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K. SEIP

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Addendum to "Curves of maximum modulus in coherent state representations"

by

K. SEIP

Division of Mathematical Sciences, University of Trondheim,
N-7034 Trondheim-NTH, Norway

ABSTRACT. – A class of analytic functions naturally related to a previous paper of the author is studied. This class consists of functions of the form $G(z)=z+\mathcal{O}(z^2)$ such that $\bar{G}(G(z))=z$ in some neighborhood of 0. First a description in terms of formal power series is given. Then it is proved that there is a bijective correspondence between this class of functions and analytic arcs through 0 with a horizontal tangent there. The G for which this arc is part of a closed curve with G meromorphic in some domain containing the curve can be associated with univalent rational functions in the unit disk. Our results are applied to the problem of finding possible curves along which the moduli of certain weighted analytic functions attain maxima.

RÉSUMÉ. – L'auteur présente une classe de fonctions analytiques résultant naturellement d'un précédent papier du même auteur. Cette classe consiste de fonctions de la forme $G(z)=z+\mathcal{O}(z^2)$ telles que $\bar{G}(G(z))=z$ aux environs de zéro. Cette classe de fonctions est d'abord décrite en termes de séries de puissance formelle, puis il est démontré qu'il existe une bijection entre cette classe de fonctions et les arcs analytiques passant par zéro et ayant une tangente horizontale en ce point. La fonction G pour laquelle cet arc est une partie d'une courbe fermée avec G meromorphique dans un domaine contenant cette courbe, peut être associée avec les fonctions rationnelles univalentes du disque unité. Les résultats sont

appliqués à la recherche d'éventuelles courbes le long desquelles les modules de certaines fonctions analytiques pondérées atteignent des maxima.

1. INTRODUCTION

The purpose of this paper is to examine a class of functions closely related to the discussion of [5], and the intention is to clarify to some extent the mathematical part of [5]. We shall close by indicating the link to [5] and by giving two applications to the problems discussed there.

The class of functions to be considered is denoted by $\mathcal{G}$. It consists of functions $G(z)$ analytic in some neighborhood of 0 with the following property. The Maclaurin series of each G is of the form

$$G(z) = z + c_2 z^2 + c_3 z^3 + \ldots, \tag{1}$$

and the inverse function G^{-1} of G defined in some neighborhood of 0 satisfies

$$G^{-1}(z) = \bar{G}(z)$$

where we have defined

$$\bar{G}(z) = \overline{G(\bar{z})}.$$

2. RESULTS ON FORMAL POWER SERIES

We start our discussion in the more general setting of formal power series [2]. That is, we consider formal power series over the field of complex numbers and restrict to elements of the group of almost units under composition of the form

$$G(x) = x + c_2 x^2 + c_3 x^3 + \ldots, \tag{2}$$

with $G^{-1} = \bar{G}$. Here we have defined

$$\bar{G}(x) = x + \bar{c}_2 x^2 + \bar{c}_3 x^3 + \ldots$$

The reader should notice the algebraic identities

$$\overline{(G^{-1})} = \bar{G}^{-1}, \qquad \text{and} \qquad \overline{GH} = \bar{G}\bar{H},$$

which will be used frequently.

For any G and any real number $r \neq 0$, define the formal power series

$$G_r(x) = \frac{1}{r} G(rx) = x + c_2 r x^2 + c_3 r^2 x^3 + \dots$$

and let G^n denote G composed with itself n times where n may be any integer, with the obvious interpretation for negative n. One observes immediately the following.

PROPOSITION 2.1. – *Let G be a formal power series of the form* (2). *Then if* $G^{-1} = \bar{G}$ *we have*

$$G_r^{-1} = \bar{G}_r \qquad \textit{and} \qquad (G^n)^{-1} = \bar{G}^n$$

for any real $r \neq 0$ *and any integer* n.

The following theorem gives a recipe for constructing all possible formal power series of the desired kind.

