Annales de l'I. H. P., section A

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Annales de l'I. H. P., section A, tome 57, nº 4 (1992), p. 361-384
<http://www.numdam.org/item?id=AIHPA_1992__57_4_361_0>

Ann. Inst. Henri Poincaré,

Vol. 57, n° 4, 1992, p. 361-384. *Physique théorique*

On the scattering matrix for perturbations of constant sign

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ABSTRACT. – Let S be the scattering matrix for a Schrödinger operator with a short-range potential q. The typical result of this paper is that there is only a finite number of eigenvalues of S on the upper (lower) semicircle if $q \geqq 0$ $(q \leqq 0)$.

RÉSUMÉ. – Soit S la matrice de diffusion pour l'opérateur de Schrödinger avec un potentiel q à courte portée. Le résultat typique du papier est qu'il n'y a qu'un nombre fini de valeurs propres de S à partie imaginaire positive (négative) si $q \geqq 0$ $(q \leqq 0)$.

INTRODUCTION

In the framework of abstract scattering theory the spectrum of the scattering matrix (SM) was for the first time considered by M. Sh. Birman and M. G. Krein in the papers ([1], [2]). In these papers perturbations of trace-class type were studied. The spectrum of the SM consists of eigenvalues $\mu_n^{\pm}$, $\mu_n^{\pm} \neq 1$, of finite multiplicites lying on the unite circle, $\operatorname{Im} \mu_n^{+} \geqq 0$, $\operatorname{Im} \mu_n^{-} < 0$, and accumulating only at the point 1. The point 1 can also be an eigenvalue of possibly infinite multiplicity. For perturbations V of definite sign, additional information about the eigenvalues $\mu_n^{\pm}$ is available. Actually, as shown in [1], eigenvalues of the SM do not accumulate at 1 from above if $V \geqq 0$ or from below if $V \leqq 0$. Further results in this direction were obtained in ([3], [4]), where the trace-class condition was also imposed.

In applications to the Schrödinger operator with a potential $q(x)$ in the space $L_2(\mathbb{R}^d)$, trace-class conditions require that $q(x) = O(|x|^{-\alpha})$ with $\alpha > d$ at infinity. We note that for the Schrödinger operator, the numbers $\delta_n^{\pm}$, related to $\mu_n^{\pm}$ by the formula

$$\mu_n^{\pm} = \exp(\pm 2 i \delta_n^{\pm}). \tag{0.1}$$

can be regarded as a natural generalization of phase shifts (see e. g. [5]) for the spherically symmetric case.

The method of [1], [2] relies on the decomposition of a trace-class perturbation into a sum of one-dimensional operators. For a one-dimensional perturbation the SM can be calculated almost explicitly. Then these "one-dimensional" results are summed up which allows one to go over to aribtrary trace-class perturbations. This "step by step" method permits one to consider general perturbations of trace-class type but can not be applied under the assumptions of the "smooth" scattering theory.

Our aim is to extend the results of [1], [2] on the spectrum of the SM by removing the trace-class assumptions. To this end it turned out to be convenient to study spectral properties of the SM relying only on the structure of its stationary representation. This allows us to consider a quite general situation and concrete assumptions of both trace-class and smooth scattering theories are easily accommodated. Special attention is paid to the Schrödinger operator for which the SM is well defined if $q(x) = O(|x|^{-\alpha})$ with arbitrary $\alpha > 1$. In particular, we extend the result on the one-sided accumulation of eigenvalues for potentials of constant sign to all $\alpha > 1$.

Another closely related problem which we treat here is the following: Suppose that the negative q_- (positive q_+) part of q vanishes at infinity more rapidly than q_+ (q_-). If q_- (q_+) is not zero, then the eigenvalues μ_n^+ (μ_n^-) may accumulate at 1 but they should be "less numerous" then the eigenvalues μ_n^- (μ_n^+). The precise statement can be given in terms of bounds on the $\mu_n^{\pm}$. In fact, it is known [6], [7] that if $q(x) = O(|x|^{-\alpha})$ then $|\mu_n^{\pm} - 1| = O(n^{-\rho})$, where $\rho = (\alpha - 1)(d-1)^{-1}$. We improve this result showing in theorem 4.16 that the bound on $|\mu_n^{\pm} - 1|$ is determined only by the bound on $q_{\mp}$. (Whenever a relation contains the signs "$\pm$" it is understood as two relations for upper and lower indices separately).

The proof of the last result relies on a kind of variational principle for the spectrum of the SM. Actually, as shown in [2], [8] eigenvalues of the SM rotate in the clockwise (counterclockwise) direction if a positive (negative) perturbation is introduced. The precise formulation of this result requires that a perturbation be small. For our purposes it is necessary to extend the result on the rotation of eigenvalues to perturbations of arbitrary magnitude (*see* Theorem 4.11).

This paper is organized as follows: Necessary information on perturbation theory for unitary operators is collected in section 1. Some basic concepts of scattering theory are discussed in section 2. The short section 3 plays the central role. There we study spectral properties of an arbitrary unitary operator S having the same structure as the SM. The main result of that section, formulated as Theorem 3.6, gives conditions of finiteness of eigenvalues of S lying on the upper (or lower) semicircle. In section 4 we go back to scattering theory and discuss applications of the results of section 3 to the SM both for abstract and differential operators. Finally, in section 5, we carry over the results of section 4 to the modified SM Σ which has the form $\Sigma=\mathrm{SJ}$, where J is some fixed unitary operator. In case H is the Schrödinger operator and J is the reflection operator, investigation of spectral properties of Σ was advocated in [9].

1. PERTURBATION THEORY FOR UNITARY OPERATORS

1. Spectral and perturbation theories for unitary operators in a Hilbert space $\mathbb{H}$ are essentially similar to those for the self-adjoint case. The spectral measure $\mathrm{E_U}(\mathrm{X})$ of a unitary operator U is defined on Borel sets X of the unit circle $\mathbb{T}: |\mathrm{X}|$ is the Lebesgue measure of X (so that $|\mathbb{T}|=2\pi$). The essential spectrum $\sigma_{\mathrm{U}}^{(\mathrm{ess})}$ of U consists of its whole spectrum $\sigma_{\mathrm{U}}=\operatorname{supp}\mathrm{E_U}$ without isolated eigenvalues of finite multiplicity. We denote by (μ_1, μ_2) and $[\mu_1, \mu_2]$, where $|\mu_i|=1$, the corresponding open and closed arcs of the unit circle $\mathbb{T}$ swept out as μ_1 moves to μ_2 in the counterclockwise (which we designate as the positive) direction. The class of compact operators is denoted by $\mathscr{K}_\infty$; $\mathscr{U}$ is the class of such unitary operators U that $\mathrm{U}-\mathrm{I}\in\mathscr{K}_\infty$, where I is the identity operator; $\mathscr{U}_0$ consists of those unitary U for which the operator $\mathrm{U}-\mathrm{I}$ has a finite rank.

Weyl's theorem on compact perturbations is formulated and proved completely in the same way as in the self-adjoint case.

PROPOSITION 1.1. – *Let operators* U_0, U *be unitary and let* $\mathrm{U}-\mathrm{U}_0\in\mathscr{K}_\infty$. *Then* $\sigma_{\mathrm{U}}^{(\mathrm{ess})}=\sigma_{\mathrm{U}_0}^{(\mathrm{ess})}$.

For finite-dimensional perturbations we can get additional information.

PROPOSITION 1.2. – *Assume that* $\dim(\mathrm{U}-\mathrm{U}_0)\,\mathbb{H}=k<\infty$. *Then for any arc* $\mathrm{X}=(\mu_1,\mu_2)\subset\mathbb{T}$

$$|\dim \mathrm{E_U}(\mathrm{X})\,\mathbb{H}-\dim \mathrm{E}_{\mathrm{U}_0}(\mathrm{X})\,\mathbb{H}|\leq k.$$

In particular, if an arc X *is a gap in the spectrum of* U_0, i.e. $\sigma_{U_0} \cap X = \varnothing$, *then the operator* U *can have only k eigenvalues (counted with their multiplicity) in this arc.*

Proposition 1.2 can easily be proved with the help of the spectral theorem.

COROLLARY 1.3. – *The spectrum of an operator* $U \in \mathcal{U}$ *consists of eigenvalues accumulating only at the point* 1. *Eigenvalues distinct from* 1 *have finite multiplicity. An operator* $U \in \mathcal{U}_0$ *has only a finite number of eigenvalues.*

In the unitary case it is natural to introduce a perturbation in the multiplicative form, that is as

$$U = TU_0 \qquad \text{or} \qquad U = U_0 T'. \tag{1.1}$$

Both forms (1.1) are equivalent since by setting $T' = U_0^{-1} T U_0$ we can rewrite the left form as the right one and vice versa. The multiplicative forms are convenient because for unitary operators U_0 and T (or T′) the operator U defined by (1.1) is automatically unitary.

In the self-adjoint case the spectrum may be shifted only by a distance not exceeding the norm of a perturbation. The following assertion may be regarded as a modification of the above for unitary operators.

THEOREM 1.4. – *Let* $U = TU_0$, *where the operators* U_0 *and* T *are unitary and* $\sigma_T \subset [\tau_1, \tau_2]$. *Then for any arc* $(\mu_1, \mu_2) \subset \mathbb{T}$ *such that*

$$|(\tau_1, \tau_2)| < |(\mu_1, \mu_2)| \tag{1.2}$$

the estimate

$$\dim E_U((\mu_1 \tau_2, \mu_2 \tau_1)) \mathbb{H} \leqq \dim E_{U_0}((\mu_1, \mu_2)) \mathbb{H} \tag{1.3}$$

holds.

Theorem 1.4 becomes more transparent for arcs centered at the point 1 when $\mu_2 = \bar{\mu}_1 =: \mu$, $\operatorname{Im} \mu \geqq 0$, and $\tau_2 = \bar{\tau}_1 =: \tau$, $\operatorname{Im} \tau \geqq 0$. This can always be achieved by rotation. Set $\mu = \exp(i\varphi)$, $\tau = \exp(i\psi)$ with $\varphi \in (0, \pi]$, $\psi \in [0, \pi)$. Then (1.2) means that $\psi < \varphi$ and

$$(\mu_1 \tau_2, \mu_2 \tau_1) = (\exp(-i(\varphi - \psi)), \exp(i(\varphi - \psi))).$$

A straightforward application of the spectral theorem gives a result weaker, than (1.3), where the left-hand side (LHS) of (1.3) is replaced by $\dim E_U((\bar{\nu}, \nu)) \mathbb{H}$, with $\nu = \exp(i\theta)$, $\sin(\theta/2) = \sin(\varphi/2) - \sin(\psi/2)$. In fact, we need only this weaker version of Theorem 1.4. The precise result (1.3) is due to M. G. Krein. Its proof can be found in [10].

