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Some rigorous results on the Pauli-Fierz model of classical electrodynamics

by

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ABSTRACT. — We consider the dynamical system describing the classical electromagnetic field interacting with an extended rigid charged particle in the non-relativistic approximation (the Pauli-Fierz model); neither the motion of the particle nor the field are given in advance. We give some preliminary mathematical results, namely: global existence and uniqueness of the solution of the Cauchy problem; existence and stability of a family of solitary wave solutions with the particle performing uniform motion; bounds on the change of the energy spectrum at high frequencies.

RÉSUMÉ. — Nous considérons le système dynamique qui décrit dans l'approximation non relativistique le champ électromagnétique classique interageant avec une particule rigide chargée. Ni le champ, ni le mouvement de la particule ne sont donnés à l'avance. Nous donnons des résultats mathématiques préliminaires, notamment : existence et unicité globale de la solution du problème de Cauchy; existence et stabilité d'une famille d'ondes solitaires avec la particule en mouvement uniforme; limite supérieure pour le changement du spectre de l'énergie aux hautes fréquences.

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1. INTRODUCTION

The aim of the present paper is to give some rigorous results on the equations describing the classical electromagnetic field interacting with matter. In fact, we consider a special model, *i. e.* that of Maxwell equations with currents due to a non relativistic extended rigid particle whose rotational degrees of freedom are neglected; the currents are not assigned *a priori*, as the particle satisfies Newton equation with Lorentz force. This is the standard model which is considered in most books and papers on quantum field theory, following Fermi [1], Heitler [2], Dirac [3], and Kramers ([4], [5]), for the case of a point particle, and Pauli-Fierz [6] for the case of an extended rigid particle considered here; following some authors ([7], [8]), we will call it the Pauli-Fierz model. A few rigorous results are known for the corresponding quantized model ([7], [8], [9]), which was studied mainly in the dipole approximation and in connection with the infrared divergence problem. Here we give some results on the corresponding classical model, without any approximation, in the spirit of some recent works where the classical electromagnetic field was reconsidered in the framework of the theory of dynamical systems [10]; *see* also [11], [12], [13]. As Pauli and Fierz, in the present paper we limit ourselves to the case of a positive (bare) mass for the charged particle.

First we prove, under suitable smoothness conditions on the charge distribution of the particle, global existence and uniqueness of the solution of the Cauchy problem in the space of finite energy states. We recall that local existence and uniqueness is a classical result for the system of coupled Maxwell and Dirac equations ([14], [15]), while global results are well known for the (linear) free system, and for the one dimensional coupled Maxwell-Dirac equations [16]; for a global result in a one dimensional model of the classical electromagnetic field interacting with matter, *see* [13].

Secondly, we give a stability result for a family of solitary wave solutions of the Pauli-Fierz model ([17], [10]), in which the particle moves with constant velocity, and the field follows rigidly the particle. Namely, we prove that, for initial data near those allowing uniform motion, the field remains for all times near a rigid field moving with the particle, and that the particle velocity remains close to the original one.

Finally, in the case of a charge distribution described by an analytic function, we consider the Fourier (space) transform of the field, and study the time evolution of the corresponding energy spectral distribution. We prove that there exists a time dependent cutoff frequency $\bar{\omega}(t)$, such that the change of the energy spectrum is exponentially small with the frequency ω for all times less than t and for all frequencies ω larger than $\bar{\omega}(t)$. The cutoff frequency turns out to increase only logarithmically with t. This result gives a precise statement corresponding to a qualitative property (freezing of the high frequencies) which was claimed informally in the classical papers on the present model [6].

The first and the third results can be easily extended to the case of several particles, interacting through a regular potential bounded from below; for example the Coulomb potential (for an extended particle).

From the technical point of view, the proof of existence and uniqueness is obtained by applying a classical theorem by Segal [18], and using energy as a Lyapunov function. For what concerns instead the stability of the solitary wave solutions described above, we first perform a change of variables which transforms such solutions into critical points of a reduced system; thus stability can be proved in the standard way, using energy as a Lyapunov function. By the way, in so doing one has to deal with the technical complication that the above change of variables is continuous, but not differentiable. Finally, the result on the change of the high frequency part of the energy spectral distribution is based on the remark that, due to the exponential decay of the Fourier coefficients of an analytic function, the interaction itself of the field oscillators with the particle decays exponentially with the frequency. This turns out to lead to a differential inequality for the energy density of a single field oscillator, which can then be solved.

The three results described above are stated in section 2, and proved in section 3. Some comments are given in section 4.

2. STATEMENT OF THE RESULTS

Following the authors quoted in the introduction, we work in the Coulomb gauge, and neglect the rotational degrees of freedom of the charged particle. So the only dynamically relevant unknowns are: for the field, the vector potential A, with the constraint $\operatorname{div} \mathrm{A}=0$, and, for the particle, the cartesian coordinates $q=(q_1, q_2, q_3)$ of its center (any reference point). The charge density at x is then given by $c\rho(x-q)$, where the "charge distribution" (or form factor) ρ is an assigned function, (the multiplication by the speed of light c has been introduced for the sake of simplicity in the form of the Hamiltonian); it is natural to assume $\rho \in \mathrm{L}^1(\mathbb{R}^3, \mathbb{R})$, just in order that the total charge be defined; we will also assume that the total charge is non-vanishing.