THEOREM 2.2. – *Let*

$$J(x) = b_2 x^2 + b_3 x^3 + \dots$$

with real coefficients b_2, b_3,... *Then there exists a unique formal power series*

$$R(x) = x + a_2 x^2 + a_3 x^3 + \dots$$

with real coefficients 1, a_2, a_3, ... *such that for*

$$G(x) = R(x) + i J(x) = x + (a_2 + i b_2) x^2 + (a_3 + i b_3) x^3 + \dots$$

we have $G^{-1} = \bar{G}$.

Proof. – The proof is constructive, that is it gives an algorithm for generating the sequence $a_2, a_3, \dots$

Put $a_1 = 1$. Suppose we have found $a_1, \dots, a_n$, and define

$$G^{(n)}(x) = x + (a_2 + i b_2) x^2 + \dots + (a_n + i b_n) x^n.$$

We then have

$$H^{(n)}(x) = (\overline{G^{(n)}}\, G^{(n)})(x) = x + c_{n+1} x^n + \dots \tag{3}$$

Define similarly $G^{(n+1)}$ with a so far unspecified a_{n+1}. Then

$$H^{(n+1)}(x) = x + d_{n+1} x^{n+1} + \dots$$

with

$$d_{n+1} = 2 a_{n+1} + c_{n+1}.$$

By the identity

$$G^{(n+1)} H^{(n+1)} = \overline{H^{(n+1)}}\, G^{(n+1)}$$

we have

$$x+(a_2+ib_2)x^2+\ldots+(a_n+ib_n)x^n+(a_{n+1}+ib_{n+1}+d_{n+1})x^{n+1}+\ldots$$
$$=x+(a_2+ib_2)x^2+\ldots+(a_n+ib_n)x^n+(a_{n+1}+ib_{n+1}+\overline{d_{n+1}})x^{n+1}+\ldots$$

This means that d_{n+1} must be real. By choosing

$$a_{n+1}=-\frac{1}{2}c_{n+1}$$

a G^{n+1} of the desired kind is produced, which obviously is unique.

Since (3) holds for $n=1$ the result follows by induction on n. □

We may characterize these formal power series in another way.

THEOREM 2.3. – *Let* G *be a formal power series of the form* (2). *Then the following statements are equivalent.*

(i) $G^{-1}=\bar{G}$

(ii) *There exists a power series* Φ *of the form*

$$\Phi(x)=x+\alpha_2 x^2+\alpha_3 x^3+\ldots$$

such that $G=\bar{\Phi}^{-1}\Phi$.

Proof. – The implication (ii) $\Rightarrow$ (i) is obvious and we concentrate on proving (i) $\Rightarrow$ (ii).

To this end, define

$$\Phi(x)=\frac{1}{2}(G(x)+x).$$

Observe that

$$\bar{\Phi}G=\Phi,$$

which shows that a Φ of the desired kind has been constructed. □

We observe that Φ is not unique. It is easily seen that it is unique up to composition $R\Phi$ with almost units

$$R(x)=x+b_2 x^2+b_3 x^3+\ldots$$

with real coefficients $b_2, b_3, \ldots$ Algebraicly we may thus associate our formal power series with the group of almost units modulo the group of almost units with real coefficients.

3. A GEOMETRIC CONSEQUENCE OF ANALYTICITY

From now on we require our power series to have a nonzero radius of convergence, that is we turn to a discussion of the class $\mathcal{G}$. We shall see how the requirement of analyticity leads us to certain geometric results.

We keep our notation from the preceding section. This means, for instance, that by definition $(GH)(z)=G(H(z))$ wherever this has a meaning.

We observe at oncė that Proposition 2.1 carries over to this restricted situation, the first part of which will be useful.

PROPOSITION 3.1. – *If* $G\in\mathcal{G}$, *then* $G_r\in\mathcal{G}$ *and* $G^n\in\mathcal{G}$ *for any real* $r\neq 0$ *and for any integer* n.

Unfortunately, Theorem 2.2 can not transferred. One may check that $J(x)=x^2$ gives an $R(x)$ with convergence radius 0.

