ANNALES DE L'I. H. P., SECTION A

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Annales de l'I. H. P., section A, tome 58, nº 2 (1993), p. 173-187
<http://www.numdam.org/item?id=AIHPA_1993__58_2_173_0>

Ann. Inst. Henri Poincaré,

Vol. 58, n° 2, 1993, p. 173-187. *Physique théorique*

Pure Jauch-Piron states on von Neumann algebras

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ABSTRACT. – We study Jauch-Piron states and two-valued measures on von Neumann algebra. We prove as the main result that, under some set-theoretical assumption, a pure state of a von Neumann algebra $\mathscr{A}$ not containing a central abelian portion is Jauch-Piron if and only if it is σ-additive. Moreover, we show that this result holds for type I factor indenpendently on the set-theoretical axiomatics. As a consequence we obtain a lucid characterization of pure Jauch-Piron states on von Neumann algebras acting on a Hilbert space with real-nonmeasurable dimension (this can be viewed as a generalization of the paper [2]). We also characterize the von Neumann algebras whose logic of projections is Jauch-Piron. Finally, we prove that every two-valued measure on the projection logic of $\mathscr{A}$, where $\mathscr{A}$ contains no type I_2 central portion, has to be concentrated at an abelian direct summand of $\mathscr{A}$.

RÉSUMÉ. – Nous étudions les états de Jauch-Piron et les mesures bi-valuées sur les algèbres de von Neumann. Notre résultat principal est que sous des hypothèses ensemblistes, un état pur de l'algèbre de von Neumann et qui ne contient pas de partie centrale abélienne est un état de Jauch-Piron si et seulement si il est σ-additif. De plus nous prouvons que ce résultat est vrai pour les facteurs de type I indépendamment de l'axiomatique ensembliste. Nous obtenons donc une caractérisation lucide des états purs de Jauch-Piron sur les algèbres de von Neumann qui agissent sur un espace de Hilbert de dimension réelle non mesurable (ceci peut être vu comme une extension de l'article [2]). Nous caractérisons aussi les algèbres

de von Neumann dont la logique des projecteurs est Jauch-Piron. Finalement nous prouvons que toute mesure bivaluée sur la logique des projecteurs de $\mathscr{A}$, où et ne contient pas de partie centrale de type I_2, est concentrée sur un somment direct abélien de $\mathscr{A}$.

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1. INTRODUCTION

In the W*-algebraic quantum mechanics formalism, a "state" of a given physical system is usually represented by a finitely additive probability measure on the quantum logic of all projections in a von Neumann algebra, $\mathscr{A}$. This attitude generalizes both the classical approach, where $\mathscr{A}$ is a commutative algebra, and the Hilbert space formalism, where $\mathscr{A}$ is an algebra of all bounded operators acting on a Hilbert space. As physicists sometimes argue (*see* e. g. [10], [11]) the measure corresponding to a state of a physical system should have the so-called Jauch-Piron property. This property has been already discussed by many authors (*see* e. g. [4], [10], [11], [12], [15], [16]).

The purpose of this paper is to contribute to this discussion by showing that a pure state of a type I factor is Jauch-Piron if and only if it is σ-additive. Moreover, it is proved that if the contimuum is of the real-nomeasurable cardinality, then the same characterization is valid for pure Jauch-Piron state of arbitrary von Neumann algebra not containing central abelian portion. There are a few consequences of this result. For instance, A. Amann proved in [2] that every Jauch-Piron state of a σ-finite von Neumann algebra without central abelian portion is normal and concentrated at central summand which is a type I factor. Our description of pure Jauch-Piron states allows (under physically acceptable assumption of the nonexistence of real-measurable cardinals) to extend this result to arbitrary von Neumann algebras. In the physical interpretation it means that the only physically meaningful representation of an individual pure state of a physical system reduces either to the classical Kolmogorov probability space, (Ω, Σ, μ), or to a Hilbert space quantum probability model, $(\mathscr{H}, \mathscr{B}(\mathscr{H}), \omega_x)$, where $\mathscr{B}(\mathscr{H})$ is an algebra of all bounded operators on a Hilbert space $\mathscr{H}$ and ω_x is a vector state of $\mathscr{B}(\mathscr{H})$. This result also seems to be of certain interest for the reason that it advocates the central position of the Hilbert-space logic within algebraic quantum mechanic approach (*see* [20]).

As a further corollary, we obtain a lucid characterization of von Neumann algebras whose projection logics have Jauch-Piron property. We show that these can be obtained from commutative algebras and finite-dimensional matrix algebras by forming finite direct sums.

Applying results on pure Jauch-Piron states, we finally examine two-valued measures on von Neumann algebras. The measures of this type (called also weak dispersion free states) become interesting in the problem of hidden variables of quantum system (*see* e.g. [9], [12], [20]). We prove that every two-valued measure on a von Neumann algebra without direct summand of type I_2 has to be concentrated at central abelian portion. This generalizes hitherto known results [1], [2], [17] and shows that, within the W*-algebraic formalism, only the classical model admits dispersion free states.

2. PRELIMINARIES

Let us first recall basic notions and agree on the notation. Throughout the paper, let $\mathscr{H}$ (resp., $\mathscr{H}_n$, $n \in \mathrm{N}$) be a complex Hilbert space (resp. n-dimensional complex Hilbert space). Let $\mathscr{B}(\mathscr{H})$ denote the algebra of all bounded operators acting on $\mathscr{H}$. In what follows, the symbol $\mathscr{A}$ will be reserved for a von Neumann algebra acting on $\mathscr{H}$. Let I be a unit of $\mathscr{A}$. For a commutant of a set $\mathscr{B} \subset \mathscr{B}(\mathscr{H})$ let us use the notation $\mathscr{B}'(=\{A \in \mathscr{B}(\mathscr{H}) \mid AB=BA \text{ for every } B \in \mathscr{B}\})$. The double commutant $(\mathscr{B}')'$ will be denoted by $\mathscr{B}''$. Let $\mathscr{Z}(\mathscr{A})$ stand for the center of $\mathscr{A}$ $(\mathscr{Z}(\mathscr{A})=\mathscr{A} \cap \mathscr{A}')$. If P is a projection of $\mathscr{A}$, then the symbol $P\mathscr{A}P$ means a von Neumann algebra $\{PAP \mid A \in \mathscr{A}\}$. The set of all projections in $\mathscr{A}$ will be denoted by $\mathscr{P}(\mathscr{A})$. It is well known that the set $\mathscr{P}(\mathscr{A})$ endowed with the ordering $\leqq$, where $P \leqq Q$ if $PQ=QP=P$ $(P, Q \in \mathscr{P}(\mathscr{A}))$, is a complete lattice [20]. The supremum and infimum of two elements P and Q of $\mathscr{P}(\mathscr{A})$ is denoted by $P \vee Q$ and $P \wedge Q$, respectively. Moreover, if we consider the orthocomplementation $P \to P^{\perp}$, where $P^{\perp}=I-P$, then $(\mathscr{P}(\mathscr{A}), \leqq, \perp)$ forms a quantum logic (*see* [20]).