As is well known ([3], [6], [7]), the Hamiltonian of the system is given by

$$\mathrm{H}=\int_{\mathbb{R}^3}\left(2\pi c^2\,\mathrm{E}(x)^2-\frac{1}{8\pi}\langle \mathrm{A}(x), \Delta\mathrm{A}(x)\rangle\right)d^3x \\ +\frac{1}{2m}\sum_{k=1}^{3}\left(p_k-\int_{\mathbb{R}^3}\rho(x-q)\,\mathrm{A}_k(x)\,d^3x\right)^2, \quad (2.1)$$

where $p=(p_1, p_2, p_3)$ is the momentum conjugate to q, and $m>0$ is the (bare) mass of the particle; $A=A(x)$ is the vector potential, playing the role of a canonical coordinate, while $E(x)=\dot{A}(x)/(4\pi c^2)$, with the dot denoting time derivative, is its conjugate momentum, coinciding up to a factor with the electric field. In the Hamiltonian we have denoted by $\langle\ ,\ \rangle$ the usual scalar product in $\mathbb{R}^3$.

We shall also use the following notations:

$H^s=H^s(\mathbb{R}^3, \mathbb{R}^3)$ is the usual Sobolev space of functions which are in L^2 together with their first s weak derivatives; in particular $H^0:=L^2(\mathbb{R}^3, \mathbb{R}^3)$;

$H^{\{s\}}\supset H^s$ is the completion of C_c^∞ (the lower index c stands for compactly supported) in the norm $\|A\|_{H^{\{s\}}}:=\||\Delta|^{s/2}A\|_{L^2}$ (we recall that, with respect to such a norm, $H^{\{s\}}$ is a Hilbert space; moreover, by the Sobolev inequality one has that $H^{\{1\}}$ is continuously imbedded in L^6); in particular $H^{\{0\}}=H^0$;

$H_*^{\{s\}}=H_*^{\{s\}}(\mathbb{R}^3, \mathbb{R}^3)$ is the subspace of $H^{\{s\}}$ constituted by the solenoidal vectors, namely is the closure in $H^{\{s\}}$ of the set of C_c^∞ vector fields with vanishing divergence;

P is the projection of $H^{\{s\}}$ onto $H_*^{\{s\}}$;

A_k is the k-th component of a vector $A\in\mathbb{R}^3$.

The appropriate phase space for the system, which makes the first integral in the Hamiltonian (2.1) well defined, and the gauge condition satisfied, is clearly

$$\mathscr{F}:=H_*^0\times H_*^{\{1\}}\times\mathbb{R}^3\times\mathbb{R}^3\ni(E, A, p, q);$$

the second integral of (2.1) is then well defined too, provided $\rho\in L^{6/5}$. The corresponding Hamilton equations of motion turn out to be

$$\left.\begin{aligned}\dot{E}&=\frac{1}{4\pi}\Delta A+P(w(A, q, p)\,\rho_q)\\ \dot{A}&=4\pi c^2 E\\ \dot{p}_k&=\int_{\mathbb{R}^3}\left\langle w(A, q, p)\,\rho(x-q), \frac{\partial}{\partial x_k}A(x)\right\rangle d^3x\\ \dot{q}&=w(A, q, p),\end{aligned}\right\}\tag{2.2}$$

where

$$\left.\begin{aligned}\rho_q(x)&:=\rho(x-q),\\ w(A, q, p)&:=\frac{1}{m}\left(p-\int_{\mathbb{R}^3}\rho(x-q)\,A(x)\,d^3x\right),\end{aligned}\right\}\tag{2.3}$$

and where one also has to assume $\rho\in L^2$ in order to make the integral at the right hand side of (2.2) convergent. Notice in particular the projector P that appears at the right hand side of the first equation, and is due to the fact that A is solenoidal. With elementary manipulations one can write system (2.2) in the more familiar form

$$\frac{1}{c^2}\ddot{A}-\Delta A=4\pi P(\dot{q}\rho_q),$$

$$m\ddot{q}_k=-\int_{\mathbb{R}^3}\rho(x-q)\dot{A}_k(x)\,d^3x+\sum_{l=1}^{3}\dot{q}_l\int_{\mathbb{R}^3}\rho(x-q)\left(\frac{\partial}{\partial x_k}A_l(x)-\frac{\partial}{\partial x_l}A_k(x)\right)d^3x. \quad (2.4)$$

The formal deduction of system (2.2) from Hamiltonian (2.1) is easily obtained according to the well known general schemes; *see* for example [19].

Our first result on this system is the following

THEOREM 2.1. – *If* $\rho\in L^1(\mathbb{R}^3,\mathbb{R})\cap H^1(\mathbb{R}^3,\mathbb{R})$, *then the vector field corresponding to system* (2.2) *generates a continuous global flow in the phase space* $\mathscr{F}$. *Moreover, one has* $p\in C^1(\mathbb{R},\mathbb{R}^3)$, $q\in C^2(\mathbb{R},\mathbb{R}^3)$.

From the proof, given in section 3, it will be apparent that the theorem can easily be generalized to the case of several particles subjected to the action of forces admitting a sufficiently smooth potential bounded from below (for example, in the case of $\rho\in C^1$ the Coulomb potential between the particles is allowed). Moreover, it can also be proved that the subspace $H^0_*\times H^1_*\times\mathbb{R}^3\times\mathbb{R}^3$ of the phase space $\mathscr{F}$ is invariant under the flow generated by (2.2).