Theorem 1.4 can be reformulated in a dual form.

COROLLARY 1.5. – *If* $|(\mu_1, \mu_2)| + |(\tau_1, \tau_2)| \leqq 2\pi$, *then*

$$\dim E_{U_0}((\mu_1, \mu_2)) \mathbb{H} \leqq \dim E_U((\mu_1 \tau_1, \mu_2 \tau_2)) \mathbb{H}. \tag{1.4}$$

For the proof we notice that $U_0 = T^* U$, where $\sigma_{T^*} \subset [\bar{\tau}_2, \bar{\tau}_1]$. Then we apply Theorem 1.4 with U_0, U, T and $\mu_j, \tau_j, j=1,2$, replaced by U, U_0, T^* and $\mu_j \tau_j, \bar{\tau}_j$ respectively.

2. In the unitary case the role of a small perturbation in the self-adjoint theory is played by a unitary operator whose spectrum lies in some small neighbourhood of the point 1. As in the self-adjoint theory, small perturbations do not change the total multiplicity of an isolated part of the spectrum of a unitary operator. In this section we suppose that $|(\mu_1, \mu_2)| < 2\pi$, i.e. $\mu_1 \neq \mu_2$.

PROPOSITION 1.6. – *Let* $U = TU_0$, $\mu_j \notin \sigma_{U_0}$, $j=1,2$, *and* $\sigma_T \subset [\tau_1, \tau_2]$. *Then for sufficiently small* $|\tau_j - 1|$, $j=1,2$,

$$\dim E_U((\mu_1, \mu_2)) \mathbb{H} = \dim E_{U_0}((\mu_1, \mu_2)) \mathbb{H}. \tag{1.5}$$

Proof. – According to (1.4), (1.3) we have that

$$\dim E_{U_0}((\mu_1 \bar{\tau}_1, \mu_2 \bar{\tau}_2)) \mathbb{H} \leqq \dim E_U((\mu_1, \mu_2)) \mathbb{H} \leqq \dim E_{U_0}((\mu_1 \bar{\tau}_2, \mu_2 \bar{\tau}_1)) \mathbb{H}. \tag{1.6}$$

Since $\mu_i \notin \sigma_{U_0}$, $j=1,2$ we can choose τ_1, τ_2 so that the LHS and the RHS (right-hand side) of (1.6) are equal to $\dim E_{U_0}((\mu_1, \mu_2)) \mathbb{H}$.

COROLLARY 1.7. – *Let* μ_0 *be an isolated eigenvalue of* U_0 *of multiplicity k. Then there are exactly k eigenvalues (with multiplicity taken into account) of the operator* $U = TU_0$ *in an arc* $(\mu_0 \bar{\zeta}, \mu_0 \zeta)$, $|\zeta| = 1$, $\operatorname{Im} \zeta > 0$, *if* $|\zeta - 1|$ *and* $|\tau_j - 1|$, $j=1,2$, *are sufficiently small.*

The spectrum of a self-adjoint operator is shifted in the positive (negative) direction if a positive (negative) perturbation is added. In the unitary case, the role of a perturbation of constant sign is played by an operator whose spectrum has a gap $(\bar{\tau}, 1)$ (or $(1, \tau)$), $\operatorname{Im} \tau > 0$. Under such multiplicative perturbations, the spectrum rotates in the counterclockwise (clockwise) direction.

We start with the case of small perturbations.

PROPOSITION 1.8. – *Let* $U = TU_0$ *and either* $\mu_1 \notin \sigma_{U_0}$ *and* $\sigma_T \subset [1, \tau]$, $\operatorname{Im} \tau > 0$, *or* $\mu_2 \notin E_{U_0}$ *and* $\sigma_T \subset [\bar{\tau}, 1]$. *Then for sufficiently small* $|\tau - 1|$

$$\dim E_U((\mu_1, \mu_2)) \mathbb{H} \leqq \dim E_{U_0}((\mu_1, \mu_2)) \mathbb{H}. \tag{1.7}$$

Proof. – Suppose, for example, that $\mu_1 \notin \sigma_{U_0}$, $\sigma_T \subset [1, \tau]$. Let us apply the second equality (1.6) with $\tau_1 = 1$, $\tau_2 = \tau$. Since $\mu_1 \notin \sigma_{U_0}$, we can choose τ so that the RHS of (1.6) is equal to the RHS of (1.7).

Remark 1.9. – If in conditions of Proposition 1.8 the RHS of (1.7) is finite, then the equality (1.5) holds. Indeed, to estimate the RHS of (1.7) by its LHS we apply the first inequality (1.6) with $\tau_1 = 1$, $\tau_2 = \tau$. The finiteness of the RHS of (1.7) ensures that $(\mu_2 \bar{\tau}, \mu_2) \cap \sigma_{U_0} = \varnothing$ for

$\operatorname{Im}\tau>0$ and sufficiently small $|\tau-1|$. Therefore the LHS of (1.6) is equal to the RHS of (1.7).

According to Proposition 1.6 the equality (1.5) can be violated only if $\mu_j\in\sigma_{U_0}$ for $j=1$ or $j=2$. If $\mu_1\notin\sigma_{U_0}$, $\mu_2\in\sigma_{U_0}$, then the total multiplicity of the spectrum in (μ_1,μ_2) may be changed only due to the rotation of the spectrum at the point μ_2. Proposition 1.8 shows that in the case $\sigma_T\subset[1,\tau]$ this point of the spectrum does not get inside the arc (μ_1,μ_2) so that the rotation in the clockwise direction is excluded.

The following consequence of Proposition 1.8 should be compared with Corollary 1.7.

COROLLARY 1.10. – *Let* μ_0 *be an isolated eigenvalue of* U_0 *and* $\sigma_T\subset[1,\tau]$ (*or* $\sigma_T\subset[\bar\tau,1]$), $\operatorname{Im}\tau>0$. *Then an arc* $(\mu_0\bar\zeta,\mu_0)$, $|\zeta|=1$, $\operatorname{Im}\zeta>0$, (*or an arc* $(\mu_0,\mu_0\zeta)$) *is a gap in the spectrum of* $U=TU_0$, *if* $|\zeta-1|$ *and* $|\tau-1|$ *are sufficiently small.*

Proposition 1.8 can be reformulated equivalently if the conditions $\mu_1\notin\sigma_{U_0}$ and $\mu_2\notin\sigma_{U_0}$ are interchanged. Then (1.7) should be replaced by the opposite inequality

$$\dim E_{U_0}((\mu_1,\mu_2))\,\mathbb{H}\leqq\dim E_U((\mu_1,\mu_2))\,\mathbb{H}.$$

3. Let us consider the rotation of the spectrum for perturbations of the class $\mathscr{U}$. By $\mathscr{U}_+$ ($\mathscr{U}_-$) is denoted the subclass of $\mathscr{U}$ consisting of operators U whose eigenvalues may accumulate at the point 1 only in the clockwise (counterclockwise) direction. In other words, eigenvalues of $U\in\mathscr{U}_+$ ($\mathscr{U}_-$) do not accumulate at 1 from below (above). Recall that according to Proposition 1.1 for $T\in\mathscr{U}$ operators U_0 and $U=TU_0$ have the same essential spectra $\sigma^{(ess)}$. In case $T\in\mathscr{U}_\pm$ the direction of the rotation of the spectrum can be taken into account.

PROPOSITION 1.11. – *Let* $U=TU_0$, *where* $T\in\mathscr{U}_-$, $(\mu_1,\mu_2)\cap\sigma^{(ess)}=\varnothing$ *and* $\mu^{(0)}\in(\mu_1,\mu_2)$. *Then*

$$\dim E_U((\mu,\mu^{(0)}))\,\mathbb{H}\leqq\dim E_{U_0}((\mu,\mu^{(0)}))\,\mathbb{H}+C_1(\mu^{(0)}),\tag{1.8}$$

$$\dim E_{U_0}((\mu^{(0)},\mu))\,\mathbb{H}\leqq\dim E_U((\mu^{(0)},\mu))\,\mathbb{H}+C_2(\mu^{(0)}),\tag{1.9}$$

where the constant $C_1(\mu^{(0)})$ ($C_2(\mu^{(0)})$) *does not depend on* $\mu\in(\mu_1,\mu^{(0)})$ (*on* $\mu\in(\mu^{(0)},\mu_2)$).

Proof. – To verify (1.8) we choose $\tau=\tau(\mu^{(0)})$, $|\tau|=1$. $\operatorname{Im}\tau>0$, so that

$$\dim E_{U_0}((\mu,\mu^{(0)}\tau))\,\mathbb{H}\leqq\dim E_{U_0}((\mu,\mu^{(0)}))\,\mathbb{H}+C'(\mu^{(0)}).\tag{1.10}$$

This is possible because $\mu^{(0)}\notin\sigma^{(ess)}$. Let us set $X_0=(1,\bar\tau)$, $X_1=[\bar\tau,1]$,

$$T_0=TE_T(X_0)+E_T(X_1),\quad T_1=TE_T(X_1)+E_T(X_0).\tag{1.11}$$

The operators T_0 and T_1 are unitary and $T=T_0T_1$. Moreover, $T_0\in\mathscr{U}_0$ since $T\in\mathscr{U}_-$ and $\sigma_{T_1}\subset[\bar\tau,1]$. Denote $U_1=T_1U_0$; then $U=T_0U_1$. By

Theorem 1.4

$$\dim E_{U_1}((\mu, \mu^{(0)}))\,\mathbb{H} \leqq \dim E_{U_0}((\mu, \mu^{(0)}\tau))\,\mathbb{H} \tag{1.12}$$

and by Proposition 1.2

$$\dim E_U((\mu, \mu^{(0)}))\,\mathbb{H} \leqq \dim E_{U_1}((\mu, \mu^{(0)}))\,\mathbb{H} + C''(\mu^{(0)}). \tag{1.13}$$

Putting inequalities (1.10), (1.12), (1.13) together, we arrive at (1.8).

The proof of (1.9) is similar. Instead of (1.10) we notice that some arc $(\mu^{(0)}, \tau\mu^{(0)})$ is a gap in the spectrum of U_0. The bound (1.12) should be replaced by

$$\dim E_{U_0}((\mu^{(0)}\tau, \mu))\,\mathbb{H} \leqq \dim E_{U_1}((\mu^{(0)}, \mu))\,\mathbb{H},$$

which is a consequence of (1.4). Finally, Proposition 1.2 allows us to estimate the RHS here by the RHS of (1.9). This concludes the proof.

