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Nielsen's theorem and the super-Teichmüller space

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ABSTRACT. – There are two ways of defining the Teichmüller space for genus g Riemann surfaces; in the ordinary ("bosonic") case these agree, but for super Riemann surfaces they differ in the number of components. This difference is described, quantified, and related to the failure of the classical Nielsen's theorem in the super category. A representation of the new large super Teichmüller space using group cohomology elements is given.

RÉSUMÉ. – On distingue deux façons de définir l'espace de Teichmüller des espaces de Riemann de genre g. Dans le cas normal (« bosonique »), elles s'identifient, mais pour les super-espaces de Riemann, elles donnent des espaces qui diffèrent par le nombre des composantes. Cette différence est calculée, et reliée au fait que le théorème classique de Nielsen est faux dans la catégorie des super-variétés. On donne une représentation de l'espace de Teichmüller nouveau (plus grand) à l'aide des éléments dans la cohomologie des groupes.

0. INTRODUCTION

The supermoduli spaces of surfaces of genus g have been studied by a number of authors [1] in connection with the various supersymmetric approaches to string theory. They also have a rôle to play in any model of super-topological gravity [2]. They are quite complicated, even for low genus g, and as usual one can obtain useful intermediate information from the super Teichmüller space $\mathrm{ST}(g)$. This is much better understood, and good accounts can be found in [3]-[5]. However, some open questions remain, and there are still some unexpected features which may be of use in field theory. The aim of this paper is to concentrate on one of these: the failure of Nielsen's theorem [6] in the super category, and its consequences for the definition of the super Teichmüller space.

Briefly, Nielsen's theorem states that a mapping of surfaces is homotopic to the identity if it induces the identity homomorphism on the fundamental group. Its importance for the Teichmüller theory is in the value it gives to a *marking* of a Riemann surface (or, set of generators for the fundamental group). Diffeomorphisms which preserve a marking are homotopic to the identity; hence, we have two equivalent approaches to Teichmüller theory, one based on markings and the other on the group Diff_0 of diffeomorphisms homotopic to the identity. (*See* for example [7].)

In this paper we shall prove that the super version of Nielsen's theorem fails. In consequence, we have two different definitions of the *super* Teichmüller space, which we call $\mathrm{ST}(g)$ and $\mathrm{S\hat{T}}(g)$ respectively, corresponding to the two approaches above. If $\mathrm{X}(g)$ is the model (body) surface of genus g, the difference between the two is easy to quantify; $\mathrm{S\hat{T}}(g)$ is the product $\mathrm{ST}(g) \times \mathrm{H}^1(\mathrm{X}(g), \mathbb{Z})$.

In an attempt to make this paper relatively self-contained, we begin (§ 1) by explaining the two approaches in the usual (bosonic) case. (We shall restrict ourselves to genus $g > 1$ for the most part, dealing with $g = 1$ in an appendix.) We treat in addition the third definition using conjugacy classes of homomorphisms – which is of course a recurring theme in gauge field theories. In paragraph 2 we develop the theory of super diffeomorphisms and their homotopy, and show that the super Nielsen theorem fails by a factor of $\mathrm{H}^1(\mathrm{X}(g), \mathbb{Z})$. In paragraph 3, we describe three definitions of the super Teichmüller space, corresponding to those in paragraph 1. We describe their relation, showing in particular that the two we have called $\mathrm{ST}(g)$ and $\mathrm{S\hat{T}}(g)$ are different.

$\mathrm{ST}(g)$ is naturally described (using a constant curvature gauge) as a space of conjugacy classes of representations – this observation is central to the methods of [3] for example. We next (in § 4) ask whether a similar description of $\mathrm{S\hat{T}}(g)$ exists. We find that, although there is no appropriate space of representations, there is an analogue in terms of non-abelian cohomology. Let SPL (2, $\mathbb{R}$) be the superlinear group (more usually known as OSp(2|1), [8]), and let SPL(2, $\mathbb{R}$)$^{\sim}$ be its universal cover. Then $S\hat{T}(g)$ has a description in terms of difference cocycles as the generic part of the non-abelian cohomology of $\pi_1(X(g))$ with values in SPL(2, $\mathbb{R}$)$^{\sim}$. In paragraph 5, we apply this to describe an exact sequence which links the groups of components of the two spaces.

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1. THE TEICHMÜLLER SPACE

Let $X = X(g)$ be an oriented, smooth surface of genus $g > 1$. There are numerous definitions of the Teichmüller space $T(g)$ of Riemann surfaces; all of them give a contractible space of (complex) dimension $3g-3$.[1] The most analytically serious involve Beltrami differentials, but these will not be discussed here. Of particular concern to us are two: the marking definition and the mapping definition.

The marking definition

Let $\Gamma_g = \mathrm{Gp}(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g; \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \ldots \beta_g^{-1})$ (surface group in genus g, *see* [9]). For any basepoint x_0 in X, a canonical system of generators for $\pi_1(X, x_0)$ defines an isomorphism

$$\psi: \quad \Gamma_g \to \pi_1(X, x_0). \tag{1}$$

Such isomorphisms fall into two classes according to their relation to the intersection product; we call ψ "positive" ("negative") if $\psi(\alpha_i)$, $\psi(\beta_i)$ have homology intersection $+1\,(-1)$ for all i. A path λ from x_0 to x_1 in X defines the usual isomorphism.

$$\lambda_{\#}: \quad \pi_1(X, x_0) \to \pi_1(X, x_1). \tag{2}$$

If we call the isomorphisms ψ and $\lambda_{\#} \circ \psi$ "equivalent" then a *marking* on the oriented surface X is an equivalence class of (positive) isomorphisms ψ. (*See* [7].) By abuse of notation we write ψ for the marking which it defines. An oriented diffeomorphism $f: X \to X'$ of surfaces maps markings of X

(1) The most important definitions in terms of depth and power for generalization involve the Beltrami differentials, for which *see* [22]. We shall not deal with these here; but *see* ([3], [17], [23]) for forms of the supersymmetric generalization.

to markings of X′, by sending ψ to $f_* \circ \psi$; this transformation depends only on the homotopy class of f.

Now let M be a Riemann surface of genus g. As such it has a natural orientation, and we define a *marked Riemann surface* to be a pair (M, ψ), with ψ a marking on [the underlying surface of] M. Call two marked Riemann surfaces (M_i, ψ_i) ($i=1, 2$) T-equivalent if there exists a conformal map $f: M_1 \to M_2$ such that $\psi_2 = f_* \circ \psi_1$. (Such a map is, of course, an oriented diffeomorphism.)

DEFINITION 1.1. — *The Teichmüller space* T (g) *is the set of* T*-equivalence classes of marked Riemann surfaces* (M, ψ) *such that* M *has genus* g.

