ANNALES DE L'I. H. P., SECTION A

EMANUELA CALICETI
STEFANO MARMI
FRANCO NARDINI

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Annales de l'I. H. P., section A, tome 56, nº 3 (1992), p. 279-305
<http://www.numdam.org/item?id=AIHPA_1992__56_3_279_0>

Ann. Inst. Henri Poincaré,

Vol. 56, n° 3, 1992, p. 279-305. *Physique théorique*

Absence of geometrical phases in the rotating stark effect

by

Emanuela CALICETI

Dipartimento di Matematica, Università di Bologna
Piazza Porta S. Donato 5, 40126 Bologna, Italy

Stefano MARMI

Dipartimento di Matematica « U. Dini », Università di Firenze
Viale Morgagni 67/a, 50134 Firenze, Italy

and

Franco NARDINI

Dipartimento di Matematica, Università di Bologna
Piazza Porta S. Donato 5, 40126 Bologna, Italy

ABSTRACT. — We prove that there are no geometrical phases in a one electron atom under a constant slowly rotating electric field.

RÉSUMÉ. — On montre l'absence des phases géométriques pour l'atome d'hydrogène sous l'action d'un champ électrique qui tourne lentement.

1. INTRODUCTION

Consider a quantum system whose self-adjoint Hamiltonian H (r) depends smoothly (in the norm resolvent sense) on some parameters r

belonging to a manifold R. Assume that a finite set of eigenvalues $E(r)$ with finite multiplicities stays separated from the rest of the spectrum when r varies in R.

Berry (1984) and Simon (1983) have shown that there exists a natural connection on the principal bundle over the parameter manifold given by the spectral subspace of $E(r)$. The holonomy associated to this connection manifests itself, in the adiabatic limit, through a phase shift of the wave function, the *Berry phase*.

After the work of Berry, holonomy effects have attracted considerable interest in chemistry and physics (*see* e. g., Berry 1989, Jackiw 1988 b, Shapere and Wilczek 1989). Moreover, Hannay (1985) and Berry (1985) have found a classical analogue to the Berry phase for classical integrable Hamiltonian systems. Their work has been generalized to the non integrable case by Montgomery (1988) and Golin, Knauf and Marmi (1989) (we refer also to the work of Marsden, Montgomery and Ratiu 1989 for further developments). Under suitable regularity hypotheses for the potential (which, however, are not verified by the model we will consider in this paper) the existence of a semiclassical expansion of the Berry phase has also been rigorously proven (Ash 1990, Gerard and Robert 1989; for a more geometrical approach to this question *see* Weinstein 1990).

In general if the Hamiltonian $H(r)$ is real the curvature vanishes identically and there should be no geometric phase (Avron, Sadun, Segert and Simon 1989).

In fact, let $P(r)$ denote the spectral projection relative to $E(r)$ defined by

$$P=(2\pi i)^{-1}\oint_{|z-E|=\varepsilon}(z-H)^{-1}\,dz. \tag{1.1}$$

If $H(r)$ is real, $P=P^*$ and $P=\bar{P}$. If $E(r)$ is non degenerate the Berry phase γ is the integral of the curvature two-form of the connection

$$\gamma=\frac{i}{2\pi}\int_S \Omega, \tag{1.2}$$

where, as shown by Simon (1983),

$$\Omega=\operatorname{Tr}(P\,dP\,dPP). \tag{1.3}$$

From the above assumptions on the projector P it follows immediately that both $\operatorname{Re}\Omega=0$ and $\operatorname{Im}\Omega=0$. Thus $\Omega=0$.

However the above considerations cannot be applied literally to conclude the absence of geometric phases in one of the most important examples of classically integrable systems, the Stark effect, when the field strength undergoes a slow rotation. This is due to the well known subtleties which come with the Stark effect (*see* e. g. Thirring 1981). In fact, as soon as the field is turned on the tunnel effect makes all hydrogen bound states unstable so that the spectrum of the Stark Hamiltonian becomes purely absolutely continuous. The hydrogen bound states turn instead into resonances (Graffi and Grecchi 1978) defined through complex scaling (or dilation analyticity: Aguilar and Combes 1971, Balslev and Combes 1971). These facts make the analysis of the Berry phase considerably more complicated. The natural Hamiltonian to start with is indeed the complex scaled one, because the role of the bound states is now played by the resonances. Since this Hamiltonian is not self-adjoint, the corresponding eigenprojections enjoy the above property no longer. Hence the above argument is not directly applicable, and the computation of the Berry phase needs some supplementary work. These difficulties will be sidestepped first by working out a very simple formula expressing the holonomy of the Berry connection in terms of the matrix elements of the angular momentum operator L_1 computed on resonance eigenvectors, and then by showing that these matrix elements are zero to all orders in perturbation theory. An explicit proof of the Borel summability of the perturbation series of the Berry holonomy will then conclude the argument.

Our result is summarized by the following

THEOREM. — *Consider the smooth family of Schrödinger operators in* $L^2(\mathbf{R}^3)$ *defined by*

$$\hat{H}(r) = -\frac{\hbar^2}{2}\Delta_x - \frac{Z}{|x|} + r_1 x_1 + r_2 x_2 + r_3 x_3, \tag{1.4}$$

and let the parameters $r=(r_1, r_2, r_3)\in\mathbf{R}^3\setminus\{0\}$ *slowly vary along a (sufficiently small) circle around the origin in the parameter space. Then the Berry holonomy—defined through dilation analyticity* (see *section* 2.2)—*is trivial* (i.e. = 1).

Let us now summarize the content of our paper. In section 2 we first briefly recall some well known facts on the Berry phase (following Avron, Sadun, Segert and Simon 1989). Then we introduce the dilation analyticity technique for the Stark Hamiltonian and we show how to define the Berry connection by means of the generalized eigenprojections on resonances. Finally we prove a formula, suggested from the reading of Ahronov and Anandan (1987), to compute the Berry holonomy by standard perturbation theory techniques. In section 3 we show that the Berry holonomy is trivial to all orders in the electric field strength and in section 4 we prove the Borel summability of the perturbative expansion of section 3, thus rigorously concluding the absence of geometrical phases in the rotating Stark effect Hamiltonian.

For the sake of completeness, in the appendix we show that the Hannay angles are zero for the corresponding classical Hamiltonian, as announced in Golin, Knauf and Marmi (1989).

ACKNOWLEDGEMENTS

We wish to thank Professor Avron for useful discussions and an anonymous referee for helping us to considerably clarify our exposition of the derivation of (2.31).

2. DILATION ANALITICITY AND THE BERRY PHASE

2.1. The Berry phase

In this section we will summarize some well-known basic facts on Berry's phase and we will follow very closely the exposition given by Avron, Sadun, Segert and Simon (1989).

Let us consider a quantum system with self-adjoint Hamiltonian $H = H(r)$ which smoothly depends on a set of parameters r which belong to a parameter manifold R. We assume that the domain D of $H(r)$ is independent of r, and indicate with $E(r)$ an eigenvalue which we suppose isolated from the rest of the spectrum for all $r \in R$. Let $P(r)$ be the associated spectral projection

$$P(r) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{dz}{z - H(r)}, \tag{2.1}$$

where the contour Γ circles $E(r)$ counterclockwise in the complex z-plane. Then $P(r)$ inherits the smoothness of $H(r)$ and has fixed dimension.

To each fixed value of r there corresponds the complex Hilbert space given by the range of $P(r)$, *i.e.* by the eigenstates of $H(r)$ with eigenvalue $E(r)$. Therefore one has a fiber bundle on the parameter space associated to the spectral subspace of $H(r)$ identified by $P(r)$. The *Berry connection* on this bundle is the operator-valued one form

$$A(P)(r) = -[(dP)(r), P(r)], \tag{2.2}$$

where d denotes exterior differentiation w.r.t. r. Note that $dP = (dP) + P\,d$, where (dP) is an operator-valued form (which does not differentiate subsequent expressions).

In the adiabatic limit (Kato 1950, Simon 1983, Avron, Seiler and Yaffe 1987) the physical evolution, generated by $H(r)$, reduces to the parallel transport w.r.t. the Berry connection. Moreover on has the following

THEOREM. – (*Kato* 1950, *Avron, Sadun, Segert and Simon* 1989) *Let* $\tau \mapsto C(\tau)$ *be a parametrized curve on* R. *Let* $U(\tau)$ *be the solution of*

$$\left(\frac{\partial}{\partial \tau} - \left[\frac{\partial P}{\partial \tau}, P\right]\right) U(\tau) = 0, \tag{2.3}$$

where $P(\tau)=P(C(\tau))$ *and with the initial condition* $U(0)=1$. *Then* $U(\tau)$ *is a unitary operator which maps the range of* $P(0)$ *to the range of* $P(\tau)$, i.e.

$$P(\tau)U(\tau)=U(\tau)P(0). \tag{2.4}$$

If one considers a closed path with $C(2\pi)=C(0)$, then $U(2\pi)$ is a unitary map from the range of $P(0)$ to itself and is an element of $\mathbf{U}(n)$, where $n=\dim(P(0))$. $U(2\pi)$ is the *holonomy* of the Berry connection (2.2). If $P(0)$ has dimension 1, (*i.e.* $E(0)$ is non degenerate) this abelian holonomy is the *Berry phase*.

The covariant derivative associated to $A(P)$ is

$$\nabla=d+A=P\,d, \tag{2.5}$$

and acts on differential forms $\omega(r)$ satisfying $P(r)\omega(r)=\omega(r)$. Note that (2.3) is just the parallel transport equation $\nabla_{\partial/\partial\tau}U=0$.

The *curvature* of the Berry connection is

$$\Omega=\nabla^2=P(dP)(dP)P. \tag{2.6}$$

The curvature two form Ω is the holonomy of Berry's connection over small closed paths. In the non degenerate (abelian) case, Berry's phase is given by the integral of the curvature over a disk bounded by the curve C.