If $T \in \mathcal{U}_+$, then $U_0 = T^* U$ where $T^* \in \mathcal{U}_-$. Therefore interchanging the roles of U_0 and U in Proposition 1.11 we obtain the dual assertion.

Proposition 1.11'. — *Let* $U = TU_0$, *where* $T \in \mathcal{U}_+$, $(\mu_1, \mu_2) \cap \sigma^{(ess)} = \varnothing$ *and* $\mu^{(0)} \in (\mu_1, \mu_2)$. *Then*

$$\begin{aligned} &\dim E_U((\mu^{(0)}, \mu))\,\mathbb{H} \leqq \dim E_{U_0}((\mu^{(0)}, \mu))\,\mathbb{H} + C_1(\mu^{(0)}), \\ &\dim E_{U_0}((\mu, \mu^{(0)}))\,\mathbb{H} \leqq \dim E_U((\mu, u^{(0)}))\,\mathbb{H} + C_2(\mu^{(0)}), \end{aligned} \tag{1.14}$$

where the constant $C_1(\mu^{(0)})$ $(C_2(\mu^{(0)}))$ *does not depend on* $\mu \in (\mu^{(0)}, \mu_2)$ (*on* $\mu \in (\mu_1, \mu^{(0)})$).

Corollary 1.12. — *Let* $U = TU_0$, *where* $T \in \mathcal{U}_+ (\mathcal{U}_-)$ *and* $(\mu_1, \mu_2) \cap \sigma_{U_0} = \varnothing$. *Then the spectrum of* U *consists in* (μ_1, μ_2) *of eigenvalues which may accumulate only at the point* μ_1 (μ_2).

For the proof it suffices to notice that the first terms in the RHS of (1.8) and (1.14) are equal to zero.

Moreover, we can obtain a bound on the number of eigenvalues of the operator U in the gap of the spectrum of U_0.

Proposition 1.13. — *Let* $U = TU_0$, *where* $T \in \mathcal{U}$, $(\mu_1, \mu_2) \cap \sigma_{U_0} = \varnothing$ *and* $\mu^{(0)} \in (\mu_1, \mu_2)$. *Then*

$$\dim E_U((\mu, \mu^{(0)}))\,\mathbb{H} \leqq \dim E_T((\mu\bar{\mu}_1, \mu^{(0)}\bar{\mu}_2))\,\mathbb{H}, \qquad \mu \in (\mu, \mu^{(0)}), \tag{1.15}$$

$$\dim E_U((\mu^{(0)}, \mu))\,\mathbb{H} \leqq \dim E_T((\mu^{(0)}\bar{\mu}_1, \mu\bar{\mu}_2))\,\mathbb{H}, \qquad \mu \in (\mu^{(0)}, \mu_2). \tag{1.16}$$

Proof. — Let us verify, for example, (1.15). Set $X_1 = [\mu^{(0)}\bar{\mu}_2, \mu\bar{\mu}_1]$, $X_0 = \mathbb{T} \setminus X_1$ and define the operators T_0, T_1 by (1.11). Then $T = T_0 T_1$, where $T_0 \in \mathcal{U}_0$ because $1 \notin \bar{X}_0$ and $\sigma_{T_1} \subset X_1$. Since (μ_1, μ_2) is a gap in the spectrum of U_0, Theorem 1.4 ensures that $(\mu, \mu^{(0)})$ is a gap in the spectrum of $U_1 = T_1 U_0$. Clearly,

$$\dim (U - U_1)\,\mathbb{H} = \dim E_T(X_0)(T - I)\,U_0\,\mathbb{H},$$

which is equal to the RHS of (1.15). Thus applying Proposition 1.2 to the pair U_1, U and the arc $(\mu, \mu^{(0)})$ we conclude the proof.

Of course, Proposition 1.13 can be naturally combined with Propositions 1.11 and 1.11′ but we do not need such a generalization.

2. SCATTERING THEORY

1. We describe here a necessary background of scattering theory for a pair of self-adjoint operators H_0, H in a Hilbert space $\mathscr{H}$. Wave operators for the pair H_0, H are introduced as strong limits

$$W_\pm = W_\pm(H, H_0) = s - \lim_{t \to \pm\infty} \exp(iHt)\exp(-iH_0 t)P_0,$$

where P_0 is the orthogonal projection onto the absolutely continuous subspace $\mathscr{H}_0^{(a)}$ of the operator H_0. If $W_\pm$ exist, then they are isometric on $\mathscr{H}_0^{(a)}$ and have the interwining property $HW_\pm = W_\pm H_0$. The scattering operator $\mathscr{S} := W_+^* W_-$ commutes with H_0. If the ranges $R(W_\pm)$ of $W_\pm$ coincide with the absolutely continuous subspace $\mathscr{H}^{(a)}$ of the operator H, then wave operators are called complete. In this case $\mathscr{S}$ is a unitary operator in $\mathscr{H}_0^{(a)}$.

Consider now the diagonalization of H_0 under the representation of $\mathscr{H}_0^{(a)}$ as a direct integral

$$\mathscr{H}_0^{(a)} \leftrightarrow \int_{\hat{\sigma}_0} \oplus \, \mathbb{H}(\lambda)\, d\lambda. \tag{2.1}$$

Here $\hat{\sigma}_0$ denotes the core of the spectrum of H_0, *i.e.* it is some set of minimal Lebesgue measure which carries the spectral measure $E_0(.)$ of H_0. The direct integral in the RHS of (2.1) is the L_2-space of vector-functions defined on $\hat{\sigma}_0$ and taking values in auxiliary Hilbert spaces $\mathbb{H}(\lambda)$. The correspondence (2.1) means that there exists a unitary operator F_0 mapping $\mathscr{H}_0^{(a)}$ onto the direct integral such that $F_0 E_0(X) F_0^*$ acts as multiplication by the characteristic function of a Borel set $X \cap \hat{\sigma}_0 \subset \mathbb{R}$. We set $F_0 f = 0$ if $f \in \mathscr{H} \ominus \mathscr{H}_0^{(a)}$. It follows that for almost all (a.a.) $\lambda \in \hat{\sigma}_0$

$$\frac{d(E_0(\lambda) f, g)}{d\lambda} = ((F_0 f)(\lambda), (F_0 g)(\lambda)), \tag{2.2}$$

where the scalar product in the RHS is evaluated in the space $\mathbb{H}(\lambda)$. We emphasize that scalar products and norms in different spaces are denoted by the same symbols. The operator $F_0 \mathscr{S} F_0^*$ acts as multiplication by an operator-function $S(\lambda) : \mathbb{H}(\lambda) \to \mathbb{H}(\lambda)$ defined for a.a. $\lambda \in \hat{\sigma}_0$ and called the scattering matrix (SM). Note that in abstract scattering theory $S(\lambda)$ is defined only up to a unitary equivalence.

Let us give sufficient conditions for the existence and completeness of wave operators $W_\pm$. These conditions permit also to obtain a convenient representation for SM. Moreover, under our assumptions the formal sum $H=H_0+V$ can be defined as a self-adjoint operator. Suppose that the "free" operator H_0 is self-adjoint and the perturbation V is factored as $V=G^* \mathscr{V} G$, where $\mathscr{V}=\mathscr{V}^*$ is bounded and G is $|H_0|^{1/2}$-bounded. It is allowed that G acts into some auxiliary Hilbert space $\mathscr{G}$; *i.e.* $G:\mathscr{H}\to\mathscr{G}$; then $\mathscr{V}$ is an operator in $\mathscr{G}$. Let $R_0(z)=(H_0-z)^{-1}$, $\operatorname{Im} z\neq 0$, be the resolvent of H_0 and the product

$$B_0(z):=GR_0(z)G^*\in\mathscr{K}_\infty. \tag{2.3}$$

Then the inverse operator

$$(I+B_0(z)\mathscr{V})^{-1} \tag{2.4}$$

exists and is bounded for $\operatorname{Im} z\neq 0$. The operator H is defined in terms of its resolvent $R(z)=(H-z)^{-1}$ which, in turn, is introduced by the relation

$$R(z)=R_0(z)-R_0(z)G^*\mathscr{V}(I+B_0(z)\mathscr{V})^{-1}GR_0(z). \tag{2.5}$$

Details of this construction can be found in [10]. We always assume that the inclusion (2.3) holds and denote by H the self-adjoint operator with the resolvent (2.5).

For scattering theory we need one of the two following assumptions. They are formulated in terms of boundary values of the operator-function $B_0(z)$ as z approches the real axis. Let us introduce the classes $\mathscr{K}_p$, $p\geqq 1$, of those compact K for which the norm

$$\|K\|_p^p=\sum_{n=1}^{\infty}\lambda_n^p(|K|)<\infty.$$

Eigenvalues $\lambda_n(|K|)$ of the operator $|K|=(K^*K)^{1/2}$ are enumerated with their multiplicities.

Assumption 2.1. There exists an open in $\mathbb{R}$ set $\Omega=\bigcup_n(\beta_n,\gamma_n)$ of full measure, such that for every n the operator-function $B_0(z)$ depends in the operator norm continuously on the parameter z, $\operatorname{Re} z\in(\beta_n,\gamma_n)$, up to the cut along $(\beta_n\gamma_n)$.

Assumption 2.2. The operator $B_0(z)\in\mathscr{K}_p$, $\operatorname{Im} z\neq 0$, for some $p<\infty$ and $B_0(z)$ has angular boundary values in $\mathscr{K}_p$ as $z\to\lambda\pm i0$ for a.a. $\lambda\in\mathbb{R}$. Besides, $\operatorname{Ker} G=\{0\}$.

Under any of these assumptions the operator (2.4) has boundary values in the operator norm as $z\to\lambda\pm i0$ for $\lambda\in\Lambda$ where the set Λ has full measure. In case of the "smooth" Assumption 2.1 the proof of this assertion can be found, for example, in [11]. Moreover, in this case Λ is a closed set. Under Assumption 2.2 we consider (*see* [10] for details) an appropriate regularized determinant $D_p(z)$ of the operator $I+B_0(z)\mathscr{V}$.

The scalar analytic function $D_p(z)$ has angular boundary values as $z \to \lambda \pm i0$ for a.a. $\lambda \in \mathbb{R}$. Therefore by Lusin-Privalov uniqueness theorem $D_p(\lambda \pm i0)$ can not vanish on a set of positive Lebesgue measure. This ensures existence and boundedness of the operator (2.4) for $z=\lambda \pm i0$ and a.a. $\lambda \in \mathbb{R}$. Now, according to the resolvent identity (2.5), it easily follows that the operator $B(z)=GR(z)G^*$ also has boundary values as $z \to \lambda \pm i0$ for a.a. $\lambda \in \mathbb{R}$.

