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Macroscopic limiting dynamics of a class of inhomogeneous mean field quantum systems

by

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ABSTRACT. – We study a class of Hamiltonian systems with inhomogeneous (*i.e.* site-dependent) mean field interactions. We define some notions of mean field limit for nets of states converging to a macroscopic limit state. We prove that the existence of such limits is preserved under the time evolution. This leads to a time evolution for the macroscopic limit states, *i.e.* to a closed set of equations for some macroscopic fields. We establish the basic properties of these equations, and their relation to the equilibrium statistical mechanics of the same systems. We discuss in detail the connection of our work to the problem of local equilibrium states, which motivated it.

RÉSUMÉ. – Nous étudions une classe de systèmes hamiltoniens avec interaction de champ moyen inhomogène (*i.e.* qui dépend du site). Nous définissons une notion de limite de champ moyen pour des familles d'états qui convergent vers un état limite macroscopique. Nous prouvons que l'existence de telles limites est préservée par l'évolution temporelle. Ceci conduit à une évolution temporelle pour l'état macroscopique, c'est-à-dire

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à un ensemble fermé d'équations pour des champs macroscopiques. Nous établissons des propriétés de ces équations et leur relation avec la mécanique statistique de l'équilibre du même système. Nous discutons en détails le rapport entre notre travail et le problème des états d'équilibre locaux qui lui a servi de motivation.

Mots clés : Mean field quantum systems; hamiltonian dynamics; macroscopic states; local equilibrium.

1. INTRODUCTION

It has by now been well established that thermodynamic systems at or near equilibrium can readily be described in the framework of C*- or W*-dynamical systems, the equilibrium states being given by KMS states. A corresponding theory of thermodynamic states far from equilibrium, e. g. stationary states of an open system describing heat conduction, is still lacking. From the experimenter's point of view, and, indeed from the point of view of phenomenological non-equilibrium thermodynamics, such a state is *locally* an equilibrium state, *i. e.* to every point x of the system one attributes (x-dependent) thermo*static* quantities like temperature, energy density, entropy density, etc. The collection of these fields constitutes one macroscopic state. The time evolution of macroscopic states is governed by the equations of phenomenological thermodynamics, but it is the task of non-equilibrium statistical mechanics to deduce these from the microscopic interaction.

The aim of this paper is twofold. We first sketch a rather general framework for the description of macroscopic states with a time evolution based on microscopic interactions. This is an introductory part, which serves as a motivation for what follows. In the main part, we shall present a class of models of mean field type, and single out a set of macroscopic states, which arise as the thermodynamic limits of nets of microscopic states. The mean field property allows us to control the thermodynamic limit of the microscopically given dynamics, and hence the time evolution of the macroscopic states. Thus we realize part of our general programme, by rigorously deriving a closed set of equations governing the time evolution of macroscopic fields. However, for mean field systems the macroscopic range of the interaction prevents the system from thermal equilibrium locally, so another part of the general programme cannot be realized in this setting. We shall discuss this point in more detail at the end of the paper.

Since we want to study x-dependent fields, it is important that our models are of the "inhomogeneous" mean field type ([3], [17]). We stress, however, that this class of models is interesting independently of the above motivation. In particular, in many applications of such models the inhomogeneity does not refer to configuration space, but, for example, to momentum space (as in the BCS-model) or to a space of random variables. A collection of examples highlighting the possible interpretations of our formalism is given at the end of section 2.

