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R-matrix brackets and their quantization

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ABSTRACT. — Let M be a manifold, g a Lie algebra acting by derivations on $C^\infty(M)$ and $R \in \Lambda^2 g$ the “canonical” modified R-matrix given by $R = \sum E_\alpha \otimes E_{-\alpha} - E_{-\alpha} \otimes E_\alpha$ for positive root vectors E_α . We construct (under some conditions on M) a corresponding Poisson bracket, and quantize it. We discuss also the “quantum plane” and a non-compact analogue of the “quantum sphere”.

RÉSUMÉ. — Soient M une variété et g un algèbre de Lie qui agit par dérivations sur $C^\infty(M)$ et $R \in \Lambda^2 g$ la R-matrice « canonique » modifiée donnée par $R = \sum E_\alpha \otimes E_{-\alpha} - E_{-\alpha} \otimes E_\alpha$ avec les vecteurs E_α correspondants aux racines positives. Nous construisons (sous quelques conditions sur M) un crochet de Poisson correspondant et nous le quantifions. Nous examinons aussi un « plan quantique » et une version non-compacte de la « sphère quantique ».

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1. INTRODUCTION

It is well-known that the “quantum sphere” arises from quantization of a certain Poisson bracket on the usual sphere. This bracket is the reduction of a Poisson bracket on a compact group. At the quantum level this means that the “quantum sphere” is as a quotient space of a quantum group (*see* for example [12]). We develop here a general approach to constructing Poisson brackets of “R-matrix type” on homogeneous spaces and to the quantization of them. This approach for \mathbb{R} a solution of the classical Yang-Baxter equations was proposed in [6]. We generalize it to \mathbb{R} a solution of the modified Yang-Baxter equations. This represents a generalization from the triangular case to the quasitriangular or braided case. We recall first the situation in [6].

Let g be a Lie algebra and $R \in \Lambda^2 g$ a “classical R-matrix”. This means that the element

$$[[R, R]]^{\text{def}} = [R^{12}, R^{13}] + [R^{12}, R^{23}] + [R^{13}, R^{23}] = 0. \tag{1}$$

This equation is called the classical Yang-Baxter equations.

We suppose that we are given a representation

$$\rho: g \rightarrow \text{Der}(C^\infty(M))$$

of g in the space of all derivations of the algebra of functions $C^\infty(M)$ on a smooth manifold M . It is evident that the map

$$\{ , \}_R: C^\infty(M)^{\otimes 2} \rightarrow C^\infty(M)$$

given by

$$\{ f_1, f_2 \}_R = \mu \langle (\rho \otimes \rho)(R), df_1 \otimes df_2 \rangle \tag{2}$$

defines a Poisson bracket. Here μ denotes the usual multiplication on $C^\infty(M)$. We call this Poisson bracket the “R-matrix bracket”.

If there exists on M another Poisson bracket $\{ , \}$ we can demand that

$$\rho: g \rightarrow \text{Der}(C^\infty(M), \{ , \})$$

where $\text{Der}(C^\infty(M), \{ , \})$ is the space of all vector fields X on M that preserve the bracket $\{ , \}$ in the form

$$X \{ f_1, f_2 \} = \{ X f_1, f_2 \} + \{ f_1, X f_2 \}.$$

In this case the brackets $\{ , \}$ and $\{ , \}_R$ are compatible in the sense that they form a Poisson pair. This means that all linear combinations

$$\{ , \}_{a,b} = a \{ , \} + b \{ , \}_R, \quad a, b \in \mathbb{C}$$

are Poisson brackets.

In [6], all the brackets $\{ , \}_{a,b}$ were quantized simultaneously in the sense of deformation quantization, and a form of “twisted” quantum mechanics was investigated. All objects of the “twisted” quantum mech-

anics live naturally in a symmetric monoidal category generated by an involutive Yang-Baxter operator S . For example, $\text{Der}(C^\infty(M))$ becomes endowed with the structure of an S-Lie algebra.

Our task in the present paper is to extend some of this work to the case where $[[R, R]]$ is not zero but merely g -invariant. More precisely, we take for $R \in \Lambda^2 g$ the famous Drinfeld-Jimbo modified R-matrix given by formula (3) below. The main difference from the situation studied in [6] and our modified case is that the bracket (2) is now a Poisson one only under some conditions on M (Proposition 2.1 below). These conditions are satisfied for example on the highest weight orbits $\mathcal{O} \subset g^*$. In this case there exists on \mathcal{O} another Poisson bracket $\{ , \}$ (the Kirillov bracket), and the two brackets $\{ , \}$ and $\{ , \}_R$ are compatible.