Moreover, it can be shown that vector-functions $GR_0(\lambda \pm i\varepsilon)f$ and $GR(\lambda \pm i\varepsilon)f$ have strong limits as $\varepsilon \to 0$ for a.a. $\lambda \in \mathbb{R}$ if f belongs to some set dense in $\mathscr{H}$. Indeed, using the results obtained about the operator (2.4) we find that it is sufficient to consider $GR_0(\lambda \pm i\varepsilon)f$ only. Under Assumption 2.1 this vector-function belongs locally to the Hardy space $H^2(\mathscr{I})$ in the half-plane (upper or lower). Therefore it has boundary values as $\varepsilon \to 0$ for a.a. $\lambda \in \mathbb{R}$. Under Assumption 2.2 the strong limit of $GR_0(\lambda \pm i\varepsilon)f$ exists if $f=G^*g$ and $g \in \mathscr{I}$ is arbitrary. Since $\operatorname{Ker} G=\{0\}$, the set of such f is dense in $\mathscr{H}$.

Thus under Assumptions 2.1 or 2.2 usual conditions of stationary schemes (*see* e.g. [12]) of scattering theory are fulfilled. It follows, in particular, that wave operators $W_\pm(H, H_0)$ exist and are complete.

To describe a stationary representation for the SM we introduce the operator $\mathscr{L}_0(\lambda): \mathscr{I} \to \mathbb{H}(\lambda)$, $\lambda \in \hat{\sigma}_0$, by the relation

$$\mathscr{L}_0(\lambda)f=(F_0 G^* f)(\lambda). \tag{2.6}$$

The RHS of (2.6) makes sense only on a set of full measure which depends on f. In order to define $\mathscr{L}_0(\lambda)$ as a bounded operator for a.a. $\lambda \in \hat{\sigma}_0$ we first consider (2.6) on linear combinations $\mathscr{D}$ of some fixed basis in $\mathscr{I}$. For all $f \in \mathscr{D}$ vectors $\mathscr{L}_0(\lambda)f$ are well defined on a common set of full measure. Moreover, the equality (2.2) and the relation between boundary values of the resolvent and the spectral density ensure that

$$2\pi i(\mathscr{L}_0(\lambda)f, \mathscr{L}_0(\lambda)g) = (B_0(\lambda+i0)f, g)-(B_0(\lambda-i0)f, g), f, g \in \mathscr{D}, \qquad \text{a.a. } \lambda \in \hat{\sigma}_0.$$

Under Assumptions 2.1 or 2.2 the operators $B_0(\lambda \pm i0)$ are bounded for a.a. $\lambda \in \hat{\sigma}_0$. Thus $\mathscr{L}_0(\lambda)$ extends by continuity from the dense set $\mathscr{D}$ to a bounded operator on the whole space $\mathscr{I}$ and

$$2\pi i \mathscr{L}_0^*(\lambda)\mathscr{L}_0(\lambda)=B_0(\lambda+i0)-B_0(\lambda-i0), \qquad \text{a.a. } \lambda \in \hat{\sigma}_0. \tag{2.7}$$

Clearly, $\mathscr{L}_0(\lambda) \in \mathscr{K}_{2p}$ if $B_0(\lambda \pm i0) \in \mathscr{K}_p$. The representation for the SM is given in the following assertion.

THEOREM 2.3. *Let Assumptions 2.1 or 2.2 be satisfied. Then the wave operators $W_\pm(H, H_0)$ exist, are complete and the SM $S(\lambda)=S(\lambda; H, H_0)$ admits the representation*

$$S(\lambda)=I-2\pi i \mathscr{L}_0(\lambda)\mathscr{V}(I+B_0(\lambda+i0)\mathscr{V})^{-1}\mathscr{L}_0^*(\lambda), \qquad \text{a.a. } \lambda \in \hat{\sigma}_0. \tag{2.8}$$

Moreover,

$$S(\lambda)-I\in\mathcal{K}_r, \qquad \text{a.a.}\,\lambda\in\hat{\sigma}_0, \tag{2.9}$$

where $r=\infty$ under Assumption 2.1 and $r=p$ under Assumption 2.2.

Though similar in form, Assumptions 2.1 and 2.2 are rather different in nature. As was already mentioned, the first of them is a usual condition of the "smooth" scattering theory. The second one can be easily verified under trace-class conditions. Indeed, theorem 2.3 ensures

THEOREM 2.4. — *Let* $GE_0(X)\in\mathcal{K}_2$ *for any bounded interval* X *and let* $G(|H_0|+I)^{-1/2}\in\mathcal{K}_{2p}$ *for some* $p<\infty$. *Then Assumption* 2.2 *is fulfilled and therefore all conclusions of Theorem* 2.3 *hold. Moreover, the inclusion* (2.9) *is valid for* $r=1$.

This assertion is somewhat different from familiar [13] trace-class results in that all its conditions are formulated in terms of H_0 and V only (but not of H). These conditions are convenient for us because they permit to obtain a representation for $S(\lambda)$ in the form (2.8). We note that under assumptions of Theorem 2.4 the operator (2.7) belongs to $\mathcal{K}_1$ which justifies (2.9) for $r=1$.

2. Let us give examples of differential operators for which Assumptions 2.1 or 2.2 hold. Set $\mathcal{H}=L_2(\mathbb{R}^d)$,

$$H_0=-\Delta+q_0(x), \qquad H=-\Delta+q_0(x)+q(x) \tag{2.10}$$

with real bounded functions q_0 and q. We suppose that q vanishes sufficiently rapidly at infinity, *i.e.*

$$|q(x)\leqq C(1+|x|)^{-\alpha} \tag{2.11}$$

where at least $\alpha>1$. The verification of Assumption 2.1 requires that the spectral analysis of the operator H_0 can be performed effectively. This is, for example, possible if q_0 is also short-range. More precisely, we introduce

Assumption 2.5. The bounds

$$|q_0(x)|\leqq C(1+|x|)^{-\alpha_0} \tag{2.12}$$

and (2.11) hold with some $\alpha_0>1$ and $\alpha>1$.

The leading particular case is $q_0=0$. We denote $H_{00}=-\Delta$. The reason to consider more general situation is that in section 4 we compare SM for different short-range potentials. This requires the study of the SM for the pair (2.10).

Let $G=G^*$ be multiplication by $(1+|x|)^{-\alpha/2}$ and $\mathcal{V}$ be multiplication by $(1+|x|)^{\alpha}\,q(x)$. Then the operator $\mathcal{V}$ is bounded and $V=G^*\mathcal{V}G$. Under Assumption 2.5 the operator $B_0(z)$ is continuous in the complex plane cut along the positive half-axis with a possible exception of the point $z=0$. This is a usual formulation of the limiting absorption principle (*see* e.g. [11], [14]). The negative spectrum of H consists of eigenvalues which may accumulate at $z=0$ only. Thus under Assumption 2.5 the pair (2.10) satisfies Assumption 2.1.

The direct integral (2.1) can be chosen as the space $\hat{\mathscr{H}}=L_2(\mathbb{R}_+;\mathbb{H})$ of vector-functions on $\mathbb{R}_+$ with values in $\mathbb{H}=L_2(S^{d-1})$. Here S^{d-1} is the unit sphere in the Euclidean space Ξ^d dual to $\mathbb{R}^d$. First, we construct a unitary mapping of $\mathscr{H}_0^{(a)}$ onto $\hat{\mathscr{H}}$ in the case $H_0=H_{00}$. Let $\hat{f}$ be the Fourier transform of a function $f\in L_2(\mathbb{R}^d)$, *i. e.*

$$\hat{f}(p)=(2\pi)^{-d/2}\int_{\mathbb{R}^d}\exp(-i\langle x,p\rangle)f(x)\,dx,\qquad p\in\Xi^d,$$

and let

$$(\Gamma_{00}(\lambda)f)(\omega)=2^{-1/2}\lambda^{(d-2)/4}\hat{f}(\lambda^{1/2}\omega),\qquad \lambda>0,\quad \omega\in S^{d-1},\qquad(2.13)$$

be (up to the numerical factor) the restriction of $\hat{f}$ onto the sphere of radius $\lambda^{1/2}$. Clearly, the operator F_{00}, defined by the relation $(F_{00}f)(\lambda)=\Gamma_{00}(\lambda)f$, maps $\mathscr{H}$ unitarily onto $\hat{\mathscr{H}}$ and $F_{00}H_{00}F_{00}^*$ acts as multiplication by the independant variable $\lambda\in\mathbb{R}_+$. In these terms the operator (2.6) takes the form

$$\mathscr{L}_{00}(\lambda)=\Gamma_{00}(\lambda)G:\mathscr{H}\to\mathbb{H}.$$

If (2.11) holds for $\alpha>1$, then it is well defined, compact and depends continuously on $\lambda>0$. This is a direct consequence of the theorem about traces in $L_2(S^{d-1})$ of functions from the Sobolev space $W_2^\beta(\Xi^d)$, $\beta>1/2\,(\beta=\alpha_0/2)$.

In the general case we consider the operator

$$\Gamma_0(\lambda)=\Gamma_{00}(\lambda)(I-V_0R_0(\lambda+i0)),\ \lambda>0,\qquad(2.14)$$

where V_0 is multiplication by $q_0(x)$. Properties of the operator Γ_{00} and the limiting absorption principle ensure that under Assumption 2.5 the product

$$\mathscr{L}_0(\lambda)=\Gamma_0(\lambda)G:\mathscr{H}\to\mathbb{H}\qquad(2.15)$$

is also well defined, compact and depends continuously on $\lambda>0$. Usual results of the stationary scattering theory (*see* e.g. [11]) show that the operator F_0, defined by the relation $(F_0f)(\lambda)=\Gamma_0(\lambda)f$, maps $\mathscr{H}_0^{(a)}$ unitarily onto $\hat{\mathscr{H}}$ and $F_0H_0F_0^*$ acts as multiplication by the independant variable $\lambda>0$. It can be verified (though we do not need this information) that $F_0=F_{00}W_-^*(H_0,H_{00})$. The choice of F_0 diagonalizing H_0 is of course not unique. We can, for example, replace $R_0(\lambda+i0)$ by $R_0(\lambda-i0)$ in (2.14). Then the corresponding operator F_0 equals $F_{00}W_+^*(H_0,H_{00})$.

Note that for the operator $\mathscr{L}_0(\lambda)$ constructed by (2.14), (2.15) the multiplication formula

$$S(\lambda;H,H_{00})=S(\lambda;H,H_0)S(\lambda;H_0,H_{00})\qquad(2.16)$$

holds. This is a consequence of the multiplication formula for the wave operators or can be proved directly (*see* e. g. [15]).