Our paper is organized as follows. A general scheme for describing space-dependent macroscopic states and their time evolutions is outlined at the end of this section. *Section* 2 contains the description of the class of inhomogeneous mean field models, together with the necessary definitions and notations. All assumptions needed later on are fully stated in this section. The basic concepts explained in this section follow the approach to mean field systems developed in [16], [17] and [18]. In *section* 3 we define two notions of thermodynamic limit for states. These are called "mean field limits" and "weak mean field limits", and we show that under an additional hypothesis of "uniform local permutation symmetry" the two are equivalent. In *section* 4 we briefly review the equilibrium thermodynamics of our models following [18]. This section also contains the new result that the property of uniform local symmetry is satisfied for the equilibrium states. We define the effective state-dependent local Hamiltonians and establish their basic properties. These Hamiltonians play an important role in both equilibrium and non-equilibrium. In *section* 5 we treat the thermodynamic limit of the dynamics. We reduce this problem to the case of homogeneous mean field models, which has been studied in various degrees of generality by several authors ([13], [5], [22], [1], [5]). We follow the treatment in [8], which is best suited to our needs. The main results here are that for any net of microscopic states the existence of a weak mean field limit, and uniform local symmetry are preserved by the microscopic time evolution. Consequently, the microscopic time evolution induces a time evolution of the limiting states. We give the form of the differential equations describing this evolution. In *section* 6 we look at properties of the solutions of these equations. In particular, we show that energy and entropy are both conserved under the limit evolution. Finally, in *section* 7 we come back to the general problem of local equilibrium, which was our motivation for this study, and we discuss possible extensions and generalizations of our results.

Let us outline the main ideas in describing macroscopic states and their dynamics.

Consider a macroscopic system contained in a (finite) region $X \subset \mathbb{R}^d$. We would like to define the "macroscopic state at the point $x \in X$" as an equilibrium state of an auxiliary C*-dynamical system. Hence to every point $x \in X$ we associate a C*-algebra $\mathscr{B}_x$, and a time evolution $\alpha_t^x \in \mathrm{Aut}\,\mathscr{B}_x$. For simplicity we assume homogeneity of the system under consideration, *i.e.* $(\mathscr{B}_x, \alpha_t^x) \cong (\mathscr{B}, \alpha_t)$. Then we define a macroscopic state as a continuous function $x \mapsto \Omega^x$ from X to the state space $K(\mathscr{B})$ of $\mathscr{B}$, and it makes sense to speak of local KMS states Ω^x, and hence of local temperatures.

In order to describe a concrete model for the microscopic interaction we consider a quantum system on the lattice $\mathbb{Z}^d$, with a C*-algebra $\mathscr{A}$ describing the observables at each site. This determines the net of local algebras $\mathscr{D}(\Lambda) = \bigotimes_{z \in \Lambda} \mathscr{A}$ for finite $\Lambda \subset \mathbb{Z}^d$, whose C*-inductive limit we denote by $\mathscr{D} \equiv \overline{\bigcup_{\Lambda} \mathscr{D}(\Lambda)}$. A corresponding net $(H(\Lambda))_{\Lambda \subset \mathbb{Z}^d}$ is defined in the usual way [20] by a translation invariant lattice potential. Measured on the microscopic scale, which is determined by the lattice spacing, we shall be looking at an increasing sequence of regions $\Lambda_l \subset \mathbb{Z}^d$ for $l \in \mathbb{N}$. To fix ideas, let us take $\Lambda_l = (lX) \cap \mathbb{Z}^d$ as the set of lattice points contained in a scaled multiple of a fixed compact region $X \subset \mathbb{R}^d$ containing the origin.

In order to establish a connection with the macroscopic view, we take X as the same region over which the macroscopic states are defined. Thus the length scale of X is the macroscopic scale, and a "macroscopic point" is represented from the microscopic point of view by points $z_l \in \Lambda_l$ such that $z_l \approx lx$. The crucial step is to identify the macroscopic algebra $\mathscr{B}$ at a point with the quasi-local algebra $\mathscr{D}$. Thus both algebras are approximated by the same net, but taken along different sequences of growing regions, in a way we shall now describe. Consider a sequence of states $\omega_l \in K(\mathscr{B}(\Lambda_l))$, a macroscopic point x in the interior of X, and a strictly local observable A, say $A \in \mathscr{D}(\Lambda_0)$ with $\Lambda_0 \subset \mathbb{Z}^d$ finite. For sufficiently large l, we have $l^{-1}\Lambda_0 + x \subset X$, and by modifying lx to a nearby lattice vector z_l we have $\Lambda_0 + z_l \subset \Lambda_l$. Therefore, the expression $\omega_l(\sigma_{z_l}(A))$ is well-defined, where σ_z denotes the automorphism of the quasi-local algebra $\mathscr{D}$ associated with the lattice translation by z. If $\{\omega_l\}$ is such that the limit