The proofs of (2.5) and (2.6) are elementary and are based on the trivial but important identity

$$P\,dPP=0, \tag{2.7}$$

which follows from

$$dP=P\,dP+dPP. \tag{2.8}$$

2.2. The Stark Hamiltonian and dilation analiticity

Our goal is to define and compute Berry's phase for the quantum Hamiltonian

$$H(r)=-\frac{\hbar^2}{2}\Delta_x-\frac{Z}{|x|}+(r_1x_1+r_2x_2+r_3x_3), \tag{2.9}$$

of a one electron atom with nuclear charge Z in a homogeneous electric field with direction $r=(r_1, r_2, r_3)\in R=\mathbf{R}^3\setminus\{0\}$ and strength $|r|$. To this end a major difficulty immediately arises, since it is well known that the Stark Hamiltonian (2.9) has no eigenvalues and its spectrum is purely absolutely continuous whereas the arguments of the previous section assume the existence of isolated eigenvalues. However, it is well known (Graffi and Grecchi 1978, Herbst 1978) that the bound states of $H(0)$ turn into resonances (long lived states) of $H(r)$ as soon as the perturbation is switched on. The resonances of H(*r*), defined through complex scaling, or dilation analyticity, appear as second sheet poles of the meromorphic matrix elements of the perturbed resolvent.

$$\lambda \mapsto \langle \psi, (\mathrm{H}(r)-\lambda)^{-1} \psi \rangle, \tag{2.10}$$

which are regular points for the corresponding elements of the unperturbed resolvent

$$\lambda \mapsto \langle \psi, (\mathrm{H}(0)-\lambda)^{-1} \psi \rangle. \tag{2.11}$$

We refer to Reed and Simon (1979) (Chapter 12, section 6) for details on (2.10) and (2.11), and to Herbst (1982) and Hunziker (1979) for a discussion of the Stark Hamiltonian.

It has been proved by Herbst (1978) that these poles are the eigenvalues of the non self-adjoint operator

$$\mathrm{H}(r, \theta)=\mathscr{U}(\theta)\,\mathrm{H}(r)\,\mathscr{U}(\theta)^{-1}, \qquad \theta \in \mathbf{C} \setminus \mathbf{R} \tag{2.12}$$

where the dilation operator

$$(\mathscr{U}(\theta)\,\varphi)(x) := e^{3\theta/2}\,\varphi(e^{\theta} x), \tag{2.13}$$

is a unitary map in $\mathrm{L}^2(\mathbf{R}^3, d^3 x)$ for all $\theta \in \mathbf{R}$. Moreover when θ is complex

$$\langle \mathscr{U}(\bar{\theta})\,\varphi, \mathscr{U}(\theta)\,\varphi \rangle = \langle \varphi, \varphi \rangle, \tag{2.14}$$

on dilation analytic vectors. The Hamiltonian (2.12) explicitly reads

$$\mathrm{H}(r, \theta)=\frac{-\hbar^2}{2} e^{-2\theta} \Delta_x - \frac{e^{-\theta} \mathrm{Z}}{|x|} + e^{\theta}\, r . x. \tag{2.15}$$

The technique of dilation analyticity has been introduced in Aguilar and Combes (1971) and Balslev and Combes (1971).

In particular it is known that when $\theta \in \mathbf{C} \setminus \mathbf{R}$ and $r \neq 0$ the operator $\mathrm{H}(r, \theta)$ has discrete spectrum and that the eigenvalues of $\mathrm{H}(0, \theta)$ are stable (in the sense of Kato 1966, chapter VIII, paragraph 1.4) w.r.t. the operator family $\mathrm{H}(r, \theta)$ as $|r| \to 0$. Moreover, both the spectrum of $\mathrm{H}(r, \theta)$ and the discrete spectrum of $\mathrm{H}(0, \theta)$ are independent of $\theta \in \mathbf{C}$ (*see* Herbst 1979).

Therefore the eigenvectors of $\mathrm{H}(r, \theta)$, which are the resonance eigenvectors of H(*r*), are the natural counterpart in our context of the discrete eigenvectors of the standard case. Thus we will consider the spectral bundles identified by the projectors $\mathrm{P}(r, \theta)$ on an individual resonance of $\mathrm{H}(r, \theta)$. As is well known (Landau and Lifchitz 1966, sections 76 and 77) their dimension is $\dim \mathrm{P}(r, \theta)=2$. Since $\mathrm{H}(r, \theta)^* = \mathrm{H}(r, \bar{\theta})$, by (2.1) it follows that $\mathrm{P}(r, \theta)^* = \mathrm{P}(r, \bar{\theta})$.

2.3. A special formula for the Berry holonomy

We now recast the formulas given in section 2.1 for the dilated Stark Hamiltonian (2.15) and we prove a simple formula which allows one to compute the holonomy of the Berry connection for closed circular paths in R.

The Berry connection on the spectral bundle of $H(r, \theta)$ identified by a given resonance projector $P(r, \theta)$ is the operator-valued one form

$$A(P)(r, \theta) = -[(dP)(r, \theta), P(r, \theta)], \tag{2.16}$$

where d denotes exterior differentiation w.r.t. r. Again $dP(r, \theta) = (dP)(r, \theta) + P(r, \theta)\, d$ and

$$P(r, \theta)\, dP(r, \theta)\, P(r, \theta) = 0. \tag{2.17}$$

The covariant derivative associated to $A(P)(r, \theta)$ is

$$\nabla(\theta) = d + A(r, \theta) = P(r, \theta)\, d, \tag{2.18}$$

and acts on differential forms $\omega(r, \theta)$ satisfying $P(r, \theta)\, \omega(r, \theta) = \omega(r, \theta)$. The *curvature* of the Berry connection is

$$\Omega_\theta = \nabla^2(\theta) = P(r, \theta)(dP)(r, \theta)(dP)(r, \theta) P(r, \theta). \tag{2.19}$$

We are interested in considering a closed circular path with $C(2\pi) = C(0)$ around $r = 0$. Without loss of generality, thanks to the spherical symmetry of the Coulomb potential, we can suppose that the electric field rotates around the x_1 axis, *i.e.*

$$\left.\begin{aligned} r_1 &= C_1(\tau) = 0, \\ r_2 &= C_2(\tau) = 2F\sin(\tau), \\ r_3 &= C_3(\tau) = 2F\cos(\tau); \end{aligned}\right\} \tag{2.20}$$

where $2F = |r| = |C(\tau)|$ denotes the constant strength of the field.

The parallel transport equation $\nabla_{\partial/\partial\tau}(\theta)\, U(\tau, \theta) = 0$ along $C(\tau)$ is

$$\left(\frac{\partial}{\partial\tau} - \left[\frac{\partial P(\tau, \theta)}{\partial\tau}, P(\tau, \theta)\right]\right) U(\tau, \theta) = 0, \tag{2.21}$$

where $P(\tau, \theta) = P(C(\tau), \theta)$ and with the initial condition $U(0, \theta) = 1$. The following slight generalization of the theorem of section 2.1 holds

THEOREM. — *The solution* $U(\tau, \theta)$ *of* (2.21) *exists and maps the range of* $P(0, \theta)$ *to the range of* $P(\tau, \theta)$, i.e.

$$P(\tau, \theta)\, U(\tau, \theta) = U(\tau, \theta)\, P(0, \theta). \tag{2.22}$$

For all eigenstates $\psi(\theta)$ $\varphi(\theta)$ *of* $P(0, \theta)$ *one also has*

$$\langle U(\tau, \bar\theta)\, \psi(\bar\theta), U(\tau, \theta)\, \varphi(\theta) \rangle = \langle \psi(\bar\theta), \varphi(\theta) \rangle. \tag{2.23}$$

Proof. — Assume for a moment that the solution exists. Then (2.22) is an immediate consequences of (2.17), from which it follows that both

$P(\tau, \theta) U(\tau, \theta)$ and $U(\tau, \theta) P(0, \theta)$ solve (2.21) with the same initial datum, and therefore they are equal. (2.23) is a consequence of the fact that, since $P(\tau, \theta)^* = P(\tau, \bar{\theta})$,

$$\left(\frac{\partial}{\partial \tau} - \left[\frac{\partial P(\tau, \bar{\theta})}{\partial \tau}, P(\tau, \bar{\theta})\right]\right) U(\tau, \bar{\theta}) = 0.$$

To show the existence and to the compute the holonomy $U(2\pi, \theta)$ we simply reduce (2.21) to a linear equation with constant coefficients by means of a unitary transformation to the co-rotating frame.

Let us denote by L_1 the first component of the angular momentum

$$L_1 = -i\hbar\left(x_2 \frac{\partial}{\partial x_3} - x_3 \frac{\partial}{\partial x_2}\right),$$

and by $S(\tau)$ the unitary group

$$S(\tau) = e^{-i\tau L_1/\hbar}. \tag{2.25}$$

Note that L_1, and consequently $S(\tau)$, is invariant under dilation.