But obviously Theorem 2.3 has a counterpart. The geometric content of this result is that there is a bijective correspondence between $\mathcal{G}$ and analytic arcs through 0 with a horizontal tangent there.

THEOREM 3.2. – *For* $G(z)$ *of the form* (1) *the following statements are equivalent.*

(i) $G\in\mathcal{G}$.

(ii) *There exists a function* Φ *of the form*

$$\Phi(z)=z+\alpha_2 z^2+\alpha_3 z^3+\ldots,$$

such that

$$G=\bar{\Phi}^{-1}\Phi$$

in some neighborhood of 0.

(iii) *There is an analytic arc* Γ_G *passing through* 0 *such that*

$$G(z)=\bar{z}$$

along Γ_G.

Proof. – (i) $\Rightarrow$ (ii): This is just a copy of the proof of Theorem 2.3.

(ii) $\Rightarrow$ (iii): For $z=\Phi^{-1}(t)$, t real and sufficiently small we have

$$G(z)=\bar{\Phi}^{-1}(t)=\overline{\Phi^{-1}(t)}=\bar{z}.$$

(iii) $\Rightarrow$ (i): In a neighborhood of 0 $\bar{G}G$ is well defined and we have

$$(\bar{G}G)(z)=z$$

along Γ_G, and thus $\bar{G}G=I$ in this neighborhood. □

Remark. – The arc Γ_G of (iii) is given by $y=Q(x)$ where Q is the real power series in

$$\Phi^{-1}(z)=z+iQ(z).$$

where we have chosen

$$\Phi(z)=\frac{1}{2}(G(z)+z).$$

For the rest of this paper we shall be interested in domains in which elements of $\mathcal{G}$ can be defined as meromorphic functions, possibly by analytic continuation. It is then natural to note that *the Möbius transformations contained in $\mathcal{G}$ are the only elements of $\mathcal{G}$ that can be continued meromorphically to the whole complex plane* (this is essentially Theorem 4.1 of [5]).

4. WHEN $G(z)=\bar{z}$ ALONG A CLOSED CURVE

We have seen that locally the elements of $\mathcal{G}$ can be associated with all possible analytic arcs (suitably normalized). We shall now take a more global point of view and look for possible *closed* curves Γ_G to which there exist functions G with $G(z)=\bar{z}$ along Γ and G meromorphic in some domain containing Γ. Such G may be associated to elements of $\mathcal{G}$ through suitable Möbius transformations.

Note that it is necessary to look for *meromorphic* G: The function $z\,G(z)$ is positive on Γ and thus equal to a constant if $z\,G(z)$ is analytic.

The result of this section reduces this problem to a study of univalent rational functions in the unit disk. The proof is essentially contained in [5] (*see* the proof of Theorem 5.4) and is therefore omitted. We have chosen to place one of the poles at 0, which is just a convenient normalization.

THEOREM 4.1. – *Let Γ be a closed curve enclosing* 0 *and let* $G(z)=\bar{z}$ *along Γ,* G *meromorphic in some domain containing Γ. Furthermore, let* G *have a k-th order pole at* 0 *and n other poles (counting multiplicities) enclosed by Γ. Then Γ is the image of the unit circle under a rational function*

$$\Psi(z)=zr(z)/s(z)$$

being univalent in the closed unit disk with $zr(z)$ and $s(z)$ relatively prime and degree $(r)=n-1$ and degree $(s)=n-k$.

Conversely, any such rational function being univalent in some domain containing the unit disk defines such a curve Γ with corresponding $G(z)$ *given by*

$$G(z)=\bar{\Psi}\left(\frac{1}{\Psi^{-1}(z)}\right).$$

Clearly, such Ψ can be found for any $k\leq n$. We can thus find G with an arbitrary number of poles enclosed by Γ.

5. RESTRICTING TO THE UNIT DISK

Now let $\mathscr{G}(\Delta)$ denote the class of functions G meromorphic in the unit disk Δ such that $G(z)=\bar{z}$ along some nondegenerate curve in Δ. $\mathscr{G}(\Delta)$ can be viewed as a subclass of $\mathscr{G}$ in view of the following simple invariance.