Assumption 2.1 is also fulfilled for the pair (2.10) if q_0 is long-range or periodic and q satisfies (2.11) for $\alpha>1$. However, in these cases constructions of the direct integral (2.1) and the corresponding mapping $\mathscr{L}_0(\lambda)$ are different.

If q_0 is an arbitrary bounded function, then for the pair (2.10) Assumption 2.2 can be verified. This, however, requires more stringent conditions on q.

Assumption 2.6. The function q_0 is bounded and the estimate (2.11) for q holds for some $\alpha>d$.

Under this assumption the inclusion $\mathrm{G}\mathrm{E}_0(\mathrm{X})\in\mathscr{K}_2$, where X is any bounded interval, was established e.g. in [16]. The same method applies for the proof of the inclusion $\mathrm{G}(|\mathrm{H}_0|+\mathrm{I})^{-1/2}\in\mathscr{K}_{2p}$, $p=p(\alpha,d)<\infty$. According to Theorem 2.4, this ensures that for the pair (2.10) Assumption 2.2 is fulfilled. We emphasize that the construction of the operator $\mathscr{L}_0(\lambda)$ defined by (2.6) relies on the spectral analysis of the operator $\mathrm{H}_0=-\Delta+q_0$. Under the only condition $q_0\in\mathrm{L}_\infty(\mathbb{R}^d)$ this analysis is implicit.

Let us summarize the results obtained.

THEOREM 2.7. — *Let* H_0, H *be given by* (2.10) *and let Assumptions* 2.5 *or* 2.6 *be satisfied. Then the wave operators* $\mathrm{W}_\pm(\mathrm{H},\mathrm{H}_0)$ *exist, are complete and the* SM $\mathrm{S}(\lambda)=\mathrm{S}(\lambda;\,\mathrm{H},\,\mathrm{H}_0)$ *admits the representation* (2.8). *Moreover, under Assumption* 2.5 *the operator* $\mathscr{L}_0(\lambda)$ *is given by* (2.13)-(2.15), *the* SM $\mathrm{S}(\lambda)$ *is continuous with respect to* $\lambda>0$ *and* (2.9) *holds for* $r=\infty$. *Under Assumption* 2.6 *the inclusion* (2.9) *holds for* $r=1$.

Note that under Assumption 2.5 the inclusion (2.9) holds actually for some $r=r(\alpha_0,\,\alpha,\,d)<\infty$.

3. SPECTRAL PROPERTIES OF THE AXIOMATIC SCATTERING MATRIX

1. In our study of the SM only the structure of its representation (2.8) is essential. Therefore it is convenient to describe its properties axiomatically.

Let $\mathrm{B}:\mathscr{J}\to\mathscr{J}$ and $\mathscr{L}:\mathscr{J}\to\mathbb{H}$ be any bounded operators in Hilbert spaces $\mathscr{J}$ and $\mathbb{H}$ satisfying the relation

$$2\pi i\mathscr{L}^*\mathscr{L}=\mathrm{B}-\mathrm{B}^*. \tag{3.1}$$

This requires of course that $\operatorname{Im}\mathrm{B}\geq 0$. Assume that $\mathscr{V}$ is a self-adjoint bounded operator in $\mathscr{J}$. We shall consider operators

$$\mathrm{S}=\mathrm{I}-2\pi i\mathscr{L}\mathscr{V}(\mathrm{I}+\mathrm{B}\mathscr{V})^{-1}\mathscr{L}^*, \tag{3.2}$$

having the structure (2.8) of the SM. An arbitrary operator (3.2), where B and $\mathscr{L}$ are connected by (3.1) and $\mathscr{V}=\mathscr{V}^*$, is called the axiomatric SM here.

The inverse operator in the RHS of (3.2) is supposed to exist. However, in case $T=I+B\mathscr{V}$ is a Fredholm operator [*i.e.* the ranges $R(T)$, $R(T^*)$ of operators T, T^* are closed and $\dim \operatorname{Ker} T=\dim \operatorname{Ker} T^*<\infty$] it can be interpreted in the following generalized sense. If $f\in R(T)$, then there exists a unique $g\in R(T^*)=\mathscr{J}\ominus \operatorname{Ker} T$ such that $f=Tg$. By definition, $T^{-1}f=g$. If $f\in \operatorname{Ker} T^*$, then we set $T^{-1}f=0$. Such an inverse operator always exists, it is bounded and TT^{-1}, $T^{-1}T$ are orthogonal projections P_T, P_{T^*} onto $R(T)$, $R(T^*)$ respectively. Clearly, $(T^{-1})^*=(T^*)^{-1}$. We note the identity

$$T_2^{-1}P_{T_1}-P_{T_2^*}T_1^{-1}=T_2^{-1}(T_1-T_2)T_1^{-1} \tag{3.3}$$

for Fredholm operators T_1 and T_2.

PROPOSITION 3.1. — *The operator* (3.2) *is unitary.*

Without going into details (*see* [7]), we note that equalities $S^*S=I$ and $SS^*=I$ can be easily deduced from (3.1) with the help of (3.3).

Basically, the operator (3.2) does not depend on the choice of an operator $\mathscr{L}$ obeying (3.1). In fact, S is the identity operator on $\operatorname{Ker}\mathscr{L}^*$. Let now $S|_{\overline{R(\mathscr{L})}}$ be the restriction of S on $\overline{R(\mathscr{L})}$. Denote $\mathscr{L}_1=(\pi^{-1}\operatorname{Im} B)^{1/2}$ and

$$S_1=I-2\pi i\mathscr{L}_1\mathscr{V}(I+B\mathscr{V})^{-1}\mathscr{L}_1^*.$$

Then $\mathscr{L}_1=|\mathscr{L}|$ for any $\mathscr{L}$ satisfying (3.1) so that $\mathscr{L}=U\mathscr{L}_1$, where U is a unitary mapping of $\overline{R(\mathscr{L}_1)}$ onto $\overline{R(\mathscr{L})}$. It follows that

$$S|_{\overline{R(\mathscr{L})}}U=US_1|_{\overline{R(\mathscr{L}_1)}}.$$

Thus we have obtained

PROPOSITION 3.2. — *Operators* $S|_{\overline{R(\mathscr{L})}}$ *are unitarily equivalent for different* $\mathscr{L}$ *obeying* (3.1).

If $B\in\mathscr{K}_\infty$, then by (3.1) $\mathscr{L}\in\mathscr{K}_\infty$ also. Therefore the second term in the RHS of (3.2) is a compact operator so that $S-I\in\mathscr{K}$. Now Proposition 1.1 justifies

PROPOSITION 3.3. — *If* $B\in\mathscr{K}_\infty$, *then the spectrum* σ_S *of* S *consists of eigenvalues lying on the unit circle and accumulating only at the point* 1. *All eigenvalues except possibly the limit point* 1 *have finite multiplicity.*

By (3.1), $\pi\|\mathscr{L}\|^2\leqq\|B\|$ so that

$$\|S-I\|\leqq 2\|B\|\,\|\mathscr{V}\|(1-\|B\|\,\|\mathscr{V}\|)^{-1}\to 0$$

as $\|B\|\,\|\mathscr{V}\|\to 0$. Explicitly, the following assertion is true.

PROPOSITION 3.4. — *If* $b=\|B\|\,\|\mathscr{V}\|<1/2$, *then* σ_S *belongs to the arc* $[\exp(-i\theta_b), \exp(i\theta_b)]\subset\mathbb{T}$ *where* $\theta_b=2\arcsin(b(1-b)^{-1})\in[0,\pi)$.

2. Our aim here is to obtain an additional information about spectral properties of the operator S in case the operator $\mathscr{V}$ has a constant sign, *i.e.* $\mathscr{V} \geqq 0$ or $\mathscr{V} \leqq 0$. To begin with, we reduce the problem to the case $\mathscr{V}=\mathrm{I}$ or $\mathscr{V}=-\mathrm{I}$. In fact, if $\pm\mathscr{V} \geqq 0$, then (3.2) can be rewritten as

$$\mathrm{S}=\mathrm{I} \mp 2\pi i \tilde{\mathscr{L}}(\mathrm{I} \pm \tilde{\mathrm{B}})^{-1} \tilde{\mathscr{L}}^* \qquad (3.4)$$

where the operators $\tilde{\mathscr{L}}=\mathscr{L}|\mathscr{V}|^{1/2}$ and $\tilde{\mathrm{B}}=|\mathscr{V}|^{1/2}\,\mathrm{B}\,|\mathscr{V}|^{1/2}$ also satisfy the relation (3.1). This is obvious if the operators $\mathrm{I}+\mathrm{B}\,\mathscr{V}$ and $\mathrm{I}\pm\tilde{\mathrm{B}}$ are invertible. In the general case we have that

$$\mathrm{P}_{\mathrm{I}\pm\tilde{\mathrm{B}}^*}|\mathscr{V}|^{1/2}(\mathrm{I}+\mathrm{B}\mathscr{V})^{-1}=(\mathrm{I}\pm\tilde{\mathrm{B}})^{-1}|\mathscr{V}|^{1/2}\mathrm{P}_{\mathrm{I}+\mathrm{B}\mathscr{V}}. \qquad (3.5)$$

With the help of (3.1) it can be verified that

$$\mathrm{P}_{\mathrm{I}+\mathrm{B}\mathscr{V}}\mathscr{L}^*=\mathscr{L}^*, \qquad \tilde{\mathscr{L}}\mathrm{P}_{\mathrm{I}\pm\tilde{\mathrm{B}}^*}=\tilde{\mathscr{L}}. \qquad (3.6)$$

Let us multiply (3.5) by $\tilde{\mathscr{L}}$ from the left and by $\mathscr{L}^*$ from the right. Taking into account (3.6) we can get rid of the projections $\mathrm{P}_{\mathrm{I}+\mathrm{B}\mathscr{V}}$ and $\mathrm{P}_{\mathrm{I}\pm\tilde{\mathrm{B}}^*}$ in the equality obtained. This proves that (3.4) coincides with (3.2).

To facilitate the study of spectral properties of the unitary operator S we introduce the self-adjoint operator

$$\mathrm{A}=\mathrm{Im}\,\mathrm{S}=(2i)^{-1}(\mathrm{S}-\mathrm{S}^*). \qquad (3.7)$$

It is easy to see that $\nu\in\sigma_{\mathrm{A}}$ if and only if $\nu=\mathrm{Im}\,\mu$ for some $\mu\in\sigma_{\mathrm{S}}$. If $\mathrm{S}-\mathrm{I}\in\mathscr{K}_\infty$, then also $\mathrm{A}\in\mathscr{K}_\infty$.

The cases of small and compact operators B are considered in Theorems 3.5 and 3.6 respectively.