Since now on, we will always assume ρ satisfies the smoothness assumptions of theorem 2.1.

We come now to a discussion concerning existence and stability of particular solutions of (2.2) with the field following rigidly the particle *i.e.* of the form

$$\left.\begin{aligned} p&=p(t), & q&=q(t),\\ A(x,t)&=X(x-q(t)), & E(x,t)&=Y(x-q(t)),\end{aligned}\right\} \quad (2.5)$$

with X and Y constant functions, and $q(t)$, $p(t)$ still undetermined. In order to discuss existence of such solutions, we consider preliminary the case in which the motion of the particle is assigned and is uniform, *i.e.* one has $q(t)=\bar{q}+vt$, with $\bar{q}$ and v constants in $\mathbb{R}^3$. In this case one obtains that the function X must satisfy the linear equation (depending on the parameter v)

$$\Delta X-\frac{1}{c^2}\sum_{i,l=1}^{3}v_i v_l\frac{\partial^2}{\partial x_i\,\partial x_l}X=-4\pi P(\rho v). \quad (2.6)$$

If $\|v\|_{\mathbb{R}^3}<c$, this is an elliptic equation which is well known to have a unique solution $X_v\in H^{\{1\}}_*$. On the other hand, for $\|v\|_{\mathbb{R}^3}>c$ the elliptic character of the equation is lost, and it is easy to show that there are no

$H^{\{1\}}$ solutions of equation (2.6). This follows from the fact that there exists a neighborhood of the origin where the Fourier transform of ρ is different from zero, due to the non-vanishing of the total charge and to ρ being L^1.

Having thus found, in the linear case, the solution $X=X_v$, by equations (2.2) (2.3) one also gets $Y=Y_v$ and $p=p_v$, with

$$\left.\begin{aligned} Y_v &:= -\sum_{l=1}^{3} \frac{v_l}{4\pi c^2} \frac{\partial X_v}{\partial x_l}, \\ p_v &:= mv + \int_{\mathbb{R}^3} \rho(x) X_v(x)\, d^3x. \end{aligned}\right\} \tag{2.7}$$

For the nonlinear problem (2.2)-(2.3), we then have

PROPOSITION 2.2. – *The only solutions of* (2.2) *with the field following rigidly the particle, i.e. of the form* (2.5), *are*

$$Y=Y_v, \qquad X=X_v, \qquad p(t)=p_v, \qquad q(t)=\bar{q}+vt \tag{2.8}$$

where v and $\bar{q}$ are constants in $\mathbb{R}^3$ with $\|v\|_{\mathbb{R}^3}<c$, X_v is the unique $H_*^{\{1\}}$ solution of equation (2.6), *while Y_v and p_v are given by* (2.7).

So, the particular solution obtained (parameterized by v, $\bar{q}$) is just uniform rectilinear motion for the particle, and the corresponding retarded potential for the field; however, in the present context this appears as a solitary-wave solution of a nonlinear problem (*see* [10]). Notice moreover that such a solution exists only if one assumes $\|v\|_{\mathbb{R}^3}<c$ (remember that we are dealing here with a *non*-relativistic theory).

We turn now to the stability properties of these particular solutions.

THEOREM 2.3. – *For $v\in\mathbb{R}^3$ with $\|v\|_{r^3}<c$, let Y_v, X_v and p_v be defined by* (2.6), (2.7). *For an initial datum* (E_0, A_0, p_0, q_0) *of the Cauchy problem for* (2.2), (*which corresponds to a unique initial particle velocity $\dot{q}_0$*), *let* $(E(t), A(t), p(t), q(t))$ *be the corresponding solution, and $\dot{q}(t)$ the corresponding particle velocity. Then, for any $\varepsilon>0$ there exists $\delta>0$ such that, if*

$$\|E_0(x)-Y_v(x-\bar{q})\|_{H^0}<\delta, \qquad \|A_0(x)-X_v(x-\bar{q})\|_{H^{\{1\}}}<\delta,$$
$$\|\dot{q}_0-v\|_{\mathbb{R}^3}<\delta, \qquad \|q_0-\bar{q}\|_{\mathbb{R}^3}<\delta,$$

one has, for all times t,

$$\begin{aligned} \|E(x,t)-Y_v(x-q(t))\|_{H^0}&<\varepsilon \\ \|A(x,t)-X_v(x-q(t))\|_{H^{\{1\}}}&<\varepsilon \\ \|p(t)-p_v\|_{\mathbb{R}^3}&<\varepsilon \\ \|\dot{q}(t)-v\|_{\mathbb{R}^3}&<\varepsilon. \end{aligned}$$

So, if the initial data are not exactly those allowing uniform motion, the theorem ensures that the speed of the particle remains close to the original one, while the field remains close to the original rigid field, centered however on the actual position $q(t)$ of the particle. In particular, as a consequence of the bound on the particle velocity, one obtains that the kinetic energy that the particle may lose by radiation, is at most of order ε. On the other hand, no bound has to be expected on the change $\| q(t)-\bar{q}\|_{\mathbb{R}^3}$ of the particle position, as is seen by analogy with the motion of a purely mechanical free particle; indeed we know from proposition 2.2 that there are nearby data allowing uniform motion with any different velocity. However, it is rather easy to show that, if one adds to the Hamiltonian an external potential, for example of the form $(q_1^2+q_2^2)/2$, then there still exists a solution of the above type with the particle moving uniformly on the q_3 axis, and that such a solution is stable in the ordinary sense.