If we drop the requirement of marking we obtain the set of conformal equivalence classes of Riemann surfaces M of genus g. This is the *Riemann moduli space* M (g), quotient of T (g) by the mapping class group (*see* below) acting discretely [10]; we shall not go into its more complicated geometry here.

The above description, which is sketchy but accurate, has said nothing about the topology of T (g), much less its complex structure. We take it for granted that these can be constructed; for details *see* for example [7], [11].

The mapping definition

For the second definition we start with a fixed (smooth) oriented surface X of genus g. For convenience we suppose X not only marked as above, but provided with a fixed basepoint x_0 and a fixed positive isomorphism ψ which we use to identify Γ_g with $\pi_1(X, x_0)$. Define a *T-map* to be an oriented diffeomorphism $f: X \to M$, where M is a Riemann surface(2). Accordingly, f defines a marking of M, which we write $[f]$. Two T-maps $f_i: X \to M_i$ ($i=1, 2$) are called 'equivalent' if there exists a conformal map $h: M_1 \to M_2$ such that $h \circ f_1$ is homotopic to f_2. Alternatively, $f_2^{-1} h f_1$ is in $\mathrm{Diff}_0(X)$, the identity component of the diffeomorphisms of X. The set of T-maps modulo this equivalence relation is another version of T (g); for the moment let us call it $\hat{T}(g)$.

By the remarks above, the homotopic maps hf_1 and f_2 define the same marking of M_2, *i.e.* $h_*[f_1] = [f_2]$. Hence, an element in $\hat{T}(g)$ determines a unique element in T (g), and we can define a map

$$\varphi: \quad \hat{T}(g) \to T(g). \tag{3}$$

The map φ is a bijection (and hence the two definitions are the same) because of a striking property of surfaces known as *Nielsen's theorem*. Among various alternative ways of stating this result, the following is as

(2) I originally used the term "Teichmüller maps", which is more suggestive but has a different use in the literature. The idea is from [24].

simple as any. Consider an oriented diffeomorphism $f: X \to X$. If we fix a basepoint $x_0 \in X$ and write simply $\pi_1(X)$ for $\pi_1(X, x_0)$, then f defines an isomorphism $f_*: \pi_1(X) \to \pi_1(X)$ by using a change of basepoint isomorphism $\lambda_\#$ to identify $\pi_1(X, f(x_0))$ and $\pi_1(X, x_0)$. This is not uniquely defined (since the change of basepoint is not), but it is unique up to conjugation in $\pi_1(X)$, *i.e.* up to inner automorphism. Hence we can assign to f a unique element f_* of $\mathrm{Aut}(\pi_1(X))/\mathrm{Inn}(\pi_1(X))$. Clearly, f_* depends only on the homotopy class of f.

THEOREM 1.1. – (*Nielsen's theorem*). *For any oriented surface* X, *this functor defines an isomorphism from the group of free homotopy classes of oriented diffeomorphisms* $f: X \to X$ *under composition* (*the mapping class group of* X) *to the quotient*

$$\mathrm{Out}^+(\pi_1(X)) = \mathrm{Aut}^+(\pi_1(X))/\mathrm{Inn}(\pi_1(X))$$

of positive automorphisms of $\pi_1(X)$ *by inner automorphismes.*

(The definition of a positive automorphism follows easily from (1): φ is positive if composition with φ sends positive isomorphisms (1) to positive ones.)

For this theorem *see* [6]; a modern account is in [12]. Of course, we can include the negative automorphisms and the orientation-reversing maps; but this isn't particularly useful here – in fact, it can make statements more complicated. It is an easy consequence of Nielsen's theorem

a) that any positive isomorphism from $\pi_1(X)$ to $\pi_1(M)$ (corresponding to a marking of M) is induced by a map, so that φ in (3) is *surjective*;

b) that if any two oriented maps from X to M induce the same marking of M, then they are homotopic, so φ is *injective*. Hence, our two definitions of Teichmüller space are the same.

We shall see that a failure in the "super" analogue of Nielsen's theorem entails the need to distinguish between the two definitions in the upper case.

The group definition

The definition of the Teichmüller space in terms of group representations is perhaps secondary in nature; but it is very well known, and in many ways the easiest to use. We shall derive it as an alternative version of the marking definition.

Let (M, ψ) be a marked Riemann surface. By uniformization, we can find a conformal map $f: M \to \mathscr{H}/\Gamma$, where $\mathscr{H}$ is the upper half plane and Γ is a discontinuous group of conformal transformations, *i.e.* a subgroup of $\mathrm{PSL}(2, \mathbb{R})$. We therefore have isomorphisms

$$\Gamma_g \xrightarrow{\psi} \pi_1(M, x_0) \xrightarrow{f_*} \pi_1(\mathscr{H}/\Gamma, f(x_0)) = \Gamma \subset \mathrm{PSL}(2, \mathbb{R}) \qquad (4)$$

associating to the marked Riemann surface (M, ψ) an embedding q of Γ_g as a subgroup of PSL(2, $\mathbb{R}$). Such an embedding is restricted to be discontinuous and "positive" in a sense which parallels our previous use; let us write

$$\mathrm{Homdis}^+ (\Gamma_g, \mathrm{PSL}(2, \mathbb{R})) \tag{5}$$

for the set of all discontinuous and positive embeddings. If we change the choice of uniformization or replace the marked Riemann surface by a T-equivalent one, the embedding q is conjugated by a fixed element of PSL(2, $\mathbb{R}$):

$$q_2(\gamma) = a^{-1} . q_1(\gamma) . a \qquad (a \in \mathrm{PSL}(2, \mathbb{R})) \tag{6}$$

From these considerations we derive the familiar expression of Teichmüller space as the quotient or orbit space

$$\mathrm{T}(g) = \mathrm{Homdis}^+ (\Gamma_g, \mathrm{PSL}(2, \mathbb{R}))/\mathrm{PSL}(2, \mathbb{R})_{\mathrm{conj}} \tag{7}$$

Notice that Nielsen's theorem was not involved in this derivation; accordingly we shall find that the formalism works in the super case to identify our first and third definitions.

2. SUPERSURFACES AND NIELSEN'S THEOREM

Having established the importance of the ordinary Nielsen's theorem in relating our various definitions, the next step is to see what happens to the theorem in the category of supermanifolds; for the basic theory of these, *see* ([8], [13]-[15]). We take the ring of scalars **B** to be a real infinite Grassmann algebra B, with augmentation ("body map") $\varepsilon : \mathrm{B} \to \mathbb{R}$; the kernel of ε is the augmentation ideal $\bar{\mathbf{B}}$.

