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# On the instability of ground states for a Davey-Stewartson system

by

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ABSTRACT. – In this work we study the instability properties of ground-states for the equation

$$iu_t + \Delta u + b\,E_1(|u|^2)\,u - a\,|u|^\alpha u = 0 \qquad (*)$$

in $\mathbb{R}^2$ and $\mathbb{R}^3$. We prove that the set

$$\Omega_\varphi = \{ e^{i\theta} \varphi(\,.\,+y) \,|\, \theta \in \mathbb{R},\ y \in \mathbb{R}^N \}$$

(where $\varphi$ is a ground-state) is unstable by the flow of the equation $(*)$ provided $a(\alpha-2) \leqq 0$.

RÉSUMÉ. – Dans cet article nous étudions l'instabilité des ground-states pour l'équation

$$iu_t + \Delta u + b\,E_1(|u|^2)\,u - a\,|u|^\alpha u = 0 \qquad (*)$$

dans $\mathbb{R}^2$ and $\mathbb{R}^3$. Nous prouvons que l'ensemble

$$\Omega_\varphi = \{ e^{i\theta} \varphi(\,.\,+y) \,|\, \theta \in \mathbb{R},\ y \in \mathbb{R}^N \}$$

(où $\varphi$ est ground-state) est instable pour le flux de l'équation $(*)$ si $a(\alpha-2) \leqq 0$.

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## 1. INTRODUCTION

In this paper we study the instability of Ground-States for the equation

$$iu_t + \Delta u + b\,\mathrm{E}_1(|u|^2)u - a|u|^\alpha u = 0 \tag{1.1}$$

where $\Delta$ is the usual Laplacian operator in $\mathbb{R}^n$, $\alpha$ and $b$ are positive constants, $a \in \mathbb{R}$ and $\mathrm{E}_1$ is the pseudo-differential operator with symbol $\sigma_1(\xi) = \xi_1^2/|\xi|^2$, $\xi \in \mathbb{R}^N$.

The equation (1.1) has its origin in fluid mechanics where, for $\alpha = N = 2$, it describes the evolution of weakly nonlinear water waves having a predominant direction of travel. More precisely, (1.1) is the N-dimensional extension of the Davey-Stewartson systems in the elliptic-elliptic case, namely

$$\left.\begin{aligned} iu_t + \lambda u_{xx} + \mu u_{yy} &= a|u|^\alpha u + b_1 uv_x \\ \nu v_{xx} + v_{yy} &= -b_2(|u|^2)_x \end{aligned}\right\} \tag{1.2}$$

($\lambda$, $\mu$ and $\nu > 0$) which describes the time evolution of two-dimensional surface of water waves having a propagation preponderantly in the $x$-direction (*see* [6]).

The Cauchy problem for the D. S. systems in all physical relevant cases has been studied in [7] by functional analytical methods. The standing waves for equation (1.1) has been treated by the author in [5], with the existence of Ground-States obtained by means of P.-L. Lions's concentration-compactness method. By standing waves we mean special periodic solutions of the form

$$u(t, x) = e^{iwt}\varphi(x), \tag{1.3}$$

where $\omega \in \mathbb{R}$ and $\varphi \in \mathrm{H}^1(\mathbb{R}^N)$. The so-called Ground-States are standing waves which minimize the action among all nontrivial solutions of the form (1.3).

The problem of stability and instability of standing waves for nonlinear Schrödinger equations has been studied by several authors. Let us mention the papers of Berestycki-Cazenave [1] on the instability of ground states, Cazenave-Lions [4] for the existence of stable standing waves and the papers of Cazenave [2], Shatah-Stauss [10] and Grillakis-Shatah-Strauss [9].

With the meaning of stability to be precised later, we may summarize our main result as follows (*see* theorem 3.15):

THEOREM. — *Let* $N \in \{2, 3\}$ *and assume* $a(\alpha - 2) \leqq 0$. *Then all Ground-States for equation* (1.1) *are unstable.*

We organize this paper as follows: in section 2 we introduce the notation and we briefly review some results on the existence of solutions for the Cauchy problem for equation (1.1) and existence and regularity for its standing waves. In section 3 we state and prove our results concerning the instability of Ground-States. Let us mention the theorem 3.12 which gives sufficient conditions for the instability; it is essentially due to Gonçalves Ribeiro (*see* [8]).

We finish this paper with an appendix where the concavity criterion of Grillakis-Shatah-Strauss is stablished for our problem in the case $N=2$.

## 2. EXISTENCE OF SOLUTIONS

In this section we shall briefly review some results about the existence of solutions for the initial value problem and the existence of standing waves for the equation (1.1).

Throughout this paper we consider the $L^p$ spaces, $1 \leqq p \leqq +\infty$, of complex functions on $\mathbb{R}^N$ with their usual norms denoted by $|.|_p$. We consider $L^2$ as a real Hilbert space endowed with the scalar product

$$(u;\, v)_2 = \int \Re\,(u(x)\,\overline{v(x)})\,dx, \tag{2.1}$$

and we denote by $H^s$ and $W^{m,p}$ the usual Sobolev spaces on $\mathbb{R}^N$. In particular we denote by $\|.\|$ the usual norm of $H^1$ related to (2.1). We also denote by $\operatorname{sob}(N)$ the Sobolev exponent, *i.e.*,

$$\operatorname{sob}(N) = \begin{cases} 2N/(N-2) & \text{if} \quad N \geqq 3, \\ +\infty & \text{if} \quad N=2. \end{cases}$$

Then we can state the following result about the local existence of weak solutions of the Cauchy problem for (1.1) in the energy space $H^1$ (*see* [3], thm. 4.3.1, p. 65).

THEOREM 2.1. – *Let* $N \in \{2, 3\}$ *and* $\alpha \in ]0, \operatorname{sob}(N)-2[$. *Then the following holds:*

(i) *For any* $u_0 \in H^1$, *there exist* $T^*$, $T_* > 0$ *and a unique maximal solution of* (1.1)

$$u \in C(]-T_*, T^*[;\, H^1) \cap C^1(]-T_*, T^*[;\, H^{-1})$$

*such that* $u(0)=u_0$. *The maximality is in the sense that* $\lim_{t \uparrow T^*} \|u(t)\| = +\infty$ *if* $T^* < +\infty$ (*resp.* $\lim_{t \downarrow -T_*} \|u(t)\| = +\infty$ *if* $T_* < +\infty$).

(ii) *We have conservation of charge and energy, that is*

$$\left.\begin{aligned} |u(t)|_2 &= |u_0|_2 \\ \mathscr{E}(u(t)) &= \mathscr{E}(u_0) \end{aligned}\right\} \tag{2.2}$$

*for all* $t \in ]-T_*, T^*[$, *where*

$$\mathcal{E}(u) = \frac{1}{2}|\nabla u|_2^2 - \frac{b}{4}\int |u|^2 E_1(|u|^2) + \frac{a}{\alpha+2}|u|_{\alpha+2}^{\alpha+2}.$$

For more specific results concerning the Cauchy problem for the Davey-Stewartson systems, we refer the reader to [7].

In order to simply the notation, from now on we will denote by $U(t)u_0$ the maximal solution $u(t)$ of (1.1) which satisfies the initial condition $u_0$.

Looking for standing waves for equation (1.1) leads us to consider the stationary equation

$$-\Delta\varphi + \omega\varphi = bE_1(|\varphi|^2)\varphi - a|\varphi|^\alpha\varphi, \tag{2.3}$$

whose solutions are critical points of the Lagrangian S defined by (*see* [5])

$$S(\varphi) = \frac{1}{2}|\nabla\varphi|_2^2 + \frac{\omega}{2}|\varphi|_2^2 - \frac{b}{4}\int |\varphi|^2 E_1(|\varphi|^2) + \frac{a}{\alpha+2}|\varphi|_{\alpha+2}^{\alpha+2}.$$

Let us introduce the following sets

$$\mathcal{X} = \{\psi \mid \psi \neq 0,\ \psi \text{ solves } (2.3)\},$$
$$\mathcal{G} = \{\varphi \in \mathcal{X} \mid S(\varphi) \leqq S(\psi),\ \forall \psi \in \mathcal{X}\},$$

($\mathcal{G}$ being the set of Ground-States).