By *a probability measure* (in short measure) μ on $\mathscr{P}(\mathscr{A})$ we mean a mapping $\mu: \mathscr{P}(\mathscr{A}) \to \langle 0, 1 \rangle$ fulfilling the following two conditions: (i) $\mu(I)=1$. (ii) $\mu(P+Q)=\mu(P)+\mu(Q)$, whenever P and Q are orthogonal projections (*i.e.* $PQ=0$). A measure μ is said to be *completely additive* (resp., *σ-additive*) if $\mu(\sum_{\alpha \in I} P_\alpha)=\sum_{\alpha \in I} \mu(P_\alpha)$, whenever $(P_\alpha)_{\alpha \in I}$ is an arbitrary system (resp., a countable system) of mutually orthogonal projections. A projection P is called a *support* of a measure μ if $\mu(Q)=1$ if and only if $Q \geqq P$.

By a *state* of $\mathscr{A}$ we mean normalized positive functional on $\mathscr{A}$. Obviously, if φ is a state, then the restriction $\varphi/\mathscr{P}(\mathscr{A})$ is a measure on $\mathscr{P}(\mathscr{A})$. A state φ is called *σ-additive* or *completely additive*, if $\varphi/\mathscr{P}(\mathscr{A})$

has the respective property. Analogously, a projection P is a support of φ if it is a support of the measure $\varphi/\mathcal{P}(\mathcal{A})$. It is well known that a state is completely additive if and only if it is normal [13]. A state φ of $\mathcal{A}$ is said to be a *vector state* if there is a unit vector $x \in \mathcal{H}$ such that $\varphi(A) = \langle Ax, x \rangle$ for every $A \in \mathcal{A}$. In this case we denote φ by the symbol ω_x. We say that a state φ is *multiplicative* if $\varphi(AB) = \varphi(A)\varphi(B)$ for every A, $B \in \mathcal{A}$. Finally, a state φ of $\mathcal{A}$ [resp., a measure μ on $\mathcal{P}(\mathcal{A})$] is called *pure* if it cannot be written as a convex combination of distinct states of $\mathcal{A}$ [resp., measures on $\mathcal{P}(\mathcal{A})$].

As we have noted, every state of $\mathcal{A}$ induces a measure on $\mathcal{P}(\mathcal{A})$. On the other hand, the theorem of Gleason, Christensen and Yeadon (*see* [5], [14], [22]) asserts that every measure μ on $\mathcal{P}(\mathcal{A})$, where $\mathcal{A}$ has no direct summand of the type I_2, can be extended to a state of $\mathcal{A}$.

We also need to introduce the following set-theoretical notions. A cardinal Γ is said to b *real-measurable* if there exists σ-additive probability measure μ on the power set 2^Γ such that $\mu(\{\gamma\}) = 0$ for every $\gamma \in \Gamma$. If a cardinal Γ is not real-measurable, then it is called *real-nonmeasurable*. The existence of real-measurable cardinals (within the ordinary set-theoretical axiomatics) is an open problem of the set theory. Nevertheless, it was proved that if Γ is real-nonmesurable, then so is its successor Γ^+ (*see* e. g. [6], [19]). So, the continuum $2^{\aleph_0}$ is real-nonmeasurable if we admit the contimuum hypothesis.

3. JAUCH-PIRON STATES

In this section we present basic facts on Jauch-Piron states. A measure μ on $\mathcal{P}(\mathcal{A})$ is said to be *Jauch-Piron*, if $\mu(P \vee Q) = 0$, whenever $\mu(P) = \mu(Q) = 0$ $(P, Q \in \mathcal{P}(\mathcal{A}))$ [or, equivalently, if the equality $\mu(P) = \mu(Q) = 1$ implies that $\mu(P \wedge Q) = 1$]. An algebra $\mathcal{A}$ is called *Jauch-Piron* if every measure on $\mathcal{P}(\mathcal{A})$ is Jauch-Piron. [In other words, $\mathcal{A}$ is a Jauch-Piron algebra if the lattice $\mathcal{P}(\mathcal{A})$ is a Jauch-Piron logic—*see* [4], [15], [16] for a study of Jauch-Piron logics.] We say that a state φ of $\mathcal{A}$ is *Jauch-Piron*, if $\varphi/\mathcal{P}(\mathcal{A})$ is a Jauch-Piron measure. Let us now illustrate the above notions by examples.

3.1. EXAMPLES. – *Every commutative von Neumann algebra $\mathcal{A}$ is Jauch-Piron.*

Proof. – Let μ be a measure on $\mathcal{P}(\mathcal{A})$. If $\mu(P) = \mu(Q) = 0$ for P, $Q \in \mathcal{P}(\mathcal{A})$, then $\mu(P \vee Q) = \mu(P + Q \wedge P^\perp) = 0$. Thus, μ is a Jauch-Piron measure.