Let us suppose that X is an oriented G^∞ [15] de Witt supermanifold over D. Our aim is to investigate the homotopy classes of diffeomorphisms $f : \mathrm{X} \to \mathrm{X}$; the first problem is to adopt a suitable definition of "homotopy" for the super category. This is not really a problem, since most sensible definitions are equivalent. We shall use the simplest.

DEFINITION 2.1. – *Let* X, Y *be supermanifolds and let* $f_0, f_1 : \mathrm{X} \to \mathrm{Y}$ *be* G^∞ *maps. Regard the unit interval* **I** *as a (trivial) supermanifold of dimension* $(1 \mid 0)$. *Then we say that* f_0, f_1 *are homotopic if there exists a* G^∞ *map*

$$\mathrm{F} : \quad \mathrm{X} \times \mathbf{I} \to \mathrm{Y}$$

such that $\mathrm{F}(x, i) = f_i(x)$ *for all* $x \in \mathrm{X}$ *and for* $i = 0, 1$.

Using this definition (which clearly gives an equivalence relation as usual), we wish to study the homotopy classes of G^∞ diffeomorphisms from a supermanifold X to itself. Let Diff(X) denote the set of such diffeomorphisms. In a sense which we could make precise by "topologizing" Diff(X), the set of homotopy classes is $\pi_0(\mathrm{Diff}(X))$. This is the analogue of the set which enters in Nielsen's theorem. Standard considerations show immediately that if two maps are homotopic so are their body maps; we are therefore able to deduce a natural mapping

$$\pi_0(\mathrm{Diff}(X)) \to \pi_0(\mathrm{Diff}(X_0)) \tag{8}$$

which is a group homomorphism if we give both sets the composition product ("mapping class group").

Now with any supermanifold X of dimension $(m|n)$ we can naturally associate an ordinary vector bundle $\mathscr{E}(X)$ of rank n over X_0; indeed, this construction is basic to Batchelor's theorem [13]. It is described by (i) taking the θ parts g_{jk} of the overlap functions which transform coordinates (x_i, θ_j) into $(\tilde{x}_i, \tilde{\theta}_j)$;

$$\tilde{\theta}_j = \sum g_{jk}(x)\,\theta_k + \text{higher terms} \tag{9}$$

and then (ii) applying the augmentation ε to $g_{jk}(x)$ to get an $(n \times n)$ real matrix function which is necessarily non-singular. And the naturality of this construction means that a diffeomorphism $f: X \to X'$ has an associated isomorphism of rank n vector bundles $\mathscr{E}(f): \mathscr{E}(X) \to \mathscr{E}(X')$ which covers the body map f_0. Our essential point is that the pair $(\mathscr{E}(f), f_0)$ gives all the information we need.

THEOREM 2.1. – *Let $f, \tilde{f}$ be diffeomorphisms from the supermanifold* X *to* X'. *Suppose that we have a homotopy of the pairs $(\mathscr{E}(f), f_0)$ and $(\mathscr{E}(\tilde{f}), \tilde{f}_0)$ through similar pairs. Then the maps themselves are* G^∞ *homotopic through diffeomorphisms as defined above.*

Proof. – It is easy to see that we can suppose that $X = X'$ and f is the identity map. Now use the fact that a supermanifold is fibred over its finite approximations – of which X_0 is the zeroth order one and $\mathscr{E}(X)$ is the first [14] – to find a homotopy from $\tilde{f}$ to a map $\tilde{g}$ which covers the given homotopy of $\mathscr{E}(\tilde{f})$ with $\mathscr{E}(f)$. We deduce that it is sufficient to prove

LEMMA 2.1. – *If f_0, $\mathscr{E}(f)$ are the identity maps of* X, $\mathscr{E}(X)$ *respectively, then f is* G^∞ *homotopic to the identity through diffeomorphisms.*

Proof. – If we filter X corresponding to quotients of the ground ring B [4] we can write a sequence of vector bundles over $\mathscr{E}(X)$, with projective limit X

$$\ldots \to \mathscr{E}_2(X) \to \mathscr{E}_1(X) \to \mathscr{E}(X) \to X_0 \tag{10}$$

which is natural, and in particular is preserved by diffeomorphisms of X. Now by repeatedly using the CHP we can find a homotopy of f with the identity through maps f_t which are the identity on $\mathscr{E}(X)$. But such a map is invertible by the general theory of G^∞ maps, its local expression being

$$f_t(x_1, \ldots, x_m, \theta_1, \ldots, \theta_n) = (x_1, \ldots, x_m, \theta_1, \ldots, \theta_n) + \text{nilpotent elements.} \quad (11)$$

This proves the lemma, hence theorem 2.1.

It follows from this theorem that the mapping class group $\pi_0(\mathrm{Diff}(X))$ is precisely the group of *homotopy classes of automorphisms of the pair* $(X_0, \mathscr{E}(X))$.

Now let X be an oriented supermanifold of real dimension $(2|2)$, whose body X_0 is an oriented compact connected surface of genus >1; we shall compute its mapping class group as defined above. As before we restrict attention to those diffeomorphisms which preserve orientation, but now we require that this should be true of both x and θ orientations. (This clearly makes sense in that a super-complex analytic map must preserve both.)

Equivalently, it must be true for both $\mathscr{E}(X)$ and X_0. We call the restricted group $\pi_0(\mathrm{Diff}^+(X))$.

We note that $\mathscr{E}(X)$ is a rank 2 oriented vector bundle over X_0 and as such is characterized by its Euler class which we can regard as an integer $\chi(\mathscr{E}(X))$.

LEMMA 2.2. – *The body homomorphism* (8) *from* $\pi_0(\mathrm{Diff}^+(X))$ *to* $\pi_0(\mathrm{Diff}^+(X_0))$ *is surjective for super surfaces.*

Proof. – Call the body homomorphism ε. Given a positive diffeomorphism $f: X_0 \to X_0$, it is sufficient by theorem 2.1 to find an oriented bundle automorphism $\mathscr{E}(f)$ of $\mathscr{E}(X)$ which covers f, so that $\varepsilon([\mathscr{E}(f), f]) = [f]$. This will be possible if $f^*(\mathscr{E}(X))$ is isomorphic to $\mathscr{E}(X)$. But f^* is the identity on $H^2(X, \mathbb{Z})$, so $\chi(f^*(\mathscr{E}(X))) = \chi(\mathscr{E}(X))$, and the two bundles are isomorphic.

We now turn to the more interesting question of characterizing the kernel of ε. Let $f: X_0 \to X_0$ be homotopic to the identity map and let $\mathscr{E}(f)$ be an automorphism of $\mathscr{E}(X)$ which covers f. By the covering homotopy property, we can find a homotopy of the pair $(\mathscr{E}(f), f)$ to a pair (h, id) through equivalences (isomorphisms + diffeomorphisms). Hence every class in the kernel of ε is represented by a pair (h, id).