PROPOSITION 5.1. – *For any* $G\in\mathscr{G}(\Delta)$ *and any Möbius self-map* T *of* Δ, $\bar{T}^{-1}GT\in\mathscr{G}(\Delta)$.

Proof. – Observe that if $G(z)=\bar{z}$ along Γ, then $(\bar{T}^{-1}GT)(z)=\bar{z}$ along $T^{-1}(\Gamma)$. □

The curves associated to $\mathscr{G}(\Delta)$ may either be closed and properly contained in Δ, or they may start and end, possibly at the same point, at the unit circle. Intersections are impossible due to the analyticity of G. We shall refer to these two kinds of curves as *closed* and *infinite curves*, respectively. It is easy to see from the result of the preceding section that in the case of closed curves, there can only be one curve associated with G. But in the case of infinite curves we note the following.

PROPOSITION 5.2. – *For any* $n\geqq 2$ *there exist* $G\in\mathscr{G}(\Delta)$ *with n distinct infinite curves along which* $G(z)=\bar{z}$.

Proof. – Let

$$\Psi(z)=z\,.\,\frac{1-\varepsilon_1 z^n}{1-\varepsilon_2 z^n}$$

which clearly is univalent in some domain containing the unit disk for sufficiently small $\varepsilon_1, \varepsilon_2>0$, $\varepsilon_1\neq\varepsilon_2$. You then observe that, as z traverses the unit circle, $\Psi(z)$ intersects the unit circle $2n$ *times*. □

It would be interesting to know if any $G\in\mathscr{G}(\Delta)$ with more than one curve arises in this way, that is as a closed curve "cut" by the unit circle. If this is so, one could argue that the case of multiple curves really is an artifact. For the applications we have in mind it is however the restriction to the disk that has a meaning.

6. CURVES OF MAXIMUM MODULUS OF ANALYTIC FUNCTIONS WITH A BERGMAN WEIGHT

We shall close by pointing out the relation to [5] and by giving two applications to the problems discussed there.

Let us recall the problem put forward in [5]. We are concerned with functions

$$S(z)=(1-|z|^2)^\alpha f(z) \tag{4}$$

where $\alpha>0$ is some fixed number and $f(z)$ is analytic in the unit disk $\Delta=\{z:|z|<1\}$. We are interested in curves of the following kind.

DEFINITION. — *A curve* Γ *is an* M-*curve of* $S\neq 0$ *if* $\frac{\partial|S|}{\partial x}=\frac{\partial|S|}{\partial y}=0$ *along* Γ, *where* S *is of the form* (4).

We also say that Γ is an M-curve of f when Γ by this definition is an M-curve of S, and S and f are related by (4).

The interested reader should consult [5] to find the quantum mechanical motivation for our interest in M-curves. The main point is that they may represent the classical orbits of certain quantum mechanical systems related to the radial harmonic oscillator and the hydrogen atom as studied by T. Paul [3]. M-curves could also represent "time-varying spectral lines" of certain classes of signals, *see* for instance [1], [4].

Straightforward arguments led us to the following results [5].

PROPOSITION 6.1. — *Suppose* S *has an* M-*curve* Γ. *Then*

(i) $|S|$ *is constant along* Γ,

(ii) $|S|$ *attains local maxima on* Γ,

(iii) S *is determined up to a constant factor by any subset of* Γ *containing an accumulation point.*

PROPOSITION 6.2. — *A curve* Γ *is an* M-*curve of* S *if and only if*

$$\frac{d}{dz}\ln f=\frac{f'(z)}{f(z)}=\frac{2\alpha\bar{z}}{1-|z|^2} \tag{5}$$

holds along Γ.

The latter proposition shows that any M-curve defines an element of $\mathscr{G}$. This is seen by putting

$$G(z)=\frac{f'(z)/f(z)}{2\alpha+zf'(z)/f(z)}$$

for which we have $G(z)=\bar{z}$ along Γ. The following theorem tells us that the opposite is true in a certain sense.