Using some constructions of Drinfeld we then proceed to quantize the bracket $\{ , \}_R$ in the form of a new associative multiplication on the linear space of $C^\infty(M)$. This is the main result of the paper.

If g is a compact real form of a simple Lie algebra, the R-matrix (3) can still be regarded (up to a factor $\iota = \sqrt{-1}$) as an element of $\Lambda^2 g$. Then the formula (2) defines a Poisson structure on all symmetric homogeneous spaces. This case will be considered elsewhere (see also the Remark below). Here we would like only to stress that we do not use the usual procedure of a reduction of the Poisson-Lie structure on a group to introduce the Poisson structures related to the R-matrices (3). We construct these structures and quantize them directly in the sense of deformation quantization.

The main idea of our approach to the quantization is to show that an associativity morphism Φ constructed by Drinfeld disappears in some special situations. This approach can be applied to obtain the "quantum plane" and some form of "quantum sphere". We consider these objects at the end of the paper.

2. R-MATRIX BRACKETS

In the sequel we suppose G to be a simple, connected and simply-connected Lie group with g the corresponding Lie algebra. We work over a field $k = \mathbb{R}$, but all our constructions remain true if $k = \mathbb{C}$, M is an analytic manifold and $C^\infty(M)$ is changed to the space of holomorphic functions. We choose for g a Cartan-Weyl basis

$$\{ E_\alpha, E_{-\alpha}, H_\alpha, \alpha > 0 \}$$

in standard notations. We denote by n_+ the nilpotent subalgebra generated by the E_α for $\alpha > 0$.

Let $R \in \Lambda^2 g$ be a modified R-matrix, *i.e.* the element $[[R]]$ introduced in formula (1) is g -invariant but not zero. Note that modified R-matrices

were classified in [1]. The most well-known one is of the form

$$R = \sum_{\alpha > 0} (E_{\alpha} \otimes E_{-\alpha} - E_{-\alpha} \otimes E_{\alpha}). \tag{3}$$

We consider this R-matrix “canonical” and will denote it R_{can} . For any modified R-matrix we can consider the bracket $\{ , \}_R$ defined in (2), but it is not always a Poisson one.

Consider also a homogeneous space M for the Lie group G . Thus, G acts transitively on M . Fix a point $x_0 \in M$ and let $G_{x_0} = \text{Stab}(x_0) \subset G$ be the stabilizer of x_0 . Let $g_{x_0} \subset g$ be the Lie algebra of G_{x_0} .

PROPOSITION 2.1. – *If $g_{x_0} \supset n_+$ then the bracket $\{ , \}_R$ is a Poisson one.*

Proof. – For this it is sufficient to see that

$$\mu^{123} \langle \rho^{\otimes 3}(\llbracket R, R \rrbracket), df_1 \otimes df_2 \otimes df_3 \rangle = 0 \tag{4}$$

where $\mu^{123} : C^{\infty}(M)^{\otimes 3} \rightarrow C^{\infty}(M)$ is the usual multiplication. The element $\llbracket R, R \rrbracket$ is a linear combination of terms of two types:

$$E_{\alpha} \wedge E_{\beta} \wedge E_{\gamma} (\alpha + \beta + \gamma = 0) \quad \text{and} \quad H_{\alpha} \wedge E_{\alpha} \wedge E_{-\alpha}$$

where α, β, γ are the roots of g . Each term contains a factor belonging to n_+ . This implies (4) at the point $x_0 \in M$. As the element $\llbracket R, R \rrbracket \in \Lambda^3 g$ is g -invariant, we have (4) at all points.

A typical example of this construction is $M = \mathcal{O} \subset g^*$, the orbit in g^* corresponding to a highest weight vector. Another example is $g = \mathfrak{sl}(n)$ and $M = k^n - \{0\}$ in the fundamental representation of g . Here k^n consists of precisely two orbits, $\{0\}$ and $k^n - \{0\}$.