Now suppose that $(h_0, \mathrm{id}) \simeq (h_1, \mathrm{id})$. Then there is a homotopy $H = \{h_t\}$ from h_0 to h_1. This is turn covers a homotopy $\{f_t\}$ from the identity map of X_0 to itself, *i.e.* a loop in $\mathrm{Diff}^+(X_0)$. However, because $g > 1$, $\pi_1(\mathrm{Diff}^+(X_0)) = 0$, (*see* [11]) so the loop $\{f_t\}$ is nullhomotopic. By the covering homotopy property again, we can cover such a nullhomotopy, and obtain a homotopy $\{\tilde{h}_t\}$ from h_0 to h_1 such that each $\tilde{h}_t$ covers the identity map (is a bundle automorphism in the usual sense). In other words, any homotopies between h_0 and h_1 can be represented by bundle map homotopies. We deduce the crucial

LEMMA 2.3. — *The kernel of* ε *is naturally identified with the group of components* $\pi_0(\mathrm{Aut}(\mathscr{E}(X))$.

Now $\mathrm{Aut}(\mathscr{E}(X))$ — the gauge group — is well known to be the space of sections of the adjoint bundle associated to the principal bundle of $\mathscr{E}(X)$. Since the group in question is, up to homotopy, SO(2) which is abelian, the adjoint bundle is trivial. We can therefore (as is easily seen directly) identify $\mathrm{Aut}(\mathscr{E}(X))$, again up to homotopy, with $\mathrm{Map}(X_0, SO(2)) = (S^1)^{X_0}$, the space of mappings from X_0 into the circle. Hence, $\pi_0(\mathrm{Aut}(\mathscr{E}(X)) = [X_0, S^1]$ which [since S^1 is a space of type $K(\mathbb{Z}, 1)$] is canonically isomorphic to $H^1(X_0, \mathbb{Z})$. Using this together with lemmas 2.2 and 2.3, we obtain our main result:

THEOREM 2.2. — *Let* X *be a super surface of genus* >1. *Then the mapping class group of* X *fits into a natural short exact sequence*:

$$0 \to H^1(X_0, \mathbb{Z}) \to \pi_0(\mathrm{Diff}^+(X)) \to \pi_0(\mathrm{Diff}^+(X_0)) \to 1. \qquad (12)$$

The action of $\pi_0(\mathrm{Diff}^+(X_0))$ *on* $H^1(X_0, \mathbb{Z})$ *is the natural one by induced homomorphisms.*

To establish the statement about the action, we go back to the topological group extension which has (12) as path component sequence:

$$(S^1)^{X_0} \to \mathrm{Aut}^+(\mathscr{E}(X)) \to \mathrm{Diff}^+(X_0) \qquad (13)$$

A map $q: X_0 \to S^1$ defines an automorphism of $\mathscr{E}(X)$ by rotating the fibre at x through an angle $q(x)$. Let $\tilde{u}$ be any fibrewise map of $\mathscr{E}(X)$ covering u in $\mathrm{Diff}^+(X_0)$, and suppose for simplicity that $\tilde{u}$ is a rotation on the fibres. Then $\tilde{u}^{-1} \circ q \circ \tilde{u}$ rotates the fibre of $\mathscr{E}(X)$ at x by an angle $q(u(x))$. Hence, the operation of $\mathrm{Diff}^+(X_0)$ on $(S^1)^{X_0}$ in (13) is by composition. The same is clearly true for the component group, the homotopy set $[X_0, S^1]$; and composition on the homotopy set corresponds to induced map in cohomology.

The next question of interest would be to determine the actual extension corresponding to a given bundle. If $\mathscr{E}(X)$ is trivial, the extension (13) has a cross-section, and (12) is split. However, the bundles of interest to us (spin bundles for genus >1) are never trivial, so this fact is of no help to us.

3. THE SUPER TEICHMÜLLER SPACES

We shall now describe the analogues for super Riemann surfaces of the three definitions of Teichmüller space, and explain why, in consequence of the description of $\pi_0(\mathrm{Diff}^+(X))$ arrived at above, the mapping definition gives a different result from the other two. Let us begin by recalling that a super Riemann surface M is a *complex analytic* supermanifold of complex dimension $(1 \mid 1)$[3], real dimension $(2 \mid 2)$, with an atlas of charts $\{z_\alpha, \theta_\alpha\}$ which is *superconformal*. In other words, the canonical odd complex derivative operator transforms between charts by

$$D_\alpha = (D_\alpha(\theta_\beta)) . D_\beta \qquad \text{where} \qquad D_\alpha = \partial/\partial\theta_\alpha + \theta_\alpha . \partial/\partial z_\alpha \tag{14}$$

(*See* [3] for all these definitions, and for the following.) It follows from complex analyticity that *a*) the body M_0 is a complex 1-manifold (Riemann surface) and *b*) the bundle $\mathscr{E}(M)$ defined in the previous section has a canonical structure of complex line bundle; while the superconformal condition implies that $\mathscr{E}(M)$ is actually a "spin structure", *i.e.* satisfies

$$(\mathscr{E}(M))^2 = K(M_0) = \text{canonical bundle.} \tag{15}$$

It is well known that the set of all such bundles is an affine space over $\mathbb{Z}_2$ with associated vector space $H^1(M_0, \mathbb{Z}_2)$, and in particular the number of spin structures is 2^{2g}. Moreover, this set is naturally in a bijective correspondence with the set of real spin structures on the underlying 2-manifold [16]. In other words, the underlying real object of $(M_0, \mathscr{E}(M))$ is a surface with a spin structure.

The marking definition

The idea of this is a simple one, and was the basis of the definition of super-Teichmüller space used in [3]. Define a marking of a super surface X to be a marking ψ of its body X_0. This is reasonable, since the triviality of the "soul" topology leads us naturally to define the fundamental group of X as that of X_0. A G^∞ diffeomorphism f of super surfaces induces a diffeomorphism f_0 at body level, and hence a transformation f_* of markings. Let (M, ψ), $(\tilde{M}, \tilde{\psi})$, be two marked super Riemann surfaces. We define them to be T-equivalent if there exists a superconformal equivalence $f: M \to \tilde{M}$ such that $f_* \circ \psi = \tilde{\psi}$. The super Teichmüller space $ST(g)$ is the set of T-equivalence classes of marked super Riemann surfaces. There is an obvious body map sending $ST(g)$ to $T(g)$. However, the information about (M, ψ) at body level—that which remains when we have factored out the kernel of ε—includes not only its Teichmüller class, but also the isomorphism class of its spin structure, *i.e.* the triple

$$(M_0, \psi, \mathscr{E}(M)) \in \{\text{Riemann surfaces, markings, spin structures}\} \tag{16}$$

The set of such triples is a bundle over $T(g)$ (local triviality is obvious) with fibre the set of spin structures, *i.e.* $H^1(M_0, \mathbb{Z}_2)$. Since this is discrete, and $T(g)$ is contractible [7], the bundle is a product which we shall call

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(3) That is, locally superholomorphically equivalent to $B \otimes C = (B_C)^{1,1}$.