The dilated Stark Hamiltonian $H(\tau, \theta) = H(C(\tau), \theta)$ with the rotating electric field is obtained from

$$\tilde{H}(F, \theta) = \frac{-\hbar^2}{2} e^{-2\theta} \Delta_x - \frac{e^{-\theta} Z}{|x|} + 2 e^{\theta} F x_3. \tag{2.26}$$

by means of $S(\tau)$

$$S(\tau) H(\tau, \theta) = \tilde{H}(F, \theta) S(\tau). \tag{2.27}$$

Let $\tilde{U}(\tau, \theta) = S(\tau) U(\tau, \theta)$. Then (2.21) becomes

$$i \frac{\partial \tilde{U}}{\partial \tau}(\tau, \theta) = \frac{1}{\hbar} L_1 \tilde{U}(\tau, \theta) + i S(\tau)\left[\frac{\partial P(\tau, \theta)}{\partial \tau}, P(\tau, \theta)\right] U(\tau, \theta). \tag{2.28}$$

From (2.27) it follows that

$$S(\tau) P(\tau, \theta) = \tilde{P}(F, \theta) S(\tau) \qquad \text{and} \qquad \tilde{P}(F, \theta) = P(0, \theta),$$

where $\tilde{P}(F, \theta)$ denotes the eigenprojection associated to $H(F, \theta)$, and also

$$\frac{\partial S}{\partial \tau}(\tau) P(\tau, \theta) + S(\tau) \frac{\partial P}{\partial \tau}(\tau, \theta) = \tilde{P}(F, \theta) \frac{\partial S}{\partial \tau}(\tau). \tag{2.29}$$

Moreover from (2.22) it follows that

$$\tilde{P}(F, \theta) \tilde{U}(\tau, \theta) = \tilde{U}(\tau, \theta) \tilde{P}(F, \theta), \tag{2.30}$$

i.e. $\tilde{U}(\tau, \theta)$ maps the range of $\tilde{P}(F, \theta)$ to itself. A simple computation gives that on the range of $\tilde{P}(F, \theta)$ (2.28) can be written as

$$i \frac{\partial \tilde{U}}{\partial \tau}(\tau, \theta) = \frac{1}{\hbar} \tilde{P}(F, \theta) L_1 \tilde{U}(\tau, \theta). \tag{2.31}$$

This is now a linear constant coefficients equation, and the existence of a solution is obvious:

$$\tilde{U}(\tau, \theta) = e^{-i\tau \tilde{P}(F, \theta) L_1/\hbar} 1. \tag{2.32}$$

It is by means of this formula that we will compute the Berry holonomy. We will prove in the next two sections that $\tilde{P}(F, \theta) L_1$ vanishes on the range of $\tilde{P}(F, \theta)$, from which it clearly follows that $\tilde{U}(2\pi, \theta) = U(2\pi, \theta) = 1$ *i.e.* the triviality of the holonomy and the vanishing of the quantal phases, which is the assertion of our theorem. Last but not least we shall prove that the r.h.s. of (2.32) is actually analytic and independent of θ, as it has to be.

3. THE BERRY HOLONOMY FOR THE STARK HAMILTONIAN

In this section we want to compute the matrix elements of $\tilde{P}(F, \theta) L_1$ and thus the holonomy of the Berry connection.

3.1. Computation of the long-lived states: separation of variables

Since we are dealing only with small values of the field strength F, perturbation theory is the natural tool to compute the eigenvectors of $\tilde{H}(F, \theta)$. (For the sake of simplicity of notations from now on we will omit the superscript $\tilde{}$). This is in principle a problem of degenerate perturbation theory: all discrete eigenvalues of $H(0, \theta)$ have multiplicity larger than one except for the ground state. However, it is well known (*see* e.g. Landau and Lifchitz 1966) that the problem can be separated in parabolic coordinates which give rise to a non degenerate perturbation problem in each single variable.

We introduce parabolic coordinates $(u, v, \varphi) \in \mathbf{R}_+^2 \times [0, 2\pi[$

$$\left.\begin{aligned} u &= \sqrt{r + y_3}, \\ v &= \sqrt{r - y_3}, \\ \varphi &= \arctan \frac{y_2}{y_1} \end{aligned}\right\} \tag{3.1}$$

where $r := |y|$. The dilated operator

$$H(F, \theta) = \frac{-\hbar^2}{2} e^{-2\theta} \Delta_y - \frac{e^{-\theta} Z}{|y|} + 2 e^{\theta} F y_3, \tag{3.2}$$

in $L^2(\mathbf{R}^3, d^3 y)$ becomes

$$H(F, \theta) = \bigoplus_{m=-\infty}^{+\infty} h_m(\theta), \tag{3.3}$$

in $\bigoplus_{m=-\infty}^{+\infty} (L^2(\mathbf{R}_+^2, (u^2+v^2)\,uv\,du\,dv) \otimes e^{im\varphi})$ where

$$h_m(F, \theta) = -\frac{1}{2} r^{-1} \frac{\hbar^{-2}}{2} e^{-2\theta}\left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2} - \frac{r}{u^2 v^2} m^2 + \frac{1}{u}\frac{\partial}{\partial u} + \frac{1}{v}\frac{\partial}{\partial v}\right) + \frac{1}{2} r^{-1}[-2e^{-\theta} Z + e^{\theta} F(u^4 - v^4)]. \quad (3.4)$$

For $F=0$ the eigenvalue problem

$$h_m(0, \theta)\,\chi_{m,\theta} = \lambda(0)\,\chi_{m,\theta}, \quad (3.5)$$

has a solution. Therefore, we can look for an eigenvector ψ_θ of the dilated operator (3.2), relative to the eigenvalue $\lambda(F)$, of the form

$$\psi_\theta(u, v, \varphi) = \chi_{m,\theta}(u, v)\, e^{im\varphi}, \quad (3.6)$$

where $\chi_{m,\theta}$ is an eigenfunction of the operator with angular part removed (3.4) with eigenvalue $\lambda(F)$

$$h_m(F, \theta)\,\chi_{m,\theta} = \lambda(F)\,\chi_{m,\theta} \quad (3.7)$$

and $\lambda(F) \to \lambda(0)$ as $F \to 0$.

Multiplying both sides of equation (3.7) by $\frac{-4r}{\hbar^2} e^{2\theta}$ and writing

$$\chi_{m,\theta}(u, v) = f_{1,\theta}(u) f_{2,\theta}(v), \quad (3.8)$$

we obtain the separation of variables: (3.7) is equivalent to

$$\left.\begin{aligned} a_m(e^{2\theta}\lambda, e^{3\theta}F) f_{1,\theta} &= e^{\theta} c_1 f_{1,\theta}, \\ a_m(e^{2\theta}\lambda, -e^{3\theta}F) f_{2,\theta} &= e^{\theta} c_2 f_{2,\theta}, \\ e^{\theta} c_1 + e^{\theta} c_2 &= -\frac{4}{\hbar^2} e^{\theta} Z \end{aligned}\right\} \quad (3.9)$$

where

$$a_m(E, F) := \frac{1}{u}\frac{d}{du}\left(\frac{1}{u}\frac{d}{du}\right) - \frac{m^2}{u^2} + \frac{2}{\hbar^2} E u^2 - \frac{2}{\hbar^2} F u^4, \quad (3.10)$$

is an ordinary differential operator in $L^2(\mathbf{R}_+, u\,du)$. For the operator-theoretic treatment and the spectral analysis of the two equations (3.9) the reader is referred to Graffi and Grecchi (1978).

In order to generate the perturbation theory it is more convenient to rewrite equations (3.9) in a θ-independent form. Multiply both sides by u^2 and perform the change of variables

$$\bar{x} = A u^2, \qquad \tilde{f}_1(\bar{x}) = \frac{1}{\sqrt{2}} e^{\bar{x}/2} \bar{x}^{|m|/2} f_{1,\theta}\left(\sqrt{\frac{\bar{x}}{A}}\right), \quad (3.11)$$

where $A := e^{\theta}\frac{\sqrt{-2\lambda(F)}}{\hbar}$. If one now multiplies by $\frac{1}{4\bar{x}}$, the first equation of (3.9) becomes

$$\mathscr{L}_m(\mathrm{B}_1, \mathrm{F})\tilde{f}_1 = \tilde{x}^2 \frac{d^2 \tilde{f}_1}{dx^2} + (|m|+1-\tilde{x})\frac{d\tilde{f}_1}{dx} + \left(\frac{\mathrm{B}_1}{4} - \frac{|m|+1}{2}\right)\tilde{f}_1 - \frac{1}{2\hbar} e^{3\theta} \frac{\mathrm{F}}{\mathrm{A}^3} \tilde{x}^2 \tilde{f}_1 = 0, \quad (3.12)$$

where $\mathrm{B}_1 = -\frac{c_1}{\mathrm{A}}$. Note that the operator $\mathscr{L}_m(\mathrm{B}, \mathrm{F})$ is actually independent of θ thanks to the definition of A and the transformation (3.11). As far as the second equation is concerned it is enough to replace F by $-$F: this replacement would obviously entail all the operator-theoretic questions taken care in Graffi and Grecchi (1978); however to the effect of generating the perturbation series, which is our only purpose, this replacement is irrelevant.

3.2. Computation of the long-lived states: perturbation theory in separated variables

It is clear that (3.12) is a Laguerre differential equation in

$$\mathrm{L}^2(\mathbf{R}_+, e^{-x} x^{|m|} dx)$$

with in addition a perturbation term $-\frac{1}{2\hbar} e^{3\theta} \frac{\mathrm{F}}{\mathrm{A}^2} \tilde{x}^2 \tilde{f}_1$. This allows us to reduce the degenerate perturbation problem for the dilated Stark effect Hamiltonian (3.2) to two non degenerate problems for the perturbed Laguerre operator (3.12) and its analogous for the v variable arising from the second equation in (3.9). To this aim we set

$$\alpha = -\frac{1}{2\hbar^2} \frac{e^{3\theta}\mathrm{F}}{\mathrm{A}^3} \quad \text{and} \quad \mathrm{N}_1 = -\frac{\mathrm{B}_1}{4} + \frac{|m|+1}{2}. \quad (3.13)$$

Thus $\tilde{f}_1$ solves (3.12) if and only if it is the eigenvector of the operator

$$\tilde{\mathscr{L}}_m(\alpha) = \tilde{x}\frac{d^2}{d\tilde{x}^2} + (|m|+1-\tilde{x})\frac{d}{d\tilde{x}} - \alpha \tilde{x}^2, \quad (3.14)$$

relative to the eigenvalue N_1 in the space $\mathrm{L}^2(\mathbf{R}_+, e^{-x} x^{|m|} dx)$. This means

$$\tilde{\mathscr{L}}_m(\alpha)\tilde{f}_1 = \mathrm{N}_1 \tilde{f}_1. \quad (3.15)$$

We have already noticed that $\tilde{\mathscr{L}}_m(\alpha)$ is the perturbation of $\tilde{\mathscr{L}}_m(0)$ corresponding to the potential $\mathrm{V}(\tilde{x}) = \tilde{x}^2$ and with coupling constant α.