Remark. – We can see that the bracket $\{ , \}_R$ is a Poisson one if an orbit $M = \mathcal{O} \subset g^*$ is “small enough”. Let us compare this case with the situation when g is a compact form of a simple Lie algebra. Similarly, the corresponding bracket $\{ , \}_R$ (after a change $R \mapsto \iota R$) is a Poisson one only on some orbits. Hence according to the results of [8], the Poisson-Lie bracket on the corresponding group can be reduced to give a Poisson bracket on all orbits $\mathcal{O} \subset g^*$. In [7] it is shown that the reduced bracket and the Kirillov one $\{ , \}_{Kir}$ form a Poisson pair iff \mathcal{O} is a hermitian symmetric space. In fact, on these orbits the bracket $\{ , \}_R$ is a Poisson one too and the reduced bracket is a linear combination of the brackets $\{ , \}_R$ and $\{ , \}_{Kir}$ forming the Poisson pair. This fact for $g = \mathfrak{su}(2)$ and M a sphere in g^* was noted in [13].

PROPOSITION 2.2. – *Let $\{ , \}$ be another Poisson bracket on M and let $\rho(g) \subset \text{Der}(C^{\infty}(M), \{ , \})$. Then the brackets $\{ , \}$ and $\{ , \}_R$ are compatible, i. e. all linear combinations $\{ , \}_{a,b} = a \{ , \} + b \{ , \}_R$ are Poisson brackets.*

Proof. – We need only to check that

$$\{\{f_1, f_2\}, f_3\}_R + \{\{f_1, f_2\}_R, f_3\} + \text{cyclic} = 0$$

where “+ cyclic” means summing over all cyclic permutations of the f_i . For the sake of convenience, we fix a basis $\{X_i\}$ of g and write $R = r^{ij} X_i \otimes X_j$. We also write $\rho(X)f$ as Xf , for $X \in g$ and $f \in C^\infty(M)$, and omit the product μ on $C^\infty(M)$ when there is no danger of confusion. Thus $\{f_1, f_2\}_R = r^{ij} (X_i f_1)(X_j f_2)$. Then

$$\begin{aligned} & \{\{f_1, f_2\}, f_3\}_R + \{\{f_1, f_2\}_R, f_3\} + \text{cyclic} \\ &= r^{ij} ((X_i \{f_1, f_2\})(X_j f_3) + \{(X_i f_1)(X_j f_2), f_3\}) + \text{cyclic} \\ &= r^{ij} ((X_i \{f_1, f_2\})(X_j f_3) + \{(X_i f_1), f_3\}(X_j f_2) + \{(X_j f_2, f_3)(X_i f_1)\}) + \text{cyclic} \\ &= r^{ij} ((X_i \{f_1, f_2\})(X_j f_3) + \{(X_i f_1), f_3\}(X_j f_2) + \{(X_j f_1, f_2)(X_i f_3)\}) + \text{cyclic} \\ &= r^{ij} ((X_i \{f_1, f_2\})(X_j f_3) + \{(X_i f_2), f_1\}(X_j f_3) - \{(X_i f_1, f_2)(X_j f_3)\}) + \text{cyclic} \end{aligned}$$

using that $\{ , \}$ is a derivation, rotation under the cyclic sum and antisymmetry of R in the form $r^{ij} = -r^{ji}$. The final expression vanishes since X_i is (by hypothesis on ρ) an element of $\text{Der}(C^\infty(M), \{ , \})$.

3. QUANTIZATION

First of all we recall that deformation quantization of a Poisson bracket $\{ , \} : C^\infty(M)^{\otimes 2} \rightarrow C^\infty(M)$ means an associative multiplication

$$f_1 \otimes f_2 \mapsto f_1 \star_h f_2, \quad f_1, f_2 \in C^\infty(M)$$

satisfying the conditions

$$f_1 \star_h f_2 = f_1 f_2 \text{ mod } \hbar, \quad f_1 \star_h f_2 - f_2 \star_h f_1 = \hbar \{f_1, f_2\} \text{ mod } \hbar^2.$$

Sometimes one uses a more general definition of quantization assuming that there is a *flat* deformation of an initial commutative algebra A_0 into a set of associative algebras A_\hbar equipped with a multiplication

$$f_1(\hbar) \otimes f_2(\hbar) \rightarrow f_1(\hbar) \star_\hbar f_2(\hbar), \quad f_1(\hbar), f_2(\hbar) \in A_\hbar$$

and satisfying similar conditions.