For $\omega$ and $b$ given positive constants, let

$$\mathcal{R}_{\omega, b} = \{(\alpha, a) \mid 0 < \alpha < \mathrm{sob}(N),\ a < a_\alpha^*\},$$

where

$$a_\alpha^* = \begin{cases} +\infty & \text{if } \alpha < 2, \\ b & \text{if } \alpha = 2, \\ b^{\alpha/2}\omega^{(2-\alpha)/2}\left(\dfrac{2}{\alpha}\right)\left(\dfrac{\alpha-2}{\alpha}\right)^{(\alpha-2)/2} & \text{if } \alpha > 2. \end{cases}$$

If we consider the functionals

$$T(\varphi) = |\nabla\varphi|_2^2,$$
$$V(\varphi) = \frac{b}{4}\int |\varphi|^2 E_1(|\varphi|^2) - \frac{a}{\alpha+2}|\varphi|_{\alpha+2}^{\alpha+2} - \frac{\omega}{2}|\varphi|_2^2,$$

then we can state the following result about the existence of Ground-States for (1.1)

THEOREM 2.2. – *Let* $N \in \{2, 3\}$ *and* $(\alpha, a) \in \mathscr{R}_{\omega, b}$. *Then the following holds:*

(i) *If* $N=2$ *then* $\varphi \in \mathscr{G}$ *if and only if* $\varphi$ *solves the following minimization problem:*

$$\left.\begin{aligned} &\varphi \in \Sigma_0, \\ &T(\varphi)=\inf\{T(\psi) \mid \psi \in \Sigma_0\}, \end{aligned}\right\} \tag{2.4}$$

*where* $\Sigma_0=\{\psi \in H^1 \mid \psi \neq 0,\ V(\psi)=0\}$.

(ii) *If* $N=3$ *then there exists a constant* $\mu_0>0$ *such that* $\varphi \in \mathscr{G}$ *if and only if* $\varphi$ *solves the following minimization problem:*

$$\left.\begin{aligned} &\varphi \in \Sigma_{\mu_0}, \\ &T(\varphi)=\inf\{T(\psi) \mid \psi \in \Sigma_{\mu_0}\}, \end{aligned}\right\} \tag{2.5}$$

*where* $\Sigma_{\mu_0}=\{\psi \in H^1 \mid \psi \neq 0,\ V(\psi)=\mu_0\}$.

(iii) *Problems* (2.4) *and* (2.5) *have solutions.*

(iv) *For all* $\varphi \in \mathscr{G}$, *there exists a real valued positive function* $\varphi_0 \in \mathscr{G}$ *such that* $\varphi=e^{i\theta}\varphi_0$.

*Proof.* – For the proofs of (i), (ii) and (iii) we refer the reader to [5], theorems 2.1 and 2.2. To prove (iv), we may proceed as in Cazenave [3], p. 171. □

We have the following result concerning the regularity of solutions of (2.3):

THEOREM 2.3. – *If* $N \in \{2, 3\}$, $\alpha \in ]0, \operatorname{sob}(N)-2[$ *and* $\varphi$ *is a solution of* (2.3), *then the following holds:*

(i) $\varphi \in W^{3,p}$, $\forall p \in [2, \infty[$.

(ii) $\lim_{|x| \to \infty} \{|\nabla \varphi(x)|+|\varphi(x)|+|E_1(|\varphi|^2)(x)|\}=0$.

(iii) *There exist positive constants* C *and* $\nu$ *such that*

$$e^{\nu|x|}\{|\varphi(x)|+|\nabla \varphi(x)|\} \leqq C, \qquad \forall x \in \mathbb{R}^N.$$

(iv) *Moreover, if* $\alpha>1$, *then* (i)-(iii) *hold for* $\partial_j \varphi$, $\forall j=1, \ldots, N$.

*Proof.* – For the proof of (i)-(iii) we refer reader to [5], thm. 2.4.

In order to prove (iv), we note that $\psi=\partial_i \varphi$ satisfies the equation

$$-\Delta\psi+\omega\psi=bE_1(|\varphi|^2)\psi+2bE_1(\Re(\varphi\bar{\psi}))\varphi \\ -\alpha a|\varphi|^{\alpha-2}\varphi\,\Re(\varphi\bar{\psi})-a|\varphi|^{\alpha}\psi,$$

and the arguments in the proof of the theorem 2.4 in [5] are applicable, provided $\alpha>1$. □

We end this section with the following

LEMMA 2.4. – *Let* $N \in \{2, 3\}$, $\omega$, *b positive constants*, $(\alpha, a) \in \mathscr{R}_{\omega, b}$ *and* $\varphi \in \mathscr{G}$. Let $\bar{\mu}=0$ *if* $N=2$ *and* $\bar{\mu}=\mu_0$ *if* $N=3$. *Then the sets*

$$\mathscr{A}_\varphi^-=\{v \in H^1 \mid S(v)<S(\varphi),\ V(v)<\bar{\mu}\}, \tag{2.6}$$
$$\mathscr{A}_\varphi^+=\{v \in H^1 \mid S(v)<S(\varphi),\ V(v)>\bar{\mu}\}, \tag{2.7}$$

*are invariant regions under the flow of* U. *Moreover, if* $u_0 \in \mathscr{A}_\varphi^-$ *then* $U(t)u_0$ *is global.*

*Proof.* – Let $u_0 \in \mathscr{A}_\varphi^\pm$. Then $S(U(t)u_0) = S(u_0) < S(\varphi)$. By the definition of $\bar{\mu}$ and Theorem 2.3 we have $V(U(t)u_0) \neq \bar{\mu}$. Since the function $t \mapsto V(U(t)u_0)$ is continuous on $]-T_*, T^*[$, we have $V(U(t)u_0) < \bar{\mu}$ if $u_0 \in \mathscr{A}_\varphi^-$ or $V(U(t)u_0) > \bar{\mu}$ if $u_0 \in \mathscr{A}_\varphi^+$, $\forall t \in ]-T_*, T^*[$.

Assuming now that $u_0 \in \mathscr{A}_\varphi^-$, we have

$$\frac{1}{2}|\nabla U(t)u_0|_2^2 = S(U(t)u_0) + V(U(t)u_0) < S(\varphi) + \bar{\mu}$$

and we conclude the proof. □

## 3. INSTABILITY

For any $X \subset H^1$ and $\delta > 0$, we define $\mathscr{V}(X, \delta)$ the $\delta$-neighborhood of X in $H^1$ by

$$\mathscr{V}(X, \delta) = \bigcup_{v \in X} B_\delta(v),$$

where $B_\delta(v)$ is the ball in $H^1$; $B_\delta(v) = \{u \in H^1 \mid \|u - v\| < \delta\}$.

DEFINITION 3.1. – *We say that* $X \subset H^1$ *is stable by the flow of* U *if and only if, for any* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that, for all* $u_0 \in \mathscr{V}(X, \delta)$, $T^*(u_0) = +\infty$ *and* $U(t)u_0 \in \mathscr{V}(X, \varepsilon)\ \forall t \geq 0$.

If $u$ is a periodic solution of (1.1), then orbital stability of $u$ means stability of the closed orbit $\{u(t) \mid t \in \mathbb{R}\}$ by the flow of U.

*Remark* 3.2. – Due to the invariances under translation and multiplication by $e^{i\theta}$, we never have orbital stability for the standing waves for equation (1.1). Indeed, let $\varphi \in \mathscr{X}$, $\omega > 0$ and consider the closed orbit $\mathcal{O}_\varphi = \{e^{i\omega t}\varphi \mid t \in \mathbb{R}\}$. Given $\varepsilon > 0$ and a unitary $y \in \mathbb{R}^N$, let $u_\varepsilon(t, x) = e^{i\varepsilon(x \cdot y - \varepsilon t) + i\omega t}\varphi(x - 2\varepsilon t y)$. Then we easily verify that $u_\varepsilon(t) = U(t)\varphi_\varepsilon$, where $\varphi_\varepsilon(x) = e^{i\varepsilon x \cdot y}\varphi(x)$. Furthermore, $\varphi_\varepsilon \to \varphi$ in $H^1$ as $\varepsilon \downarrow 0$, but for $\varepsilon > 0$,

$$|u_\varepsilon(t) - e^{i\theta}\varphi|_2^2 \geq 2|\varphi|_2^2 - 2(|\varphi| * |\tilde{\varphi}|)(2\varepsilon t y),$$

where $\tilde{\varphi}(x) = \varphi(-x)$. Since $\varphi(x) \to 0$ as $|x| \to \infty$ (*see* theorem 2.3), we infer from Lebesgue's theorem that $(|\varphi| * |\tilde{\varphi}|)(sy) \to 0$ as $s \uparrow \infty$ and we conclude that $\underline{\lim}_{t \to \infty} \operatorname{dist}(u_\varepsilon(t); \mathcal{O}_\varphi) > |\varphi|_2$.