$$\Omega^x(A) \equiv \lim_{l \to \infty} \omega_l(\sigma_{z_l}(A))$$

exists for all strictly local A, and all sequences z_l such that $\lim_l l^{-1} z_l = x$, then we may consider the state Ω^x on $\mathscr{B}_x \cong \mathscr{B} = \overline{\bigcup \mathscr{D}(\Lambda_0)}$ defined by this limit as the local macroscopic state at the point x.

It is easy to see from this definition that the function $x \mapsto \Omega^x$ must be continuous when the state $K(\mathscr{B})$ is equipped with the weak*-topology. Moreover, each Ω^x will be a translation invariant state, since for each fixed $z \in \mathbb{Z}^d$ the condition $l^{-1} z_l \to x$ implies $l^{-1}(z_l + z) \to x$. It is only slightly more difficult to prove that any function $x \mapsto \Omega^x$ with these properties can be realized as a limit of a suitable sequence ω_l. In particular, we can make Ω^x be a $\beta(x)$-KMS state for α_t^x (provided these exist), where $x \mapsto \beta(x)$ is a given positive continuous function.

We remark that rather than starting from a lattice model we could also have used a continuous system as the microscopic model.

The net $\{H(\Lambda)\}$ of Hamiltonians should define not only the local, or microscopic, but also the macroscopic time evolution, *i.e.* the evolution $\Omega \mapsto \Omega_t = \{\Omega_t^x\}$. In order to achieve this, it is usually necessary to rescale the time replacing the microscopic time by $\tilde{t} = \kappa_l t$, where t is the macroscopic time. If Ω^x is given by the above equation, a candidate for the time-evolved macroscopic state is then $\Omega_t^x(A) = \lim_l \omega_l(\alpha_{\kappa_l t} \circ \sigma_{z_l}(A))$. Its existence can be proved for a simple model [10]: take a free Fermi gas and an initial state Ω which is locally $\beta(x)$-KMS, where β is an arbitrary continuous function, and choose $\kappa_l = l$. Then Ω_t^x as defined by the above limit exists. This model is unphysical, in the sense that in a free Fermi gas there is no interaction and, consequently, the limit states Ω_t^x are no longer KMS states. It does support the view, however, that if disturbances propagate under the microscopic interaction with a finite velocity, the choice $\kappa_l = l$ seems natural, yielding a finite macroscopic propagation in macroscopic times.

A typical feature of the mean field models we study here is that the length scale of the interactions is itself macroscopic. This long range interaction makes it impossible for the system to reach equilibrium locally. At the same time this has the effect that disturbances propagate over macroscopic distances in unscaled "microscopic" time, so that we shall have to set $\kappa_l = 1$. With this choice, we will indeed obtain a well-defined dynamics for the macroscopic states.