The proof of theorem 2.3, given in the next section, will be obtained by performing preliminarily a canonical transformation such that the particular solution (parameterized by v, $\bar{q}$) described above appears as a stable fixed point of a suitable reduced system. Notice that a reduction of this type is needed also in the special case $v=0$.

In order to state our third result, we consider the Fourier (space) transforms of the fields, defined by

$$\left.\begin{aligned} \mathrm{E}(x) &= \frac{1}{c\sqrt{2\pi}(2\pi)^{3/2}} \sum_{j=1,2} \int_{\mathrm{S}} e^j(k)(\hat{\mathrm{E}}^j(k)\cos\langle k,x\rangle \\ &\qquad + \hat{\mathrm{E}}^j(-k)\sin\langle k,x\rangle)\, d^3k \\ \mathrm{A}(x) &= \frac{c\sqrt{8\pi}}{(2\pi)^{3/2}} \sum_{j=1,2} \int_{\mathrm{S}} e^j(k)(\hat{\mathrm{A}}^j(k)\cos\langle k,x\rangle \\ &\qquad + \hat{\mathrm{A}}^j(-k)\sin\langle k,x\rangle)\, d^3k, \end{aligned}\right\} \tag{2.9}$$

where S is a half space of $\mathbb{R}^3$, and $\{e^j(k)\}_{j=1,2}$ are unit polarization vectors perpendicular to k. As $\hat{\mathrm{E}}^j(.)$, $\hat{\mathrm{A}}^j(.)$ are defined almost everywhere, all the following equalities and inequalities are intended to be valid almost everywhere. Denote by $\eta^j(k)$ the energy density corresponding to a field oscillator with wave vector k, frequency $\omega(k):=c\|k\|_{\mathbb{R}^3}$, and polarization $e^j(k)$, namely

$$\eta^j(k):=\frac{1}{2}(\hat{\mathrm{E}}^j(k)^2+\omega(k)^2\,\hat{\mathrm{A}}^j(k)^2).$$

Define also the energy spectral distribution density (per frequency) $\eta(\omega)$ by

$$\eta(\omega):=c\sum_{j=1,2}\int_{\|k\|_{\mathbb{R}^3}=\omega/c} \eta^j(k)\, d\mathscr{S},$$

where $d\mathscr{S}$ is the surface element of the sphere of radius $\|k\|_{\mathbb{R}^3}$ in k-space, so that the total energy of the free electromagnetic field (namely the first integral at the right hand side of (2.1)) can be written as

$$\int_0^\infty \eta(\omega)\, d\omega.$$

Then we have

THEOREM 2.4. — *Consider the Cauchy problem for system* (2.2). *Assume that the charge distribution* ρ *can be extended to a complex analytic function on a complex strip of width* σ *in each of the space variables* (i.e., *for* $|\operatorname{Im} x_k| < \sigma$); *denote* $\rho_* := \operatorname{Sup}|\rho|$ *on such a complex strip,* $\omega_* := (2c)/\sigma$, *and introduce an arbitrary dimensional parameter* τ *having the dimensions of time. Then there exists a time-dependent cutoff frequency*

$$\bar{\omega}(t) := \omega_* \log \frac{|t|}{\tau},$$

such that along the solutions of the problem one has

$$\operatorname*{Sup}_{t' \leqq t} |\eta(\omega, t') - \eta(\omega, 0)| \leqq K(\omega)\, \rho_*\, e^{-\omega/\omega_*} \qquad \textit{for all} \quad \omega \geqq \bar{\omega}(t),$$

with

$$K(\omega) = \tau K_1 \omega^2;$$

here K_1 *is a constant depending on the initial data, which is given explicitly in the proof* [see *eq.* (3.16)].

A similar (although weaker) result can be obtained also for ρ less regular than analytic. In particular, we have

PROPOSITION 2.5. — *Consider the Cauchy problem for system* (2.2), *and assume* $\rho \in C^\infty(\mathbb{R}^3, \mathbb{R})$. *Then there exist a time-dependent cutoff frequency* $\bar{\omega}(t)$ *and a function* $\bar{\eta}(\omega)$ *satisfying the properties*

$$\lim_{t \to \infty} \frac{\bar{\omega}(t)}{t^\alpha} = 0 \qquad \textit{for all} \quad \alpha > 0,$$

$$\lim_{\omega \to \infty} \omega^\alpha \bar{\eta}(\omega) = 0 \qquad \textit{for all} \quad \alpha > 0,$$

such that along the solutions of the problem one has

$$\operatorname*{Sup}_{t' \leqq t} |\eta(\omega, t') - \eta(\omega, 0)| \leqq \bar{\eta}(\omega) \qquad \textit{for all} \quad \omega \geqq \bar{\omega}(t).$$

Notice that while theorem 2.4 requires a function ρ with an unbounded support, proposition 2.5 instead applies also to localized particles.

The proofs of theorem 2.4 and of proposition 2.5 are explicitly given in section 3. However one could also state analogous results which control the changes of the quantities $\eta^j(k)$ rather than $\eta(\omega)$. Moreover, one could also very easily prove as a corollary the following

PROPOSITION 2.6. – *In the same hypotheses of theorem* 2.4, *assume also that the initial data* A_0 *and* E_0 *are analytic in a complex strip with* $|\mathrm{Im}\, x_k| < \tilde{\sigma}$. *Then, for all times* t, *the projections* $A(t)$, $E(t)$ *of the solution of the equations of motion are analytic in a complex strip with* $|\mathrm{Im}\, x_k| < \min\{\tilde{\sigma}, 2\sigma - \varepsilon\}$, *for any positive* ε.