For each $y \in \mathbb{R}^N$, let $\tau_y$ the translation operator defined by $\tau_y v = v(. + y)$. If $Y \subset L^p$, we denote $\Omega_Y = \{e^{i\theta}\tau_y v \mid \theta \in \mathbb{R}, y \in \mathbb{R}^N, v \in Y\}$. With this notation we introduce the following set which is a natural extension of the closed orbit $\mathcal{O}_\varphi$ in the study of stability in our context:

$$\Omega_\varphi = \{ e^{i\theta} \tau_y \varphi \mid \theta \in \mathbb{R},\ y \in \mathscr{R}^N \}.$$

In fact, it is well known that (*see* [3]), in the pure power case (which corresponds in our context to the case $b=0$ and $a<0$), $\Omega_\varphi$ is stable by the flow if $0<\alpha<4/N$. Moreover, $\Omega_\varphi$ is unstable if $\alpha \geqq 4/N$ provided $\varphi \in \mathscr{G}$.

LEMMA 3.3. – *For each* $Y \in H^1$ *and* $\delta > 0$, *we have* $\mathscr{V}(\Omega_Y, \delta) = \Omega_{\mathscr{V}(Y, \delta)}$.

*Proof.* – Let $u \in \mathscr{V}(\Omega_Y, \delta)$. Then there exist $\theta \in \mathbb{R}$, $y \in \mathbb{R}^N$ and $w \in Y$ such that

$$\| u - e^{i\theta} \tau_y w \| = \| e^{-i\theta} \tau_{-y} u - \omega \| < \delta.$$

Hence $e^{-i\theta} \tau_{-y} u \in \mathscr{V}(Y, \delta)$ and consequently $u \in \Omega_{\mathscr{V}(Y, \delta)}$.

Conversely, if $u \in \Omega_{\mathscr{V}(Y, \delta)}$, then $u = e^{i\theta} \tau_y v$ for some $\theta \in \mathbb{R}$, $y \in \mathbb{R}^N$ and $v \in \mathscr{V}(Y, \delta)$. Hence, there exists $w \in Y$ such that $\| w - v \| < \delta$ and the conclusion follows since $\| u - e^{i\theta} \tau_y w \| = \| e^{-i\theta} \tau_{-y} - w \| = \| v - w \| < \delta$. $\square$

For each $v \in H^1$, we introduce the symmetric matrix $\mathscr{A}(v) = (a_{jk})_{j,k=0}^N$, where

$$a_{00} = | v |_2^2, \qquad a_{0k} = (iv;\ \partial_k v)_2 \qquad \text{and} \qquad a_{jk} = (\partial_j v;\ \partial_k v)_2.$$

LEMMA 3.4. – *Let* $\varphi \in H^2$ *such that* $\mathscr{A}(\varphi)$ *is strictly positive definite. Then there exists* $\varepsilon_0 > 0$ *such that, for all* $v \in \mathscr{V}(\Omega_\varphi, \varepsilon_0)$, *there exists a unique* $\mathscr{N}(v) \in \Omega_\varphi$ *satisfying*

$$| v - \mathscr{N}(v) |_2 \leqq | v - w |_2, \qquad \forall w \in \Omega_\varphi. \tag{3.1}$$

*Moreover, the function* $\mathscr{N} : \mathscr{V}(\Omega_\varphi, \varepsilon_0) \to \Omega_\varphi$ is $C^2$ *with derivative* $\mathscr{N}' : \mathscr{V}(\Omega_\varphi, \varepsilon_0) \to \mathscr{L}(H^1;\ H^1)$ *given by*

$$\mathscr{N}'(v)\phi = \langle \Lambda_0'(v);\ \phi \rangle i \mathscr{N}(v) + \sum_{j=1}^N \langle \Lambda_j'(v);\ \phi \rangle \partial_j \mathscr{N}(v), \tag{3.2}$$

*where* $\Lambda_j : \mathscr{V}(\Omega_\varphi, \varepsilon_0) \to \mathbb{R}$ *are* $C^2$ *functionals satisfying for* $j=1, \ldots, N$, $\theta \in \mathbb{R}$, $y \in \mathbb{R}^N$, $v \in \mathscr{V}(\Omega_\varphi, \varepsilon_0)$:

$$\Lambda_0(e^{i\theta} \tau_y v) = \Lambda_0(v) + \theta \pmod{2\pi},$$
$$\Lambda_j(e^{i\theta} \tau_y v) = \Lambda_j(v) + y_j.$$

*In particular, for* $v \in \Omega_\varphi$, $j=0, \ldots, N$ *and* $k=1, \ldots, N$ *we have*

$$(\Lambda_j'(v);\ v)_2 = 0, \tag{3.3}$$
$$(\Lambda_0'(v);\ iv)_2 = 1, \qquad (\Lambda_k'(v);\ iv)_2 = 0, \tag{3.4}$$
$$(\Lambda_j'(v);\ \partial_k v)_2 = \delta_{jk}. \tag{3.5}$$

*Proof.* – Let $\varepsilon > 0$ and consider the function

$$F: \quad B_\varepsilon(\varphi) \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R},$$

$$(v, \theta, y) \mapsto \frac{1}{2} | e^{i\theta} \tau_y \varphi - v |_2^2.$$

Since $\varphi \in H^2$, we easily verify that $F \in C^3$ in all its variables and

$$\partial_\theta F(v, \theta, y) = -(ie^{i\theta}\tau_y \varphi; v)_2, \tag{3.6}$$
$$\partial_{y_j} F(v, \theta, y) = -(e^{i\theta}\tau_y \partial_j \varphi; v)_2, \tag{3.7}$$
$$\partial^2_{\theta\theta} F(v, \theta, y) = (e^{i\theta}\tau_y \varphi; v)_2,$$
$$\partial^2_{\theta y_j} F(v, \theta, y) = -(ie^{i\theta}\tau_y \partial_j \varphi; v)_2,$$
$$\partial^2_{y_j y_k} F(v, \theta, y) = -(e^{i\theta}\tau_y \partial_{jk} \varphi; v)_2.$$

Denoting by $\text{Grad } F = (\partial_\theta F, \partial_{y_1} F, \ldots, \partial_{y_N} F)$ and by Hess F the hessian matrix of F in the variables $(\theta, y)$, we get

$$\left.\begin{aligned} \text{Grad } F(\varphi, 0, 0) &= 0, \\ \text{Hess } F(\varphi, 0, 0) &= \mathscr{A}(\varphi). \end{aligned}\right\}$$

It follows from the implicit function theorem that there exist $\varepsilon_0 > 0$, $\theta_0 > 0$ and $R > 0$ and $N+1$ functions of class $C^2$

$$\left.\begin{aligned} \Lambda_0 :\ & B_{\varepsilon_0}(\varphi) \to ]-\theta_0, \theta_0[, \\ \Lambda_j :\ & B_{\varepsilon_0}(\varphi) \to ]-R, R[, \qquad j = 1, \ldots, N, \end{aligned}\right\} \tag{3.8}$$

such that, for all $v \in B_{\varepsilon_0}(\varphi)$ we have:

$$\text{Grad } F(v, \Lambda_0(v), \Lambda(v)) = 0, \tag{3.9}$$
$$\text{Hess } F(v, \Lambda_0(v), \Lambda(v)) \text{ is a strictly positive definite matrix,} \tag{3.10}$$

where $\Lambda(v) = (\Lambda_1(v), \ldots, \Lambda_N(v))$. Therefore, if we denote by

$$\mathscr{N}(v) = e^{i\Lambda_0(v)}\tau_{\Lambda(v)}\varphi, \tag{3.11}$$

then, from (3.9) and (3.10), we infer that, for $v \in B_{\varepsilon_0}(\varphi)$, $\mathscr{N}(v)$ is the unique element of $B_{\varepsilon_0}(\varphi)$ satisfying

$$|v - \mathscr{N}(v)|_2 \leqq |v - w|_2,$$

for all $w \in \Omega_\varphi \cap B_{\varepsilon_0}(\varphi)$. Moreover, since

$$F(e^{i\theta'}\tau_{y'} v, \theta' + \theta, y + y') = F(v, \theta, y)$$

for all $\theta', \theta \in \mathbb{R}$, $y', y \in \mathbb{R}^N$, it follows from Lemma 3.3 that $\Lambda_0$ and $\Lambda$ may be extended to $\mathscr{V}(\Omega_\varphi, \varepsilon_0)$ in such a way that $\Lambda_0(e^{i\theta}\tau_y v) = \Lambda_0(v) + \theta \pmod{2\pi}$, $\Lambda_j(e^{i\theta}\tau_y v) = \Lambda_j(v) + y_i$, $\forall v \in B_{\varepsilon_0}(\varphi)$, $\forall \theta \in \mathbb{R}$, $\forall y \in \mathbb{R}^N$ and (3.1) holds.