In this section we construct such a deformation quantization of the R-matrix bracket $\{ , \}_R$ (for $R = R_{\text{can}}$) using some of Drinfeld’s results in [3]. Generalizing the construction in [6], the deformed multiplication is introduced by means of an element $F \in U(g)^{\otimes 2} \llbracket \hbar \rrbracket$ quantizing R . In the case when R is a “classical” R-matrix (obeys the classical Yang-Baxter equations), this F can be constructed to satisfy the so-called “cocycle condition”, with the result that the deformed multiplication is associative. If R is a modified R-matrix, as in our case, this is no longer so. Instead, the “coboundary” of F is a non-trivial element $\Phi \in U(g)^{\otimes 3} \llbracket \hbar \rrbracket$. This Φ

corresponds in Drinfeld’s work to a break-down of associativity in a deformed category of representations $U(g)$ [3]. In our work, we find that it disappears for $R=R_{can}$ in some situations and is absent from its quantization. Therefore the result of the quantization is an associative algebra in this case too. Our approach can be applied to quantize $\{ , \}_R$ for any R for which Drinfeld’s construction in [3] can be carried out.

We recall now some of Drinfeld’s results from [3], in a form convenient for our purposes. Let R_{can} be defined as in (3). Then there exists

$$F_{\hbar} \in U(g)^{\otimes 2} \llbracket \hbar \rrbracket$$

such that

- 1) $F_{\hbar} - F_{\hbar}^{21} = (1/2)\hbar R_{can} \text{ mod } \hbar^2$ where F_{\hbar}^{21} is the image of F_{\hbar} under the transposition $\sigma(X \otimes Y) = Y \otimes X$
- 2) $F_{\hbar} = 1 \text{ mod } \hbar$
- 3) $(\varepsilon \otimes \text{id})F = (\text{id} \otimes \varepsilon)F = 1$ where $\varepsilon: U(g) \llbracket \hbar \rrbracket \rightarrow k \llbracket \hbar \rrbracket$ is the usual counit on $U(g)$
- 4) The “associativity defect”

$$\Phi = (\Delta^{23} F) F^{23} (F^{-1})^{12} (\Delta^{12} F^{-1}) \in U(g) \llbracket \hbar \rrbracket^{\otimes 3}$$

(where Δ is obtained from the usual coproduct on $U(g)$) is of the form $\Phi = \exp P(\hbar t^{12}, \hbar t^{23})$. Here $t \in g \otimes g$ is the split Casimir element corresponding to the inverse of the Killing form and $t^{12} = t \otimes \text{id}$, $t^{23} = \text{id} \otimes t$ and P is a Lie (*i.e.* commutator) formal power-series with coefficients in k .

The last observation means that Φ can be expanded in the form

$$\Phi = 1 + \sum_i P_i(\hbar t^{12}, \hbar t^{23}) \llbracket \hbar t^{12}, \hbar t^{23} \rrbracket Q_i(\hbar t^{12}, \hbar t^{23}) \tag{5}$$

where P_i and Q_i are polynomials of $\hbar t^{12}$ and $\hbar t^{23}$. When R is a classical R -matrix there is a series F_{\hbar} obeying the same conditions but with $\Phi = 1$ (*see* [2]). The condition 4) with $\Phi = 1$ has been called the “cocycle condition” for F . In our case, although Φ differs from 1, let us note that it is manifestly ad-invariant since t is.

Let $\varphi = \llbracket R, R \rrbracket$ and define $\varphi_{\rho}, \Phi_{\rho}: C^{\infty}(M)^{\otimes 3} \rightarrow C^{\infty}(M)^{\otimes 3}$ by

$$\begin{aligned} \varphi_{\rho}(f_1 \otimes f_2 \otimes f_3) &= \langle (\rho \otimes \rho \otimes \rho)(\varphi), df_1 \otimes df_2 \otimes df_3 \rangle \\ \Phi_{\rho}(f_1 \otimes f_2 \otimes f_3) &= (\rho \otimes \rho \otimes \rho)(\Phi)(f_1 \otimes f_2 \otimes f_3). \end{aligned}$$

Here and further on we denote by ρ the extension of the usual action of $\rho: g \rightarrow \text{Der}(C^{\infty}(M))$ to an action of $U(g)$ on $C^{\infty}(M)$.

Denote by $\mu: C^{\infty}(M)^{\otimes 2} \rightarrow C^{\infty}(M)$ the usual commutative multiplication and put $\mu^{123} = \mu\mu^{12} = \mu\mu^{23}: C^{\infty}(M)^{\otimes 3} \rightarrow C^{\infty}(M)$. Our Proposition 2.1 means that $\mu^{123}\varphi_{\rho} = 0$. We now prove under some conditions, a quantum analogue of this statement.