In accordance with the standard functional analysis notation, write $l^\infty = \{ (x_n) \mid x_n \in \mathrm{C} \text{ for every } n \in \mathrm{N} \text{ and } \sup_{n \in \mathrm{N}} |x_n| < \infty \}$ and

$$c_0 = \{ (x_n) \in l^\infty \mid \lim_{n \to \infty} x_n = 0 \}.$$

3.2. EXAMPLE. – *Consider the tensor product* $\mathscr{A} = l^\infty \otimes \mathscr{B}(\mathscr{H}_n)$, *where* $n \geqq 2$. *Let* φ_1 *be a state of* l^∞ *such that* $\varphi_1/c_0 = 0$ *and let* φ_2 *be a pure state of* $\mathscr{B}(\mathscr{H}_n)$. *Then the product state* $\varphi = \varphi_1 \otimes \varphi_2$ *of* $\mathscr{A}$ *is not Jauch-Piron.*

Proof. – The support R of φ_2 is an atom (*i.e.* one-dimensional projection) in $\mathscr{P}(\mathscr{B}(\mathscr{H}_n))$. Suppose that R is representable by the following $n \times n$-matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \cdot^{\cdot^{\cdot}} & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Put $\mathrm{P} = \mathrm{I} \otimes \mathrm{R}$, where I is the identity in l^∞. If we identify $\mathscr{A}$ with a direct sum $\sum_{j \in \mathrm{N}} \oplus \mathscr{B}(\mathscr{H}_n)$ (by means of the mapping $(a_j) \otimes \mathrm{A} \to \sum_{j \in \mathrm{N}} \oplus a_j \mathrm{A}$ – *see* [18], Cap. IV.7]), we can set $\mathrm{Q} = \sum_{j \in \mathrm{N}} \oplus \mathrm{Q}_j$, where each Q_j is representable by the following $n \times n$-matrix

$$\begin{pmatrix} \cos^2 1/j & 1/2 \sin 2/j & 0 & \dots & 0 \\ 1/2 \sin 2/j & \sin^2 1/j & 0 & \dots & 0 \\ \vdots & \vdots & & \cdot^{\cdot^{\cdot}} & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

(Q_j'*s* are one-dimensional projections converging to P.)

Obviously, $\mathrm{P} = \sum_{j \in \mathrm{N}} \oplus \mathrm{R}$ and $\mathrm{P} \wedge \mathrm{Q} = \sum_{j \in \mathrm{N}} \oplus (\mathrm{Q}_j \wedge \mathrm{R}) = 0$. Thus, $\varphi(\mathrm{P} \wedge \mathrm{Q}) = 0$. On the other hand,

$$\varphi(\mathrm{P}) = \varphi_1(\mathrm{I})\, \varphi_2(\mathrm{R}) = 1.$$

and

$$\varphi(\mathrm{Q}) = \varphi(\mathrm{PQP}) = \varphi((\cos^2 1/j) \otimes \mathrm{R}) = \varphi_1((\cos^2 1/j))\, \varphi_2(\mathrm{R}) = 1.$$

[we have used the fact that the sequence $(1 - \cos^2 1/j)$ belongs to c_0 and therefore $\varphi_1((\cos^2 1/j)) = 1$]. This concludes the proof. (It should be noted that for $n = 2$ this example has already appeared in [2].)

Let us notice that Example 3.2 introduces a typical class of states which are not Jauch-Piron. Indeed, let φ be a state of $\mathscr{A}$. Then φ is a Jauch-Piron state if and only if so is its restriction to every von Neumann subalgebra generated by two noncommutative projections. Such a subalgebra has to be $*$-isomorphic to a direct sum of a commutative algebra and a type I_2 algebra (*see* [18]). Since φ induces a Jauch-Piron state on the commutative part, we see that φ is Jauch-Piron if the same is true of its normalized restriction to any type I_2 subalgebra. According to the structural theory of type I algebras, every type I_2 von Neumann algebra is $*$-isomorphic to a tensor product $M \otimes \mathscr{B}(\mathscr{H}_2)$, where $\mathscr{M}$ is commutative. Obviously, this is the case of Example 3.2.

Jauch-Piron states and Jauch-Piron measures have been investigated by many authors and certain criteria of Jauch-Piron property have been found. In [4], the authors proved that every σ-additive state (and so every σ-additive measure on the algebra not containing type I_2 direct summand) has to be σ-Jauch Piron, thus in particular Jauch-Piron. It has been also observed (*see* e.g. [14]) that every state admitting a support is Jauch-Piron. On the other hand, A. Amann proved in [2] that every singular state on the σ-finite factor is not Jauch-Piron.

4. PURE JAUCH-PIRON STATES

In this section we characterize Jauch-Piron states which are pure. This allows us to generalize the Amann's description of pure Jauch-Piron states of σ-finite algebras [2].

Let us start with the following technical lemma. [Let us recall that two states φ and ψ are unitarily equivalent if there is a unitary mapping $\mathscr{U} \in \mathscr{A}$ such that $\psi(\mathscr{U} A \mathscr{U}^*) = \varphi(A)$ whenever $A \in \mathscr{A}$.]

4.1. LEMMA. – *Let φ be a pure Jauch-Piron state of $\mathscr{A}$. Then the following statements hold:*

(i) *If $P \in \mathscr{P}(\mathscr{A})$ and $\varphi(P) \neq 0$, then there is a pure Jauch-Piron state ψ of $\mathscr{A}$ such that $\psi(P) = 1$ and ψ is unitarily equivalent with φ.*

(ii) *Let (P_n) be an orthogonal sequence of projections of $\mathscr{A}$ such that $\varphi(\sum_{n \in N} P_n) \neq \sum_{n \in N} \varphi(P_n)$. Then there is a pure Jauch-Piron state ψ of $\mathscr{A}$ unitary equivalent with φ, such that $\psi(\sum_{n \in N} P_n) = 1$ and $\psi(P_n) = 0$ for all $n \in N$.*

Proof. – (i) Let $(\pi_\varphi, \mathscr{H}_\varphi, x_\varphi)$ be the GNS-representation engendered by φ. Then $\varphi = \omega_{x_\varphi} \circ \pi_\varphi$ and x_φ is a cyclic vector of π_φ. Since $\pi_\varphi(P) \neq 0$, we can take a unit vector, x, in the range of $\pi_\varphi(P)$ and define a state ψ by putting $\psi = \omega_x \circ \pi_\varphi$. Since the representation π_φ is irreducible, the vector x is cyclic for the representation π_φ. The essential uniqueness of the GNS-construction (*see* [13], 4.5.3, p. 279]) yields that π_φ is equivalent with the GNS-representation π_ψ associated with ψ. By virtue of [13]. Theorem 10.2.6, p. 730, the states φ and ψ are unitarily equivalent. It follows that ψ is again a pure Jauch-Piron state. Finally, $\psi(\mathrm{P})=\langle\pi_\varphi(\mathrm{P})x, x\rangle=1$ and the proof is complete.