From the proofs, it will be apparent that theorem 2.4, proposition 2.5 and proposition 2.6 hold also in the case of several particles, subject to the action of forces admitting a C^2 potential bounded from below.

3. PROOFS

Before entering in the details of the proofs we will spend a few words in order to clarify what we mean by "solution" of a differential equation.

Given a semilinear equation

$$\dot{u} = B u + f(u), \tag{3.1}$$

where B is the generator of a linear C^0 group e^{Bt} on a Banach space $\mathscr{B}$, and $f: \mathscr{B} \to \mathscr{B}$ is continuous, we consider three kinds of solutions:

1. $u \in C^1([-T, T], \mathscr{B})$ is said to be a *strict* (or classical) solution of (3.1) if it satisfies equation (3.1);

2. $u \in C^0([-T, T], \mathscr{B})$ is said to be a *mild* solution of (3.1) if it satisfies the equation

$$u(t) = e^{Bt} u(0) + \int_0^t e^{B(t-s)} f(u(s))\, ds;$$

3. $u \in C^0((-T, T), \mathscr{B})$ is said to be a *strong* solution of (3.1) if $\forall t \in (-T, T)$ there exists $\varepsilon > 0$ and a sequence $u_n^{\varepsilon, t} \in C^1([t-\varepsilon, t+\varepsilon], \mathscr{B})$ of strict solutions of (3.1) such that $u_n^{\varepsilon, t} \to u$ in $C^0([t-\varepsilon, t+\varepsilon], \mathscr{B})$ as $n \to \infty$.

We recall that, if $f \in C^1(\mathscr{B}, \mathscr{B})$, then $\forall u_0 \in \mathscr{B}$ eq. (3.1) has a unique (local) mild solution continuously dependent (in the $C^0([-T, T], \mathscr{B})$ topology) on u_0. Moreover, for $u_0 \in D(B)$ (the domain of B), this mild solution is also strict (*see* [20] theorems 1.2 and 1.5 of sect. 6). Using these facts it is easy to prove that, if $f \in C^1(\mathscr{B}, \mathscr{B})$, then each mild solution is also a strong solution and viceversa. In what follows this property will play a relevant role.

We come now to the proofs of the theorems.

Proof of Theorem 2.1. – We begin by proving local existence. Define the norm of a point $u \in \mathscr{F} := H_*^0 \times H_*^{\{1\}} \times \mathbb{R}^3 \times \mathbb{R}^3$ by

$$\|u\|^2 := 2\pi c^2 \|E\|_{L^2}^2 - \int_{\mathbb{R}^3} \frac{1}{8\pi} \langle A(x), \Delta A(x) \rangle\, d^3 x + \frac{1}{m} \|p\|_{\mathbb{R}^3}^2 + \mu^2 \|q\|_{\mathbb{R}^3}^2; \tag{3.2}$$

here μ is an arbitrary parameter, which can be set equal to 1, and is introduced just in order to have dimensional homogeneity. The r.h.s. of system (2.2) can be decomposed into the sum of a linear part

$$\mathrm{B}(\mathrm{E}, \mathrm{A}, p, q) := \left(\frac{1}{4\pi}\Delta\mathrm{A},\, 4\pi c^2\,\mathrm{E},\, 0,\, 0\right), \tag{3.3}$$

and the remaining C^1 nonlinear part (notice that $q \mapsto \rho_q$ is C^1 as an L^2-valued map, and that $\mathrm{L}^1 \cap \mathrm{H}^1 \subset \mathrm{L}^{6/5}$); the linear operator B is clearly skew-adjoint (see [15]). It follows that there exists a local flow (of mild, and so also strong, solutions) solving (2.2), which leaves invariant the subspace $\mathrm{H}^1_* \times (\mathrm{H}^{\{1\}}_* \cap \mathrm{H}^{\{2\}}_*) \times \mathbb{R}^3 \times \mathbb{R}^3$ (*i.e.* in the domain of the operator B).

It is immediate to see that, for any strict solution u, we have $d\mathrm{H}(u(t))/dt = 0$; from this we have $\mathrm{H}(u(t)) = \mathrm{H}(u_0)$, for all $u_0 \in \mathscr{F}$. So we get the *a priori* estimate

$$2\pi c^2 \|\mathrm{E}\|^2_{\mathrm{L}^2} - \int_{\mathbb{R}^3} \frac{1}{8\pi} \langle \mathrm{A}(x), \Delta\mathrm{A}(x) \rangle\, d^3x + \frac{m}{2}\|w\|^2_{\mathbb{R}^3} = \mathrm{H}(u_0); \tag{3.4}$$

then, using eqs. (2.2)-(2.3), and the Sobolev inequality, we obtain

$$\left.\begin{aligned} \|q(t)\|_{\mathbb{R}^3} &\leqq \|q_0\|_{\mathbb{R}^3} + t\sqrt{\frac{2\,\mathrm{H}(u_0)}{m}}, \\ \|p(t)\|_{\mathbb{R}^3} &\leqq (\sqrt{2m} + \mathrm{C}_1 \|\rho\|_{\mathrm{L}^{6/5}})\sqrt{\mathrm{H}(u_0)}. \end{aligned}\right\} \tag{3.5}$$

for some positive constant C_1. Inequalities (3.4) and (3.5) then ensure existence for all times. □