We obtain (3.2) easily by taking derivatives in (3.11).

Let $v \in \mathscr{V}(\Omega_\varphi, \varepsilon_0)$, $\phi \in H^1$. Then, we infer from (3.6) and (3.9) that $(v + s\phi; i\mathscr{N}(v + s\phi))_2 = 0$, for all $s \in \mathbb{R}$ sufficiently small. By differentiating this last identity in $s$, we obtain

$$(\mathscr{N}'(v)\phi; iv)_2 = (i\mathscr{N}(v); \phi)_2, \qquad \forall \phi \in H^1,$$

which, by (3.2), gives us

$$i\mathcal{N}(v)=(v;\,\mathcal{N}(v))_2\,\Lambda_0'(v)+\sum_{j=1}^{N}(iv;\,\partial_j\mathcal{N}(v))_2\,\Lambda_j'(v), \tag{3.12}$$

for all $v\in\mathcal{V}(\Omega_\varphi,\,\varepsilon_0)$.

The same arguments with (3.7), (3.9) and (3.2) gives us

$$\partial_j\mathcal{N}(v)=(iv;\,\partial_j\mathcal{N}(v))_2\,\Lambda_0'(v)+\sum_{k=1}^{N}(\partial_j v;\,\partial_k\mathcal{N}(v))_2\,\Lambda_k'(v). \tag{3.13}$$

In particular, for $v\in\Omega_\varphi$ we obtain

$$\left.\begin{aligned} iv&=|v|_2^2\,\Lambda_0'(v)+\sum_{j=1}^{N}(iv;\,\partial_j v)_2\,\Lambda_j'(v),\\ \partial_k v&=(iv;\,\partial_k v)_2\,\Lambda_0'(v)+\sum_{j=1}^{N}(\partial_j v;\,\partial_j v)_2\,\Lambda_j'(v).\end{aligned}\right\} \tag{3.14}$$

Since $(iv;\,v)_2=(\partial_j v;\,v)_2=0$ and $\mathcal{A}(v)=\mathcal{A}(\varphi)$ is invertible for all $v\in\Omega_\varphi$, we deduce (3.3)-(3.5) easily from (3.14). □

If $\varphi\in\mathcal{G}$, then it follows from Theorem 2.2-(iv) that $(i\varphi;\,\partial_j\varphi)_2=0$, $\forall j=1,\,\ldots,\,N$ and we easily verify from (3.14) that $\Lambda_0'(v)=\dfrac{iv}{|\varphi|_2^2}$ for $v\in\Omega_\varphi$. Moreover, by solving the system (3.14) we obtain $\Lambda_j'(v)$ as a linear combination of $\partial_1 v,\,\ldots,\,\partial_N v$ for all $v\in\Omega_\varphi$. In particular, for $N=2$ we obtain explicitly from (3.14) that, for every $v\in\Omega_\varphi$:

$$\Lambda_j'(v)=\frac{|\partial_k\varphi|_2^2\,\partial_j v-(\partial_1\varphi;\,\partial_2\varphi)_2\,\partial_k v}{|\partial_1\varphi|_2^2\,|\partial_2\varphi|_2^2-(\partial_1\varphi;\,\partial_2\varphi)_2^2},$$

where $(j,\,k)=(1,\,2),\,(2,\,1)$.

LEMMA 3.5. – *Let* $N\in\{2,\,3\}$. *If* $\varphi\in\mathcal{G}$, *then* $\mathcal{A}(\varphi)$ *is strictly positive definite.*

*Proof.* – $\mathcal{A}(\varphi)$ is always positive definite because, $\forall\,\vartheta\in\mathbb{R}^{N+1}$, we may write

$$\mathcal{A}(\varphi)\,\vartheta.\vartheta=|\vartheta_0\,i\varphi+\vartheta_1\,\partial_1\varphi+\ldots+\vartheta_N\,\partial_N\varphi|_2^2.$$

Assume that $\mathcal{A}(\varphi)\,\vartheta.\vartheta=0$ for some $\vartheta=(\vartheta_0,\,\vartheta')\neq 0$ and consider $f(t)=e^{i\vartheta_0 t}\,\varphi(t\vartheta')$. Since $\partial_t f(t)=e^{i\vartheta_0 t}(i\vartheta_0\,\varphi(t\vartheta')+\vartheta'.\nabla\varphi(t\vartheta'))$, we have

$$|\partial_t f(t)|_2^2=|i\vartheta_0\,\varphi(t\vartheta')+\vartheta'.\nabla\varphi(t\vartheta')|_2^2=t^{-N}\mathcal{A}(\varphi)\,\vartheta.\vartheta=0,$$

which is in contradiction with theorem 2.3. □

*Remark* 3.6. – It follows directly from (3.6), (3.7) and (3.9) that, for all $v\in\mathcal{V}(\Omega_\varphi,\,\varepsilon_0)$:

$$(v;\,i\mathcal{N}(v))_2=(v;\,\partial_j\mathcal{N}(v))_2=0.$$

In particular, from (3.2) we obtain for all $v \in \mathcal{V}(\Omega_\varphi, \varepsilon_0)$ and all $\phi \in H^1$:

$$(v; \mathcal{N}'(v)\phi)_2 = (\mathcal{N}(v); \mathcal{N}'(v)\phi)_2 = 0.$$

Consider the operator $\mathcal{P} : L^2 \times (L^2 \setminus \{0\}) \to L^2$ defined by

$$\mathcal{P}(v, w) = v - \frac{(v; w)_2}{|w|_2^2} w.$$

Note that $\mathcal{P}$ is the orthogonal projection (in the $L^2$ setting) of $v$ on the closed hyperplane $\{u \in L^2 \mid (u; w)_2 = 0\}$.

LEMMA 3.7. – *For all* $v \in \mathcal{V}(\Omega_\varphi, \varepsilon_0)$, the following holds:

$$|\mathcal{P}(v, \mathcal{N}(v))|_2 < \varepsilon_0, \tag{3.15}$$

$$|(\mathcal{N}(v); v)_2 - |\varphi|_2^2| < \varepsilon_0 |\varphi|_2. \tag{3.16}$$

*Proof.* – From the definition of $\mathcal{P}$ we obtain

$$v - \mathcal{N}(v) = \mathcal{P}(v, \mathcal{N}(v)) + \left(\frac{(v; \mathcal{N}(v))_2}{|\varphi|_2^2} - 1\right)\mathcal{N}(v).$$

From Lemma 3.4 and Pitagoras' Theorem, we have

$$\varepsilon_0^2 > |v - \mathcal{N}(v)|_2^2 = |\mathcal{P}(v, \mathcal{N}(v))|_2^2 + \left|\frac{(v; \mathcal{N}(v))_2}{|\varphi|_2^2} - 1\right|^2 |\mathcal{N}(v)|_2^2$$

and the conclusion follows since $|\mathcal{N}(v)|_2 = |\varphi|_2$. □

*Remark* 3.8. – It follows directly from Remark 3.6 that $(\mathcal{P}(v, \mathcal{N}(v)); \mathcal{N}'(v)\varphi)_2 = 0$ for every $v \in \mathcal{V}(\Omega_\varphi, \varepsilon_0)$ and $\phi \in H^1$. Moreover, from Theorem 2.2-(iv), the functions in the following sets are mutually orthogonal in $L^2$:

$$\{v, i\mathcal{N}(v), \partial_j \mathcal{N}(v)\},$$

$$\{i\mathcal{N}(v), \partial_j \mathcal{N}(v), \mathcal{P}(v, \mathcal{N}(v))\}.$$

For each $\psi \in H^1$ and $v \in \mathcal{V}(\Omega_\varphi, \varepsilon_0)$, we define

$$\mathcal{M}_\psi(v) = -ie^{i\Lambda_0(v)} \tau_{\Lambda(v)} \psi. \tag{3.17}$$

Note that $\mathcal{M}_\varphi(v) = -i\mathcal{N}(v)$. Therefore, we infer from (3.2) that

$$\mathcal{M}'_\psi(v)\phi = \langle \Lambda'_0(v); \phi \rangle i\mathcal{M}_\psi(v) + \sum_{j=1}^{N} \langle \Lambda'_j(v); \phi \rangle \partial_j \mathcal{M}_\psi(v). \tag{3.18}$$