2. DEFINITION OF THE MODELS

We shall be concerned throughout with the thermodynamic limit of a family of physical systems. The systems will be labelled by the elements l of some directed set $(I, <)$. Sequences (*i.e.* the case $I = \mathbb{N}$) will be sufficient for most purposes, but by allowing general nets, we can also treat at no extra cost examples, where I is e. g. the set of regions in $\mathbb{R}^d$, tending to $\mathbb{R}^d$ in the sense of van Hove. Before we discuss the limiting properties of these systems as l becomes large, we shall describe the structures going into the definition of each single system. For each l, we consider a system of N_l particles. A **single particle** is characterized by its observable algebra $\mathscr{A}$, which is a C*-algebra with identity **1**. In many applications, we may take $\mathscr{A}$ as the algebra of $d \times d$-matrices, and think of each "particle" as one "spin". The observable algebra of the l-th system is hence $\mathscr{A}^{N_l}$, where we use the notation $\mathscr{A}^{N}$ for the N-fold minimal C*-tensor product [24] $\mathscr{A} \otimes \mathscr{A} \otimes \ldots \otimes \mathscr{A}$. The C*-inductive limit of the net $(\mathscr{A}^{k})_{k \in \mathbb{N}}$ will be denoted by $\mathscr{A}^{\infty}$. Each one of the N_l spins will be considered as "**located**" at a site $x \in X$ in some compact space X. Often X will be a subregion of $\mathbb{R}^d$. We denote the site of the j-th particle by $\xi_{l,j} \in X$ for $1 \leqq j \leqq N_l$. Thus the collection of all sites is specified by the N_l-tuple $\xi_l \in X^{N_l}$. For example, the sites may form a lattice with spacing $l^{-1} \in \mathbb{R}$, where $I = \mathbb{R}^+$, $X \subset \mathbb{R}^d$, and $\xi_{l,1}, \ldots, \xi_{l,N_l}$ is some enumeration of the lattice points in $X \cap (\mathbb{Z}^d/l)$. The **time evolution** will be implemented by a unitary group generated by a Hamiltonian belonging to the observable algebra $\mathscr{A}^{N_l}$. Note that this implies that the Hamiltonian is bounded, which is a rather severe technical restriction. We shall give some indication later how this restriction may be relaxed, and shall make this assumption now in order to concentrate on other, more essential points. The Hamiltonian will depend on the locations $\xi_{l,j}$, and we take it to be of the form $N_l . H_l(\xi_l)$, where $H_l : X^{N_l} \to \mathscr{A}^{N_l}$ is a continuous function, with respect to the product topology on X^{N_l} and the norm topology on $\mathscr{A}^{N_l}$. The space of such functions will be denoted by $\mathscr{C}(X^{N_l}, \mathscr{A}^{N_l})$. The factor N_l is taken out of the Hamiltonian for later convenience, *i.e.* $H_l(\xi_l)$ denotes the Hamiltonian density of the system, and the time evolution is given by the automorphisms

$$\alpha_l^t(A) = e^{itN_l H_l(\xi_l)} A \, e^{-itN_l H_l(\xi_l)} \tag{1}$$

for $A \in \mathscr{A}^{N_l}$ and $t \in \mathbb{R}$.

We shall often have to consider subalgebras of $\mathscr{A}^{N_l}$ of the form $\mathscr{A}^k$ for some $k < N_l$. We shall use the following notation: by a **(k, l)-embedding** we mean an injective map $\eta : \{1, \ldots, k\} \to \{1, \ldots, N_l\}$. With any such embedding we associate a homomorphism $\hat{\eta} : \mathscr{A}^k \to \mathscr{A}^{N_l}$, by identifying the j-th tensor factor of $\mathscr{A}^k$ with the $\eta(j)$-th tensor factor of $\mathscr{A}^{N_l}$. More explicitly, $\hat{\eta}(A_1 \otimes \ldots \otimes A_k) = B_1 \otimes \ldots \otimes B_{N_l}$, where $B_i = A_j$ for $i = \eta(j)$, and $B_i = \mathbf{1}$ otherwise. For the composition $\omega \circ \hat{\eta}$ we shall simply write $\omega\hat{\eta}$. With each (k, l)-embedding η we also associate the k-tuple $\xi_l \eta \in X^k$ with $(\xi_l \eta)_j = \xi_{l, \eta(j)}$. Since every permutation $\pi : \{1, \ldots, N_l\} \to \{1, \ldots, N_l\}$ is injective, we can consider it as an (N_l, l)-embedding. In this case the associated homomorphism $\hat{\pi}$ is an automorphism of $\mathscr{A}^{N_l}$.