(ii) We proceed similarly as above. If $\varphi(\sum_{n\in\mathrm{N}}\mathrm{P}_n)\neq\sum_{n\in\mathrm{N}}\varphi(\mathrm{P}_n)$, then $\pi_\varphi(\sum_{n\in\mathrm{N}}\mathrm{P}_n)\neq\sum_{n\in\mathrm{N}}\pi_\varphi(\mathrm{P}_n)$ and, obviously, $\sum_{n\in\mathrm{N}}\pi_\varphi(\mathrm{P}_n)\leqq\pi_\varphi(\sum_{n\in\mathrm{N}}\mathrm{P}_n)$. It enables us to find a unit vector x in the range of $\pi_\varphi(\sum_{n\in\mathrm{N}}\mathrm{P}_n)$ such that $(\sum_{n\in\mathrm{N}}\pi_\varphi(\mathrm{P}_n))x=0$. Let us again define a state ψ by the formula $\psi=\omega_x\circ\pi_\varphi$. Then $\psi(\sum_{n\in\mathrm{N}}\mathrm{P}_n)=1$ and $\psi(\mathrm{P}_n)=0$ for all $n\in\mathrm{N}$. By the same reasoning as in the above part (i) we see that ψ is a pure Jauch-Piron state unitarily equivalent with φ. This concludes the proof.

4.2. Proposition. – *Let us assume that the continuum is a real-nonmeasurable cardinal. Then every pure Jauch-Piron state φ of the algebra $\mathscr{A}=\mathscr{M}\otimes\mathscr{B}(\mathscr{H}_n)$, where $\mathscr{M}$ is a von Neumann algebra and $n\geqq 2$, has to be σ-additive.*

Proof. – Let us assume that φ is a pure Jauch-Piron state of $\mathscr{A}$. Since $\mathscr{H}_n$ is finitely dimensional, there is a one-dimensional projection $\mathrm{P}\in\mathscr{P}(\mathscr{B}(\mathscr{H}_n))$ such that $\varphi(\mathrm{I}\otimes\mathrm{P})\neq 0$. Applying Lemma 4.1 (i), one can find a pure Jauch-Piron state ψ of $\mathscr{A}$ which is unitarily equivalent with φ and $\psi(\mathrm{I}\otimes\mathrm{P})=1$. It is easy to verify that ψ is a product state. Making use of [13]. Prop. 1.3.2. p. 848, we have $\psi=\psi_1\otimes\psi_2$, where ψ_1 and ψ_2 is a pure state of $\mathscr{M}$ and $\mathscr{B}(\mathscr{H}_n)$, respectively. We shall prove that ψ_1 is a σ-additive state. Assume that it is not the case and derive a contradiction.

Knowing that ψ_1 is not σ-additive, we can find an orthogonal sequence $(\mathrm{P}_n)\subset\mathscr{M}$ such that $\psi(\sum_{n\in\mathrm{N}}\mathrm{P}_n\otimes\mathrm{P})\neq\sum_{n\in\mathrm{N}}\psi(\mathrm{P}_n\otimes\mathrm{P})$. Applying Lemma 4.1 (ii), we can construct another pure Jauch-Piron state ρ of $\mathscr{A}$, which is unitarily equivalent with ψ, and which fulfils $\rho(\sum_{n\in\mathrm{N}}\mathrm{P}_n\otimes\mathrm{P})=1$, while $\rho(\mathrm{P}_n\otimes\mathrm{P})=0$ for all $n\in\mathrm{N}$. We see that $\rho(\mathrm{I}\otimes\mathrm{P})=1$ and so ρ is again a product state. Let us express ρ in the form $\rho=\rho_1\otimes\rho_2$, where ρ_1 and ρ_2 are pure states of $\mathscr{M}$ and $\mathscr{B}(\mathscr{H}_n)$, respectively. The W*-algebra generated by $\{\mathrm{P}_n\otimes\mathrm{I}\,|\,n\in\mathrm{N}\}$ is $*$-isomorphic to l^∞ in such a way that the projections $\mathrm{P}_n\otimes\mathrm{I}$ correspond to atoms in $\mathscr{P}(l^\infty)$. Identifying these two algebras, we see that $\rho_1/c_0=0$ and ρ cannot be a pure Jauch-Piron state (Example 3.2). We have obtained a contradiction.

We now prove that ψ is a σ-additive state. Let us take an increasing sequence $(\mathrm{Q}_n)\subset\mathscr{P}(\mathscr{A})$ such that $\mathrm{Q}_n\nearrow\mathrm{Q}$ in the weak operator topology. Identifying $\mathrm{A}\otimes\mathrm{B}$ wth the matrix $[b_{i,j}\mathrm{A}]_{ij}$, where $[b_{ij}]_{ij}$ is a (numerical) matrix of B, we can view $\mathscr{A}$ as the algebra of all $n \times n$ matrices with entries in $\mathscr{M}$ (*see* [13]. Example 11.2.2, p. 813). Moreover, we can assume that $I \otimes P$ is representable by the following matrix:

$$\begin{pmatrix} I & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}.$$

Then we have

$$\psi(Q_n) = \psi\left(\begin{pmatrix} I & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix} Q_n \begin{pmatrix} I & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}\right)$$

$$= \psi\left((Q_n)_{1,1} \otimes \begin{pmatrix} I & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}\right)$$

$$= \psi_1((Q_n)_{1,1})\, \psi_2\left(\begin{pmatrix} I & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \dots & 0 \end{pmatrix}\right) = \psi_1((Q_n)_{1,1}).$$

