

2. It remains to analyze V_L. We shall use a procedure similar to that described in section 8 of [5] and prove that under the hypothesis (H.2) the following estimate is true:

$$\int_1^\infty \left\| \frac{\langle Q \rangle}{\langle Q \rangle + r} [A, V_L] \right\|_{\mathscr{X}} \frac{dr}{r} < \infty \tag{4.3}$$

so that part (L) of Theorem 2.3 will imply that V_L is of class $\mathscr{C}^1(A; \mathscr{G}, \mathscr{G}^*)$. We begin by computing the commutator:

$$\begin{aligned}[A, V_L] &= [\tilde{A}, V_L] + i[G\cdot\alpha\beta, V_L] \\ &= [Q\cdot F, V_L] - \frac{i}{2}[f, V_L] + i[G\cdot\alpha\beta, V_L] \\ &= [Q, V_L]\cdot F + Q\cdot[F, V_L] - \frac{i}{2}[f, V_L] + i\alpha\beta\cdot[G, V_L] + i[\alpha\beta, V_L]\cdot G.\end{aligned}$$

We set $\varphi_r(x) := \dfrac{\langle x \rangle}{\langle x \rangle + r}$ and observe that in order to prove the finiteness of the integral in (4.3) we have to estimate the following types of norms:

(i) $\| \varphi_r(Q)[\xi(P), V_L] \|_{\mathscr{X}}$

(ii) $\| \varphi_r(Q)\, Q[\xi(P), V_L] \|_{\mathscr{X}}$

(iii) $\| \varphi_r(Q)[\alpha_j\beta, V_L] \|_{\mathscr{X}}$

where $\xi \in C_0^\infty(\mathbb{R}^3)$. We have used the facts that F_j, G_j, $f \in C_0^\infty(\mathbb{R}^3)$, $F_j(P)$ and $G_j(P)$ are bounded operators on $\mathscr{G}$ and $\varphi_r(Q)$ commutes evidently with $\alpha_j\beta$. Using a functional calculus based on the Fourier transform, in section 8 of [5] is proven the following commutator estimate, for φ a C^∞ function polynomially bounded together with its derivatives and ξ in $\mathscr{S}(\mathbb{R}^3)$:

$$\begin{aligned}&\| \varphi(Q)\, Q[\xi(P), T] \|_{\mathscr{X}} \\ &\quad \leqq (2\pi)^{-3/2} \sum_{j=1}^{3} \int_0^1 \int_{\mathbb{R}^3} |\mathscr{F}(\partial_j \xi)(x)|\, \| \varphi(Q - xt)[P_j, T] \|_{\mathscr{X}}\, dx\end{aligned}$$

We take $\varphi = \langle Q \rangle^a (\langle Q \rangle + r)^{-1}$, with $a = 1, 2$ so that it satisfies the conditions imposed to φ for any $r \in \mathbb{R}_+^*$ and observe that:

$$\begin{aligned}&\| \varphi(Q - xt)[P_j, V_L] \|_{\mathscr{X}} \\ &\quad \leqq \left\| \frac{\langle Q - xt \rangle^a}{\langle Q \rangle^a} \right\|_{B(\mathscr{G}^*)} \left\| \frac{\langle Q \rangle + r}{\langle Q - xt \rangle + r} \right\|_{B(\mathscr{G}^*)} \| \varphi(Q)[P_j, V_L] \|_{\mathscr{X}}\end{aligned}$$

Using these formulas and hypothesis (H.2) it is easy to see that all the above terms are bounded by integrable functions of r. Thus V_L is also of class $\mathscr{C}^1(A; \mathscr{G}, \mathscr{G}^*)$. ■

To conclude, from Proposition 4.1 and the conclusion of the previous section, the regularity condition is verified for H with respect to the unitary group generated by A. It still remains to prove that A is conjugate with respect to H in form sense on a given interval $I \subset (m, +\infty)$. For that we have to prove the Mourre estimate, which due to the results concerning H_0 is reduced to prove that certain operators are compact.

PROPOSITION 4.2. — *If* V *satisfies* (H.1)-(H.2) *and* A *is defined by* (3.3) (I *being a bounded, open interval in* $(m, +\infty)$) *then the following estimate holds:*

$$E(I)[iH, A]E(I) \geqq aE(I) + K \tag{4.4}$$

with $a>0$ *and* K *a compact operator in* $B(\mathcal{H})$.