THEOREM 3.5. – *If* $\mathscr{V}\geqq 0$ (*or* $\mathscr{V}\leqq 0$) *and* $\|\mathrm{B}\|\,\|\mathscr{V}\|<1$, *then* σ_{S} *belongs to the lower* (*upper*) *semicircle.*

Proof. – Supposing that $\mathscr{V}=\pm\mathrm{I}$ we find that the operator (3.7) is equal to

$$\mathrm{A}=\mp 2\pi\mathscr{L}\,\mathrm{Re}\,(\mathrm{I}\pm\mathrm{B})^{-1}\mathscr{L}^*. \qquad (3.8)$$

Clearly, $\pm\mathrm{Im}\,\sigma_{\mathrm{S}}\geqq 0$ if and only if $\pm\mathrm{A}\geqq 0$. Thus it suffices to show that for arbitrary $f\in\mathbb{H}$

$$\mathrm{Re}\,((\mathrm{I}\pm\mathrm{B})^{-1}\mathscr{L}^*f,\,\mathscr{L}^*f)\geqq 0. \qquad (3.9)$$

Denote $g=(\mathrm{I}\pm\mathrm{B})^{-1}\mathscr{L}^*f$. Then the LHS of (3.9) is equal to $\mathrm{Re}\,((\mathrm{I}\pm\mathrm{B})\,g,\,g)$, which is bounded from below by $(1-\|\mathrm{B}\|)\,\|g\|^2$. Since $\|\mathrm{B}\|<1$, this concludes the proof.

THEOREM 3.6. – *If* $\mathrm{B}\in\mathscr{K}_\infty$ *and* $\mathscr{V}\geqq 0$ (*or* $\mathscr{V}\leqq 0$), *then eigenvalues of* S *may accumulate at the point* 1 *only from below* (*from above*).

Proof. – It is sufficient to show that eigenvalues of the operator (3.7) do not accumulate at zero from the right (from the left). Suppose again that $\mathscr{V}=\pm\mathrm{I}$. Let us represent $\mathrm{B}\in\mathscr{K}_\infty$ as a sum $\mathrm{B}=\mathrm{K}+\mathrm{B}_1$, where the operator K has a finite rank and $\| B_1 \| < 1$. Similarly to (3.9) we have that

$$\mp A_1 := 2\pi \mathscr{L} \operatorname{Re}(I \pm B_1)^{-1} \mathscr{L}^* \geqq 0. \tag{3.10}$$

According to (3.3)

$$(I+B_1)^{-1} - (I+B)^{-1} = (I+B)^{-1} K (I+B_1)^{-1} + Q (I+B_1)^{-1} \tag{3.11}$$

where Q is the orthogonal projection onto the subspace $\operatorname{Ker}(I+B)$. The RHS of (3.11) is an operator of a finite rank. Comparing (3.8) and (3.10) we find that the difference $A - A_1$ also has a finite rank. Therefore the operator A has only a finite number of positive (negative) eigenvalues if $A_1 \leqq 0$ ($A_1 \geqq 0$).

4. APPLICATIONS TO THE SCATTERING MATRIX

In this section we use the results of section 3 to obtain an information about the SM for a pair of self-adjoint operators.

1. Under the assumptions of Theorem 2.3 the SM $S(\lambda) = S(\lambda; H, H_0)$ admits for a.a. $\lambda \in \hat{\sigma}_0$ the representation (2.8). It obviously has the form (3.2) with $\mathscr{L}$ and B playing the roles of $\mathscr{L}_0(\lambda)$ and $B_0(\lambda)$ respectively. According to (2.7) the relation (3.1) is also satisfied. Now $B_0(\lambda + i0) \in \mathscr{K}_\infty$ and the inverse operator in (2.8) exists for a.a. $\lambda \in \hat{\sigma}_0$. Therefore we can apply the results of the previous section to $S(\lambda)$.

In our study of spectral properties of the SM we can avoid any reference to the direct integral (2.1) in the definition of $S(\lambda)$. Actually, denote by $\mathscr{L}_0(\lambda)$ any bounded operator obeying the relation (2.7) and let $S(\lambda)$ be constructed by the formula (2.8). According to Propositions 3.1 and 3.2 the operator $S(\lambda)$ is unitary and it has the same (with multiplicity taken into account) eigenvalues, which do not coincide with 1, as the SM introduced above. Thus such an operator $S(\lambda)$ can be accepted for the SM. It is defined for a.a. $\lambda \in \mathbb{R}$ and $S(\lambda) = I$ if λ is a regular point of H_0. For example, we can set

$$\mathbb{H}(\lambda) = \mathscr{J}, \qquad \mathscr{L}_0(\lambda) = (\pi^{-1} \operatorname{Im} B_0(\lambda + i0))^{1/2}.$$

Note that for this choice of $\mathscr{L}_0(\lambda)$ the operator (2.8) depends in the operator norm continuously on $\lambda \in \Omega$ if Assumption 2.1 is fulfilled.

We emphasize that the results below hold for all those λ, where $S(\lambda)$ is defined. This set of points λ has full measure. Moreover, under Assumption 2.1 this set is open. In particular, under Assumption 2.5 the results on the SM $S(\lambda)$ for the pair (2.10) are valid for all $\lambda > 0$.

The following auxiliary assertion is a direct combination of Theorem 2.3 with Proposition 3.3.

PROPOSITION 4.1. — *Let Assumptions* 2.1 *or* 2.2 *be satisfied. Then the spectrum of the* SM S *consists of eigenvalues accumulating only at the point* 1. *All eigenvalues except possibly the limit point* 1 *have finite multiplicity.*

We recall that under Assumptions 2.5 or 2.6 the pair of Schrödinger operators (2.10) satisfies Assumptions 2.1 or 2.2 respectively. Therefore Proposition 4.1 ensures

PROPOSITION 4.2. — *Let* H_0, H *be given by* (2.10) *and let Assumptions* 2.5 *or* 2.6 *be satisfied. Then all conclusions of Proposition* 4.1 *about the spectrum of the* SM S *hold.*

Our main concern here is the study of the spectrum of the SM for perturbations of constant sign. We accept, by definition, that a perturbation $V=G^* \mathscr{V} G$ is positive (negative) if $\mathscr{V} \geqq 0$ ($\mathscr{V} \leqq 0$).

Applying Theorem 3.6 we immediately obtain

THEOREM 4.3. — *Let Assumptions* 2.1 *or* 2.2 *be satisfied and* $\mathscr{V} \geqq 0$ (*or* $\mathscr{V} \leqq 0$). *Then eigenvalues of* S *may accumulate at the point* 1 *only from below* (*from above*).

Let us formulate explicitly the particular case of this theorem for the pair (2.10).

THEOREM 4.4. — *Let Assumptions* 2.5 *or* 2.6 *be satisfied and* $q \geqq 0$ (*or* $q \leqq 0$). *Then eigenvalues of* S *may accumulate at the point* 1 *only from below* (*from above*).

Small perturbations are easily considered with the help of Proposition 3.4 and Theorem 3.5. We formulate the results only for the pair (2.10).

THEOREM 4.5. — *Suppose that*

$$|q(x)| \leqq \varepsilon (1+|x|)^{-\alpha} \tag{4.1}$$

where $\alpha > 1$ *if* q_0 *satisfies* (2.12) *for* $\alpha_0 > 1$ *and* $\alpha > d$ *if* $q_0 \in L_\infty(\mathbb{R}^d)$. *Then*

$$\sigma_S \subset [\bar{\tau}_\varepsilon, \tau_\varepsilon],\ |\tau_\varepsilon| = 1,\ \operatorname{Im} \tau_\varepsilon > 0, \qquad S = S(H, H_0), \tag{4.2}$$

where $|\tau_\varepsilon - 1| \to 0$ *as* $\varepsilon \to 0$. *If, moreover,* $q \geqq 0$ (*or* $q \leqq 0$), *then* $\sigma_S \subset [\bar{\tau}_\varepsilon, 1]$ ($\sigma_S \subset [1, \tau_\varepsilon]$).

Remark 4.6. — Let (2.12) and (4.1) hold for $\alpha_0 > 1$ and $\alpha > 1$. Then $\tau_\varepsilon = \tau_\varepsilon(\lambda) \to 1$ as $\varepsilon \to 0$ uniformly in $\lambda \in [\lambda_0, \infty)$, $\lambda_0 > 0$ and $\tau_\varepsilon(\lambda) \to 1$ as $\lambda \to \infty$ uniformly in $\varepsilon \in (0, \varepsilon_0)$. For the proof it suffices to use the familiar (*see* e.g. [14]) bound $\| B_0(\lambda \pm i0) \| = O(\lambda^{-1/2})$.

2. Here we take $H_{00} = -\Delta$ as the free operator and compare the SM

$$S_0(\lambda) = S(\lambda; H_0, H_{00}), \qquad S(\lambda) = S(\lambda; H, H_{00}) \tag{4.3}$$

for two Schrödinger operators (2.10). We suppose now that q_0 and q are short-range, that is the Assumption 2.5 holds. Then operators (4.3) are well-defined for all $\lambda > 0$, are unitary and differ from the identity operator by a compact term. Therefore the spectra of these operators satisfy the conclusions of Proposition 4.1. We shall show that eigenvalues of SM rotate in the clockwise (counterclockwise) direction if a perturbation is increased (decreased), *i.e.* $q \geqq 0$ ($q \leqq 0$). Denote by $N_0(\mu_1, \mu_2)$ and $N(\mu_1, \mu_2)$ the numbers of eigenvalues (with their multiplicity taken into account) of the operators S_0 and S in an arc $(\mu_1, \mu_2) \subset \mathbb{T}$, $1 \notin [\mu_1, \mu_2]$.

We start with the case of small perturbations. Note that by a somewhat different method it was considered earlier by T. Kato [8]. First we show that the spectrum of the SM depends continuously on a perturbation.

PROPOSITION 4.7. — *Let the bounds* (2.12) *and* (4.1) *hold for some* $\alpha_0 > 1$ *and* $\alpha > 1$. *If* $\mu_j \notin \sigma_{S_0}$, $j = 1, 2$, *then for sufficiently small* ε

$$N(\mu_1, \mu_2) = N_0(\mu_1, \mu_2). \tag{4.4}$$

Proof. — The operators S_0, S and $S(H, H_0)$ are connected by the multiplication formula (2.16). According to (4.2) we can apply Proposition 1.6 with U_0, U and T playing the roles of S_0, S and $S(H, H_0)$ respectively. Therefore (4.4) is a consequence of (1.5).