Let $\mathscr{N}$ be a von Neumann subalgebra of $\mathscr{M}$ generated by countably many operators (Q_n). Then $\mathscr{N}$ is $*$-isomorphic to a von Neumann algebra acting on a Hilbert space whose dimension is at most continuum. Indeed, let $\mathscr{R}$ be a separable C*-algebra of $\mathscr{N}$ generated by (Q_n). It follows from the GNS-construction that a Hilbert space $\mathscr{H}_\rho$ associated with ρ is separable, whenever ρ is a state of $\mathscr{R}$. Since the dual $\mathscr{R}^*$ of $\mathscr{R}$ has cardinality at most continuum, we see that the universal representation of $\mathscr{R}$ acts on the Hilbert space $\mathscr{H}_u = \sum_{\rho \in (\mathscr{R}^*)^+} \mathscr{H}_\rho$ whose dimension is at most continuum. Now we can use [13]. Theorem 10.1.12, p. 719, asserting that $\mathscr{N}$ has a faithful representation on some subspace of $\mathscr{H}_u$. Thus, (after an obvious identification) the state ψ_1 can be regarded as a σ-additive state of the algebra acting on $2^{\aleph_0}$-dimensional Hilbert space. Following the technique of [7], we can easily prove that every σ-additive state of an algebra acting on a real-nonmeasurably-dimensional Hilbert spac is completely additive. It follows that ψ_1 is a normal state of $\mathscr{N}$ and so $\psi_1((Q_n)_{1,1}) \nearrow \psi_1((Q)_{1,1})$. Therefore ψ is σ-additive. The unitary equivalence of φ and ψ now yields that φ is σ-additive.

The main result of this paper is the following theorem.

4.3. THEOREM. — *Let us assume that the continuum is a real-nonmeasurable cardinal. Then a pure state* φ *of* $\mathscr{A}$ *is Jauch-Piron if and only if* φ *is either* σ*-additive or there is a central abelian projection* P *of* $\mathscr{A}$ *such that* $\varphi(P)=1$.

Proof. — The state φ is concentrated either at the commutative or at the noncommutative central part of $\mathscr{A}$. Considering now Example 3.1, we may (and shall) assume that $\mathscr{A}$ is the von Neumann algebra with no commutative central summand. Let us prove that pure Jauch-Piron state φ of $\mathscr{A}$ is then σ-additive. The decomposition of $\mathscr{A}$ into direct sum with respect to the types enables us to devide the proof into the following steps.

Let $\mathscr{A}$ be either a properly infinite type I algebra or let $\mathscr{A}$ be one of the type II and III. Employing [13], Lemma 6.5.6, p. 426, and the halving lemma [13], Lemma 6.3.3 p. 412, we find a projection $P\in\mathscr{P}(\mathscr{A})$ such that $P\sim P^{\perp}$. This allows us to represent $\mathscr{A}$ in the form $P\mathscr{A}P\otimes\mathscr{B}(\mathscr{H}_2)$ (*see* [18]). By Proposition 4.2 φ has to be σ-additive.

Let $\mathscr{A}$ be a finite type I algebra. By the structural theory of von Neumann algebras (*see* [13], $\mathscr{A}$ can be decomposed into direct sum $\sum_{j\in J}\oplus(\mathscr{A}_j\otimes\mathscr{B}(\mathscr{H}_{n_j}))$, where each $\mathscr{A}_j\,(j\in J)$ is a commutative von Neumann algebra and (n_j) is an increasing sequence (finite or infinite) of positive integers. We can now assume that either all n_j's are even or all n_j's are odd (otherwise we split the sequence in question into the odd and the even part).

Let us consider the first case. Then each algebra $\mathscr{A}_j\otimes\mathscr{B}(\mathscr{H}_n)$ is $\star$-isomorphic to an algebra $(\mathscr{A}\otimes(\mathscr{B}(\mathscr{H}_{n_{j/2}}))\otimes\mathscr{B}(\mathscr{H}_2)$, $j\in J$. Hence,

$$\sum_{j\in N}\oplus(\mathscr{A}_j\otimes\mathscr{B}(\mathscr{H}_{n_{j/2}}))\otimes\mathscr{B}(\mathscr{H}_2)=(\sum_{j\in J}\oplus\mathscr{A}_j\otimes\mathscr{B}(\mathscr{H}_{n_{j/2}}))\otimes\mathscr{B}(\mathscr{H}_2)$$

and the assertion of the theorem follows from Proposition 4.2.

We now take up the case of n_j's odd. Let P_j be an (n_j-1)-dimensional projection in $\mathscr{B}(\mathscr{H}_{n_j})$ and let Q_j be two-dimensional projections in $\mathscr{B}(\mathscr{H}_{n_j})$ such that $Q_j\geqq P_j^{\perp}\,(j\in J)$. Put $P_1=\sum_{j\in J}\oplus(I_j\otimes P_j)$, where I_j is the identity in $\mathscr{A}_j$. Put further $P_2=\sum_{j\in J}\oplus(I_j\otimes Q_j)$. Then $P_2\geqq P_1^{\perp}$ and so we have either $\varphi(P_1)\neq 0$ or $\varphi(P_2)\neq 0$. Using Lemma 4.1 (i), we can construct a pure Jauch-Piron state ψ of $\mathscr{A}$ which is unitarily equivalent with φ and which satisfies either $\psi(P_1)=1$ or $\psi(P_2)=1$. In both cases the reasoning of the previous paragraph yields that ψ is σ-additive. Thus, φ is σ-additive, too. Since the converse implication follows from [4] the proof is complete (it should be mention that these tensor product representations had already been used by A. Amann (*see* [2])).

Whether the continuum is real-nonmeasurable or not might depend on the model of set-theory we work in (*see* [6]). However, in the important case of a type I factor we have the following "absolute" result.