Proof. — From Proposition 4.1 we know that $[H, A] \in \mathcal{X}$ so that the left-hand side of (4.4) is evidently in $B(\mathcal{H})$. For $\varphi \in C_0^\infty(\mathbb{R})$ we denote $\Phi := \varphi(H)$ and $\Phi_0 := \varphi(H_0)$. Let us choose φ to be equal to 1 on I and have support in a small neighbourhood of I in $(m, +\infty)$. Then $\Phi u = E(I)u = u$ so that

$$\Phi[H, A]\Phi = (\Phi - \Phi_0)[H, A]\Phi_0 + \Phi[H, A](\Phi - \Phi_0) + \Phi_0[H, A]\Phi_0$$

The first two terms are compact because

$$\Phi - \Phi_0 \in B_\infty(\mathcal{H}, \mathcal{H}^{1/2}) \cap B_\infty(\mathcal{H}^{-1/2}, \mathcal{H}),$$

Φ and Φ_0 belong to $B(\mathcal{H}^{-1/2}, \mathcal{H}^{1/2})$ and $[H, A] \in B(\mathcal{H}^{1/2}, \mathcal{H}^{-1/2})$. The last term is the sum $\Phi_0[H_0, A]\Phi_0 + \Phi_0[V, A]\Phi_0$, so we just have to show that $\Phi_0[V, A]\Phi_0 \in B_\infty(\mathcal{H})$. Since $\Phi_0 \in B(\mathcal{H}, \mathcal{H}^{+1}) \cap B(\mathcal{H}^{-1}, \mathcal{H})$, it is enough to prove that $[V, A] \in B_\infty(\mathcal{H}^1, \mathcal{H}^{-1})$. But

$$[V, A] = \lim_{\varepsilon \to 0} \frac{W_\varepsilon V W_{-\varepsilon} - V}{i\varepsilon}$$

exists as a norm limit in $B(\mathcal{H}^{1/2}, \mathcal{H}^{-1/2})$, because $V \in \mathscr{C}^1(A, \mathcal{X}) \subset C_n^1(A, \mathcal{X})$ (*see* [5]). In particular, it also exists in norm in $B(\mathcal{H}^1, \mathcal{H}^{-1})$. Recall that $V \in B_\infty(\mathcal{H}^1, \mathcal{H}^{-1/2}) \subset B_\infty(\mathcal{H}^1, \mathcal{H}^{-1})$ and a norm-limit of compact operators is compact. ■

Thus from Propositions 4.1 and 4.2 we see that the hypotheses of Theorem 2.3 are verified for $I \subset (m, +\infty)$. For the negative part of the essential spectrum $(-\infty, -m)$, one can take $-A$ as conjugate operator and the hypotheses of Theorem 1.3 will also be verified. Finally, observe that $\mathcal{H}_1^{-1/2} \subset D(A; \mathcal{H}^{-1/2})$ in our case. Hence, by interpolation we obtain that $\mathcal{H}_{1/2,1}^{-1/2} = (\mathcal{H}^{-1/2}, \mathcal{H}_1^{-1/2})_{1/2,1} \subset \mathcal{K}$ and $\mathcal{K}^* \subset \mathcal{H}_{-1/2,\infty}^{1/2}$. Hence, by duality we have the following theorem.

THEOREM 4.3. — *Let* H *be the Dirac operator on* $\mathcal{H} = L^2(\mathbb{R}^3) \otimes E$ *with* $D(|H|^{1/2}) = \mathcal{H}^{1/2}$ *defined at the beginning of this section with* $V: \mathcal{H}^{1/2} \to \mathcal{H}^{-1/2}$ *symmetric, small at infinity and satisfying the conditions* (H.1) *and* (H.2). *Then, if* I *is an open, bounded interval in* $\mathbb{R}$, *not containing the points* $\pm m$ *in its closure, the spectrum of* H *in* I *has only a finite number* *of eigenvalues which are all of finite multiplicity and no singular continuous part. Moreover, the function*

$$\mathbb{C}\setminus\sigma(\mathrm{H})\ni z\mapsto(\mathrm{H}-z)^{-1}\in\mathrm{B}(\mathscr{H}^{-1/2}_{1/2,1},\mathscr{H}^{1/2}_{-1/2,\infty})$$

can be extended to a weak-continuous function on*

$$\tilde{\mathrm{I}}_{\pm}:=\{z\in\mathbb{C}\,|\,\pm\operatorname{Im}z\geqq 0;\ \operatorname{Re}z\in\mathrm{I};\ z\notin\sigma_{pp}(\mathrm{H})\}.$$

We have stated in the introduction the assumptions (H.1) and (H.2) in a different, more intuitive form. In order to prove that (H.1), (H.2) are consequences of the corresponding conditions of that theorem, we shall use theorem 3.1 of [5].