$T^{spin}(g)$;

$$T^{spin}(g) = T(g) \times H^1(M_0, \mathbb{Z}_2) \tag{17}$$

By analogy with formula (7), we can also define this set group-theoretically:

$$T^{spin}(g) = \text{Homdis}^+(\Gamma_g, SL(2, \mathbb{R}))/SL(2, \mathbb{R})_{conj} \tag{18}$$

And the "classical" result on super Teichmüller space, for whose proof *see* ([3], [4]) is

THEOREM 3.1. – *The super Teichmüller space* $ST(g)$ *is the quotient "super orbifold"* $S\tilde{T}(g)/\sigma$, *where* $S\tilde{T}(g)$ *is a de Witt supermanifold of complex dimension* $(3g-3|2g-2)$ *with body* $T^{spin}(g)$, *and* σ *acts on* $S\tilde{T}(g)$ *by changing the sign of all* θ_i*'s.*

Note. – We have referred to "complex dimension", and it follows from various sources ([3]-[5], [17]) that *a*) the body and soul of $ST(g)$ have complex structures, separately; *b*) there is a version of the Bers embedding in the complex superspace $(B_{\mathbb{C}})^{3g-3, 2g-2}$. However, the question of the complex structure on $ST(g)$ still awaits a full treatment. There is an existence proof [18]; for comments on this and for a different approach *see* [19].

The group definition

The marking definition is, as before, naturally linked to the group theoretic one. For this reason, and because it raises no problems, we deal with the latter next. Let $SPL(2, \mathbb{R})$ be the supergroup of superlinear transformations of the super half plane $S\mathcal{U}$, *i.e.*

$$F(z, \theta) = \Big((az+b)/(cz+d) + \theta(\gamma z+\delta)/(cz+d)^2,$$
$$(\gamma z+\delta)/(cz+d) + \theta\left(1+\frac{1}{2}\delta\gamma\right)/(cz+d)\Big) \tag{19}$$

where $a, b, c, d \in B_0$, $\gamma, \delta \in B_1$, and $ad-bc=1$.

[For this group, also called $OSp(2, 1)$, and its matrix expression, *see* ([3], [8]).]

Let SCf be the larger (infinite dimensional) supergroup of *all* superconformal transformations of $S\mathcal{U}$. In contrast with the bosonic case, these two groups are not the same, although they do have the same body $SL(2, \mathbb{R})$. Fortunately, this does not matter. In fact, from ([3], [4], [20]) we can get

THEOREM 3.2. — *The super Teichmüller space* ST(g) *is naturally identified with either of the two (diffeomorphic) representation spaces*

$$\mathrm{Homdis}^{+}(\Gamma_g, \mathrm{SPL}(2, \mathbb{R}))/\mathrm{SPL}(2, \mathbb{R})_{\mathrm{conj}} = \mathrm{Homdis}^{+}(\Gamma_g, \mathrm{SCf})/\mathrm{SCf})_{\mathrm{conj}} \quad (20)$$

Here a homomorphism of supergroups is called discrete if its body is. The proof that these two spaces are naturally identified is the main result of [20].

The mapping definition

The interesting question arises when we introduce the mapping definition, since this gives a different result. Choose as before a fixed (real) marked super surface X of genus g with basepoint x_0. Define a *super* T-*map* to be an oriented G^∞ diffeomorphism $f: X \to M$, where M is a super Riemann surface. Again, f defines a marking $[f]$ of M. We say that two super T-maps $f_1: X \to M_i$ $(i=1, 2)$ are equivalent if there exists a superconformal $h: M_1 \to M_2$ such that $h \circ f_1$ is homotopic to $f_2: f_2^{-1} h f_1$ is in the group $\mathrm{Diff}_0(X)$ of positive G^∞ diffeomorphisms homotopic to the identity. Notice that it is a consequence of theorem 2.2 that this group is *smaller* than the group of diffeomorphisms whose *body* is homotopic to the identity; and it is the latter group which was used in obtaining the two previous definitions. Let us write $\mathrm{S\hat{T}}(g)$ for the set of equivalence classes of super T-maps. We have a natural mapping as before

$$\varphi: \quad \mathrm{S\hat{T}}(g) \to \mathrm{ST}(g). \quad (21)$$

And our main result is

THEOREM 3.3. — $H^1(X, \mathbb{Z})$ *acts freely on* $\mathrm{S\hat{T}}(g)$ *and* φ *is constant on the orbits of this action. The induced mapping*

$$\bar{\varphi}: \quad \mathrm{S\hat{T}}(g)/H^1(X, \mathbb{Z}) \to \mathrm{ST}(g) \quad (22)$$

is a diffeomorphism (of super orbifolds).

Proof. — Let $\mathrm{Diff}_0\hat{}(X)$ be the group of diffeomorphisms whose body is homotopic to the identity. Clearly,

$$\pi_0(\mathrm{Diff}^{+}(X_0)) = \mathrm{Diff}^{+}(X_0)/\mathrm{Diff}_0(X_0) = \mathrm{Diff}^{+}(X)/\mathrm{Diff}_0\hat{}(X),$$

while $\pi_0(\mathrm{Diff}^{+}(X)) = \mathrm{Diff}^{+}(X)/\mathrm{Diff}_0(X)$. Now $\mathrm{Diff}^{+}(X)$ acts on T-maps in an obvious way by composition, and hence on $\mathrm{S\hat{T}}(g)$; and the stabilizer of a point in $\mathrm{S\hat{T}}(g)$ is by definition $\mathrm{Diff}_0(X)$. On the other hand, f and $f \circ \tilde{f}$ give the same marking of M, and hence define the same element of $\mathrm{ST}(g)$, if and only if $\tilde{f} \in \mathrm{Diff}_0\hat{}(X)$. We deduce that the group $\mathrm{Diff}_0\hat{}(X)/\mathrm{Diff}_0(X)$ acts freely on $\mathrm{S\hat{T}}(g)$, and $\varphi[f_1] = \varphi[f_2]$ if and only if $[f_1]$, $[f_2]$ are in the same orbit of the group.