It is well known that for $\alpha = 0$ the operator $\tilde{\mathscr{L}}_m(0)$ has simple eigenvalues

$$\mathrm{N}_1 = -n_1, n_1 = 0, 1, 2, \ldots \quad (3.16)$$

with corresponding orthonormal eigenvectors

$$\mathcal{J}_1(\bar{x}) = \sqrt{\frac{n_1!}{(n_1+|m|)!}} \mathrm{L}_{n_1}^{(|m|)}(\bar{x}), \tag{3.17}$$

where

$$\mathrm{L}_{n_1}^{(|m|)}(\bar{x}) = \frac{1}{n_1!} e^{\bar{x}} \bar{x}^{-|m|} \frac{d^{n_1}}{d\bar{x}^{n_1}} \left(e^{-\bar{x}} \bar{x}^{-n_1+|m|} \right), \tag{3.18}$$

is a Laguerre polynomial.

We are now ready to look for the perturbative solution of (3.15) in the form

$$\left.\begin{aligned} \mathrm{N}_1 &= \mathrm{N}_1^{(0)} + \mathrm{N}_1^{(1)} + \mathrm{N}_1^{(2)} + \dots, \\ \mathcal{J}_1 &= \mathcal{J}_1^{(0)} + \mathcal{J}_1^{(1)} + \mathcal{J}_1^{(2)} + \dots, \end{aligned}\right\} \tag{3.19}$$

where $\mathrm{N}_1^{(0)}$ and $\mathcal{J}_1^{(0)}$ are the solutions for the unperturbed problem (3.16), (3.17) and (3.18). $\mathrm{N}_1^{(j)}$ and $\mathcal{J}_1^{(j)}$ are the j-th order perturbative corrections due to the term $\alpha \bar{x}^2$ in (3.14) (*see* Landau and Lifchtiz 1966).

The first order solutions are

$$\mathrm{N}_1^{(1)} = \mathrm{V}_{n_1, n_1}, \tag{3.20}$$

$$\mathcal{J}_1^{(1)} = \sum_{k \neq n_1} \frac{\mathrm{V}_{n_1, k}}{k - n_1} \sqrt{\frac{k!}{(k-|m|)!}} \mathrm{L}_k^{(|m|)}, \tag{3.21}$$

where

$$\mathrm{V}_{n_1, k} := \sqrt{\frac{k!}{(k-|m|)!}} \sqrt{\frac{n_1!}{(n_1-|m|)!}} \langle \mathrm{L}_{n_1}^{(|m|)}, \alpha \bar{x}^2 \mathrm{L}_k^{(|m|)} \rangle, \tag{3.22}$$

and the inner product in (3.22) is to be understood in the space $\mathrm{L}^2(\mathbf{R}_+, e^{-x} x^{|m|} dx)$.

By the recurrence relation valid for the Laguerre polynomials (*see* Gradshteyn and Ryzhik 1980, paragraph 8.97) it is easy to see that only four terms of the sum (3.21) are different form zero. More precisely

$$\mathcal{J}_1^{(1)} = \alpha \left(c_2^{(1)} \mathrm{L}_{n_1+2}^{(|m|)} + c_1^{(1)} \mathrm{L}_{n_1+1}^{(|m|)} + c_{-1}^{(1)} \mathrm{L}_{n_1-1}^{(|m|)} + c_{-2}^{(1)} \mathrm{L}_{n_1-2}^{(|m|)} \right). \tag{3.23}$$

The correction $\mathrm{N}_1^{(1)}$ to the eigenvalue in the first approximation (3.20) and the coefficients $c_j^{(1)}$ in (3.23) can be easily computed by (3.22) together with the recurrence relations for the Laguerre polynomials. We do not write them down because their explicit expressions are not relevant for our purposes.

The higher order approximations can be obtained in the same way: for instance the second order correction to the eigenfunction is the sum of eight terms

$$\begin{aligned} \mathcal{J}_1^{(2)} = {} & \alpha^2 \left(c_4^{(2)} \mathrm{L}_{n_1+4}^{(|m|)} + c_3^{(2)} \mathrm{L}_{n_1+3}^{(|m|)} + c_2^{(2)} \mathrm{L}_{n_1+2}^{(|m|)} + c_1^{(2)} \mathrm{L}_{n_1+1}^{(|m|)} \right) \\ & + \alpha^2 \left(c_{-1}^{(2)} \mathrm{L}_{n_1-1}^{(|m|)} + c_{-2}^{(2)} \mathrm{L}_{n_1-2}^{(|m|)} + c_{-3}^{(2)} \mathrm{L}_{n_1-3}^{(|m|)} + c_{-4}^{(2)} \mathrm{L}_{n_1-4}^{(|m|)} \right). \end{aligned} \tag{3.24}$$

Once more the coefficients in (3.24) can be easily computed from the matrix elements (3.22) by well known formulas (*see* again Landau and Lifchitz 1966).

3.3. Perturbative form for the long lived states

Now we are in a position to write an explicit perturbative expression for the eigenvectors of the operator $H(F, \theta)$ (*i.e.* the long-lived states of $H(F)$ in parabolic coordinates). For the second order approximation it is sufficient to insert (3.23) and (3.24) into (3.12) and then perform the inverse coordinate transform of (3.11).

In the same way it is easy to see that the general form of the eigenvectors up to the order k in F is

$$\psi_\theta(u, v, \varphi) = 2 e^{-A(u^2+v^2)/2} (A u^2)^{|m|/2} (A v^2)^{|m|/2} \times \sum_{h=0}^{k} \left(F^h \sum_{j=-n(h)}^{n(h)} c_j'^{(h)} L_{n_1+j}^{(|m|)} (A u^2) \right) \times \sum_{h=0}^{k} \left(F^h \sum_{j=-n(h)}^{n(h)} c_j''(h) L_{n_2+j}^{(|m|)} (A v^2) \right) e^{im\varphi}, \quad (3.25)$$

where $n(h)$ is an h-dependent positive integer and the coefficients $c_j'^{(h)}$, $c_j''(h)$ depend only on n_1, n_2, $|m|$, Z and $\hbar$, but are independent of F and θ.

3.4. Computation of the Berry holonomy

With the results of the previous subsections we can compute the holonomy of the Berry connection introduced in section 2.2. We consider long lived states (3.25) relative to a fixed resonance which appears near the eigenvalue

$$\lambda(0) = -\frac{1}{2}\frac{Z^2}{n^2 \hbar^2} \qquad (n = n_1 + n_2 + |m| + 1)$$

of the unperturbed operator $H(0, 0)$ for small values of the field strength F. As explained in 2.3 we shall compute (2.31). More precisely we will show that the matrix elements

$$\langle \varphi_{\bar{\theta}}, L_1 \psi_\theta \rangle = 0, \qquad (3.26)$$

for any φ_θ and ψ_θ in the range of $P(F, \theta)$. The expression (3.25) for ψ_θ and φ_θ guarantees that the functions $\theta \mapsto \psi_\theta(u, v, \varphi)$ and $\theta \mapsto \varphi_\theta(u, v, \varphi)$ are holomorphic in θ. On the other hand L_1 is invariant under dilation. By a standard argument of dilation analyticity technique, (Aguilar and Combes 1971, Balslev and Combes 1971, Reed and Simon 1978, Chapter 13, section 10), the above remark allows us to prove that the l.h.s. of (3.26) is independent of θ since it is holomorphic and constant on the real axis: if $\theta \in \mathbf{R}$

$$\langle \varphi_\theta, L_1 \psi_\theta \rangle = \langle \mathcal{U}(\theta)\varphi, L_1 \mathcal{U}(\theta)\psi \rangle = \langle \varphi, L_1 \psi \rangle. \qquad (3.27)$$

The eigenfunctions ψ_θ and φ_θ are given by (3.25) in parabolic coordinates; thus, it is natural to express the angular momentum operator in the same coordinates

$$L_1 = \frac{1}{2} v \sin\varphi \frac{\partial}{\partial u} - \frac{1}{2} u \sin\varphi \frac{\partial}{\partial v} + \frac{1}{2}\left(\frac{v}{u} - \frac{u}{v}\right)\cos\varphi \frac{\partial}{\partial \varphi}. \qquad (3.28)$$

Inserting (3.25) and (3.28) into (3.26) we obtain that the r.h.s. is a sum of terms of the following three kinds (thanks to its independence of θ we can assume A real)

$$\left\langle e^{-1/2\, Au^2} (A u^2)^{|m|/2} L_{n_1+j'}^{(|m|)}(A u^2), \frac{\partial}{\partial u}\left(e^{-1/2\, Au^2}(A u^2)^{|m|/2} L_{n_1+i'}^{(|m|)}(A u^2)\right)\right\rangle$$
$$\times \langle e^{-1/2\, Av^2}(A v^2)^{|m|/2} L_{n+j''}^{(|m|)}(A v^2), v e^{-1/2\, Av^2}(A v^2)^{|m|/2} L_{n_2+i''}^{(|m|)}(A v^2)\rangle$$
$$\times \langle e^{im\varphi}, \sin\varphi\, e^{im\varphi}\rangle \qquad (3.29)$$

$$\langle e^{-1/2\, Au^2}(A u^2)^{|m|/2} L_{n_1+j'}^{(|m|)}(A u^2), u e^{-1/2\, Au^2}(A u^2)^{|m|/2} L_{n_1+i'}^{(|m|)}(A u^2)\rangle$$
$$\left\langle e^{-1/2\, Av^2}(A v^2)^{|m|/2} L_{n_2+j''}^{(|m|)}(A v^2), \frac{\partial}{\partial v}\left(e^{-1/2\, Av^2}(A v^2)^{|m|/2} L_{n_2+i''}^{(|m|)}(A v^2)\right)\right\rangle$$
$$\times \langle e^{im\varphi}, \sin\varphi\, e^{im\varphi}\rangle \qquad (3.30)$$

$$\left\langle e^{-1/2\, Au^2}(A u^2)^{|m|/2} L_{n_1+j'}^{(|m|)}(A u^2), \frac{1}{u} e^{-1/2\, Au^2}(A u^2)^{|m|/2} L_{n_1+i'}^{(|m|)}(A u^2)\right\rangle$$
$$\times \langle e^{-1/2\, Av^2}(A v^2)^{|m|/2} L_{n_2+j''}^{(|m|)}(A v^2), v e^{-1/2\, Av^2}(A v^2)^{|m|/2} L_{n_2+i''}^{(|m|)}(A v^2)\rangle$$
$$\langle e^{im\varphi}, im \cos\varphi\, e^{im\varphi}\rangle. \qquad (3.31)$$

It is immadiately evident that all these terms are null because of the third factor, *i.e.* the integral in $d\varphi$. This shows that the Berry holonomy is trivial at all orders of perturbation theory.