Let us consider again the operator $\Lambda=\langle Q\rangle$ and the group $e^{i\Lambda t}$ that it generates. This group is unitary in $\mathscr{H}$, leaves invariant all the Sobolev spaces $\mathscr{H}^s$ and for integer $s\geqq 0$ one easily gets:

$$\|e^{i\Lambda t}\|_{\mathrm{B}(\mathscr{H}^s)}\leqq c_s\langle t\rangle^s$$

for a constant c_s and all $t\in\mathbb{R}$. By interpolation and duality, this estimate will remain true for any $s\in\mathbb{R}$. In particular, $e^{i\Lambda t}$ growth like $\langle t\rangle^{1/2}$ in $\mathscr{G}^*=\mathscr{H}^{-1/2}$ and we may take $\mathrm{N}\geqq 1/2$ in section 3 of [5]. Let $l\geqq\mathrm{N}$ an integer and $\rho(\lambda)=[\lambda(\lambda+i)^{-1}]^l$ for $\lambda\in\mathbb{R}$. Fix some real $a<1$ and $\theta\in\mathrm{C}_0^\infty(\mathbb{R})$ with $\theta(\lambda)>0$ if $a<|\lambda|<a^{-1}$, $\theta(\lambda)=0$ otherwise and $\sum_{j\in\mathbb{Z}}\theta(a^j\lambda)=1$ if $\lambda\neq 0$. If $\theta(\lambda)=\sum_{j=0}^{+\infty}\theta(a^j\lambda)$, then $\theta_0\in\mathrm{C}^\infty(\mathbb{R})$, $\theta_0(x)=0$ (resp. $=1$) if $|\lambda|\leqq a$ (resp. $|\lambda|\geqq 1$). Theorem 3.1 of [5] shows that there is a constant $c<\infty$ such that for all $u\in\mathscr{H}^{-1/2}$ and $0<\varepsilon<1$ (with $\|\cdot\|_{-1/2}$ the norm in $\mathscr{H}^{-1/2}$):

$$\|\rho(\varepsilon\Lambda)u\|_{-1/2}\leqq c\,\varepsilon^{\mathrm{N}}\|u\|_{-1/2}+c\|\theta_0(\varepsilon\Lambda)u\|_{-1/2}+c\,\varepsilon^l\int_{a\varepsilon}^1\|\theta(\tau\Lambda)u\|_{-1/2}\frac{d\tau}{\tau^{1+l}}.$$

From this estimate one easily gets for any $\mathrm{T}\in\mathscr{X}$:

$$\left\|\left[\frac{\varepsilon\langle Q\rangle}{\varepsilon\langle Q\rangle+i}\right]^l\mathrm{T}\right\|_{\mathscr{X}}\leqq c\,\varepsilon^{\mathrm{N}}\|\mathrm{T}\|_{\mathscr{X}}+c\|\theta_0(\varepsilon\langle Q\rangle)\mathrm{T}\|_{\mathscr{X}}+c\,\varepsilon^l\int_{a\varepsilon}^1\|\theta(\varepsilon\langle Q\rangle)\mathrm{T}\|_{\mathscr{X}}\frac{d\tau}{\tau^{1+l}}$$

Since $(\varepsilon\langle Q\rangle+1)(\varepsilon\langle Q\rangle+i)^{-1}$ is bounded in $\mathrm{B}(\mathscr{H}^{-1/2})$ by a constant independent of ε, one can replace above $\varepsilon\langle Q\rangle+i$ by $\varepsilon\langle Q\rangle+1$. By an easy argument (*see* Corollary 3.2 in [5]) one gets for any $0\leqq\sigma<\mathrm{N}$ a constant $c<\infty$ such that for any $\mathrm{T}\in\mathscr{X}$:

$$\int_0^1\left\|\left[\frac{\varepsilon\langle Q\rangle}{\varepsilon\langle Q\rangle+1}\right]^l\mathrm{T}\right\|_{\mathscr{X}}\frac{d\varepsilon}{\varepsilon^{1+\sigma}}\leqq c\|\mathrm{T}\|_{\mathscr{X}}+c\int_0^1\|\theta_0(\varepsilon\langle Q\rangle)\mathrm{T}\|_{\mathscr{X}}\frac{d\varepsilon}{\varepsilon^{1+\sigma}}.$$

If ξ is a function of class C^∞ on $\mathbb{R}^3$ equal to zero near $x=0$ and equal to one near infinity, and if we take $\varepsilon=\frac{1}{r}$ above, we get for some $c<\infty$ and all $T\in\mathscr{X}$:

$$\int_0^1 \left\| \left[\frac{\langle Q\rangle}{\langle Q\rangle+r} \right]^l T \right\|_{\mathscr{X}} r^\sigma \frac{dr}{r} \leqq c \| T \|_{\mathscr{X}} + c \int_0^1 \left\| \xi\left(\frac{Q}{r}\right) T \right\|_{\mathscr{X}} r^\sigma \frac{dr}{r}.$$

We use this estimate in two situations: $l=2$, $\sigma=1$, $N=\frac{3}{2}$ for the short-range part and $l=1$, $\sigma=0$, $N=\frac{1}{2}$ for the long-range part.

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