4.4. THEOREM. – *A pure state of a type I factor $\mathscr{A}$ is Jauch-Piron if and only if it is σ-additive.*

Proof. – Let φ be a pure Jauch-Piron state of $\mathscr{A}$. Let us identify $\mathscr{A}$ with $\mathscr{B}(\mathscr{H})$. If $\mathscr{H}$ is separable, then we can use [2] or the proof of Theorem 4.3. Suppose that $\mathscr{H}$ has uncountable dimension. Assume that φ is not Jauch-Piron and seek a contradiction. There is a sequence (P_n) of orthogonal projections in $\mathscr{H}$ such that $\sum_{n \in N} P_n = I$ and $\sum_{n \in N} \varphi(P_n) \neq 1$. Thus, we can choose a projection P_{n_0} such that $\dim P_{n_0}(\mathscr{H}) = \dim \mathscr{H}$. Let us split P_{n_0} into two orthogonal projections, $P^1_{n_0}$ and $P^2_{n_0}$, such that $\dim P^1_{n_0}(\mathscr{H}) = \dim P^2_{n_0}(\mathscr{H}) = \dim P_{n_0}(\mathscr{H})$. Setting $P_1 = P^1_{n_0} + \sum_{n \in N, n \neq n_0} P_{2n-1}$, $P_2 = P^2_{n_0} + \sum_{n \in N, n \neq n_0} P_{2n}$, we see that either

$$\varphi(P_1) \neq \varphi(P^1_{n_0}) + \sum_{n \in N, n \neq n_0} \varphi(P_{2n-1}) \text{ or } \varphi(P_2) \neq \varphi(P^2_{n_0}) + \sum_{n \in N, n \neq n_0} \varphi(P_{2n}).$$

Let us consider the first case. Since $\dim P_1(\mathscr{H}) = \dim \mathscr{H}$, we have $\mathscr{B}(\mathscr{H}) = \mathscr{B}(P_1(\mathscr{H})) \otimes \mathscr{B}(\mathscr{H}_2)$. Making use of Lemma 4.1, we can construct a pure Jauch-Piron state ψ unitarily equivalent with φ and satisfying the following conditions: $\psi(P_1 \otimes Q) = 1$, $\psi(P^1_{n_0} \otimes Q) = 0$ and $\psi(P_{2n-1} \otimes Q) = 0$ for all $n \in N$ [by Q we do note a suitable one dimensional projection of $\mathscr{B}(\mathscr{H}_2)$]. Arguing as in the proof of Proposition 4.2, we obtain a contradiction with Lemma 3.2 which completes the proof. (The converse implication follows again from [4].)

Applying the Gleason-Christensen-Yeadon theorem and assuming the continuum hypothesis, we see that a pure measure μ on a von Neumann algebra $\mathscr{A}$ without a direct summand of the type I_2 is Jauch-Piron if and only if either μ is σ-additive or μ is concentrated at central abelian projection. As a consequence of Theorem 4.3, we further obtain the following characterization of pure Jauch-Piron states. This result shows that the pure Jauch-Piron states essentially exist only on the commutative algebras and the algebras $\mathscr{B}(\mathscr{H})$.

4.5. THEOREM. – *Let $\mathscr{A}$ be a von Neumann algebra acting on a Hilbert space $\mathscr{H}$. Let* $\dim \mathscr{H}$ *be real-nonmeasurable. Let us assume that φ is a pure Jauch-Piron state of $\mathscr{A}$. Then either $\varphi(P) = 1$ for some central abelian projection of $\mathscr{A}$ or φ is a normal state with the following property: there is an atom Q in the center of $\mathscr{P}(\mathscr{A})$ such that $\varphi(Q) = 1$ and $Q\mathscr{A}$ is a type* I *factor.*

Proof. — Similarly as in the proof of Theorem 4.3, we can assume that $\mathscr{A}$ has no abelian direct summand. Since every σ-additive state of $\mathscr{A}$ is normal, we obtain by an obvious modification of the proof of Proposition 4.2 (and by mimicking the proof of Theorem 4.3), that φ is a normal state. It remains to prove that φ is concentrated at central portion which is a type I factor.

By [13], $\varphi/\mathscr{Z}(\mathscr{A})$ is a normal pure state of $\mathscr{Z}(\mathscr{A})$. Thus, φ is multiplicative on $\mathscr{Z}(\mathscr{A})$ (*see* [13]) and so $\varphi/\mathscr{P}(\mathscr{Z}(\mathscr{A}))$ is a two-valued measure. Let $Q \in \mathscr{Z}(\mathscr{A})$ be a support of $\varphi/\mathscr{Z}(\mathscr{A})$. We show that Q is an atom in $\mathscr{P}(\mathscr{Z}(\mathscr{A}))$. For this, let us take a nonzero central projection R such that $R \leqq Q$. Thus, $\varphi(R) \neq 0$. This implies that $\varphi(R) = 1$ and so $Q \leqq R$. Therefore $R = Q$ and Q is an atom. Obviously, $Q\mathscr{A}$ is a factor. We have to prove that $\mathscr{M} = Q\mathscr{A}$ is of the type I. We can easily check that $\varphi/\mathscr{M}$ is a pure state. Let $\pi_\varphi : \mathscr{M} \to \mathscr{B}(\mathscr{H}_\varphi)$ be the GNS-representation engendered by the state $\varphi/\mathscr{M}$. Then π_φ is irreducible and so $\pi_\varphi(\mathscr{M})'' = \mathscr{B}(\mathscr{H}_\varphi)$. Making use of [18] p. 127, π_φ is σ-weakly continuous and so the set $\pi_\varphi(\mathscr{M}_1)$, where $\mathscr{M}_1$ is the closed unit ball of $\mathscr{M}$, is compact in the weak operator topology. Since $\pi_\varphi(\mathscr{M}_1) = (\pi_\varphi(\mathscr{M}))_1$ [13], Cor. 10.1.8, p. 716 and $\mathscr{B}(\mathscr{H}_\varphi)_1$ is contained in the weak operator closure of $(\pi_\varphi(\mathscr{M}))_1$ (Kaplansky density theorem), we see that $\mathscr{B}(\mathscr{H}_\varphi)_1 \subset (\pi_\varphi(\mathscr{M}))_1$ and so $\mathscr{B}(\mathscr{H}_\varphi) = \pi_\varphi(\mathscr{M})$. Since π has to be one-to-one (a factor does not contain any nontrivial two-sided and σ-weak-operator closed ideal), we obtain that $\mathscr{M}$ is $*$-isomorphic to $\mathscr{B}(\mathscr{H}_\varphi)$. Thus $\mathscr{M}$ is of the type I and this completes the proof.