In section 4 we will prove the strong L^2 Borel-summability of the above perturbative series, thus rigorously proving that the Berry holonomy is trivial.

3.5. The two dimensional case

It is worth remarking that the Berry holonomy is trivial at all orders of perturbation theory also in the two dimensional case, *i.e.* when one considers the operator (2.9) in $L^2(\mathbf{R}^2, d^2x)$. This (non physical) situation may have some interest since the triviality of the Berry holonomy has the same origin as the classical case.

We can adapt the above arguments and obtain a perturbative expression for the two dimensional ψ_θ which is similar to (3.25) except for the substitution of Laguerre with Hermite polynomials. Again, the l.h.s of (3.26) is given by a sum of terms of the form (3.29) and (3.30) where the φ integral does not appear any more. However it is immediate to check that the integrals in du and dv are integrals of odd functions over $\mathbf{R}$ and hence they are zero as well.

4. BOREL SUMMABILITY FOR THE BERRY HOLONOMY

Taking into account the F-dependence of $A = \frac{e^\theta \sqrt{-2\lambda(F)}}{\hbar}$, section 3.3 shows that the Berry holonomy is trivial at all finite orders of perturbation theory since the perturbation series of the matrix elements is

$$\langle \varphi_{\bar\theta}, L_1 \psi_\theta \rangle = \sum_{k=1}^{N-1} \delta_k(\theta) F^k + F^N R_N(F, \theta), \tag{4.1}$$

where φ_θ and ψ_θ belong to the range of $\tilde{P}(F, \theta)$, $\delta_k(\theta) = 0$ for all k and $R_N(F, \theta)$ is bounded for all N.

The proof of our results is now achieved showing that the series (4.1) is Borel summable.

PROPOSITION. — $\langle \varphi_{\bar\theta}, L_1 \psi_\theta \rangle$ *is analytic in the sector*

$$S_\delta := \left\{ F \in C \,|\, -\frac{\pi}{2} + \varepsilon < \arg F < \frac{3\pi}{2} - \varepsilon,\ 0 < |F| < \delta \right\}, \tag{4.2}$$

and there exist $C > 0$ *and* $\sigma > 0$ *such that*

$$|R_N(F, \theta)| \leq C \sigma^N N!, \qquad \text{for all } N = 1, 2, \ldots \tag{4.3}$$

Proof. — Let $H(F, \theta)$ be the family of closed operators in $L^2(\mathbf{R}^3)$ defined by (3.2) with domain $D(H(F, \theta)) = D(-\Delta_y) \cap D(y_3)$ and $F, \theta \in \mathbf{C}$ such that

$$0 < \varepsilon \leqq 3 \operatorname{Im} \theta + \arg F \leqq \pi - \varepsilon. \tag{4.4}$$

Each eigenvalue $\lambda(0)$ of $H(0, \theta)$ is stable w.r.t. $H(F, \theta)$ as long as $\lambda(0)$ does not belong to the "numerical range at infinity" (*see* Hunziker and Vock 1982), *i.e.* F and θ must satisfy the following relationship as well

$$\varepsilon \leqq \operatorname{Im} \theta + \arg F \leqq \pi - \varepsilon. \tag{4.5}$$

The already proved independence of the l.h.s. of (3.26) of θ, together with (4.4) and (4.5) prove the analyticity of the matrix elements (4.1) in the sector (4.2).

The proof of (4.3) is obtained by showing thtat both terms of the scalar product appearing in (3.26) satisfy an analogous estimate (*see* Hunziker and Pillet 1983, section 4, Auberson and Mennessier 1981 and Graffi, Grecchi, Harrel and Silverstone 1985, appendix A).

Step 1. — We first recall the notion of stability for a fixed eigenvalue $\lambda(0)$ of $H(0, \theta)$ w.r.t. $H(F, \theta)$ as $F \to 0$, with θ fixed. Let R_θ denote the region of uniform boundedness for the resolvents, *i.e.*

$$R_\theta = \{ z \in \mathbf{C} \mid (z - H(F, \theta))^{-1} \text{ exists and is uniformly bounded for } F \to 0 \}, \tag{4.6}$$

where F and θ satisfy (4.4) and (4.5). Then $\lambda(0)$ is stable w.r.t. $H(F, \theta)$ if the following two conditions are satisfied:

(*a*) for $r > 0$ sufficiently small $\{ z \in \mathbf{C} \mid 0 < |z - \lambda| < r \} \subset R_\theta$;

(*b*) $\lim_{F \to 0} \| \mathscr{P}(F, \theta) - \mathscr{P}(0, \theta) \| = 0$,

where $\mathscr{P}(F, \theta) := (2\pi i)^{-1} \oint_\Gamma dz (z - H(F, \theta))^{-1}$ and Γ is a circle centered at λ and radius less than r. $\mathscr{P}(0, \theta)$ is the eigenprojection of $H(0, \theta)$ relative to eigenvalue λ.

It follows from (*b*) that inside Γ there are exactly m eigenvalues of $H(F, \theta)$ (counting molteplicities) as $F \to 0$, if the multiplicity of λ is m.

We now apply to our model the degenerate asymptotic perturbation theory developed by Hunziker and Pillet (1983). In order to obtain the standard reduction to an operator acting on the finite dimensional spectral subspace we introduce some notation in analogy with (Hunziker and Pillet 1983).

First of all notice that $\mathscr{P}(F, \theta)$ is the spectral projection for the $\lambda(0)$-group

$$\lambda_s(F) := \lambda(0) + \Delta\lambda_s(F), \qquad s = 1, \ldots, m, \tag{4.7}$$

where $\Delta\lambda_s(F)$ are the eigenvalues of

$$\Delta H(F, \theta) = \mathscr{P}(F, \theta)[H(F, \theta) - \lambda(0)]\mathscr{P}(F, \theta), \tag{4.8}$$

considered as an operator on $M(F, \theta) = \operatorname{Ran}\mathscr{P}(F, \theta)$. For small F this can be transformed into an equivalent eigenvalue problem in the space $M(0, \theta) = \operatorname{Ran}\mathscr{P}(0, \theta)$ as follows. Let us consider the operator on $M(0, \theta)$ defined by

$$D(F, \theta) := \mathscr{P}(0, \theta)\mathscr{P}(F, \theta)\mathscr{P}(0, \theta). \tag{4.9}$$

From (*b*) we have that $\| D(F, \theta) - 1 \| < 1$ for small F, so that D^{-1} and $D^{-1/2}$ are well defined operators. Then

$$S(F, \theta) = D(F, \theta)^{-1/2}\mathscr{P}(0, \theta)\mathscr{P}(F, \theta), \tag{4.10}$$

is an operator from $M(F, \theta)$ to $M(0, \theta)$ with inverse

$$S(F, \theta)^{-1} = \mathscr{P}(F, \theta)\mathscr{P}(0, \theta)D(F, \theta)^{-1/2}. \tag{4.11}$$

The eigenvalue problem for $\Delta H(F, \theta)$ is equivalent to the eigenvalue problem for the following operator in $M(0, \theta)$

$$E(F, \theta) = S(F, \theta)\Delta H(F, \theta)S(F, \theta)^{-1} \tag{4.12}$$

$$= D(F, \theta)^{-1/2}N(F, \theta)D(F, \theta)^{-1/2}, \tag{4.13}$$

where $N(F, \theta) = \mathscr{P}(0, \theta)\mathscr{P}(F, \theta)[H(F, \theta) - \lambda(0)]\mathscr{P}(F, \theta)\mathscr{P}(0, \theta)$ again acts on $M(0, \theta)$. The eigenprojections and eigennilpotents of $E(F, \theta)$ and $H(F, \theta)$ are also related by the similarity transformation (4.12).

Let us first analyze the family $E(F, \theta)$ in $M(0, \theta)$. With the same symbol we will denote its matrix representation w.r.t. the basis

$$B_\theta = \{\varphi_\theta^i = \mathscr{U}(\theta)\varphi^i, i = 1, \ldots, m\}, \tag{4.14}$$

where $\{\varphi^i, i = 1, \ldots, m\}$ is an orthonormal basis for $M(0, 0)$ consisting of eigenfunctions relative to $\lambda(0)$, and $U(\theta)$ is the transformation defined in (2.13).

Our goal is to apply theorems 4.1 and 4.2 of (Hunziker and Pillet 1983). It will then follow that the eigenvalues and the eigenprojections of $E(F, \theta)$ are Borel summable and the eigennilpotents vanish identically for small F.

To this end we must verify the following two conditions:

(i) $E(F, \theta) \in B^m$, *i.e.* $E(F, \theta)$, as an $m \times m$ matrix in **C**, is analytic in S_δ and has an asymptotic power series in S_δ:

$$E(F, \theta) \sim \sum_{k=0}^{\infty} E_k(\theta) F^k, \tag{4.15}$$

and there exist positive constants C and σ such that

$$\| E(F, \theta) - \sum_{k=0}^{N-1} E_k(\theta) F^k \| \leq C \sigma^N F^N N!, \tag{4.16}$$

for all $F \in S_\delta$ and $N = 1, 2, \ldots$

(ii) The matrix $E_k(\theta)$ is self-adjoint.

The proof of (ii) and the analyticity in the sector S_δ easily follow from standard dilation analicity arguments (*see*, e. g. Aguilar and Combes 1971), implying not only that the matrix $E_k(\theta)$ is self-adjoint but also that it is independent of θ. Indeed, as will become clear throughout the proof of (i), the elements of $E_k(\theta)$, $\langle \varphi^j_{\bar\theta}, E_k(\theta)\varphi^i_\theta\rangle$, do not depend on θ for $|\operatorname{Im}\theta|<\frac{\pi}{2}$ as also observed in subsection 3.3, thus (ii) follows from the self-adjointness of the operator $E_k(0)$.