In order to come to the proof of proposition 2.2 and theorem 2.3, we first introduce a suitable change of variables $(\mathrm{E}, \mathrm{A}, p, q) \mapsto (\mathrm{Y}, \mathrm{X}, \Pi, q)$ in the phase space $\mathscr{F}$. This is defined, for given (E, A, p, q), by

$$\left.\begin{aligned} \mathrm{E}(x) &= \mathrm{Y}(x-q) \\ \mathrm{A}(x) &= \mathrm{X}(x-q) \\ p_k &= \Pi_k + \int_{\mathbb{R}^3} \left\langle \mathrm{Y}(x), \frac{\partial}{\partial x_k}\mathrm{X}(x) \right\rangle d^3x \\ q_k &= q_k, \end{aligned}\right\} \tag{3.6}$$

and turns out to be continuous and invertible, but not differentiable.

Formally the equations of motion in the new variables are

$$\left.\begin{aligned} \dot{\mathrm{Y}} &= \sum_{l=1}^{3} w_l(\Pi, \mathrm{Y}, \mathrm{X})\,\partial_l \mathrm{Y} + \frac{1}{4\pi}\Delta \mathrm{X} + \mathrm{P}(w(\Pi, \mathrm{Y}, \mathrm{X})\,\rho) \\ \dot{\mathrm{X}} &= \sum_{l=1}^{3} w_l(\Pi, \mathrm{Y}, \mathrm{X})\,\partial_l \mathrm{X} + 4\pi c^2 \mathrm{Y} \\ \dot{\Pi} &= 0 \\ \dot{q} &= w(\Pi, \mathrm{Y}, \mathrm{X}), \end{aligned}\right\} \tag{3.7}$$

where

$$w_k(\Pi, \mathrm{Y}, \mathrm{X}) = \frac{1}{m}\left(\Pi_k + \int_{\mathbb{R}^3} \left\langle \mathrm{Y}(x), \frac{\partial}{\partial x_k}\mathrm{X}(x) \right\rangle d^3 x - \int_{\mathbb{R}^3} \rho(x)\,\mathrm{X}_k(x)\, d^3 x\right). \tag{3.8}$$

These equations are Hamiltonian with Hamiltonian function

$$\mathrm{H} = \int_{\mathbb{R}^3} \left(2\pi c^2\, \mathrm{Y}(x)^2 - \frac{1}{8\pi}\langle \mathrm{X}(x), \Delta \mathrm{X}(x)\rangle\right) d^3 x + \frac{m}{2}\| w(\Pi, \mathrm{Y}, \mathrm{X})\|^2_{\mathbb{R}}. \tag{3.9}$$

LEMMA 3.1. – *If* $u(t) = (\mathrm{E}(t), \mathrm{A}(t), p(t), q(t))$ *is a strong solution of* (2.2), *then the corresponding* $(\mathrm{Y}(t), \mathrm{X}(t), \Pi(t), q(t))$ *defined by* (3.6) *is a strong solution of* (3.7), *and vice versa.*

Proof. – In order to prove the direct statement, namely that $(\mathrm{Y}(t), \mathrm{X}(t), \Pi(t), q(t))$ is a solution of (3.7) if u is a solution of (2.2), it is enough to check it for local strict solutions, and then to exploit the continuity of (3.6) in order to extend the result to strong solutions; this is in fact easily done in a standard way. The converse statement is proved in an analogous way. □

By the above lemma, system (3.7) generates a global flow in $\mathscr{F}$ with the Hamiltonian (3.9) as a conserved quantity. So we can study system (3.7) as equivalent to the original Pauli-Fierz model, and the change of variables (3.6) can be said to be canonical. Since Π (which coincides with the total momentum of the system) is a constant of motion for system (3.7), Hamiltonian (3.9) defines a reduced system, described by the first two equations of (3.7), which thus contain all information on the Pauli-Fierz model.

Proof of proposition 2.2. – The key remark is that to each critical point of the above reduced system there corresponds a particular solution of (2.2) of the form (2.5), and vice versa. Furthermore, one observes that, if (Y, X) is a critical point of the reduced system, then w, as defined by (3.8), is constant, say $w(t)=v$. So the condition for (Y,X) to be an equilibrium is obtained by setting equal to zero the r.h.s. of the first two equations of (3.7), with $w(\Pi, Y, X)=v$. By eliminating Y, this gives equation (2.6), which has a unique solution $X_v \in H_*^{\{1\}}$ if and only if $\|v\|_{\mathbb{R}^3}<c$, as discussed above. From the second equation of (3.7), one then gets eq. (2.7), which gives the value Y_v of Y. Finally, from (3.8), with $w=v$, $Y=Y_v$, $X=X_v$, one obtains the value Π_v of Π, and then the corresponding value p_v of p. It is also easy to check that (Y_v, X_v, Π_v) depends continuously on v. □

In order to prove theorem 2.3 we need the following

LEMMA 3.2. – *Fix $v\in\mathbb{R}^3$ with $\|v\|_{\mathbb{R}^3}<c$, and let $\Pi_v\in\mathbb{R}^3$ be as above (namely defined by* (3.8) *with $w=v$, $Y=Y_v$, $X=X_v$). Then the critical point (Y_v, X_v) of the dynamical system with Hamiltonian* (3.9) *(where Π is considered as a parameter) is stable.*