Let us remark that if the real-measurable cardinals exist (within the set theory including the generalized contiuum hypothesis), they must be extremely huge. From this point of view, the assumption that $\mathscr{A}$ acts on a Hilbert space $\mathscr{H}$ of real non-measurable dimension is a technical condition whose validity seems to be plausible as far as a potential physical situations are concerned.

As a further corollary of Theorem 4.3 we obtain the following characterization of Jauch-Piron von Neumann algebras. It asserts that the only examples of Jauch-Piron algebras are the commutative algebras, the finite-dimensional factors $\mathscr{B}(\mathscr{H}_n)$ and direct sums of the latter two types.

4.6. THEOREM. — *Let $\mathscr{A}$ be a von Neumann algebra not containing a type* I_2 *direct summand. Then $\mathscr{A}$ is Jauch-Piron if and only if $\mathscr{A}$ is the direct sum of a commutative von Neumann algebra and finitely many finite-dimensional factors.*

Proof. — According to [18], p. 50, it is sufficient to prove that every Jauch-Piron von Neumann algebra $\mathscr{A}$ with no direct summand of the type I_1, I_2 has to be finite-dimensional. Looking for a contradiction, let us suppose that $\dim \mathscr{A}$ is infinite. First, let us prove that $\mathscr{A}$ contains a von Neumann subalgebra $\mathscr{M}$ which is σ-finite, infinite-dimensional and not containing a direct summand of the type I_1 or I_2. Let us first assume that $\mathcal{A}$ does not contain type I direct summand. Then every cyclic projection P of $\mathcal{A}$ has to be infinite-dimensional and so we can set $\mathcal{M}=P\mathcal{A}P$. This is a σ-finite von Neumann algebra (*see* [13], 5.5.15, p. 338). Suppose that $\mathcal{A}$ is of the type I. Then either $\mathcal{A}$ contains a direct summand. $\mathcal{N}$ of the type, I_γ, $(\gamma \geqq \aleph_0)$ or $\mathcal{A}$ is a direct sum of at most countably many finite I_{n_j} algebras ($n_j \in N$). In the former case $\mathcal{N}$ is $*$-isomorphic to a tensor product $\mathcal{B} \otimes \mathcal{B}(\mathcal{H})$, where $\mathcal{B}$ is commutative and $\dim \mathcal{H} \geqq \aleph_0$. Thus we can choose $\mathcal{M}$ as an subalgebra generated by the set $\{I \otimes A \mid A \in \mathcal{B}(\mathcal{K})\}$, where $\mathcal{K}$ is a separable infinite-dimensional subspace of $\mathcal{H}$. In the latter case we can identify $\mathcal{A}$ with a direct summand of the tensor products $\mathcal{A}_{n_j} \otimes \mathcal{B}(\mathcal{H}_{n_j})$ and define $\mathcal{M}$ as the algebra generated by the set $\{I_{n_j} \otimes A \mid A \in \mathcal{B}(\mathcal{H}_{n_j})\}$. Since every state of $\mathcal{M}$ can be extended to a state of $\mathcal{A}$, $\mathcal{M}$ is a Jauch-Piron algebra. Thus, we can assume that $\mathcal{A}$ is σ-finite (we can set $\mathcal{A}=\mathcal{M}$). The fact that $\dim \mathcal{A}=\infty$ implies now existence of an orthogonal sequence (P_n) of nonzero projections such that $\sum_{n \in N} P_n = I$. The von Neumann subalgebra generated by (P_n) can be identified with l^∞, which is an algebra admitting a pure state ρ such that $\rho/c_0=0$. (We take any pure state ρ of the C*-algebra l^∞/c_0 and then define $\rho=\rho \circ i$, where i is the canonical mapping of l^∞ onto l^∞/c_0.) Extending ρ to a pure state of $\mathcal{A}$ (*see* [13], Theorem 4.3.13, p. 266), we obtain a non σ-additive pure state of $\mathcal{A}$ which is not a Jauch-Piron state (Theorem 4.5). This contradiction completes the proof.

5. TWO-VALUED MEASURES

In this section we examine two-valued measures on von Neumann algebras. By a two-valued measure on $\mathcal{P}(\mathcal{A})$ we understand any measure $m: \mathcal{P}(\mathcal{A}) \to \{0, 1\}$. Obviously, every multiplicative state φ of $\mathcal{A}$ induces a two-valued measure $\varphi/\mathcal{P}(\mathcal{A})$. We first prove that the reverse implication holds, too.

5.1. LEMMA. – *Let $\mathcal{A}$ be a von Neumann algebra without a type I_2 direct summand. Then every two-valued measure on $\mathcal{P}(\mathcal{A})$ can be extended to a multiplicative state of $\mathcal{A}$.*

Proof. – Let μ be a two-valued measure on $\mathcal{P}(\mathcal{A})$. Using the Gleason-Christensen-Yeadon theorem, we can extend μ to a state φ of $\mathcal{A}$. Let $\pi_\varphi : \mathcal{A} \to \mathcal{H}_\varphi$ be the GNS-representation engendered by φ. Let x_φ be a unit cyclic vector of π_φ such that $\varphi=\omega_{x_\varphi} \circ \pi_\varphi$. For every $P \in \mathcal{P}(\mathcal{A})$ we have that $\mu(P)=\langle \pi_\varphi(P) x_\varphi, x_\varphi \rangle$. We see that $\mu(P)$ is either 0 or 1. It follows that either $\pi_\varphi(P) x_\varphi = x_\varphi$ or $\pi_\varphi(P) x_\varphi = 0$. Hence,