Let us turn then to the proof of (i). First of all we show that $N(F, \theta)\in B^m$ and determine its asymptotic series. The elements of the matrix $N(F, \theta)$ w.r.t. the fixed basis B_θ are given by the scalar products $\langle \varphi^j_{\bar\theta}, N(F, \theta)\varphi^i_\theta\rangle$, which can be written more explicitely as follows

$$\langle \varphi^j_{\bar\theta}, N(F, \theta)\varphi^i_\theta\rangle=\langle \varphi^j_{\bar\theta}, [H(F, \theta)-\lambda]\mathscr{P}(F, \theta)\varphi^i_\theta\rangle$$
$$=(2\pi i)^{-1}\oint_\Gamma dz\,(z-\lambda)\langle \varphi^j_{\bar\theta}, [z-H(F, \theta)]^{-1}\varphi^i_\theta\rangle. \quad (4.17)$$

We now proceed in the standard way by expanding the resolvent $R(F, \theta)=[z-H(F, \theta)]^{-1}$ up to the N-th order, *i.e.*

$$R(F, \theta)=R(0, \theta)\sum_{k=0}^{N}[F y_3 e^\theta R(0, \theta)]^k$$
$$+R(F, \theta)[F y_3 e^\theta R(0, \theta)]^{N+1}. \quad (4.18)$$

We will show that

$$\langle \varphi^j_{\bar\theta}, N(F, \theta)\varphi^i_\theta\rangle\sim\sum_{k=0}^{\infty}\langle \varphi^j_{\bar\theta}, N_k(\theta)\varphi^i_\theta\rangle F^k, \quad (4.19)$$

with $N_k(\theta):=(2\pi i)^{-1}\oint_\Gamma (z-\lambda)R(0, \theta)[e^\theta y_3 R(0, \theta)]^k dz$, by proving the estimate

$$\|R(F, \theta)[e^\theta y_3 R(0, \theta)]^N\varphi^i_\theta\|\leqq C\sigma^N N!, \quad (4.20)$$

for some positive N-independent constants C and σ and for all $F\in S_\delta$ and $N=1, 2, \ldots$.

From now on the argument is the same as the one used by Herbst (1979) to prove that the Rayleigh-Schrödinger series for the resonance corresponding to the ground state of $H(0, 0)$ is asymptotic. Since it is quite brief, we re-propose it here, for sake of completeness.

Since $\Gamma\subset R_\theta$, $R(F, \theta)$ is uniformly bounded for $z\in\Gamma$ and F sufficiently small. Thus, it suffices to bound $\|[y_3 R(0, \theta)]^N\varphi^i_\theta\|$ by $\sigma^N N!$. Now it is well-known that there exists $\alpha>0$ such that $\varphi^i_\theta\in D(e^{\alpha|y_3|})$, *i.e.* $\|e^{\alpha|y_3|}\varphi^i_\theta\|<\infty$, and $e^{\beta y_3}R(0, \theta)e^{-\beta y_3}$ is uniformly bounded in β and z

for $|\beta| \leqq \alpha$, $z \in \Gamma$. Following the notation of Herbst (1979) we have

$$(y_3 \mathrm{R}(0, \theta))^{\mathrm{N}} \varphi_\theta^i = (y_3 j)(j^{-1} \mathrm{R} j)(y_3 j)(j^{-2} \mathrm{R} j^2) \times \ldots \times (y_3 j)(j^{-\mathrm{N}} \mathrm{R} j^{\mathrm{N}}) e^{\alpha |y_3|} \varphi_\theta^i, \quad (4.21)$$

where $\mathrm{R} = \mathrm{R}(0, \theta)$, $j(y_3) = e^{-\alpha |y_3|/\mathrm{N}}$. Then, if we set $\beta_k = \dfrac{\alpha k}{\mathrm{N}}$, $k = 0, \ldots, \mathrm{N}$, we have $|\beta_k| \leqq \alpha$ and hence $\| e^{\beta_k |y_3|} \mathrm{R} e^{-\beta_k |y_3|} \| \leqq \mathrm{C}'$. Moreover $|y_3 e^{-\alpha |y_3|/\mathrm{N}}| \leqq \mathrm{C}'' \mathrm{N}$. Therefore

$$\| [y_3 \mathrm{R}(0, \theta)]^{\mathrm{N}} \varphi_\theta^i \| \leqq \mathrm{const}\, (\mathrm{C}')^{\mathrm{N}} (\mathrm{C}'' \mathrm{N})^{\mathrm{N}} \leqq \mathrm{C}_1 \sigma^{\mathrm{N}} \mathrm{N}!, \quad (4.22)$$

which is the required bound.

The very same argument can be used to prove that

$$\mathrm{D}(\mathrm{F}, \theta) = \mathscr{P}(0, \theta) \mathscr{P}(\mathrm{F}, \theta) \mathscr{P}(0, \theta)$$

belongs to B^m. In fact, the only difference in the integral appearing in (4.17) is that now the term $(z - \lambda)$ is missing, and indeed it plays no role in obtaining (4.20). From this one can easily show that $\mathrm{D}(\mathrm{F}, \theta)^{-1/2} \in \mathrm{B}^m$, whence finally $\mathrm{E}(\mathrm{F}, \theta) \in \mathrm{B}^m$, by using (4.13).

As anticipated above we can now apply theorem 4.2 of (Hunziker and Pillet 1983) to $\mathrm{E}(\mathrm{F}, \theta)$ in order to obtain that its eigenvalues and eigenprojections are Borel summable, and that the eigennilpotents vanish identically for small F.

Step 2. – Using again the argument presented in Step 1 we obtain a Gevrey-1 type estimate (Ramis 1978, 1980) for $\mathscr{P}(\mathrm{F}, \theta) u$ for any $u \in \mathrm{L}^2(\mathbf{R}^3)$ such that there exists $\alpha > 0$ with $u \in \mathrm{D}(e^{\alpha |y_3|})$. More precisely

$$\| \mathscr{P}(\mathrm{F}, \theta) u - \sum_{k=0}^{\mathrm{N}-1} u_k(\theta) \mathrm{F}^k \| \leqq \mathrm{C} \sigma^{\mathrm{N}} \mathrm{N}! \mathrm{F}^{\mathrm{N}}, \quad (4.23)$$

for all N, (C and σ positive and independent of N) with

$$u_k(\theta) = \frac{1}{2\pi i} \left(\oint_\Gamma \mathrm{R}(0, \theta) [e^\theta y_3 \mathrm{R}(0, \theta)]^k dz \right) u. \quad (4.24)$$

In particular we can take $u = \varphi_\theta^i$ for all $i = 1, \ldots, m$. From now on we will call an operator (or a vector, or a scalar) Gevrey-1 if it only satisfies an estimate of type Gevrey-1, without being necessarily analytic in the required region that guarantees Borel summability. So we will first show that, if $\lambda(\mathrm{F})$ is one of the m eigenvalues of $\mathrm{H}(\mathrm{F}, \theta)$ in the $\lambda(0)$-group (*see* (4.7)), then the corresponding eigenprojection $\mathrm{P}(\mathrm{F}, \theta)$, as an operator from $\mathrm{M}(0, \theta)$ to $\mathrm{M}(\mathrm{F}, \theta)$, is Gevrey-1. Since the basis of $\mathrm{M}(\mathrm{F}, \theta)$ given by $\{ \mathscr{P}(\mathrm{F}, \theta) \varphi_\theta^i, i = 1, \ldots, m \}$ consists of Gevrey-1 vectors by the above result, it is enough to prove a Gevrey-1 estimate for the scalar products

$$\langle \mathscr{P}(\bar{\mathrm{F}}, \bar{\theta}) \varphi_{\bar{\theta}}^j, \mathrm{P}(\mathrm{F}, \theta) \varphi_\theta^i \rangle \quad \text{for all } i, j = 1, \ldots, m. \quad (4.25)$$

In fact if $A^{lj}(F, \theta)$ denotes the matrix element of $P(F, \theta)$ w.r.t. the bases B_θ of $H(0, \theta)$ and $\{\mathscr{P}(F, \theta)\varphi_\theta^i, i=1, \ldots, m\}$ of $M(F, \theta)$ we have:

$$\langle \mathscr{P}(\bar{F}, \bar{\theta})\varphi_{\bar{\theta}}^i, P(F, \theta)\varphi_\theta^l\rangle = \sum_{j=1}^{m} A^{lj}(F, \theta)\langle \mathscr{P}(\bar{F}, \bar{\theta})\varphi_{\bar{\theta}}^i, \mathscr{P}(F, \theta)\varphi_\theta^j\rangle. \quad (4.26)$$

For every fixed $l=1, \ldots, m$, system (4.26) can be solved for $A^{lj}(F, \theta)$ with the Cramer formula, since the associated matrix has elements

$$\langle \mathscr{P}(\bar{F}, \bar{\theta})\varphi_{\bar{\theta}}^i, \mathscr{P}(F, \theta)\varphi_\theta^j\rangle = \langle \varphi_{\bar{\theta}}^i, \mathscr{P}(0, \theta)\mathscr{P}(F, \theta)\mathscr{P}(0, \theta)\varphi_\theta^j\rangle. \quad (4.27)$$

But (4.27) is precisely the matrix that represents $D(F, \theta)$ w.r.t. the basis B_θ, thus it is invertible. Hence, if we prove that (4.25) is Gevrey-1, so will be $A^{lj}(F, \theta)$, for all $l, j=1, \ldots, m$. In this context we can work indifferently with the operator $H(F, \theta)$ or with $\Delta H(F, \theta)$ (*see* (4.8)), thus with abuse of notation we will still call $P(F, \theta)$ the eigenprojector for $\Delta H(F, \theta)$ relative to $\Delta\lambda(F)=\lambda(F)-\lambda(0)$.