LEMMA 3.9. – *Let* $\varphi \in H^2$ *be such that* $\mathcal{A}(\varphi)$ *is strictly positive definite. Then there exists* $\varepsilon_0 > 0$ *such that* $\forall u_0 \in \mathcal{V}(\Omega_\varphi, \varepsilon_0)$, *we can find* $T_{\varepsilon_0} > 0$ *such that*, $\forall \psi \in H^1$, $\exists C > 0$ *for which the solution* $u(t) = U(t)u_0$ *of* (1.1) *satisfies*

$$|(u(t); \mathcal{M}_\psi(u(t)))_2| \leq C, \qquad \forall t \in [0; T_{\varepsilon_0}[.$$

*Proof.* — Let $\varepsilon_0$ given in Lemma 3.4 and define

$$T_{\varepsilon_0}=\sup\{t>0 \mid U(\tau)u_0\in\mathscr{V}(\Omega_\varphi,\varepsilon_0),\ \forall\tau\in[0;t]\}.$$

From the continuity of $U(t)u_0$, we have $T_{\varepsilon_0}>0$ and the result follows easily from (2.2) and $|\mathscr{M}_\psi(v)|_2=|\psi|_2$. □

For each $\psi\in H^1$ we consider the functional

$$\mathscr{H}_\psi(v)=(v;\mathscr{M}_\psi(v))_2,\qquad \forall v\in\mathscr{V}(\Omega_\varphi,\varepsilon_0).$$

Then the following holds:

LEMMA 3.10. — *For each* $\psi\in H^1$, $\mathscr{H}_\psi$ is a $C^2$-*functional satisfying:*

$$\mathscr{H}_\psi(e^{i\theta}\tau_y v)=\mathscr{H}_\psi(v),\qquad \forall\theta\in\mathbb{R},\quad\forall y\in\mathbb{R}^N \tag{3.19}$$

$$(\mathscr{H}'_\psi(v);iv)_2=0,\qquad \forall v\in\Omega_\varphi \tag{3.20}$$

$$(\mathscr{H}'_\psi(v);\partial_j v)_2=0,\qquad \forall v\in\Omega_\varphi \tag{3.21}$$

*Proof.* — (3.19) follows from the properties of $\Lambda_0$ and $\Lambda$ (*see* Lemma 3.4). By differentiating $\mathscr{H}_\psi$ we obtain from (3.18) that

$$\mathscr{H}'_\psi(v)=\mathscr{M}_\psi(v)+(v;i\mathscr{M}_\psi(v))_2\Lambda'_0(v)+\sum_{j=1}^N(v;\partial_j\mathscr{M}_\psi(v))_2\Lambda'_j(v). \tag{3.22}$$

In particular, for $v=\varphi$ we have

$$\mathscr{H}'_\psi(\varphi)=-i\psi+(\varphi;\psi)_2\Lambda'_0(\varphi)+\sum_{j=1}^N(i\varphi;\partial_j\psi)_2\Lambda'_j(\varphi) \tag{3.23}$$

and we infer from (3.4) and (3.5) that

$$(\mathscr{H}'_\psi(\varphi);i\varphi)_2=(\mathscr{H}'_\psi(\varphi);\partial_j\varphi)_2=0,\qquad \forall j=1,\ldots,N. \tag{3.24}$$

We obtain (3.20) and (3.21) from (3.24) and the well known formula $\mathscr{H}_\psi(Lv)'=L^*\mathscr{H}'_\psi(Lv)$, $\forall L\in\mathscr{L}(H^1,H^1)$. □

*Remark* 3.11. — Note that (3.23) may be written as

$$i\mathscr{H}'_\psi(\varphi)=\mathscr{P}(\psi,\varphi)+\sum_{j=1}^N(i\varphi;\partial_j\psi)_2\, i\Lambda'_j(\varphi).$$

Let us point out that $iv$ and $\partial_j v$ being tangent to $\Omega_\varphi$ at, $v$, the identities (3.20) and (3.21) say that, for all $\psi\in H^1$, the functional $\mathscr{H}_\psi:\mathscr{V}(\Omega_\varphi,\varepsilon_0)\to\mathbb{R}$ defines a field that is transversal to $\Omega_\varphi$. This indicates that the trajectories created by $\mathscr{H}'_\psi$ and starting near $\Omega_\varphi$ may scape $\mathscr{V}(\Omega_\varphi,\varepsilon)$ in a finite time if $\varepsilon$ is small enough. This is the key idea for the following result which is essentially due to Gonçalves Ribeiro [8].

THEOREM 3.12. — *Let* $\varphi\in\mathscr{G}$. *If* $\alpha\in]1,\mathrm{sob}(N)-2[$ *and*

$$\inf_{\psi\in H'}\langle S''(\varphi)\mathscr{H}'_\psi(\varphi);\mathscr{H}'_\psi(\varphi)\rangle<0, \tag{3.25}$$

*then there exists* $\varepsilon>0$ *and a sequence* $\{u_j\}$ *in* $\mathscr{V}(\Omega_\varphi,\varepsilon)$ *satisfying*

(i) $u_j \to \varphi$ *in* $H^1$ *as* $j \to \infty$.

(ii) *For all* $j \in \mathbb{N}$, $U(t) u_j$ *is global but scapes* $\mathcal{V}(\Omega_\varphi, \varepsilon)$ *in a finite time.*

*Proof.* – First of all, let $\varepsilon_0$ given in Lemma 3.4 and consider $\psi \in H^1$ such that

$$\langle S''(\varphi) \mathcal{H}'_\psi(\varphi); \mathcal{H}'_\psi(\varphi) \rangle < 0. \tag{3.26}$$

For each $v_0 \in \mathcal{V}(\Omega_\varphi, \varepsilon_0)$, we consider the initial value problem

$$\left. \begin{array}{c} \dot{v}(s) = i \mathcal{H}'_\psi(v(s)), \\ v(0) = v_0 \end{array} \right\} \tag{3.27}$$

A straightfoward calculation with the identities (3.12) and (3.13) leads us to get the estimates:

$$\| \Lambda''_j(\varphi) \|_{\mathcal{L}(H^1, H^{-1})} \leqq C_0, \qquad \forall j = 0, \ldots, N,$$

where $C_0$ depends on $|\varphi|_2$, $\|\varphi\|$ and $\|\varphi\|_{H^2}$.

Since $\Lambda_j : B_{\varepsilon_0}(\varphi) \to \mathbb{R}$ is $C^2$, $\forall j = 0, \ldots, N$ (*see* Lemma 3.4), by taking $\tilde{\varepsilon}_0 \leqq \varepsilon_0$ if necessary, we may assume that there exists $C_1 > 0$ such that, for all $v \in \mathcal{V}(\Omega_\varphi, \varepsilon_0)$

$$\| \Lambda''_j(v) \|_{\mathcal{L}(H^1, H^{-1})} \leqq C_1, \qquad \forall j = 0, \ldots, N. \tag{3.28}$$

Therefore, by differentiating the identity (3.23) we get

$$\| \mathcal{H}''_\psi(v) \|_{\mathcal{L}(H^1, H^{-1})} \leqq C, \qquad \forall v \in \mathcal{V}(\Omega_\varphi, \varepsilon_0), \tag{3.29}$$

where C depends only on $C_1$ and $\|\psi\|$.

It follows from (3.29) that the initial value problem (3.27) admits a unique maximal solution $v \in C^2(]-\sigma_0, \sigma_0]; \mathcal{V}(\Omega_\varphi, \varepsilon_0))$, where $\sigma_0 = \sigma_0(v_0)$ is some positive constant. Moreover, for each $\varepsilon_1 < \varepsilon_0$, there exists $\sigma_1 > 0$ such that $\sigma_0(v_0) \geqq \sigma_1$, $\forall v_0 \in \mathcal{V}(\Omega_\varphi, \varepsilon_1)$.

Taking $\varepsilon_1 < \varepsilon_0$ and $\sigma_1 > 0$ as above, we may define the nonlinear semi-group

$$\begin{aligned} \mathcal{U} : \ ]-\sigma_1, \sigma_1[ \times \mathcal{V}(\Omega_\varphi, \varepsilon_1) &\to \mathcal{V}(\Omega_\varphi, \varepsilon_0) \\ (s, v_0) &\mapsto \mathcal{U}(s) v_0, \end{aligned}$$

where $\mathcal{U}(s) v_0$ is the unique maximal solution of (3.27).

Note that $\mathcal{U}$ is a $C^1$-function in both variables and, for $v_0 \in \mathcal{V}(\Omega_\varphi, \varepsilon_1)$, the function $s \mapsto \mathcal{U}(s) v_0$ is $C^2$. Moreover, it follows from the invariances of $\Lambda_j$, $j = 0, \ldots, N$ (*see* Lemma 3.4), that $\mathcal{U}(s)$ comutes with $e^{i\theta} \tau_y$ for each $\theta \in \mathbb{R}$, $y \in \mathbb{R}^N$.