$\mathcal{H}_\varphi=\overline{\mathrm{sp}}\{\pi_\varphi(\mathrm{A})x_\varphi \mid \mathrm{A}\in\mathcal{A}\}=\overline{\mathrm{sp}}\{\pi_\varphi(\mathrm{P})x \mid \mathrm{P}\in\mathcal{P}(\mathcal{A})\}=\mathrm{sp}\{x_\varphi\}$. Therefore, for every $\mathrm{A}\in\mathcal{A}$ there is a complex number λ_A such that $\pi_\varphi(\mathrm{A})x_\varphi=\lambda_\mathrm{A}x_\varphi$. Obviously, $\lambda_{\mathrm{AB}}=\lambda_\mathrm{A}\lambda_\mathrm{B}$ ($\mathrm{A}, \mathrm{B}\in\mathcal{A}$) and therefore

$$\varphi(\mathrm{AB})=\langle \pi_\varphi(\mathrm{AB})x_\varphi, x_\varphi\rangle=\lambda_{\mathrm{AB}}=\lambda_\mathrm{A}\lambda_\mathrm{B}$$
$$=\langle \pi_\varphi(\mathrm{A})x_\varphi, x_\varphi\rangle\langle \pi_\varphi(\mathrm{B})x_\varphi, x_\varphi\rangle=\varphi(\mathrm{A})\varphi(\mathrm{B}).$$

for every $\mathrm{A}, \mathrm{B}\in\mathcal{A}$. This concludes the proof.

5.2. Corollary. – *Every two-valued measure on $\mathcal{P}(\mathcal{A})$, where $\mathcal{A}$ is a von Neumann algebra without a type I_2 direct summand, is Jauch-Piron.*

Proof. – Let μ be a two-valued measure on $\mathcal{P}(\mathcal{A})$ and let φ be a multiplicative state extending μ. Reasoning as in the remark following Example 3.2 we may assume that $\mathcal{A}$ is of the form $\mathcal{N}\oplus(\mathcal{M}\otimes\mathcal{B}(\mathcal{H}_2))$, where $\mathcal{M}$ and $\mathcal{N}$ are commutative algebras. We prove that $\varphi/\mathcal{M}\otimes\mathcal{B}(\mathcal{H}_2)=0$. Indeed, φ induces a multiplicative functional on $\mathcal{B}(\mathcal{H}_2)$ (we can set $\varphi'(\mathrm{A})=\varphi(\mathrm{I}\otimes\mathrm{A})$, $\mathrm{A}\in\mathcal{B}(\mathcal{H}_2)$ and find φ' to be zero (*see* e.g. [3])). Thus, φ is concentrated at $\mathcal{N}$ and so it is Jauch-Piron.

In the following theorem we describe a general form of two-valued measures on von Neumann algebras. This generalizes hitherto known results about two-valued measures on $\mathcal{B}(\mathcal{H})$ ([1], [2], [17]).

5.3. Theorem. – *Let $\mathcal{A}$ be a von Neumann algebra without a type I_2 direct summand. Suppose that μ is a two-valued measure on $\mathcal{P}(\mathcal{A})$. Then there is a central abelian projection Q such that $\mu(\mathrm{Q})=1$.*

Proof. – By Lemma 5.1 and Corollary 5.2, we can extend μ to a pure Jauch-Piron multiplicative state φ of $\mathcal{A}$. Arguing as in Theorem 4.3, we can assume that $\mathcal{A}$ is of a fixed type.

Let $\mathcal{A}$ be one of the types I_∞, II, III. Let us prove that $\mathcal{A}$ does not admit any multiplicative state. Suppose that $\mathrm{P}\in\mathcal{P}(\mathcal{A})$ and $\mathrm{P}\sim\mathrm{P}^\perp$. Then $\mathcal{A}$ is $*$-isomorphic to $\mathrm{P}\mathcal{A}\mathrm{P}\otimes\mathcal{B}(\mathcal{H}_2)$. But such an algebra has no multiplicative states (*see* the proof of Corollary 5.2).

Let $\mathcal{A}$ be a finite algebra of the type I. It means that $\mathcal{A}$ is $*$-isomorphic to a direct sum $\sum_{j\in \mathrm{J}}\oplus\mathcal{N}_j\otimes\mathcal{B}(\mathcal{H}_n))$, where (n_j) is an increasing (finite or infinite) sequence of natural numbers. Let us prove that φ is concentrated at the abelian part of $\mathcal{A}$. Suppose that it is not the case and derive a contradiction. Let us consider a subalgebra $\mathcal{N}$ of $\mathcal{A}$ generated by the set $\{\mathrm{I}_j\otimes\mathrm{A} \mid \mathrm{A}\in\mathcal{B}(\mathcal{H}_n)\}$. Then φ includes a two valued measure on a σ-finite algebra $\mathcal{N}$. Employing Theorem 4.5, this restriction has to be σ-additive and so φ is concentrated at a summand $\mathcal{M}\otimes\mathcal{B}(\mathcal{H}_{n_{j_0}})$. This is only possible if $n_{j_0}=1$, in which case φ is supported by the central abelian portion I_1. The proof is complete.

5.4. Remark. – The assumption in the previous theorem that $\mathcal{A}$ does not contain any direct summand of the type I_2 is essential, because every type I_2 von Neumann algebra $\mathscr{A}$ admits a two-valued measure. Indeed, let us represent $\mathscr{A}$ as the tensor product $C(X)\otimes\mathscr{B}(\mathscr{H}_2)$, where X is an extremelly disconnected compact Hausdorff space. Then we can identify $C(X)\otimes\mathscr{B}(\mathscr{H}_2)$ with an algebra of all $\mathscr{B}(\mathscr{H}_2)$-valued continuous functions on X (the operations are defined poitwise—*see* [13]. Example 11.1.6, p. 809). Pick up an $x\in X$ and take a two-valued measure μ_0 on $\mathscr{P}(\mathscr{B}(\mathscr{H}_2))$. One can easily verify that the formula

$$\mu(P)=\mu_0(P(x)), \qquad P\in\mathscr{P}(\mathscr{A}),$$

defines a two-valued measure on $\mathscr{P}(\mathscr{A})$.

ACKNOWLEDGEMENTS

The author would like to thank Prof. P. Pták for the constant interest in his research.

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(*Manuscript received September* 27, 1991.)