Now we have seen that this group is the same as $\operatorname{Ker}\{\pi_0(\operatorname{Diff}^+(X)) \to \pi_0(\operatorname{Diff}^+(X_0))\}$, which by theorem 2.2 is isomorphic to $H^1(X, \mathbb{Z})$; so we deduce the existence of the required action, that it is free, and that φ factors through the space of orbits. To see that φ is surjective (the only remaining question) let M be a marked SRS of genus g. Then using Nielsen's theorem, there is an oriented diffeomorphism of bodies $f_0: X_0 \to M_0$ which induces the given marking on M_0. As in paragraph 2, we can now use the homotopy lifting theorem for vector bundles, plus Batchelor's theorem on the reducibility of the G^∞ supermanifold category to that of vector bundles, to construct an oriented diffeomorphism $f: X \to M$, *i.e.* a super T-map. Hence, φ takes the class of (M, f) to the marked SRS M – that is, it is surjective.

Note 1. – At this point we should mention that the different definitions of super-Teichmüller space do not result in different supermoduli spaces. In fact, the supermoduli space $SM(g)$ is defined (invariantly) as the set of superconformal equivalence classes of super Riemann surfaces of genus g. As such, it is easy to see that it is the quotient of $ST(g)$ by the usual mapping class group $\operatorname{Diff}^+(X_0)/\operatorname{Diff}_0(X_0) = \operatorname{Diff}^+(X)/\operatorname{Diff}_0\hat{}(X)$; while it is the quotient of $S\hat{T}(g)$ by the larger *super* mapping class group $\operatorname{Diff}^+(X)/\operatorname{Diff}_0(X)$. In other words, we have different versions of the "modular group", but the same moduli space.

Note 2. – The space $S\hat{T}(g)$ is the appropriate one for the super Beltrami theory as developed in [17]. In fact, it is shown there that the components of the space of allowable super Beltrami coefficients are in bijective correspondence with $H^1(X_0, \mathbb{Z})$.

4. COCYCLES

A good way to express the above results, one might think, would be to return to the group definition and write $S\hat{T}(g)$ as the representation space $\operatorname{Homdis}^+(\Gamma_g, G)/G_{\mathrm{conj}}$ for some suitable group G. There is only one group G which is really an appropriate choice for this, and that is the universal cover $\tilde{G} = SCf^\sim$ of SCf, with associated exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{j} SCf^\sim \xrightarrow{\pi} SCf \to 1 \tag{23}$$

The extension is central, in fact SCf is $\operatorname{Ad}(SCf^\sim)$, the quotient by the centre (4). Unfortunately, the homomorphisms from Γ_g to SCf with which

(4) We could (as explained above) use either the group $SCf^\sim$ or its subgroup $SPL(2, R)^\sim$, with the same result. The statements made about $SCf^\sim$ are consequences of the fact that its body, like that of $SPL(2, R)$, is $SL(2, R)$.

we are concerned do not lift to the universal cover, so such a construction is impossible. In fact, any such homomorphism λ has as body $\lambda_0 : \Gamma_g \to \mathrm{SL}(2, \mathbb{R})$; and this in turn has a classifying map

$$\mathrm{B}\lambda_0 : \mathrm{B}\Gamma_g \to \mathrm{BSL}(2, \mathbb{R}) = \mathrm{K}(\mathbb{Z}, 2) \tag{24}$$

Since $\mathrm{B}\Gamma_g = \mathrm{X}$ (the surface) we derive a Chern class $c(\lambda) \in \mathrm{H}^2(\mathrm{X}, \mathbb{Z})$. Clearly, λ lifts to the cover $\mathrm{SCf}^{\sim}$ if and only if $c(\lambda) = 0$. On the other hand, a properly discontinuous positive action of Γ_g on the half-plane corresponds to a hyperbolic structure on the tangent bundle of X (with the appropriate orientation). This implies that the $\mathrm{SL}(2, \mathbb{R})$-bundle induced on X by $\mathrm{B}\lambda_0$ is simply the tangent bundle of X, and $c(\lambda)$ for any such λ is the Euler class $\chi(\mathrm{X})$ which is not 0.

This rules out any obvious identification of $\mathrm{S}\hat{\mathrm{T}}(g)$ with a homomorphism space as above. However, we can produce a relative version in terms of difference classes which we shall construct in non-abelian cohomology [21] (note that a homomorphism is itself a non-abelian cocycle, and that the conjugation relation corresponds to that of homology [4]). To see how this works, we begin by picking a fixed "basepoint" $f : \mathrm{X} \to \mathrm{M}$ from which to measure distance. As noted before, f defines a marked SRS and hence a homomorphism $q = q(f) : \Gamma_g \to \mathrm{SCf}$, up to conjugation. Fix q once for all; we can then consider Γ_g as acting on both SCf and $\mathrm{SCf}^{\sim}$ by inner automorphisms. The class $\tilde{d}((\mathrm{M}, f), (\mathrm{M}', f'))$ which we shall find will be in the cohomology $\mathrm{H}^1(\Gamma_g, \mathrm{SCf}^{\sim})$, with respect to this action. Note that the (central) extension (23) gives rise to an exact sequence in non-abelian cohomology:

$$\xrightarrow{\delta} \mathrm{H}^1(\Gamma_g, \mathbb{Z}) \xrightarrow{j_*} \mathrm{H}^1(\Gamma_g, \mathrm{SCf}^{\sim}) \xrightarrow{\pi_*} \mathrm{H}^1(\Gamma_g, \mathrm{SCf}) \xrightarrow{\delta} \mathrm{H}^2(\Gamma_g, \mathbb{Z}) \tag{25}$$

where the mappings of groups/sets are defined as usual, and interpreted in the appropriate way.

Now a second homomorphism $q' : \Gamma_g \to \mathrm{SCf}$ defines a cocycle in $\mathrm{Z}^1(\Gamma_g, \mathrm{SCf})$ by

$$\gamma \mapsto \varphi(q')(\gamma) = q'(\gamma)\, q(\gamma^{-1}). \tag{26}$$

And the cocycles $\varphi(q_1)$, $\varphi(q_2)$ are homologous if and only if q_1, q_2 are conjugate. A super T-map $f' : \mathrm{X} \to \mathrm{M}'$ defines a homomorphism q' as above, up to conjugation; and hence, a unique cohomology class $[\varphi(q')]$ in $\mathrm{H}^1(\Gamma_g, \mathrm{SCf})$. We shall call this class $d((\mathrm{M}, f), (\mathrm{M}', f'))$; our aim is to prove:

THEOREM 4.1. – *For each* $(\mathrm{M}', f') \in \mathrm{S}\hat{\mathrm{T}}(g)$, *there is a canonical "difference class"* $\tilde{d}((\mathrm{M}, f), (\mathrm{M}', f')) \in \mathrm{H}^1(\Gamma_g, \mathrm{SCf}^{\sim})$ *which lifts* $d((\mathrm{M}, f), (\mathrm{M}', f'))$; *and this defines a bijection from* $\mathrm{S}\hat{\mathrm{T}}(g)$ *to an open dense subset of* $\mathrm{H}^1(\Gamma_g, \mathrm{SCf})^{\sim}$.