Let $V_{\alpha} \subset C^{\infty}(M)$ be an irreducible representation of the Lie algebra g (where α is a highest weight of the representation). Suppose that the linear

span of all the V_α is dense in $C^\infty(M)$ and that

$$f_1 f_2 = \mu(f_1 \otimes f_2) \in V_{\alpha+\beta}, \quad f_1 \in V_\alpha, f_2 \in V_\beta.$$

PROPOSITION 3.1. — *In this setting we have $\mu^{123} \Phi_p = \mu^{123}$.*

Proof. — Let C be the Casimir element of g . Then

$$t = (\Delta C - C \otimes \text{id} - \text{id} \otimes C)/2$$

and one can see that in the setting above we have

$$\mu t(f_1 \otimes f_2) = c(\alpha, \beta) \mu(f_1 \otimes f_2)$$

if $f_1 \in V_\alpha$ and $f_2 \in V_\beta$. This is because all irreducible g -submodules of $V_\alpha \otimes V_\beta$ give zero after the multiplication μ , apart from $V_{\alpha+\beta}$. Proceeding, we therefore have

$$\begin{aligned} \mu^{123} t^{12}(f_1 \otimes f_2 \otimes f_3) &= c(\alpha, \beta) \mu^{123}(f_1 \otimes f_2 \otimes f_3), \\ \mu^{123} t^{23}(f_1 \otimes f_2 \otimes f_3) &= c(\beta, \gamma) \mu^{123}(f_1 \otimes f_2 \otimes f_3) \end{aligned}$$

if $f_1 \in V_\alpha, f_2 \in V_\beta, f_3 \in V_\gamma$. Take now three such functions f_1, f_2, f_3 and apply the operator $\mu^{123} \Phi_p$ to them:

$$\begin{aligned} \mu^{123} \Phi_p(f_1 \otimes f_2 \otimes f_3) &= \\ (\mu^{123} + \sum_i \mu^{123} P_i(\hbar t^{12}, \hbar t^{23}) [\hbar t^{12}, \hbar t^{23}] Q_i(\hbar t^{12}, \hbar t^{23})) &(f_1 \otimes f_2 \otimes f_3) \\ = (\mu^{123} + \sum_i \mu^{123} P_i(\hbar c_{12}, \hbar c_{23}) [\hbar c_{12}, \hbar c_{23}] Q_i(\hbar c_{12}, \hbar c_{23})) & \\ (f_1 \otimes f_2 \otimes f_3) &= \mu^{123}(f_1 \otimes f_2 \otimes f_3) \end{aligned}$$

where $c_{12} = c(\alpha, \beta), c_{23} = c(\beta, \gamma)$. This completes the proof.

We leave the reader to check that the setting above is satisfied for $M = \mathcal{O}$ where $\mathcal{O} \subset V$ is the orbit of a highest weight vector in a vector space V .

Proposition 3.1 means that we can proceed with the deformation quantization by means of F_\hbar even though F_\hbar is not a cocycle. Thus for all manifolds M satisfying the conditions of Proposition 2.1 we deform the usual multiplication μ to μ_F defined by

$$\mu_F(f_1 \otimes f_2) = \mu(\rho \otimes \rho)(F)(f_1 \otimes f_2).$$

The identity element is not deformed.

COROLLARY 3.2. — *When M obeys the conditions in Proposition 3.1, then $\star_\hbar = \mu_F$ is associative and*

$$\mu_F(f_1 \otimes f_2) - \mu_F(f_2 \otimes f_1) = (1/2) \hbar \{f_1, f_2\}_R \text{ mod } \hbar^2$$

where $R = R_{\text{can}}$. We denote this deformation quantization of $C^\infty(M)$ by $C^\infty(M, \mu_F)$.

Proof. — We verify associativity. For brevity, we write simply F for $(\rho \otimes \rho)(F_\hbar): C^\infty(M)^{\otimes 2} \rightarrow C^\infty(M)^{\otimes 2}$. Then $\mu_F \mu_F^{12} = \mu \ F \ \mu^{12} \ F^{12} = \mu \mu^{12}$

$(\Delta^{12} F) F^{12} = \mu \mu^{12} \Phi_\rho(\Delta^{12} F) F^{12} = \mu \mu^{12} (\Delta^{23} F) F^{23} = \mu F \mu^{23} F^{23} = \mu_F \mu_F^{23}$. We used the properties of F_\hbar summarized above, and Proposition 3.1. The property 1) of F_\hbar evidently also implies that we have quantized $\{ , \}_R$ as stated.