Proof. – Make the trivial translation $Y':=Y-Y_v$, $X':=X-X_v$. Omitting primes, constant terms and the argument of the field variables, the Hamiltonian takes the form

$$H=\int\left(2\pi c^2 Y^2-\frac{1}{8\pi}\langle X, \Delta X\rangle\right)d^3x+\left\langle v, \int\sum_{l=1}^{3} Y_l \nabla X_l d^3x\right\rangle$$
$$+\frac{1}{2m}\left\|\int\left(\sum_{l=1}^{3}(Y_l\nabla X_{vl}+Y_{vl}\nabla X_l)-\rho X+\sum_{l=1}^{3} Y_l\nabla X_l\right)d^3x\right\|_{\mathbb{R}^3}^2. \quad (3.10)$$

This is a good Lyapunov function. Indeed, with norms defined by

$$\|Y\|^2:=2\pi c^2\|Y\|_{L^2}^2, \qquad \|X\|^2:=-\frac{1}{8\pi}\int\langle X,\Delta X\rangle d^3x,$$

one has that the modulus of the sum of the terms appearing in the first line of (3.10) is larger than

$$\|Y\|^2+\|X\|^2-\frac{2\|v\|_{\mathbb{R}^3}}{c}\|X\|\,\|Y\|$$

(use Schwartz inequality); this in turn is easily seen to be larger than

$$(\|X\|^2+\|Y\|^2)\left(1-\frac{\|v\|_{\mathbb{R}^3}}{c}\right).$$

Since $\|v\|_{\mathbb{R}^3}<c$, there follows that zero is an isolated minimum of H, and also that H satisfies the so called "potential well' condition [21]. So zero is a stable critical point for the considered dynamical system (see Marsden Hughes [21], theorem 6.3). □

We come now to the

Proof of theorem 2.3. – Take $v' := \dot{q}_0$, and construct $X_{v'}$, $Y_{v'}$, $\Pi_{v'}$ as above; by continuity of the application $v \mapsto (X_v, Y_v, \Pi_v)$, we have that $X_{v'}$, $Y_{v'}$, $\Pi_{v'}$ is near X_v, Y_v, Π_v. Then, by lemma 3.2, under the hypotheses of theorem 2.3, for $\| v - v' \|_{\mathbb{R}^3} < \delta$ and δ small enough, we have

$$\| X(t) - X_v \| \leqq \| X(t) - X_{v'} \| + \| X_{v'} - X_v \| < \varepsilon, \tag{3.11}$$

and similarly for $Y(t)$. Using the last equation of (3.7) and continuity of w as a $\mathbb{R}^3$-valued function on phase space $\mathcal{F}$, we get

$$\| \dot{q}(t) - v' \|_{\mathbb{R}^3} \leqq C_2 (\| Y(t) - Y_{v'} \| + \| X(t) - X_{v'} \|),$$

from which, going back to the variables (E, A, p, q), the proof is easily obtained. □

Finally, we give the

Proof of theorem 2.4 *and of proposition* 2.5. – Introduce the Fourier transform $\hat{\rho}_q(k)$ of the charge distribution $\rho_q(x) = \rho(x - q)$:

$$\rho_q(x) = \frac{\sqrt{2}}{(2\pi)^{3/2}} \int_S (\hat{\rho}_q(k) \cos \langle k, x \rangle + \hat{\rho}_q(-k) \sin \langle k, x \rangle) \, d^3 k,$$

and write the Hamiltonian in terms of the Fourier transforms $\hat{E}^j(k)$, $\hat{A}^j(k)$ of the fields [*see* (2.9)]:

$$H = \sum_{j=1,2} \int_{\mathbb{R}^3} \frac{1}{2} (\hat{E}^j(k)^2 + \omega(k)^2 \hat{A}^j(k)^2) \, d^3 k + \frac{1}{2m} \left[p - \sum_{j=1,2} \int_{\mathbb{R}^3} 2\sqrt{\pi} \, c \, \hat{\rho}_q(k) \hat{A}^j(k) e^j(k) \, d^3 k \right]^2; \tag{3.12}$$

here $\hat{E}^j(k)$, $\hat{A}^j(k)$ play the role of canonical conjugate coordinates (*see* [2]). Calculating the Poisson brackets of $\eta^j(k)$ with the Hamiltonian, one obtains

$$\dot{\eta}^j(k) = \frac{2\sqrt{\pi} \, c}{m} \hat{\rho}_q(k) \hat{E}^j(k) \langle w', e^j(k) \rangle,$$

where w' is the vector appearing in the square brackets in (3.12). Define now a function $\rho_\sharp(\omega)$ in such a way that

$$\operatorname{Sup}_{\| k \|_{\mathbb{R}^3} = \omega / c} | \hat{\rho}_q(k) | \leqq \rho_\sharp(\omega);$$

then, by energy conservation one gets

$$\| w' \|_{\mathbb{R}^3} \leqq \sqrt{2 m \mathscr{E}_0},$$

where $\mathscr{E}_0$ is the initial value of the total energy; moreover, one has $|\hat{E}^j(k)| \leqq \sqrt{2\eta^j(k)}$. From this one gets

$$\dot{\eta}^j(k) \leqq 2\,C\,\rho_\sharp(c\,\|k\|_{\mathbb{R}^3})\sqrt{\eta^j(k)}$$
$$-\dot{\eta}^j(k) \leqq 2\,C\,\rho_\sharp(c\,\|k\|_{\mathbb{R}^3})\sqrt{\eta^j(k)},$$

where

$$C := 2c\sqrt{\frac{\pi\mathscr{E}_0}{m}}; \tag{3.13}$$

solving the above pair of differential inequalities, one obtains

$$\left|\sqrt{\eta^j(k,t)} - \sqrt{\eta^j(k,0)}\right| \leqq t\,C\,\rho_\sharp(c\,\|k\|_{\mathbb{R}^3}).$$