COROLLARY 4.8. — *Let* μ_0, $\mu_0 \neq 1$, *be an eigenvalue of the operator* S_0 *of multiplicity k. Then for sufficiently small* $|\zeta - 1|$, $|\zeta| = 1$, $\operatorname{Im} \zeta > 0$, *and* $\varepsilon > 0$ *there are exactly k eigenvalues of the operator* S *in the arc* $(\mu_0 \bar{\zeta}, \mu_0 \zeta)$.

Quite similarly, combining Theorem 4.5 with Proposition 1.8 and Remark 1.9 we obtain the result about the rotation of the spectrum.

THEOREM 4.9. — *Let the bounds* (2.12) *and* (4.1) *hold for some* $\alpha_0 > 1$ *and* $\alpha > 1$. *If* $q \geqq 0$ *and* $\mu_2 \notin \sigma_{S_0}$ *or* $q \leqq 0$ *and* $\mu_1 \notin \sigma_{S_0}$, *then* $N(\mu_1, \mu_2) = N_0(\mu_1, \mu_2)$ *for sufficiently small* ε.

COROLLARY 4.10. — *Let* $\mu_0, \mu_0 \neq 1$, *be an eigenvalue of the operator* S_0. *Then for sufficiently small* $|\zeta - 1|$, $|\zeta| = 1$, $\operatorname{Im} \zeta > 0$, *and* $\varepsilon > 0$ *there are no eigenvalues of the operator* S *in the arc* $(\mu_0, \mu_0 \zeta)$ *if* $q \geqq 0$ *and in the arc* $(\mu_0 \bar{\zeta}, \mu_0)$ *if* $q \leqq 0$.

As was explained in section 1, p. 2, Theorem 4.9 gives the precise formulation of the notion of the rotation of the spectrum. Remark also that, under its assumptions, $N(\mu_1, \mu_2) \geqq N_0(\mu_1, \mu_2)$ if $q \geqq 0$ and $\mu_1 \notin \sigma_{S_0}$ or $q \leqq 0$ and $\mu_2 \notin \sigma_{S_0}$.

Let us mention a particular case of the results obtained. Suppose that $H_0 = -\Delta$, $H_\gamma = -\Delta + \gamma q$, where the coupling constant $\gamma \geqq 0$ and q satisfies the bound (2.11) for $\alpha > 1$. Then as γ increases the spectrum of the $S(H_\gamma, H_0)$ rotates in the clockwise (counterclockwise) direction if $q \geqq 0$ ($q \leqq 0$).

We emphasize that in Theorem 4.9 the parameter ε depends on the points μ_j, $j = 1, 2$. In particular, ε may tend to zero as $\mu_1 \to 1$ or $\mu_2 \to 1$.

Thus even for small ε Theorem 4.9 does not give us information on a displacement of all eigenvalues of the SM. To remedy this drawback we shall now consider the movement of eigenvalues in the neighbourhood of the accumulation point $1 \in \mathbb{T}$. A perturbation q is no longer assumed to be small. Since in contrast to Theorem 4.9 and Corollary 4.10 the roles of operators H_0 and H are now symmetric it is sufficient to consider, for example, the case $q \geqq 0$.

THEOREM 4.11. – *Let Assumption* 2.5 *hold,* $q \geqq 0$ *and let* $\mu^{(0)} \neq 1$ *be some fixed point of* $\mathbb{T}$. *Then*

$$N(\mu, \mu^{(0)}) \leqq N_0(\mu, \mu^{(0)}) + C_1(\mu^{(0)}), \tag{4.5}$$

$$N_0(\mu^{(0)}, \mu) \leqq N(\mu^{(0)}, \mu) + C_2(\mu^{(0)}), \tag{4.6}$$

where the constant $C_1(\mu^{(0)})$ $(C_2(\mu^{(0)}))$ *does not depend on* $\mu \in (1, \mu^{(0)})$ *(on* $\mu \in (\mu^{(0)}, 1)$).

Proof. – We use again the multiplication formula (2.16). By Theorem 4.4 the operator $S(H, H_0) \in \mathcal{U}_-$. Therefore we can apply Proposition 1.11 with the operators U_0, U and T playing the roles of the operators S_0, S and $S(H, H_0)$. Thus (4.5) and (4.6) are direct consequences of (1.8) and (1.9) respectively.

Remark 4.12. – The bound (4.5) ((4.6)) is non-trivial only in the limit $\mu \to 1 + i0$ $(\mu \to 1 - i0)$ when the function $N(\mu, \mu^{(0)})$ $(N_0(\mu^{(0)}, \mu))$ may tend to infinity.

Remark 4.13. – If $q \leqq 0$, then N and N_0 should be interchanged in (4.5) and (4.6) so that

$$N_0(\mu, \mu^{(0)}) \leqq N(\mu, \mu^{(0)}) + C_1(\mu^{(0)}),$$

$$N(\mu^{(0)}, \mu) \leqq N_0(\mu^{(0)}, \mu) + C_2(\mu^{(0)}). \tag{4.7}$$

Remark 4.14. – Choosing in (4.5) and (4.7) $q_0 = 0$, we recover Theorem 4.4.

Theorem 4.11 gives a kind of variational principle for scattering phases $\delta_n^\pm$ connected with eigenvalues (counted with their multiplicities) $\mu_n^\pm$, $\mu_n^\pm \neq 1$, of the SM $S(H, H_{00})$ by the formula (0.1). We assume that $\pi/2 \geqq \delta_1^+ \geqq \ldots \geqq \delta_n^+ \geqq \delta_{n+1}^+ > 0$ and $\pi/2 > \delta_1^- \geqq \ldots \geqq \delta_n^- \geqq \delta_{n+1}^- > 0$. Similarly, we denote by $\delta_{0,n}^\pm$ scattering phases related to $S(H_0, H_{00})$. Then (4.5) and (4.6) are equivalent to the bounds

$$\delta_{n+k_+}^+ \leqq \delta_{0,n}^+, \; \delta_{0,n+k_-}^- \leqq \delta_n^-,$$

where $k_\pm$ are some fixed finite numbers.

3. Let again $\mu_n^\pm$ be eigenvalues of the SM for the pair $H_0 = H_{00} = -\Delta$, $H = -\Delta + q$. Here we obtain a bound on $|\mu_n^\pm - 1|$ (or equivalently on $\delta_n^\pm$)

in terms of $q_{\mp}$ where

$$q_+(x)=\max\{q(x),0\},\ q_-(x)=q_+(x)-q(x).$$

We proceed from the familiar [7] bound on $|\mu_n^{\pm}-1|$ in terms of q.

PROPOSITION 4.15. — *Under the condition* (2.11), *where* $\alpha>1$,

$$|\mu_n^{\pm}-1|\leqq Cn^{-\rho},\ \rho=(\alpha-1)(d-1)^{-1}. \tag{4.8}$$

Theorem 4.11 allows us to improve this result.

THEOREM 4.16. — *Suppose that* (2.11) *holds for some* $\alpha>1$. *Let for one of the signs*

$$q_{\pm}(x)\leqq C(1+|x|)^{-\alpha_{\pm}},\qquad \alpha_{\pm}>\alpha. \tag{4.9}$$

Then for the same (*upper or lower*) *sign*

$$|\mu_n^{\mp}-1|\leqq Cn^{-\rho_{\pm}},\ \rho_{\pm}=(\alpha_{\pm}-1)(d-1)^{-1}. \tag{4.10}$$

Proof. — Let, for example, (4.9) hold for the lower sign. Denote $H_-=-\Delta-q_-$. By Theorem 4.4 eigenvalues $\tilde{\mu}_n$ of the SM S (H_-,H_{00}) may accumulate at the point 1 only from the above and by (4.8) they satisfy the bound $|\tilde{\mu}_n-1|\leqq Cn^{-\rho_-}$. This is equivalent to the bound

$$\dim E_{S(H_-,H_{00})}((\mu,\mu^{(0)}))\,\mathbb{H}\leqq C|\mu-1|^{-\tau_-},\qquad \tau_-=\rho_-^{-1}, \tag{4.11}$$

where $\mu^{(0)}\neq 1$ is some fixed point of $\mathbb{T}$ and $\mu\to 1+i0$. Let us now apply Theorem 4.11 to the SM S(H,H_{00}) and S(H_-,H_{00}). Since $q\geqq -q_-$, it follows from (4.5) that

$$\dim E_{S(H,H_{00})}((\mu,\mu^{(0)}))\,\mathbb{H}\leqq \dim E_{S(H_-,H_{00})}((\mu,\mu^{(0)}))\,\mathbb{H}+C(\mu^{(0)}), \tag{4.12}$$

where $C(\mu^{(0)})$ does not depend on $\mu\to 1+i0$. Therefore (4.11) ensures that the LHS of (4.12) is also bounded by $|\mu-1|^{-\tau_-}$. This is equivalent to the bound (4.10) for $|\mu_n^+-1|$.

Remark 4.17. — If $q_{\pm}(x)=o(|x|^{-\alpha_{\pm}})$, $|x|\to\infty$, then $|\mu_n^{\mp}-1|=o(n^{-\rho_{\pm}})$. The proof is the same but instead of Proposition 4.14 we should use the bound $|\mu_n^{\pm}-1|=o(n^{-\rho})$ which holds if $q=o(|x|^{-\alpha})$.

Remark 4.18. — The bounds (4.8) and hence (4.10) are uniform in the spectral parameter $\lambda\in[\lambda_0,\lambda_1]$ where $0<\lambda_0<\lambda_1<\infty$.

5. MODIFIED SCATTERING MATRIX

1. Let $\Sigma=SJ$ be the product of the SM $S=S(\lambda;H,H_0)$ with some unitary operator J. Such an operator Σ will be called the modified SM. Here we shall carry over the results of section 4 about S to the operator $\Sigma=\Sigma(\lambda;H,H_0)$. In case $H_0=-\Delta$, $H=-\Delta+q$ it is natural to choose J

as the reflection operator in the space $\mathbb{H}=L_2(S^{d-1})$, *i.e.*

$$(J f)(\omega)=f(-\omega), \qquad \omega\in S^{d-1}. \tag{5.1}$$

Then solutions of the Schrödinger equation, behaving as standing waves at infinity, are described [9] in terms of eigenvalues and eigenfunctions of such an operator Σ.

Clearly, the spectrum of the operator (5.1) consists of eigenvalues 1 and -1 with corresponding eigenfunctions being even and odd. If not specified otherwise, we suppose only that the spectrum of J has some gap (γ_+, γ_-) and study the spectrum of the operator Σ in this gap. In conditions of Theorem 2.3

$$\Sigma - J \in \mathscr{K}_\infty$$

so that by Proposition 1.1 the spectrum of Σ is discrete in (γ_+, γ_-). To take a sign of a perturbation into account we combine Theorem 4.4 with Corollary 1.12. In the last assertion U_0, U and T play the roles of J, Σ and S respectively. Thus we obtain

THEOREM 5.1. – *Under Assumptions* 2.1 *or* 2.2 $\sigma_\Sigma^{(ess)} \cap (\gamma_+, \gamma_-)=\varnothing$. *If, moreover,* $\mathscr{V}\geqq 0$ (*or* $\mathscr{V}\leqq 0$), *then eigenvalues of* Σ *may accumulate only at the point* $\gamma_-(\gamma_+)$.