We now discuss the problem of quantizing all the brackets $\{ , \}_{a,b}$. The strategy used in the unbraided case [6] was to begin by deformation quantizing the bracket $\{ , \}$. Suppose that this is done as an associative algebra $C^\infty(M, \star_{\hbar_1})$ with g -invariant multiplication \star_{\hbar_1} . We can then proceed to deform this in the same way as above using the series F_{\hbar_2} . In the setting of [6] this F_{\hbar_2} satisfies the cocycle condition and we obtain the two-parameter family of associative multiplications

$$f_1 \star_{\hbar_1, \hbar_2} f_2 = \star_{\hbar_1} F_{\hbar_2}(f_1 \otimes f_2)$$

with

$$f_1 \star_{a\hbar, b\hbar} f_2 - f_2 \star_{a\hbar, b\hbar} f_1 = \hbar \{f_1, f_2\}_{a,b} \text{ mod } \hbar^2.$$

Thus the algebra $C^\infty(M, \mu_{a,b})$ where $\mu_{a,b} = \star_{a\hbar, b\hbar}$ is the quantization of $\{ , \}_{a,b}$.

The same approach can be used in our present braided case. However, in general the deformed multiplication is not associative because the analogue of Proposition 3.1 in this case is not true.

4. PROPERTIES OF THE QUANTIZED ALGEBRAS

First of all, we note that the data F_\hbar in the last section is used by Drinfeld in [3] in a different way, namely to define a quasitriangular Hopf algebra (quantum group) $H = (U(g), \Delta_F = F^{-1} \Delta(\cdot) F, \mathcal{R} = (F^{-1})^{21} e^{\hbar/2} F)$. It is isomorphic to the famous quantum group $U_q(g)$, see [3].

This quantum group H plays for us the role of a symmetry group for the quantum algebra $C^\infty(M, \mu_F)$ in the sense

$$h \triangleright \mu_F(f_1 \otimes f_2) = \mu_F(h \triangleright (f_1 \otimes f_2)). \tag{6}$$

Here $f_1, f_2 \in C^\infty(M, \mu_F)$, $h \in H$ and \triangleright denotes the action $h \triangleright f_i = \rho(h) f_i$ extended to tensor powers of $C^\infty(M, \mu_F)$ in the usual way by $h \triangleright (f_1 \otimes f_2) = \sum_i h_i \triangleright f_1 \otimes h^i \triangleright f_2$ for $\Delta_F h = \sum_i h_i \otimes h^i$. The proof is evident.

This H -invariance of μ_F corresponds in the undeformed case to g -invariance of the initial product μ .

Consider now the braided monoidal category ${}_H\mathcal{M}$ of all H -modules. The observation (6) means that the map μ_F is a morphism in ${}_H\mathcal{M}$. Functoriality of the braiding Ψ in the category then implies such identifies as

$$S \mu_F^{12} = \mu_F^{23} S^{23} S^{12}$$

Here $S = \Psi_{C^\infty(M, \mu_F), C^\infty(M, \mu_F)} = \sigma \mathcal{R}_\rho$ is the braiding applied to the object $C^\infty(M, \mu_F)^{\otimes 2}$. See [10] for an introduction to braided categories in the context of quantum groups.

Other g -invariant constructions on $C^\infty(M)$ can likewise be deformed in a H -invariant way by means of F , to morphisms in the category ${}_H\mathcal{M}$. For example, let $f \mapsto \int f d\Omega$ be a g -invariant integration on M , and

$f_1 \otimes f_2 \mapsto \langle f_1, f_2 \rangle = \int f_1 f_2 d\Omega$ the corresponding pairing. Putting $\langle f_1, f_2 \rangle_F = \langle \cdot, \cdot \rangle_F$ we obtain a morphism in ${}_H\mathcal{M}$.

We consider in the same way a new transposition obtained by deforming the usual one by F ,

$$\tilde{S} = \tilde{S}_F = F^{-1} \sigma F : C^\infty(M)^{\otimes 2} \rightarrow C^\infty(M)^{\otimes 2}.$$

In our braided situation this \tilde{S} does not obey the quantum Yang-Baxter equations. But it is involutive ($\tilde{S}^2 = \text{id}$) and plays an important role in what follows. It is a morphism in the category ${}_H\mathcal{M}$. It is evident that the algebra $C^\infty(M, \mu_F)$ is \tilde{S} -commutative in the sense

$$\mu_F \tilde{S} = \mu_F. \tag{7}$$

In the involutive setting in [6] $\tilde{S} = S$ and μ_F was S -commutative, *i.e.* commutative in the (symmetric) monoidal category.