Let

$$P(v) = \langle S'(v); i \mathcal{H}'_\psi(v) \rangle$$

and

$$R(v) = \langle S''(v) \mathcal{H}'_\psi(v); \mathcal{H}'_\psi(v) \rangle - \langle S'(v); \mathcal{H}''_\psi(\mathcal{H}'_\psi(v)) \rangle.$$

Since $s \mapsto \mathcal{U}(s) v_0$ is $C^2$, we may apply Taylors' theorem to get

$$S(\mathcal{U}(s) v_0) = S(v_0) + P(v_0) s + \frac{1}{2} R(\mathcal{U}(\nu s) v_0) s^2,$$

for some $\nu \in ]0, 1[$. Since R is continuous and $R(\varphi) < 0$, there exists $\varepsilon_2 \leqq \varepsilon_1$ and $\sigma_2 \leqq \sigma_1$ such that

$$S(\mathcal{U}(s) v_0) < S(v_0) + P(v_0) s$$

for all $s \in ]-\sigma_2, \sigma_2[$, $\forall v_0 \in B_{\varepsilon_2}(\varphi)$. Since $\mathcal{U}(s)$ comutes with $e^{i\theta} \tau_y$, we obtain from Lemma 3.3:

$$\left.\begin{aligned} &\forall v_0 \in \mathcal{V}(\Omega_\varphi, \varepsilon_2), \qquad \forall s \in ]-\sigma_2, \sigma_2[, \quad s \neq 0, \\ &\qquad S(\mathcal{U}(s) v_0) < S(v_0) + P(v_0) s. \end{aligned}\right\} \tag{3.30}$$

In particular, for $v_0 = \mathcal{U}(\tau)\varphi$, with $\tau \neq 0$ sufficiently small, we get

$$S(\mathcal{U}(s)\mathcal{U}(\tau)\varphi) \leqq S(\mathcal{U}(\tau)\varphi) + P(\mathcal{U}(\tau)\varphi) s. \tag{3.31}$$

So, by taking $s = -\tau < 0$ we obtain

$$S(\varphi) \leqq S(\mathcal{U}(\tau)\varphi) - P(\mathcal{U}(\tau)\varphi)\tau. \tag{3.32}$$

Furthermore, from (3.30) we have

$$S(\mathcal{U}(\tau)\varphi) < S(\varphi), \qquad \forall \tau \in ]-\sigma_2, \sigma_2[, \quad \tau \neq 0, \tag{3.33}$$

and we infer from (3.32) and (3.33) that, for some $\sigma_3 \leqq \sigma_2$:

$$P(\mathcal{U}(\tau)\varphi) < 0, \qquad \forall \tau \in ]0, \sigma_3[. \tag{3.34}$$

On the other hand we have

$$\langle V'(\varphi); i\mathcal{H}'_\psi(\varphi) \rangle \neq 0 \tag{3.35}$$

[because otherwise we would have $i\mathcal{H}'_\psi(\varphi)$ tangent to $\Sigma_{\mu_0}$ at $\varphi$ ($\mu_0 = 0$ if $N=2$) and since $\varphi$ minimizes S on $\Sigma_{\mu_0}$, we would have $\langle S''(\varphi)\mathcal{H}'_\psi(\varphi); \mathcal{H}'_\psi(\varphi) \rangle \geqq 0$ in contradiction with (3.26)]. Moreover, since $\mathcal{H}'_{-\psi}(\varphi) = -\mathcal{H}'_\psi(\varphi)$ [*see* (3.23)], we may assume without loss of generality that $\langle V'(\varphi); i\mathcal{H}'_\psi(\varphi) \rangle < 0$. So, for $\tau > 0$ small enough we have

$$V(\mathcal{U}(\tau)\varphi) = V(\varphi) + \int_0^\tau \langle V'(\mathcal{U}(\zeta)\varphi); i\mathcal{H}'_\psi(\mathcal{U}(\zeta)\varphi) \rangle \, d\zeta < \bar{\mu} \tag{3.36}$$

(where $\bar{\mu} = 0$ if $N=2$ and $\bar{\mu} = \mu_0$ if $N=3$). Define

$$\mathbb{D} = \mathcal{A}_\varphi^- \cap \{ v \in H^1 \mid P(v) < 0 \}$$

[*see* (2.6)]. Then $\mathbb{D}$ is invariant by the flow of U and we have from (3.33), (3.34) and (3.36) that

$$\mathcal{U}(\tau)\varphi \in \mathbb{D}, \qquad \forall \tau \in ]0, \sigma_3[. \tag{3.37}$$

Let $\tau_j \in ]0, \sigma_3[$, $j \in \mathbb{N}$, such that $\tau_j \downarrow 0$ as $j \uparrow \infty$ and consider $u_j = \mathcal{U}(\tau_j)\varphi$. Then $u_j \to \varphi$ in $H^1$ as $j \uparrow \infty$ which proves (i). Moreover, from (3.37) and

Lemma 2.4 we infer that $U(t)u_j$ is local, for all $j$. In order to conclude the proof, we need only to verify that $U(t)u_j$ escapes $\mathscr{V}(\Omega_\varphi, \varepsilon_3)$ for some $\varepsilon_3>0$, for all $j\in\mathbb{N}$.

Consider the function $A: ]-\sigma_1, \sigma_1[\times \mathscr{V}(\Omega_\varphi, \varepsilon_1)\to\mathbb{R}$ defined by $A(s, v_0)=V(\mathscr{U}(s)v_0)$. Since [*see* (3.35)] $A(0, v)=\bar{\mu}$ and

$$\partial_s A(0, v)=\langle V'(\varphi);\, i\mathscr{H}'_\psi(\varphi)\rangle\neq 0 \qquad \text{for all} \qquad v\in\Omega_\varphi,$$

it follows from the implicit function theorem that there exists $\tilde{\varepsilon}_2\leqq\varepsilon_2$ such that, for every $v_0\in\mathscr{V}(\Omega_\varphi, \tilde{\varepsilon}_2)$, there exists a unique $s(v_0)$ satisfying

$$V(\mathscr{U}s(v_0))v_0)=\bar{\mu}. \tag{3.38}$$

From (3.30) and (3.38) we obtain (with $\varepsilon_3=\min\{\varepsilon_2, \tilde{\varepsilon}\}$):

$$\left.\begin{array}{l}\forall v_0\in\mathscr{V}(\Omega_\varphi, \varepsilon_3), \qquad \exists s_0\in]-\sigma_3, \sigma_3[;\\ S(\varphi)\leqq S(\mathscr{U}(s_0)v_0)\leqq S(v_0)+P(v_0)s_0\end{array}\right\} \tag{3.39}$$

If we introduce $T_j=\sup\{t_1>0\,|\,U(t)u_j\in\mathscr{V}(\Omega_\varphi, \varepsilon_3),\ \forall t\in]0, t_1[\}$, it follows from (3.39) that $\forall j\in\mathbb{N}$, $\forall t\in]0, T_j[$, $\exists s_j^t\in]-\sigma_3, \sigma_3[$ satisfying

$$S(\varphi)\leqq S(U(t)u_j)+P(U(t)u_j)s_j^t.$$

From the conservation laws (2.2) and (3.37) we infer that

$$P(U(t)u_j)\geqq\frac{S(\varphi)-S(u_j)}{\sigma_3}=C_j>0, \qquad \forall t\in]0, T_j[.$$

On the other hand, we know that

$$\mathscr{H}_\psi(U(t)u_j)=\mathscr{H}_\psi(u_j)+\int_0^t P(U(\zeta)u_j)\,d\zeta,$$

which gives

$$\mathscr{H}_\psi(U(t)u_j)\geqq\mathscr{H}_\psi(u_j)+t\,C_j, \qquad \forall t\in]0, T_j[. \tag{3.40}$$

From Lemma 3.9 and (3.40) we infer that $T_j<\infty$, which completes the proof. □

In order to give conditions to assure (3.25), we need the following ellementary Lemma:

LEMMA 3.13. – *For all* $\phi\in\mathscr{S}(\mathbb{R}^N;\mathbb{R})$, *we have the following identities:*

(i) $$\int\phi\, x.\nabla\phi=-\frac{N}{2}\int|\phi|^2.$$

(ii) $$\int|\phi|^\alpha\phi\, x.\nabla\phi=-\frac{N}{\alpha+2}\int|\phi|^{\alpha+2}.$$

(iii) $$\int E_1(|\phi|^2)\phi\, x.\nabla\phi=-\frac{N}{4}\int E_1(|\phi|^2)|\phi|^2.$$

*Proof.* – The identities (i) and (ii) are trivial consequences of Gauss' Theorem. In order to prove (iii), we remember the following well known identity: $\mathcal{F}\{x.\nabla f\} = -\operatorname{div}(\xi\mathcal{F}\{f\})$, where $\mathcal{F}$ denotes the Fourier transform.