THEOREM 6.3. — *For any* $G\in\mathscr{G}$ G_r *defines an* M-*curve for all sufficiently small* r.

Proof. — For sufficiently small r G_r will be analytic in $z\in\Delta$ so the only thing that could prevent it from defining an M-curve, are possible poles of $G_r(z)/(1-zG_r(z))$, thus points in Δ at which

$$zG_r(z)-1=0.$$

Now pick some $\varepsilon>0$ and choose r such that $G_r(z)$ is analytic in $|z|\leqq 1+\varepsilon$. Then for sufficiently small r we have

$$|z^2-1-(zG_r(z)-1)|<|z^2-1|$$

on $|z|=1+\varepsilon$. Thus by Rouché's theorem $zG_r(z)-1$ has precisely two roots in $|z|<1+\varepsilon$. Also for sufficiently small r there are two intersections between Γ_{G_r} and the unit circle, and it is clear that the two roots of $zG_r(z)-1$ correspond to these intersections. □

In principle we now know how to generate all possible M-curves. A somewhat systematic way to do it could be as follows. Divide the elements of $\mathcal{G}$ into equivalence classes by identifying G with H whenever there exists a real $r\neq 0$ such that $G(z)=\frac{1}{r}H(rz)$. Then pick out one normalized representative G for each equivalence class, for instance with $|G''(0)|=1$. To each equivalence class we can identify a "spectrum" defined by

$$\sigma(G)=\{r>0: G_r \text{ defines an M-curve}\}$$

By Theorem 6.3 $\sigma(G)$ is nonempty for all G, it consists at least of some interval with 0 as the one endpoint. If G is not a Möbius transformation we also know that $\sup\sigma(G)<\infty$. We urge the reader to compute $\sigma(G)$ for the nontrivial equivalence class of Möbius transformations. In this case $\sigma(G)$ can be divided into two parts, an infinite and bounded discrete set and an interval as described above.

M-curves may be closed and they may be infinite as shown in [5]. As suggested by Proposition 5.2 we may have several infinite curves. We prove this result as a supplement to the examples of [5].

PROPOSITION 6.4. — *For any $n\geqq 2$ there are f having n distinct infinite* M-*curves.*

Proof. — Take G as in the proof of Proposition 5.2. We first show that $1-zG(z)$ is zero-free in Δ. Notice that by construction $1-zG(z)$ has $2n$ distinct zeros along $\Psi(\partial\Delta)$ since we have $2n$ intersections between $\Psi(\partial\Delta)$ and $\partial\Delta$. Then observe that the equation

$$zG(z)=1$$

for z in the closure of $\Psi(\Delta)$ is equivalent to the equation

$$\Psi(\zeta)\bar{\Psi}\left(\frac{1}{\zeta}\right)=1 \tag{6}$$

for ζ in the closure of Δ with $z=\Psi(\zeta)$. (6) can be written as a $2n$-th degree equation in ζ. The $2n$ roots of this equation correspond to the $2n$ intersections identified above.

It remains to be checked if the parameters in G can be chosen such that the residue of

$$F(z)=\frac{2\alpha G(z)}{1-zG(z)}$$

at the simple pole 0 equals a positive integer. To this end we compute

$$\operatorname{Res}(F)_{z=0} = \lim_{\zeta \to 0} \frac{2\alpha\Psi(\zeta)\bar{\Psi}(1/\zeta)}{1-\Psi(\zeta)\bar{\Psi}(1/\zeta)} = \frac{2\alpha(\varepsilon_1/\varepsilon_2)}{1-(\varepsilon_1/\varepsilon_2)}.$$

Finally you may choose

$$\frac{\varepsilon_1}{\varepsilon_2} = \frac{n}{n+2\alpha}$$

for some given positive integer n to complete the proof. □

In addition to the remark following Proposition 5.2 some questions related to the above result remain unsolved. In the case of infinite M-curves, does f have a zero if and only if it has several M-curves? And, does $|S(z)|$ obtain global maxima along its M-curve(s)?

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