Note that in [9] was introduced a notion braided-commutative algebra of functions on a “braided group”. There, like here, the braided-commutativity was not given simply by the braiding but by a variant of it, denoted Ψ' [9].

In a similar way to that above, we can deform the vector fields $\rho(X)$ for $X \in \mathfrak{g}$. We introduce this deformed representation ρ_F by $\text{ev}(\rho_F \otimes \text{id})(X \otimes f) = \text{ev}(\rho \otimes \text{id})F(X \otimes f)$. Writing $F = \sum F^{(1)} \otimes F^{(2)}$ say, $F^{(1)}$ acts on X by the adjoint action $F^{(1)}(X) = \sum F^{(1)}_i X_s F^{(1)}_{(2)}$. Here s is the antipode of $U(\mathfrak{g})$ and $\Delta F^{(1)} = \sum F^{(1)}_i \otimes F^{(1)}_{(2)}$. ev is the evaluation map and $F^{(2)}$ acts directly on f (also by ρ). This is a general feature of deformations of representations of \mathfrak{g} or $U(\mathfrak{g})$ to ones of H . In our case we apply the theory to $\rho : \mathfrak{g} \rightarrow \text{Der}(C^\infty(M))$. We denote the deformed vector field by $\rho_F(X)$, so

$$\rho_F(X)f = \rho\left(\sum (F^{(1)}(X)) F^{(2)}\right)f. \tag{8}$$

From this formula, it is clear that the “deformed vector fields” are not vector fields in the usual sense. Instead we see that the infinitesimal transformation generated by X has been deformed to a finite transformation by the action of the element $\sum (F^{(1)}(X)) F^{(2)}$. Thus the “deformed vector fields” are like the finite difference operators arising in the theory of q -differentiation and q -deformed special functions.

Let us note that in the involutive situation studied in [6] we have $\tilde{S}=S$ and all the above constructions live naturally in a symmetric monoidal category generated by S , the deformed vector fields are S -derivations and g_F is an S -Lie algebra in a straightforward sense. *See* [6] for details.

5. EXAMPLES: THE QUANTUM PLANE AND QUANTUM SPHERE

To conclude the paper we will consider the “quantum plane” and a non-compact version of the “quantum sphere” in the framework of our approach to quantization.

Consider \mathbb{R}^2 with the co-ordinate functions q, p . Let

$$\{f, g\} = \frac{\partial f}{\partial q} \frac{\partial g}{\partial p} - \frac{\partial f}{\partial p} \frac{\partial g}{\partial q},$$

the familiar Poisson bracket. Denote by X_f the Hamiltonian vector field corresponding to $f \in C^\infty(\mathbb{R}^2)$ by $X_f = \{f, \cdot\}$. The Hamiltonian vector fields $H = X_{pq}$, $X = X_{-p^2/2}$, $Y = X_{q^2/2}$ form the Lie algebra \mathfrak{sl}_2 ,

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

So $g = \mathfrak{sl}_2$ acts on \mathbb{R}^2 , which consists of two orbits $\{0\}$ and $\mathbb{R}^2 - \{0\}$. Let $R = R_{\text{can}} = X \otimes Y - Y \otimes X$ be the unique (up to isomorphism) modified R -matrix on \mathfrak{sl}_2 . The R -matrix bracket is

$$\{f, g\}_R = (Xf)(Yg) - (Yf)(Xg) = \left\{ -\frac{p^2}{2}, f \right\} \left\{ \frac{q^2}{2}, g \right\} - \left\{ \frac{q^2}{2}, f \right\} \left\{ -\frac{p^2}{2}, g \right\}.$$

After quantization of $\{ \cdot, \cdot \}_R$ we obtain the algebra $C^\infty(\mathbb{R}^2, \mu_F)$. We compute the relations for q, p in this algebra. We remark that the operator $\tilde{S} = F_\hbar^{-1} \sigma F_\hbar$ determining these relations [according to (7)], has the same eigenspaces as the braiding operator $S = F_\hbar^{-1} \sigma e^{\hbar u/2} F_\hbar$. The image of $S \in U(\mathfrak{g})^{\otimes 2}[[\hbar]]$ in the two-dimensional representation is well known, see for example [5]. Our representation on $C^\infty(\mathbb{R}^2)$ is of course infinite-dimensional, but it is determined by the action on the generators q, p , which form the two-dimensional one.