Then, a simple calculation shows that, for any positive $\delta < 1$, one has

$$|\eta^j(k,t) - \eta^j(k,0)| \leqq C\,\tau'\,\rho_\sharp(\omega(k))^\delta\,[2\sqrt{\eta^j(k,0)} + C\,\tau'\,\rho_\sharp(\omega(k))^\delta], \tag{3.14}$$

for all times t with

$$|t| \leqq \tau'\,\rho_\sharp(\omega(k))^{\delta-1}, \tag{3.15}$$

and τ' an arbitrary dimensional parameter. Considering separately the cases where the argument of the modulus at the l.h.s. of (3.14) is positive or negative, integrating both sides of the so obtained inequalities over a sphere of radius ω/c in k-space, and using Schwartz inequality, one obtains, for times satisfying (3.15), the bound

$$|\eta(\omega,t) - \eta(\omega,0)| \leqq \frac{8\,C\,\tau'}{c^2}\,\omega^2\,\rho_\sharp(\omega)^\delta\left(\sqrt{\frac{\pi\eta(\omega,0)}{c}} + \pi\,C\,\tau'\,\rho_\sharp(\omega)^\delta\right).$$

A simple reformulation of this statement, the choice $\delta = 1/2$, the remark that for an analytic function one can take

$$\rho_\sharp(\omega) = \rho_*\exp\left(-\frac{\sigma}{c}\,\omega\right),$$

and the further choice $\tau' = \tau\sqrt{\rho_*}$ (with ρ_*, σ and τ as in the statement of the theorem), gives theorem 2.4 with

$$K_1 := \frac{16}{c}\,\rho_*\sqrt{\frac{\pi\mathscr{E}_0}{m}}\left[\sqrt{\frac{\pi\eta(\omega,0)}{c}} + 2\,\pi\tau\rho_*\,c\sqrt{\frac{\pi\mathscr{E}_0}{m}}\exp\left(-\frac{\sigma}{2c}\,\omega\right)\right]; \tag{3.16}$$

we recall that $\mathscr{E}_0$ is the total energy of the system.

The remark that, for C^∞ functions $\rho_\sharp(\omega)$, decreases more rapidly than any power of ω gives proposition 2.5. □

4. CONCLUSIONS

We add now some comments. First of all, we would like to make clear which is the philosophy of the present paper. The aim is to deal with classical electrodynamics as a dynamical system; so, for example, we renounce to deal with the field "created" by the particle as an "*a priori* object", and prominence is given to the Cauchy problem for the whole system, where the field as a dynamical object has the same dignity as the particle. Obviously one would like to discuss relativistic models. The nonrelativistic Pauli-Fierz model was chosen here just as a concrete simple definite model, having however historical relevance, from which to start out.

From such a point of view one might think that some significance can be attributed to the simple result just found that, even in the present non relativistic model, uniform rectilinear motions can exist only if the particle has velocity smaller than c; indeed, this seems to indicate that such a limitation on the particle velocity is somehow a dynamical consequence of the particle's interaction with the electromagnetic field (*see* also sect. 35 of ref. [22]). Concerning such solutions with uniform motion, another interesting remark is the following: while in a relativistic version of the theory their existence would be a trivial consequence of the Lorentz invariance, here instead such motions appear as solitary waves solutions of a nonlinear problem, so that their dynamical character is stressed. On the other hand, even in a relativistic framework, where through Lorentz transformation the problem is reduced to the study of the particular solution with $v=0$, the stability properties of the latter solution are to be established in a nontrivial way, for example by proving stability in the ordinary sense for the critical point of a suitable reduced system, as was done here.

Furthermore, the stability properties of the solutions describing uniform motion might also be of interest in connection with the problem of the so called run-away solutions ([23], [24]): namely, one should make clear whether the existence of run-away solutions is indeed a dynamical property of the complete system. In fact, this discussion would require the study of an Hamiltonian of the type (2.1), with however a negative value for the bare mass m of the charged particle, while in the present paper we limited ourselves to the case of positive m. We intend to discuss such a problem elsewhere.

Finally, we add a remark concerning the freezing of the high frequencies. This point is usually quoted as evidently expected *a priori* [6]. But only a rigorous analytical treatment can give the quantitative information obtained here, namely that the cutoff frequency, above which freezing is guaranteed, increases logarithmically with time. By the way, this seems to be in agreement with some qualitative statements made by Jeans at the turn of the century [25].

Note. – After completing this work we got knowledge, from Andrea Posilicano, of some papers on the stability of solitary waves ([26], [27], [28]). We point out that since the theorem of ref. [27] gives a necessary and sufficient condition for the stability of a solitary wave, it applies also to our particular solution. However, in the present case it seems to be simpler to prove stability by our method, rather than to check the hypotheses of that theorem.

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