This theorem can be directly applied to the pair (2.10) if Assumptions 2.5 or 2.6 are satisfied. In particular, if J is given by (5.1) then eigenvalues of Σ may accumulate only at the points 1 and -1. If $q\geqq 0$ ($q\leqq 0$), then there is no accumulation at 1 from above (below) and at -1 from below (above).

2. The results on the rotation of the spectrum of the SM can straightforwardly be extended to the modified SM. To this end we use the multiplication formula

$$\Sigma(\lambda; H, H_{00})=S(\lambda; H, H_0)\,\Sigma(\lambda; H_0, H_{00}) \tag{5.2}$$

which is a consequence of (2.16). As in section 4, p. 2, we suppose now that $H_{00}=-\Delta$, and H_0 and H are given by (2.10), where q_0 and q are short-range potentials.

Eigenvalues of the modified SM rotate in the clockwise (counterclockwise) direction if $q\geqq 0$ ($q\leqq 0$). To give precise formulations denote by $M_0(\mu_1, \mu_2)$ and $M(\mu_1, \mu_2)$ the numbers of eigenvalues (with their multiplicities taken into account) of the operators $\Sigma(H_0, H_{00})$ and $\Sigma(H, H_{00})$ in an arc (μ_1, μ_2), $[\mu_1, \mu_2]\subset(\gamma_+, \gamma_-)$. First we consider the case of small perturbations and formulate, for example, a modification of Theorem 4.9.

THEOREM 5.2. – *Let the bounds* (2.12) *and* (4.1) *hold for some* $\alpha_0>1$ *and* $\alpha>1$. *If* $q\geqq 0$ *and* $\mu_2\notin\sigma_{\Sigma_0}$ *or* $q\leqq 0$ *and* $\mu_1\notin\sigma_{\Sigma_0}$, *then* $M(\mu_1, \mu_2)=M_0(\mu_1, \mu_2)$ *for sufficiently small* ε.

Next we formulate a modification of Theorem 4.11 for perturbations of arbitrary magnitude.

THEOREM 5.3. — *Let Assumption* 2.5 *hold,* $q \geqq 0$ *and let* $\mu^{(0)}$ *be some fixed point of* (γ_+, γ_-). *Then*

$$M(\mu, \mu^{(0)}) \leqq M_0(\mu, \mu^{(0)}) + C_1(\mu^{(0)}), \tag{5.3}$$
$$M_0(\mu^{(0)}, \mu) \leqq M(\mu^{(0)}, \mu) + C_2(\mu^{(0)}),$$

where the constant $C_1(\mu^{(0)})$ $(C_2(\mu^{(0)}))$ *does not depend on* $\mu \in (\gamma_+, \mu^{(0)})$ (*on* $\mu \in (\mu^{(0)}, \gamma_-)$).

Proofs of Theorems 5.2 and 5.3 are quite similar to those of Theorems 4.9 and 4.11. Again we should apply the results of section 4, p. 1, on $S(H, H_0)$ and use Proposition 1.8 (combined with Remark 1.9) and Proposition 1.11 respectively. Now the operators U_0, U and T play the roles of $\Sigma(H_0, H_{00})$, $\Sigma(H_0, H_{00})$ and $S(H, H_0)$.

3. Bounds on eigenvalues of the operator $\Sigma = \Sigma(H, H_0)$, where $H_0 = H_{00} = -\Delta$, can be deduced from those for the SM $S = S(H, H_0)$. Denote by ν_n^+ (ν_n^-) eigenvalues of Σ accumulating at γ_+ (γ_-). Now we use Proposition 1.13 with U_0, U and T playing the roles of J, Σ and S respectively. Thus the following assertion is a direct consequence of Theorem 4.16.

PROPOSITION 5.4. — *Under the assumptions of Theorem* 4.16

$$|\nu_n^{\mp} - \gamma_{\mp}| \leqq C n^{-\rho_\pm}.$$

This estimate can be improved if J is given by (5.1). In this case there are two series of eigenvalues ν_n^+ (and ν_n^-) accumulating at the points 1 and -1 in the clockwise (counterclockwise) direction. All the estimates below hold for both series. We proceed from the result of [9] which shows that eigenvalues of Σ can be estimated in terms of the even part

$$q^{(e)}(x) = 2^{-1}(q(x) + q(-x))$$

of q only.

PROPOSITION 5.5. — *Assume that* $q^{(e)}(x) = O(|x|^{-a})$, $|x| \to \infty$, $a > 1$, *and the bound* (2.11) *is fulfilled for some* $\alpha > (a+1)/2$. *Then*

$$|\nu_n^{\pm} - \nu_0| = O(n^{-r}), \qquad r = (a-1)(d-1)^{-1}, \qquad \nu_0 = 1, -1.$$

Here we shall show with the help of Theorem 5.3 that only the fall-off of the even part of $q_{\mp}$ is essential.

THEOREM 5.6. — *Let the bound* (2.11) *be fulfilled for some* $\alpha > 1$. *Assume that for one of the signs*

$$(q_\pm)^{(e)}(x) \leqq C(1 + |x|)^{-a_\pm} \tag{5.4}$$

and

$$q_\pm(x) \leqq C(1 + |x|)^{-\alpha_\pm}, \alpha_\pm > (a_\pm + 1)/2. \tag{5.5}$$

Then for the same sign

$$|\nu_n^{\mp}-\nu_0|\leqq C n^{-r_\pm}, \qquad r_\pm=(a_\pm-1)(d-1)^{-1}, \qquad \nu_0=1,-1. \quad (5.6)$$

Proof. — Let, for example, (5.4), (5.5) hold for the lower sign. Denote $H_-=-\Delta-q_-$. By Theorem 5.1 eigenvalues $\tilde{\nu}_n$ of the operator $\Sigma_-=\Sigma(H_-, H_{00})$ may accumulate at the points $\nu_0=1$ and $\nu_0=-1$ in the clockwise direction only. By Proposition 5.5 they satisfy the relation $|\tilde{\nu}_n-\nu_0|=O(n^{-r_-})$ which is equivalent to the bound

$$\dim E_{\Sigma_-}((\mu, \mu^{(0)}))\,\mathbb{H}\leqq C|\mu-\nu_0|^{-t_-}, \qquad t_-=r_-^{-1}. \qquad (5.7)$$

Here $\mu\to\pm1\pm i0$ for $\pm\operatorname{Im}\mu^{(0)}>0$. According to (5.3) $\dim E_\Sigma((\mu, \mu^{(0)}))\,\mathbb{H}$, $\Sigma=\Sigma(H, H_{00})$, is estimated (up to some fixed constant) by the LHS of (5.7) and hence by $C|\mu-\nu_0|^{-t_-}$. This is equivalent to the bound (5.6) for $|\nu_n^+-\nu_0|$.

REFERENCES

[1] M. Sh. BIRMAN and M. G. KREIN, On the Theory of Wave Operators and Scattering Operators, *Dokl. Acad. Nauk. U.S.S.R.*, T. **144**, 1962, pp. 475-478 (Russian).

[2] M. Sh. BIRMAN and M. G. KREIN, Some Problems of the Theory of Wave Operators and Scattering Operators, *Proc. of the joint Sov.-Amer. Symp on Partial Diff. Equations*, Novosibirsk, 1963 (Russian).

[3] L. S. KOPLIENKO, On the Spectrum of the Scattering Matrix, *Izv. VUZov, Math.*, No. 5, 1971, pp. 54-61 (Russian).

[4] S. J. ROTFELD, On Spectral Properties of the Scattering Matrix, II, *Topics in Math. Phys.*, Leningrad Univ, T. 7, 1974, pp. 135-149 (Russian).

[5] L. LANDAU et E. LIFSHITZ, *Mécanique Quantique. Théorie non relativiste*, Moscou, MIR, 1967.

[6] S. T. KURODA, Some Remarks on Scattering for Schrödinger Operators, *J. Fac. Sci. Univ. Tokyo*, Sec 1 A, T. **17**, 1970, pp. 315-329.

[7] M. Sh. BIRMAN and D. R. YAFAEV, Asymptotics of the Spectrum of a Scattering Matrix, *Zap. Nauchn. Sem. Leningrad Otdel Mat. Inst. Steklov* (*LOMI*), T. **110**, 1981, pp. 3-29 (Russian).

[8] T. KATO, Monotonicity Theorems in Scattering Theory, *Hadronic J*, T. **1**, 1978, pp. 134-154.

[9] D. R. YAFAEV, On the Asymptotics of Scattering Phases for the Schrödinger Equation, *Ann. Inst. H. Poincaré*, T. **53**, 1990, pp. 283-299.

[10] D. R. YAFAEV, *Mathematical Scattering Theory: General Theory*, Amer. Math. Soc., 1992.

[11] S. T. KURODA, Scattering Theory for Differential Operators, *J. Math. Soc. Japan*, T. **25**, 1973, pp. 75-104 and 222-234.

[12] M. Sh. BIRMAN and D. R. YAFAEV, A General Scheme in Stationary Scattering Theory, *Topics in Math. Phys.*, Leningrad Univ., T. **12**, 1987, pp. 89-117 (Russian).

[13] M. Sh. BIRMAN, A Local Condition of the Existence of Wave Operators, *Izv. Acad. Sci. U.S.S.R.*, S. Math. T. **32**, 1968, pp. 914-942 (Russian).

[14] L. HÖRMANDER, *The Analysis of Linear Partial Differential Operators, II*, Springer-Verlag, 1983.

[15] D. R. YAFAEV, The Eikonal Approximation and the Asymptotics of the Total Scattering Cross-Section for the Schrödinger Equation, *Ann. Inst. H. Poincaré*, T. **44**, 1986, pp. 397-425.

[16] M. Sh. BIRMAN and D. R. YAFAEV, On the Trace-Class Method in Potential Scattering Theory, *Zap. Nauchn. Sem. Leningrad Otdel. Mat. Inst. Steklov (LOMI)*, T. **171**, 1989, pp. 12-35 (Russian).

[17] A. V. SOBOLEV and D. R. YAFAEV, Spectral Properties of the Abstract Scattering Matrix, *Trudy MIAN*, T. **188**, 1990, pp. 125-149 (Russian).

(*Manuscript received march* 22, 1991.)