If $P^E(F, \theta)$ denotes the eigenprojector for $E(F, \theta)$, which is Borel summable by the result obtained in step 1, we have by (4.10), (4.11) and (4.12)

$$\begin{aligned} P^E(F, \theta) &= S(F, \theta)P(F, \theta)S(F, \theta)^{-1} \\ &= D(F, \theta)^{-1/2}\mathscr{P}(0, \theta)P(F, \theta)\mathscr{P}(0, \theta)D(F, \theta)^{-1/2}. \end{aligned} \quad (4.28)$$

The last inequality follows from

$$P(F, \theta)\mathscr{P}(F, \theta)=\mathscr{P}(F, \theta)P(F, \theta)=P(F, \theta).$$

Thus

$$\mathscr{P}(0, \theta)P(F, \theta)\mathscr{P}(0, \theta)=D^{1/2}(F, \theta)P^E(F, \theta)D^{1/2}(F, \theta)$$

and (4.25) becomes

$$\begin{aligned} \langle \mathscr{P}(\bar{F}, \bar{\theta})\varphi_{\bar{\theta}}^j, P(F, \theta)\varphi_\theta^i\rangle &= \langle \mathscr{P}(0, \bar{\theta})\varphi_{\bar{\theta}}^j, \mathscr{P}(F, \theta)P(F, \theta)\mathscr{P}(0, \theta)\varphi_\theta^i\rangle \\ &= \langle \varphi_{\bar{\theta}}^j, \mathscr{P}(0, \theta)P(F, \theta)\mathscr{P}(0, \theta)\varphi_\theta^i\rangle \\ &= \langle \varphi_{\bar{\theta}}^j, D^{1/2}(F, \theta)P^E(F, \theta)D^{1/2}(F, \theta)\varphi_\theta^i\rangle. \end{aligned} \quad (4.29)$$

Now the required estimate follows form the Borel summability of $D^{1/2}(F, \theta)$ and $P^E(F, \theta)$ proved in step 1.

In particular an eigenfunction $\varphi_\theta(F)$ of $H(F, \theta)$ relative to $\lambda(F)$ can be taken of the form $\varphi_\theta(F)=P(F, \theta)\varphi_0$, with $\varphi_0\in M(0, \theta)$; therefore $\varphi_\theta(F)$ is Gevrey-1 and this provides the required estimate for the first term of the scalar product in (3.26).

Step 3. – Finally we need to prove that $L_1\psi_\theta(F)$ is Gevrey-1. To this end we set

$$\begin{aligned} L_1\psi_\theta(F) = {} & [L_1(H(0, \theta)+(1+|y|^2)^{1/2}-\lambda(0))^{-1}] \\ & \times (H(0, \theta)+(1+|y|^2)^{1/2}-\lambda(0))\psi_\theta(F). \end{aligned} \quad (4.30)$$

Since the term inside the square brackets in the r.h.s. of (4.30) is a bounded operator in $L^2(\mathbf{R}^3)$ independent of F, it suffices to obtain a Gevrey-1 estimate for $(H(0, \theta)+(1+|y|^2)^{1/2}-\lambda(0))\,\psi_\theta(F)$. We have

$$(H(0, \theta)-\lambda(0)+(1+|y|^2)^{1/2})\,\psi_\theta(F)=(\lambda(F)-\lambda(0))\,\psi_\theta(F)$$
$$-2e^\theta F y_3\,\psi_\theta(F)+(1+|y|^2)^{1/2}\,\psi_\theta(F). \quad (4.31)$$

Since the eigenvalue $\lambda(F)$ is Gevrey-1 it remains to study the last two terms of the r.h.s. of (4.31). In particular we will analyze $(1+|y|^2)^{1/2}\,\psi_\theta(F)$; a similar argument works for $y_3\,\psi_\theta(F)$.

Again it suffices to show that the operator

$$(1+|y|^2)^{1/2}\,P(F, \theta): M(0, \theta) \to (1+|y|^2)^{1/2}\,M(F, \theta) \quad (4.32)$$

verifies a Gevrey-1 estimate. Mimicking the proof presented in step 2, let $B^{lj}(F, \theta)$ denote its matrix-elements w.r.t. the bases B_θ of $M(0, \theta)$ and $\{(1+|y|^2)^{1/2}\,\mathscr{P}(F, \theta)\,\varphi_\theta^i,\ i=1, \ldots, m\}$ of $(1+|y|^2)^{1/2}\,M(F, \theta)$. Then we have

$$\langle (1+|y|^2)^{1/2}\,\mathscr{P}(\bar{F}, \bar{\theta})\,\varphi_{\bar{\theta}}^i,\ (1+|y|^2)^{1/2}\,P(F, \theta)\,\varphi_\theta^l \rangle$$
$$=\sum_{j=1}^{m} B^{lj}(F, \theta)\,\langle (1+|y|^2)^{1/2}\,\mathscr{P}(\bar{F}, \bar{\theta})\,\varphi_{\bar{\theta}}^i, (1+|y|^2)^{1/2}\,\mathscr{P}(F, \theta)\,\varphi_\theta^j \rangle. \quad (4.33)$$

Assuming for the moment that for each fixed l this system is invertible for $B^{lj}(F, \theta)$, it remains to prove that the l.h.s. of (4.33) and the vectors $(1+|y|^2)^{1/2}\,\mathscr{P}(F, \theta)\,\varphi_\theta^i$ are Gevrey-1 for all i, $l=1, \ldots, m$. Now, since the l.h.s. of (4.33) can be written as $\langle (1+|y|^2)\,\mathscr{P}(\bar{F}, \bar{\theta})\,\varphi_{\bar{\theta}}^i,\ P(F, \theta)\,\varphi_\theta^l \rangle$, both results are achieved by proving a Gevrey-1 estimate for $(1+|y|^2)\,\mathscr{P}(F, \theta)\,\varphi_\theta^i$ for all $i=1, \ldots, m$.

To show the invertibility of (4.33) notice that the scalar products

$$\langle (1+|y|^2)^{1/2}\,\mathscr{P}(\bar{F}, \bar{\theta})\,\varphi_{\bar{\theta}}^i,\ (1+|y|^2)^{1/2}\,\mathscr{P}(F, \theta)\,\varphi_\theta^j \rangle$$
$$=\langle \varphi_{\bar{\theta}}^i,\ \mathscr{P}(F, \theta)(1+|y|^2)\,\mathscr{P}(F, \theta)\,\varphi_\theta^j \rangle, \quad (4.34)$$

are the matrix-elements of the operator

$$\mathscr{P}(0, \theta)\,\mathscr{P}(F, \theta)(1+|y|^2)\,\mathscr{P}(F, \theta)\,\mathscr{P}(0, \theta): M(0, \theta) \to M(0, \theta) \quad (4.35)$$

w.r.t. B_θ. Since it tends in norm to $\mathscr{P}(0, \theta)(1+|y|^2)\,\mathscr{P}(0, \theta)$ as $F \to 0$, it is invertible for suitably small F.

So let us turn to $(1+|y|^2)\mathscr{P}(\mathrm{F}, \theta)\varphi_\theta^i$; since $\mathscr{P}(\mathrm{F}, \theta)(1+|y|^2)^{1/2}\varphi_\theta^i$ is Gevrey-1 from step 2, we only need to examine the commutator

$$\begin{aligned}[\mathscr{P}(\mathrm{F}, \theta), (1+|y|^2)]\varphi_\theta^i &= \frac{1}{2\pi i}\oint_\Gamma \mathrm{R}(\mathrm{F}, \theta)\left[\frac{\hbar}{2}e^{-2\theta}\Delta_y, |y|^2\right]\mathrm{R}(\mathrm{F}, \theta)\,dz\,\varphi_\theta^i \\ &= \hbar e^{-2\theta}\frac{1}{4\pi i}\oint_\Gamma \mathrm{R}(\mathrm{F}, \theta)(6+4y.\nabla_y)\mathrm{R}(\mathrm{F}, \theta)\,dz\,\varphi_\theta^i \\ &= 3\hbar e^{-2\theta}\frac{1}{2\pi i}\oint_\Gamma \mathrm{R}(\mathrm{F}, \theta)\mathrm{R}(\mathrm{F}, \theta)\,dz\,\varphi_\theta^i \\ &\quad + \hbar e^{-2\theta}\frac{1}{\pi i}\oint_\Gamma \mathrm{R}(\mathrm{F}, \theta)(\nabla_y.y+[y, \nabla_y]) \\ &\qquad \times \mathrm{R}(\mathrm{F}, \theta)\,dz\,\varphi_\theta^i, \qquad (4.36)\end{aligned}$$

where, as above, $\mathrm{R}(\mathrm{F}, \theta)=(z-\mathrm{H}(\mathrm{F}, \theta))^{-1}$. Now $\mathrm{R}(\mathrm{F}, \theta)$ is uniformly bounded on Γ as $\mathrm{F}\to 0$, thus the first summand in (4.36) is Gevrey-1, since so is $\mathrm{R}(\mathrm{F}, \theta)\varphi_\theta^i$. As for the second summand, since $[y, \nabla_y]=-3$ we are only left with the term

$$\begin{aligned}&\sum_{j=1}^{3}\hbar e^{-2\theta}\frac{1}{\pi i}\oint_\Gamma \mathrm{R}(\mathrm{F}, \theta)\frac{\partial}{\partial y_j}y_j\mathrm{R}(\mathrm{F}, \theta)\,dz\,\varphi_\theta^i \\ &\quad = \sum_{j=1}^{3}\hbar e^{-2\theta}\frac{1}{\pi i}\oint_\Gamma \mathrm{R}(\mathrm{F}, \theta)\frac{\partial}{\partial y_j}\mathrm{R}(\mathrm{F}, \theta)\,dz y_j\varphi_\theta^i \\ &\quad + \sum_{j=1}^{3}\hbar e^{-2\theta}\frac{1}{\pi i}\oint_\Gamma \mathrm{R}(\mathrm{F}, \theta)\frac{\partial}{\partial y_j}\left[y_j, \frac{\hbar}{2}e^{-2\theta}\Delta_y\right]\mathrm{R}(\mathrm{F}, \theta)\,dz\,\varphi_\theta^i \\ &\quad = \sum_{j=1}^{3}\hbar e^{-2\theta}\frac{1}{\pi i}\oint_\Gamma \mathrm{R}(\mathrm{F}, \theta)\frac{\partial}{\partial y_j}\mathrm{R}(\mathrm{F}, \theta)\,dz y_j\varphi_\theta^i \\ &\quad + \sum_{j=1}^{3}\hbar^2 e^{-2\theta}\frac{1}{\pi i}\oint_\Gamma \mathrm{R}(\mathrm{F}, \theta)\frac{\partial^2}{\partial y_j^2}\mathrm{R}(\mathrm{F}, \theta)\,dz\,\varphi_\theta^i. \qquad (4.37)\end{aligned}$$

Now the final assert follows from the uniform boundedness of both $\mathrm{R}(\mathrm{F}, \theta)\frac{\partial}{\partial y_j}$ and $\mathrm{R}(\mathrm{F}, \theta)\Delta_y$ as $\mathrm{F}\to 0$, and the fact that $\mathrm{R}(\mathrm{F}, \theta)y_j\varphi_\theta^i$ is Gevrey-1, since $y_j\varphi_\theta^i\in \mathrm{D}(e^{\alpha|y|})$ for some $\alpha>0$. This concludes the proof of the proposition.