Note 1. – The bijection so defined is clearly continuous in a sense which it is easy to specify. Its dependence on the basepoint (M, *f*) is coded in the description of the cohomology set (which depends on the action of Γ_g on $\mathrm{SCf}^{\sim}$).

Note 2. – The "dense subset" is a signpost to an important point which might be overlooked; that the super Teichmüller space ST(*g*) is *not* a component (or union of components) in $H^1(\Gamma_g, \mathrm{SCf})$. The latter also contains classes coming from *non-discontinuous* homomorphisms, corresponding in some broad sense to singular (degenerate) surfaces on the boundary of ST(*g*). What that boundary might be is beyond the scope of this paper, but it is easy to deduce from what we know for Riemann surfaces that ST(*g*) is open and dense in the appropriate subset of $H^1(\Gamma_g, \mathrm{SCf})$.

We begin by showing that the classes we are concerned with do indeed have lifts.

LEMMA 4.1. – *The image of the projection* π_* *contains the set of all classes* $d((\mathrm{M}, f), (\mathrm{M}', f'))$.

Proof. – The usual exact sequence for central extensions (*see* [21]) tells us that the image of π_* consists precisely of all $x \in H^1(\Gamma_g, \mathrm{SPL}(2, \mathbb{R}))$ with $\delta(x) = 0$. If x is the class $d((\mathrm{M}, f), (\mathrm{M}', f'))$ corresponding to the cocycle $\varphi(q')$ as in (26) above, then the discussion at the beginning of this chapter shows that $\delta(x) = c(q') - c(q) \in H^2(\Gamma_g, \mathbb{Z}) = \mathbb{Z}$. Hence, for x to lift, we must have $c(q') = c(q)$. But as we have seen this is true in this case – both are equal to $\chi(\mathrm{X})$.

Note. – The set of all difference classes $d((\mathrm{M}, f), (\mathrm{M}', f'))$ can be naturally identified with the super Teichmüller space; simply forget (M, *f*) and apply the group theoretic definition to consider $[q']$ as a point in ST(*g*).

LEMMA 4.2. – *In the sequence* (25), *the mapping*

$$j_*: \quad H^1(\Gamma_g, \mathbb{Z}) \to H^1(\Gamma_g, \mathrm{SCf}^{\sim})$$

is injective, and π_* *gives a regular covering of* $\mathrm{ST}(g) \subset H^1(\Gamma_g, \mathrm{SCf})$ *with group* $H^1(\Gamma_g, \mathbb{Z})$.

Proof. – From the general theory of non-abelian cohomology, plus lemma 4.1, π_* is a regular covering as stated, and its group is the image of j_*, or the cokernel of the preceding homomorphism δ. To show that δ is zero, we shall show that π_* is an epimorphism on H^0. Here we are dealing with groups – the centralizers of $q(\Gamma_g)$ in $\mathrm{SCf}^{\sim}$ and SCf respectively. These need rather careful consideration. At the body level, the centralizer $H^0(\Gamma_g, \mathrm{SL}(2, \mathbb{R}))$ is precisely the centre $\mathbb{Z}_2 \subset \mathrm{SL}(2, \mathbb{R})$. By considering the spectral sequence for the solvable group SCf_1 (the kernel of the body map on SCf, compare [4]), we find $H^0(\Gamma_g, \mathrm{SCf}_1) = 0$. There is therefore an exact sequence

$$0 \to H^0(\Gamma_g, SCf) \to H^0(\Gamma_g, SL(2, \mathbb{R})) \to H^1(\Gamma_g, SCf_1) \to H^1(\Gamma_g, SCf) \to \dots \quad (27)$$

from which $H^0(\Gamma_g, SCf)$ must be either 0 or $\mathbb{Z}_2$ ([5]).

If the group is 0, then immediately $\delta = 0$ and j_* is injective as claimed. If on the other hand it is $\mathbb{Z}_2$, then the body map $\varepsilon : H^0(\Gamma_g, SCf) \to H^0(\Gamma_g, SL(2, \mathbb{R}))$ is an isomorphism. It follows using the five lemma that ε carries π_* for $SCf^\sim$ isomorphically onto π_* for $SL(2, \mathbb{R})^\sim$. The latter is surjective, hence so is the former as required.

We are now in a position to define the difference class. First note that, as seen above, we can consider $ST(g)$ as an open dense subset of $H^1(\Gamma_g, SCf)$, namely the set of all classes $d((M, f), (M', f'))$. $ST(g)$ itself is a disjoint union of 2^{2g} components, each with body equal to $T(g)$. The covering π_* must be a product on each of these (simply connected) components. By Theorem 3.3, $S\hat{T}(g) \to ST(g)$ is a regular covering with group $H^1(\Gamma_g, \mathbb{Z})$; by lemma 2.2, π_* is the same. Hence we can find a map $\tilde{d} : S\hat{T}(g) \to H^1(\Gamma_g, SCf^\sim)$ which is an $H^1(\Gamma_g, \mathbb{Z})$-equivariant embedding lifting $d : ST(g) \to H^1(\Gamma_g, SCf)$. Moreover, this will be unique once we have specified the image of a single point. The natural way to do this is to take (M, f) as the basepoint of $S\hat{T}(g)$, and map it into the trivial element 1 of $H^1(\Gamma_g, SPL(2, \mathbb{R})^\sim)$. We accordingly have a difference class $\tilde{d}((M, f), (M', f'))$ for all (M', f') in $S\hat{T}(g)$, and $\tilde{d}$ is an injective map on $S\hat{T}(g)$; also, $\tilde{d}((M, f), (M, f)) = 1$.

Lastly, the image of $\tilde{d}$ is the subset $\pi_*^{-1}(\text{Image}(d))$, which is open and dense in $H^1(\Gamma_g, SCf^\sim)$ since (*a*) Image (d) is open dense in $H^1(\Gamma_g, SCf)$ and (*b*) π_* is a covering. This observation completes the proof of theorem 4.1.

Note. — From the above, we have in fact proved more than the existence of the difference class, since we have shown it to be uniquely defined subject to certain natural conditions.

5. THE COMPONENT EXACT SEQUENCES

The sequence (25) in cohomology is of course not a sequence of groups — its terms cannot be given group structures in any reasonable way.