By denoting $\psi = |\phi|^2$, we have from Parseval identity and the definition of $E_1$:

$$2\int E_1(|\phi|^2)\phi x.\nabla\phi = \int E_1(\psi) x.\nabla\psi$$

$$= \int \sigma_1(\xi)\mathcal{F}\{\psi\}\overline{\mathcal{F}\{x.\nabla\psi\}} = -\int \sigma_1(\xi)\mathcal{F}\{\psi\}\operatorname{div}\overline{\xi\mathcal{F}\{\psi\}}$$

$$= -\frac{N}{2}\int \sigma_1(\xi)|\mathcal{F}\{\psi\}|^2 - \frac{1}{2}\int \sigma_1(\xi)\mathcal{F}\{\psi\}\xi.\nabla\overline{\mathcal{F}\{\psi\}}$$

$$= -\frac{N}{2}\int E_1(\psi)\psi - \frac{1}{4}\int \sigma_1(\xi)\xi.\nabla(|\mathcal{F}\{\psi\}|^2)$$

$$= -\frac{N}{2}\int E_1(|\phi|^2)|\phi|^2 + \frac{1}{2}\int \operatorname{div}(\xi\sigma_1(\xi))|\mathcal{F}\{\psi\}|^2$$

and the conclusion follows because $\operatorname{div}(\xi\sigma_1(\xi)) = N\sigma_1(\xi)$. □

PROPOSITION 3.14. – *Let* $N\in\{2, 3\}$, $\omega$, $b>0$, $(\alpha, a)\in\mathcal{R}_{\omega, b}$ *with* $\alpha>1$ *and* $\varphi\in\mathcal{G}$. *Then we have*

$$\inf_{\psi\in H^1} \langle S''(\varphi)\mathcal{H}'_\psi(\varphi); \mathcal{H}'_\psi(\varphi)\rangle \leq \frac{N^2\alpha}{4(\alpha+2)} a(\alpha-2)|\varphi|_{\alpha+2}^{\alpha+2} + \frac{(2-N)}{2}|\nabla\varphi|_2^2.$$

*Proof.* – From Theorem 2.2 we may assume without loss of generality that $\varphi$ is real. Let $\psi\in H^1$ real valued. It follows from Remark 3.11 that $i\mathcal{H}'_\psi(\varphi) = \mathcal{P}(\psi, \varphi)$, wherewith we get

$$\langle S''(\varphi)\mathcal{H}'_\psi(\varphi); \mathcal{H}'_\psi(\varphi)\rangle = J_1 - 2\frac{(\varphi;\psi)_2}{|\varphi|_2^2}J_2 + \frac{(\varphi;\psi)_2^2}{|\varphi|_2^4}J_3, \quad (3.41)$$

where $J_1 = \langle S''(\varphi)\psi; \psi\rangle$, $J_2 = \langle S''(\varphi)\psi; \varphi\rangle$ and $J_3 = \langle S''(\varphi)\varphi; \varphi\rangle$.

Since

$$S''(\varphi)\phi = -\Delta\phi + \omega\phi - bE_1(|\varphi|^2)\phi - 2bE_1(\Re(\varphi\bar\phi))\varphi + a\alpha|\varphi|^{\alpha-2}\phi\Re(\varphi\bar\phi) + a|\varphi|^\alpha\phi,$$

for all $\phi\in H^1$ and remembering that $\varphi$ and $\psi$ are real, we get:

$$J_1 = |\nabla\psi|_2^2 + \omega|\psi|_2^2 - b\int E_1(|\varphi|^2)|\psi|^2 - 2b\int E_1(\varphi\psi)\varphi\psi + a(\alpha+1)\int|\varphi|^\alpha|\psi|^2,$$

$$J_2 = (\nabla \varphi; \nabla \psi)_2 + \omega (\varphi; \psi)_2 - b \int E_1 (|\varphi|^2) \varphi\psi$$
$$- 2b \int E_1 (\varphi\psi) |\varphi|^2 + a(\alpha+1) \int |\varphi|^\alpha \varphi\psi,$$
$$J_3 = |\nabla \varphi|_2^2 + \omega |\varphi|_2^2 - 3b \int E_1 (|\varphi|^2) |\varphi|^2 + a(\alpha+1) \varphi|_{\alpha+2}^{\alpha+2}.$$

Taking into account that $\varphi$ solves (2.3), we have

$$J_2 = -2b \int E_1 (\varphi\psi) |\varphi|^2 + a\alpha \int |\varphi|^\alpha \varphi\psi,$$
$$J_3 = -2b \int E_1 (|\varphi|^2) |\varphi|^2 + a\alpha |\varphi|_{\alpha+2}^{\alpha+2}.$$

Let $\psi = x.\nabla\varphi$. From Theorem 2.3 we have that $\psi \in H^1$ and since $\int E_1 (\varphi\psi) |\varphi|^2 = \int E_1 (|\varphi|^2) \varphi\psi$, it follows from Lemma 3.13 that

$$J_2 = \frac{Nb}{4} \int E_1 (|\varphi|^2) |\varphi|^2 - \frac{Na\alpha}{\alpha+2} |\varphi|_{\alpha+2}^{\alpha+2}.$$

Since $(\varphi; \psi)_2 = -\frac{N}{2} |\varphi|^2$, we may rewrite (3.41) as:

$$\langle S''(\varphi) \mathcal{H}'_\psi(\varphi); \mathcal{H}'_\psi(\varphi) \rangle = \frac{N^2 \alpha}{4(\alpha+2)} a(\alpha-2) |\varphi|_{\alpha+2}^{\alpha+2} + J_1. \quad (3.42)$$

In order to calculate $J_1$, let $\phi_n \in \mathcal{D}(\mathbb{R}^2; \mathbb{R})$ such that $\varphi_n \to x.\nabla\varphi$ in $H^1$ and consider

$$J_{1n} = \int \nabla (x.\nabla\varphi).\nabla\varphi_n + \omega \int (x.\nabla\varphi) \phi_n - b \int E_1 (|\varphi|^2)(x.\nabla\varphi) \varphi_n$$
$$- 2b \int E_1 (\varphi\phi_n) \varphi x.\nabla\varphi + a(\alpha+1) \int |\varphi|^\alpha x.\nabla\phi_n.$$

(Remark that $J_{1n} \to J_1$ *as* $n \to \infty$.) Taking into account that $\varphi$ solves (2.3) and applying Gauss theorem, we obtain

$$J_{1n} = \int \nabla (x.\nabla\varphi).\nabla\phi_n + \int \nabla\varphi.\nabla(x.\nabla\phi_n) - 2\omega \int \varphi\phi_n$$
$$- 2a \int |\varphi|^\alpha \varphi\phi_n + 2b \int E_1 (|\varphi|^2) \varphi\phi_n$$
$$+ b \int x.\nabla(E_1(|\varphi|^2)) \varphi\phi_n - 2b \int E_1 (\varphi\phi_n) \varphi x.\nabla\varphi.$$

Integration by parts gives

$$\int \nabla (x.\nabla \varphi).\nabla \phi_n + \int \nabla \varphi.\nabla (x.\nabla \phi_n) = -\int \nabla \varphi.\nabla \phi_n$$

and we obtain

$$J_{1n} = -\omega \int \varphi \phi_n - a \int |\varphi|^\alpha \varphi \phi_n + b \int E_1(|\varphi|^2)\varphi \phi_n$$
$$+ b \int x.\nabla (E_1(|\varphi|^2))\varphi x.\nabla \varphi - 2b \int E_1(\varphi x.\nabla \varphi)\varphi x.\nabla \varphi.$$

Letting $n \to \infty$ and using again Lemma 3.13, we get:

$$J_1 = \frac{Na}{\alpha+2}|\varphi|_{\alpha+2}^{\alpha+2} + \frac{N\omega}{2}|\varphi|_2^2 - \frac{Nb}{4}\int E_1(|\varphi|^2)|\varphi|^2$$
$$+ b \int x.\nabla (E_1(|\varphi|^2))\varphi x.\nabla \varphi - 2b \int E_1(\varphi x.\nabla \varphi)\varphi x.\nabla \varphi.$$

Now, applying Proposition 2.5-(iii) of [5] and remembering that $\varphi x.\nabla \varphi = \frac{1}{2} x.\nabla(|\varphi|^2)$ we get

$$J_1 = \frac{b}{2}\int x.\nabla (E_1(|\varphi|^2))x.\nabla(|\varphi|^2)$$
$$- \frac{b}{2}\int E_1(x.\nabla(|\varphi|^2))x.\nabla(|\varphi|^2) + \frac{(2-N)}{2}|\nabla \varphi|_2^2.$$