APPENDIX

The Hannay angles for the Stark Hamiltonian

Three dimensional case. — In this appendix we are concerned with the motion of a classical particle in the potential $\mathrm{V}(x)=-1/|x|+2\mathrm{F}(t).x$, where F (t) is a slowly rotating constant force field. We will show that the Hannay angles are zero. This model is of some relevance in celestial mechanics (*see*, e. g., Béletski 1977), as a limit of the two centers of force problem.

The Hamiltonian of our model is given by

$$\mathrm{H}(p, x, \omega t)=\frac{|p|^2}{2}-\frac{1}{|x|}+2\mathrm{F}[x_2 \cos(\omega t)+x_3 \sin(\omega t)], \qquad (\mathrm{A}.1)$$

where $(x, p)\in \mathrm{T}^* \mathrm{M}_0$, $\mathrm{M}_0=\mathbf{R}^3 \setminus \{0\}$. F is the intensity of the constant force field and $\omega \ll 1$ is its (small) angular velocity. $\mathrm{H}(p, x, \omega t)$ is the classical counterpart of the quantum Hamiltonian (2.9).

In the co-rotating frame one obtains the new Hamiltonian

$$\mathrm{H}_\omega(\mathrm{P}, y)=\frac{|\mathrm{P}|^2}{2}-\frac{1}{|y|}+2\mathrm{F}y_3-\omega(y_1 \mathrm{P}_2-y_2 \mathrm{P}_1). \qquad (\mathrm{A}.2)$$

We introduce parabolic coordinates $(u, v, \varphi)\in \mathbf{R}_+^2 \times [0, 2\pi[$ as in (3.1) and we proceed by extending the phase space. Let s be the new time parametrization defined by

$$\frac{dt}{ds}=u^2+v^2>0. \qquad (\mathrm{A}.3)$$

Then

$$\mathscr{H}_\omega := \frac{dt}{ds}(\mathrm{H}_\omega-\mathrm{E})=\mathrm{H}_1+\mathrm{H}_2+\omega \hat{\mathrm{H}}, \qquad (\mathrm{A}.4)$$

where

$$\mathrm{H}_1 := \frac{p_u^2}{2}+\frac{p_\varphi^2}{2u^2}-\frac{1}{2}+\mathrm{F}u^4-\mathrm{E}u^2, \qquad (\mathrm{A}.5)$$

$$\mathrm{H}_2 := \frac{p_v^2}{2}+\frac{p_\varphi^2}{2v^2}-\frac{1}{2}-\mathrm{F}v^4-\mathrm{E}v^2, \qquad (\mathrm{A}.6)$$

$$\hat{\mathrm{H}} := \frac{u^2+v^2}{2}\left[(up_v-vp_u)\cos\varphi-\frac{u^2-v^2}{2uv}p_\varphi \sin\varphi\right]. \qquad (\mathrm{A}.7)$$

$\mathscr{H}_\omega$ generates the time evolution of the original Hamiltonian H_ω on the submanifold $\mathscr{H}_\omega=0$ w.r.t. the new time parametrisation s (Thirring 1978). E is the energy of H_ω.

Note that if $\omega=0$ this coordinate transformation has separated the problem. Thus the system is integrable and the first integrals of the motion are $\mathscr{H}_0=0$, H_1 and p_φ.

To compute the Hannay angles we will follow the averaging procedure discussed in Golin, Knauf and Marmi (1989), Golin and Marmi (1990) and we will average $\hat{\mathrm{H}}$ over the three-dimensional invariant tori. Since φ is a cyclic coordinate for the integrable problem ($\omega=0$), by averaging w.r.t. φ the result is zero.

TWO DIMENSIONAL CASE. — For the classical problem considered here one can also study the planar problem corresponding to polar orbits, *i. e.* orbits with the initial condition $\varphi=0$, $p_\varphi=0$. These orbits verify an exact resonance condition, and will remain in the plane y_2, y_3 forever. To compute Hannay angles partial averaging must be applied (Arnol'd 1983, Arnol'd, Kozlov and Neishtadt 1988, Lochak and Meunier 1988).

The two-dimensional configuration space $M_1 := \{y \in M_0 \mid y_1 = 0\}$ is now parametrized by the parabolic coordinates $(u, v) \in \mathbf{R}^2_+$, and the first integrals of the problem are $H_\omega = E < 0$ and $H_1 = K$. We will denote the restriction of phase space functions to $T^* M_1$ with the same notation used before.

The action variables corresponding to the regions of bounded motion are given by

$$I_u := \frac{1}{2\pi}\oint p_u\, du = \frac{1}{\pi}\int_{u_-}^{u_+} \sqrt{2\left[K + \frac{1}{2} + E u^2 - F u^4\right]}\, du, \tag{A.8}$$

$$I_v := \frac{1}{2\pi}\oint p_v\, dv = \frac{1}{\pi}\int_{v_-}^{v_+} \sqrt{2\left[K + \frac{1}{2} + E v^2 + F v^4\right]}\, dv, \tag{A.9}$$

where u_-, u_+ and v_-, v_+ are the classical turning points

$$u_+ := \sqrt{\frac{E + \sqrt{E^2 + 4F(K + (1/2))}}{2F}}, \qquad u_- = -u_+, \tag{A.10}$$

$$v_+ := \sqrt{\frac{-E - \sqrt{E^2 - 4F(1/2) - K)}}{2F}}, \qquad v_- = -v_+. \tag{A.11}$$

Note also that

$$\frac{\partial (I_u, I_v)}{\partial (E, K)} = \begin{pmatrix} \frac{1}{2\pi}\int \frac{u^2\, du}{p_u} & \frac{1}{2\pi}\int \frac{du}{p_u} \\ \frac{1}{2\pi}\int \frac{v^2\, dv}{p_v} & \frac{-1}{2\pi}\int \frac{dv}{p_v} \end{pmatrix}, \tag{A.12}$$

and

$$\frac{\partial (p_u, p_v)}{\partial (E, K)} = \begin{pmatrix} \frac{u^2}{p_u} & \frac{1}{p_u} \\ \frac{v^2}{p_v} & \frac{-1}{p_v} \end{pmatrix}. \tag{A.13}$$

We now follow Golin, Knauf and Marmi (1989) to state the prescription for averaging over the invariant two-tori defined by the constants E and K. We must average $\hat{H}\, ds/dt$ w.r.t. the physical time t. Let

$\mathrm{S}:=s(\mathrm{T})=\int_0^{\mathrm{T}} \frac{ds}{dt'}\,dt'$; we have

$$\left\langle \frac{ds}{dt}\hat{\mathrm{H}} \right\rangle := \lim_{\mathrm{T}\to+\infty} \frac{1}{\mathrm{T}} \int_0^{\mathrm{T}} \frac{ds}{dt}\hat{\mathrm{H}}\,dt = \lim_{\mathrm{S}\to+\infty} \frac{\mathrm{S}}{\mathrm{T}(\mathrm{S})} \frac{1}{\mathrm{S}} \int_0^{\mathrm{S}} \hat{\mathrm{H}}\,ds$$
$$= \frac{\lim_{\mathrm{S}\to+\infty} (1/\mathrm{S}) \int_0^{\mathrm{S}} \hat{\mathrm{H}}\,ds}{\lim_{\mathrm{S}\to+\infty} (1/\mathrm{S}) \int_0^{\mathrm{S}} (dt/ds)\,ds}. \qquad (\mathrm{A}.14)$$

By the adiabatic assumption we can replace time averages with averaging over two-tori, thus

$$\left\langle \frac{ds}{dt}\hat{\mathrm{H}} \right\rangle = \frac{\oint \hat{\mathrm{H}}\,d\varphi_u\,d\varphi_v}{\oint (dt/ds)\,d\varphi_u\,d\varphi_v}, \qquad (\mathrm{A}.15)$$

where φ_u, φ_v are the angle variables canonically conjugated to I_u, I_v.

To simplify our computation we will now evaluate (A.15) without re-expressing $\hat{\mathrm{H}}$ and $\frac{dt}{ds}$ in action-angle coordinates. To this purpose note that

$$d\varphi_u\,d\varphi_v = \det\left(\frac{\partial(\mathrm{E},\mathrm{K})}{\partial(\mathrm{I}_u,\mathrm{I}_v)}\right)\det\left(\frac{\partial(p_u,p_v)}{\partial(\mathrm{E},\mathrm{K})}\right)du\,dv$$
$$= \frac{((u^2+v^2)/p_u p_v)\,du\,dv}{\frac{1}{4\pi^2}\oint((u^2+v^2)/p_u p_v)\,du\,dv}, \qquad (\mathrm{A}.16)$$

by (A.12) and (A.13). Finally

$$\left\langle \frac{ds}{dt}\hat{\mathrm{H}} \right\rangle = 2\pi^2 \frac{\oint (up_v - vp_u)((u^2+v^2)^2/p_u p_v)\,du\,dv}{\oint ((u^2+v^2)^2/p_u p_v)\,du\,dv} = 0, \qquad (\mathrm{A}.17)$$

since the integrand function at the numerator is odd whereas the turning points are symmetric.

For more examples on the relation existent between symmetries and zero Hannay angles we also refer to Golin and Marmi (1989).

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(*Manuscript received October* 5, 1990;
Revised version received March 11, 1991.)