([5]) The two cases are distinguished (following the non-abelian cohomology exact sequence through) by the image of the centre of SL(2, R) in $H^1(\Gamma_g, SCf_1)$, where it corresponds to the θ-reversal. It follows from the results of [4] that $H^0(\Gamma_g, SCf)$ is Z_2 precisely when the homomorphism q (or the SRS $q(\Gamma_g)\backslash\mathscr{H}$) is split, *i.e.* $q(\Gamma_g)\backslash\mathscr{H}$ is the canonical extension of an ordinary Riemann surface.

However, if we apply the component functor π_0, we do obtain groups, and these fit together in a way which, while it is not strictly concerned with supermanifolds, is interesting in terms of the previous results. In order to clarify what is happening, we fix as before a "basepoint" (M, f) in $\hat{ST}(g)$. M is in particular a marked super Riemann surface, with a spin structure. And we have

PROPOSITION 5.1. – *The sets of components* $\pi_0(\hat{ST}(g))$, $\pi_0(ST(g))$ *can be identified (once a basepoint is chosen) with the groups* $H^1(\Gamma_g, \mathbb{Z})$, $H^1(\Gamma_g, \mathbb{Z}_2)$ *respectively, in such a way that the regular covering* φ *of* (21) *gives rise on components to the exact sequence of abelian groups*:

$$0 \to H^1(\Gamma_g, \mathbb{Z}) \to H^1(\Gamma_g, \mathbb{Z}) \to H^1(\Gamma_g, \mathbb{Z}_2) \to 0 \tag{28}$$

associated with the usual coefficient sequence.

Proof. – The simplest way to see the group structures is as follows. First, embed $ST(g)$, $\hat{ST}(g)$ in non-abelian cohomology groups $H^1(\Gamma_g, SCf)$, $H^1(\Gamma_g, SCf^{\sim})$ as before. We write $H^1(\Gamma_g, SCf)_{dis}$ for the image of $ST(g)$, and $H^1(\Gamma_g, SCf^{\sim})_{dis}$ similarly (reflecting the use of discontinuous homomorphisms in the characterization of these sets). We are therefore looking at the sequence

$$H^1(\Gamma_g, \mathbb{Z}) \to \pi_0(H^1(\Gamma_g, SCf)_{dis}) \to \pi_0(H^1(\Gamma_g, SCf^{\sim})_{dis}) \tag{29}$$

(*a priori* not a group exact sequence). To proceed further, we need to leave SCf behind. Since π_0 is a functor purely at the body level – and since the characterization of "discontinuous" homomorphisms is the same – we know that $\pi_0(H^1(\Gamma_g, SCf)_{dis})$ maps bijectively under ε to $\pi_0(H^1(\Gamma_g, SL(2, \mathbb{R}))_{dis})$; similarly for $SCf^{\sim}$ and $SL(2, \mathbb{R})^{\sim}$. We can therefore replace the sequence (29) by the corresponding one with special linear groups.

Now $\pi_0(H^1(\Gamma_g, SL(2, \mathbb{R}))_{dis})$ can be identified with the set of spin structures on X, and so – after fixing a basepoint – with $H^1(\Gamma_g, \mathbb{Z}_2)$. More explicitly, consider the sequence

$$H^1(\Gamma_g, \mathbb{Z}_2) \xrightarrow{j_*} \pi_0(H^1(\Gamma_g, SL(2, \mathbb{R}))_{dis}) \xrightarrow{\pi_*} \pi_0(H^1(\Gamma_g, PSL(2, \mathbb{R}))_{dis}) \tag{30}$$

By a similar argument (but simpler) to that used in Lemma 4.2, j_* is injective. Since $\pi_0(H^1(\Gamma_g, PSL(2, \mathbb{R}))_{dis}) = \pi_0(T(g))$ is trivial, j_* is bijective, and so $ST(g) = \pi_0(H^1(\Gamma_g, SL(2, \mathbb{R}))_{dis})$ can be identified with $H^1(\Gamma_g, \mathbb{Z}_2)$ using our fixed "basepoint". Exactly the same argument makes it possible to identify $\hat{ST}(g) = \pi_0(H^1(\Gamma_g, SL(2, \mathbb{R})^{\sim})_{dis})$ with $H^1(\Gamma_g, \mathbb{Z})$. Because of the naturality of these constructions, we can deduce that the map from $\hat{ST}(g)$ to $ST(g)$ (which is induced by the covering homomorphism of groups) is identified with the reduction mod 2 in cohomology; and the rest of Proposition 5.1 follows immediately.

APPENDIX

The case $g=1$

We here consider what corrections have to be brought to the theory in the case of the super-tori, where $g=1$. (The case $g=0$ is trivial as usual.) The major part of the previous argument goes through as before; the only point where we specifically used the assumption $g>1$ was in the identification of the kernel of $\varepsilon : \pi_0(\mathrm{Diff}^+(\mathrm{X})) \to \pi_0(\mathrm{Diff}^+(\mathrm{X}_0))$. If we now suppose that X_0 is a torus, we can once again reduce the question to the conditions for two maps (h_0, id), (h_1, id) which are fibrewise homotopic to be homotopic through isomorphisms. Here, $\pi_1(\mathrm{Diff}(\mathrm{X}_0))$ is not trivial, but equals $\pi_1(\mathrm{X}_0)$, since $\mathrm{Diff}_0(\mathrm{X}_0)$ is contractible to the subgroup of group translations by [11]. This might raise problems; but in our case, we are dealing with the spin bundle of the torus which is trivial (as a *real* vector bundle). Hence, the maps h_t which make up the homotopy of h_0, h_1 can be written

$$h_t(x, z) = (f_t(x), g_t(x) . z), \qquad (x \in \mathrm{X}_0, z \in \mathbb{R}^2)$$

with $f_0 = f_1 = \mathrm{id}$. Now write $\tilde{h}_t(x, z) = (x, g_t(z))$; $\tilde{h}_t$ is the required homotopy through isomorphisms.

We can therefore once again identify the kernel of ε with $[\mathrm{X}, \mathrm{S}^1] = \mathrm{H}^1(\mathrm{X}, \mathbb{Z})$. The main lines of the argument thereafter are the same.

However, it should be remembered that (as in [3]), the description of ST(1) in theorem 3.1 needs altering. In fact, ST(1) has four components. The three which correspond to the odd spin structures have dimension $(1 \mid 0)$, while the even spin one has dimension $(1 \mid 1)$. Similarly within $\mathrm{S}\tilde{\mathrm{T}}(1)$, the odd spin components have dimension $(1 \mid 0)$, the even ones $(1 \mid 1)$.

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