From the explicit form for this standard braiding, one obtains the relations

$$qp - \alpha pq = 0 \tag{9}$$

for some $\alpha \in k$. This algebra is the well-known “quantum plane”. It is isomorphic to the subalgebra $C^\infty(\mathbb{R}^2, \mu_F)$ of polynomials in the generators.

Next we consider the quantization of all the $\{ \cdot, \cdot \}_{a, b}$. First we consider the quantization of the bracket $\{ \cdot, \cdot \}$. The multiplication in the quantized

algebra is viewed as a deformation of the usual one on $C^\infty(\mathbb{R}^2)$ and is defined by the formula

$$f_1 \star_h f_2 = \mu e^{(1/2)h((\partial/\partial q) \otimes (\partial/\partial p) - (\partial/\partial p) \otimes (\partial/\partial q))} (f_1 \otimes f_2)$$

where μ is the usual product.

One can also show the multiplication \star_{h_1, h_2} to be associative in this case. The corresponding algebra is generated by $p, q, 1$ with relations of the form

$$pq - \alpha qp = \beta \tag{10}$$

where β is a second parameter. The reader can easily check that this two-parameter deformation is flat.

For the sake of comparison, we recall also the result [6] for the classical R-matrix

$$R = H \otimes X - X \otimes H$$

for $g = \mathfrak{sl}_2$. The result of the quantization of $\{ , \}_{a, b}$ is the algebra with generators p, q satisfying the relation

$$qp - pq - \alpha p^2 = \beta$$

in contrast to (10).

Consider now the Lie algebra $g = \mathfrak{sl}_2$ and let $M = \mathcal{O} \subset g^*$ be the orbit of a highest weight vector, *i. e.* the cone $2xy = h^2$ (without the point 0). For the sake of convenience we fix here a base $\{X, Y, H\} \in g$ with the relations

$$[H, X] = X, \quad [H, Y] = -Y, \quad [X, Y] = H$$

and put $x = \langle X, \xi \rangle, y = \langle Y, \xi \rangle, h = \langle H, \xi \rangle$ for $\xi \in g^*$. Then the Kirillov bracket between the co-ordinate functions is

$$\{h, x\} = x, \quad \{h, y\} = -y, \quad \{x, y\} = h$$

while the R-matrix bracket for $R = X \otimes Y - Y \otimes X$ is equal to $\{f, g\}_R = \{f, g\}h$ for $f, g \in C^\infty(g^*)$.

We consider the restriction of these two brackets on M and quantize the bracket $\{ , \}_R$ into the algebra $C^\infty(M, \mu_F)$. We would like to obtain all relations between the generators of this algebra.

To do this we use the result of [4] where all quadratic relations compatible with the action of the quantum group $H = (U_q(\mathfrak{sl}_2), \Delta_F, \mathcal{R})$ (*see* Section 4) are computed. These relations are

$$-q^{-1} \varphi_+ \varphi_- - q \varphi_- \varphi_+ + \varphi_z \varphi_z = \varepsilon \tag{11}$$

(the ‘‘quantum Casimir relation’’) and

$$\left. \begin{aligned} \varphi_+ \varphi_z &= q^2 \varphi_z \varphi_+ + \mu \varphi_+, & \varphi_z \varphi_- &= q^2 \varphi_- \varphi_z + \mu \varphi_-, \\ \varphi_+ \varphi_- &= \varphi_- \varphi_+ + (q - q^{-1}) \varphi_z \varphi_z + q^{-1} \mu \varphi_z. \end{aligned} \right\} \tag{12}$$

We use here the notation of [4]. It is clear that the case $\varepsilon = 0$ and $\mu = 0$ corresponds to the case of the algebra $C^\infty(M, \mu_F)$.

Thus we have a set of associative algebras A_q with one parameter q . Consider now a set of algebras $A_{q,\mu}$ obtained as quotient algebras of $\varphi_+, \varphi_-, \varphi_z$ with the relations (12). It is easy to show that this two-parameter deformation is flat.

To conclude the paper we would like to note that the case $\varepsilon \neq 0$ is not, however, embraced by the conditions of Proposition 3.1. We will examine this case in another paper where we develop a similar but more complicated approach to quantization on symmetric spaces.

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