Denoting by $f(\xi) = \mathscr{F}\{|\varphi|^2\}(\xi)$ and remembering that

$$\mathscr{F}\{x.\nabla g\} = -\operatorname{div}(\xi \mathscr{F}\{g\}),$$

we obtain from Parseval identity:

$$J_1 = \frac{b}{2}\int (\operatorname{div}(\xi\sigma_1(\xi)f(\xi)) - \sigma_1(\xi)\operatorname{div}(\xi f(\xi)))\operatorname{div}(\xi f(\xi))$$
$$+ \frac{(2-N)}{2}|\nabla \varphi|_2^2. \quad (3.43)$$

Since

$$\operatorname{div}(\xi\sigma_1(\xi)f(\xi)) - \sigma_1(\xi)\operatorname{div}(\xi f(\xi)) \equiv 0,$$

the conclusion follows by merging (3.42) and (3.43). □

With Theorem 3.12 and Proposition 3.14 we are ready to stabilish our main result, namely:

THEOREM 3.15. – *Let* $N \in \{2, 3\}$, $\omega$, $b > 0$, $(a, \alpha) \in \mathscr{R}_{\omega, b}$ and $\varphi \in \mathscr{G}$. *If* $\alpha > 1$ *and* $a(\alpha - 2) \leqq 0$ *then* $\Omega_\varphi$ *is unstable by the flow of* U.

The proof follows directly from Theorem 3.12 and Proposition 3.14 excepted the case $N=2$ and $a(\alpha-2)=0$ for which the condition (3.25) is not assured. This is a special case for which a very simple idea of Weinstein [11] is applicable and that we treat as the next proposition.

PROPOSITION 3.16. – *Let* $\alpha=2$, $N=2$, $a<b$ *and* $\varphi\in\mathscr{X}$. *Then* $u(t, x)=e^{i\omega t}\varphi(x)$ *is a global solution of* (1.1) *which is unstable in the sense that there exists* $\{\varphi_n\}\subset H^1$ *such that* $\varphi_n\to\varphi$ *in* $H^1$ *as* $n\uparrow\infty$ *and* $U(t)\varphi_n$ *blows up in a finite time.*

*Proof.* – Since $\varphi\in\mathscr{X}$ we have that $\mathscr{E}(\varphi)=0$. Hence $\mathscr{E}(\lambda\varphi)<0$ for all $\lambda>1$. Let $\varphi_n=\lambda_n\varphi$, where $\lambda_n>1$, $\lambda_n\downarrow 1$ and consider $u_n(t)=U(t)\varphi_n$. Since $|.|\varphi\in L^2$ (*see* Theorem 2.2), it follows that the function $I_n(t)=\int|x|^2|u_n(t)|^2dx$ is well defined and we have (*see* [7], thm. 3.2, p. 495):

$$d_{tt}I_n(t)=16\,\mathscr{E}(u_n(t)).$$

But by conservation of energy we have $\mathscr{E}(u_n(t))=\mathscr{E}(\varphi_n)<0$, which concludes the proof. □

## 4. APPENDIX

In a well known paper (*see* [9]), stability and instability of stationary states for abstract Hamiltonian systems are obtained by means of the convexity of a real valued function $d(\omega)$ which is related to the action of the system. In our context, this approach leads us to consider the function $\omega\mapsto I_\omega$, where (in the case $N=2$ for instance)

$$I_\omega=\inf\left\{\frac{1}{2}T(\psi)\,|\,V_\omega(\psi)=0\right\},$$

$$V_\omega(\psi)=\frac{b}{4}\int E_1(|\psi|^2)|\psi|^2-\frac{a}{\alpha+2}|\psi|_{\alpha+2}^{\alpha+2}-\frac{\omega}{2}|\psi|_2^2.$$

This point of view does not seem to work in our context because we do not have a precise characterisation for the kernel of $S''(\varphi)$. Indeed, equation (2.3) does not have radial solutions (*cf.* [5]) and $S''(\varphi)\,\partial_j\varphi=\partial_j S'(\varphi)=0$ for $j=i,\ldots,N$ if $\varphi\in\mathscr{X}$. Nevertheless, it is interesting to remark that the same condition on the parameters $a$ and $\alpha$ which implies instability of Ground-States, assures the concavity of the function $\omega\mapsto I_\omega$. More precisely,

THEOREM 4.1. – *Let* $N=2$, $\omega$, $b>0$. *If* $a(\alpha-2)<0$, *then the function* $\omega\mapsto I_\omega$ *is concave on* $]0,+\infty[$.

*Proof.* — First of all, note that if $a(\alpha-2)<0$, then $I_\omega$ is well defined on $]0, +\infty[$. Indeed, if $\alpha<2$ then $(a, \alpha)\in\mathscr{R}_{\omega, b}$, $\forall a\in\mathbb{R}$, $\forall\omega>0$. On the other hand, if $\alpha>2$ and $a<0$ then $(a, \alpha)\in\mathscr{R}_{\omega, b}$, $\forall\omega>0$. Assume $a(\alpha-2)<0$ and let $\omega>0$. From Theorem 2.3 the minimisation problem (2.4) has a solution $\varphi_\omega$ which we choose from now on. The following ellementary facts hold:

(i) For each $h\in\mathbb{R}$ we have

$$V_{\omega+h}(\varphi_\omega)=-\frac{h}{2}|\varphi_\omega|_2^2; \tag{4.1}$$

(ii) $\inf\{\tau>0\,|\,V_\omega(\tau\varphi_\omega)=0\}=1$ $\left(\text{because otherwise } I_\omega<\frac{1}{2}|\nabla\varphi_\omega|_2^2\right)$.

By continuity, we have

$$V_\omega(\tau\varphi_\omega)<0, \qquad \forall\tau\in]0, 1[. \tag{4.2}$$

Let us consider the function

$$F(\tau, h)=f(\tau)-h, \qquad \text{where } f(\tau)=\frac{2V_\omega(\tau\varphi_\omega)}{\tau^2|\varphi_\omega|_2^2}.$$

Since $F(1, 0)=0$ and $\partial_\tau F(1, 0)=f'(1)=4I_\omega/|\varphi_\omega|_2^2>0$, it follows from the implicity function theorem that there exists $\varepsilon>0$ and a $C^\infty$-function $\sigma:]-\varepsilon, \varepsilon[\to\mathbb{R}$ such that

$$f(\sigma(h))-h=0, \qquad \forall h\in]-\varepsilon, \varepsilon[. \tag{4.3}$$

From (4.1), (4.2) and the definition of F, we have

$$V_{\omega+h}(\sigma(h)\varphi_\omega)=0, \qquad \forall h\in]-\varepsilon, \varepsilon[, \tag{4.4}$$

$$(\sigma(h)-1)h>0, \qquad h\neq 0. \tag{4.5}$$

In particular, from the definition of $I_\omega$ and (4.4) we obtain

$$I_{\omega+h}\leqq\sigma^2(h)I_\omega, \qquad \forall h\in]-\varepsilon, \varepsilon[,$$

from which we deduce

$$\frac{I_{\omega+h}-2I_\omega+I_{\omega-h}}{h^2}\leqq\left(\frac{\sigma^2(h)+\sigma^2(-h)-2}{h^2}\right)I_\omega.$$

Letting $h\downarrow 0$ and taking into account the regularity of $\sigma(h)$, we obtain

$$\lim_{h\to 0}\left(\frac{I_{\omega+h}-2I_\omega+I_{\omega-h}}{h^2}\right)\leqq(d_{hh}\sigma^2(h))|_{h=0})I_\omega.$$

But from (4.3) we see that $\sigma(h)=f^{-1}(h)$ and a straightforward calculation gives us

$$d_{hh}\sigma^2(h)|_{h=0}=\frac{|\varphi_\omega|_2^4|\varphi_\omega|_{\alpha+2}^{\alpha+2}}{2I_\omega^3(\alpha+2)}a(\alpha-2)<0,$$

which completes the proof. □

*Remark* 4.2. – The result of Theorem 4.1 seems to be true for $N=3$ although the arguments used in its proof do not work. Indeed, according to Theorem 2.2, $\varphi_\omega$ is a ground state if and only if $\varphi_\omega$ is a solution of problem (2.5), where $\mu_0=\mu_\omega=(I_\omega/3)^{3/2}$ (*cf.* [5]). This strong dependence on $\omega$ does not lead us to proceed as before.

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