In order to describe the connection between the systems for different l, we have to recall here some basic definitions from [16], [18], slightly modified to suit the structure under investigation. If $k \leqq N_l$, and $A \in \mathscr{A}^k$, we shall denote by $\mathrm{sym}_l(A)$ the average of $\hat{\eta}(A)$ over all (k, l)-embeddings, *i.e.*

$$\mathrm{sym}_l(A) = \frac{(N_l - k)!}{N_l!} \sum_{\eta} \hat{\eta}(A), \tag{2}$$

where the sum runs over all (k, l)-embeddings η. The normalization factor is chosen such that $\mathrm{sym}_l(\mathbf{1})=\mathbf{1}$, and $\|\mathrm{sym}_l(A)\|\leqq\|A\|$. One can also obtain $\mathrm{sym}_l(A)$ by first embedding A into $\mathscr{A}^{N_l}$ as $A\otimes\mathbf{1}\otimes\ldots\otimes\mathbf{1}$ with N_l-k tensor factors $\mathbf{1}$, and then symmetrizing over all permutations. It is easy to check that $\mathrm{sym}_{l'}(\mathrm{sym}_l(A))=\mathrm{sym}_{l'}(A)$ for $N_{l'}\geqq N_l\geqq k$, and $A\in\mathscr{A}^k$. A basic concept in the theory of mean field systems is the following space of nets:

DEFINITION 2.1. – *Let $\mathscr{A}$ be a C*-algebra with unit* $\mathbf{1}$, *and let $(N_l)_{l\in I}$ be a net of natural numbers diverging to ∞. Then a net $(A_l)_{l\in I}$ with $A_l\in\mathscr{A}^{N_l}$ is called* **strictly symmetric of degree** *k if there is some $\tilde{A}\in\mathscr{A}^k$ and $l_0\in I$ such that $N_l\geqq k$ and $A_l=\mathrm{sym}_l(\tilde{A})$, whenever $l>l_0$.*

A net $(A_l)_{l\in I}$ is called **approximately symmetric**, *if for all $\varepsilon>0$ there are $l_\varepsilon\in I$, $k\in\mathbb{N}$, and $A\in\mathscr{A}^k$, such that for all $l>l_\varepsilon$ we have $N_l\geqq k$, and*

$$\|A_l-\mathrm{sym}_l(\tilde{A})\|\leqq\varepsilon.$$

The set of strictly symmetric nets will be denoted by $\mathscr{Y}(\mathscr{A})$, and the set of approximately symmetric nets by $\tilde{\mathscr{Y}}(\mathscr{A})$.

As an example consider the net (H_l) of Hamiltonians given by

$$N_l H_l=\sum_{i=1}^{N_l}\hat{\eta}_i(\varepsilon)+\frac{1}{N_l-1}\sum_{i\neq j}^{N_l}\hat{\eta}_{ij}(V), \tag{3}$$

where $\eta_i:\{1\}\to\{1,\ldots,N_l\}$ is the $(1, l)$-embedding taking 1 to $i\leqq N_l$, and η_{ij} is the $(2, l)$-embedding with $1\mapsto i$ and $2\mapsto j$. Thus $\hat{\eta}_i(\varepsilon)$ represents the one-particle energy contribution of the i-th particle, and $\hat{\eta}_{ij}(V)$ represents the interaction between the i-th and j-th particle. Dividing by N_l it is clear that $H_l=\mathrm{sym}_l(H_2)$ is strictly symmetric of degree 2 with $H_2=\varepsilon\otimes\mathbf{1}+V$. When the normalization factor for the double sum is replaced by $1/N_l$, which is more customary for a mean field interaction, the resulting net will be only approximately symmetric. Each net $H\in\tilde{\mathscr{Y}}(\mathscr{A})$ specifies the Hamiltonian density of a homogeneous generalized mean field system [16]. We call these systems "homogeneous", because the Hamiltonian does not depend on the location parameters ξ_l.

The Hamiltonians which we consider are not of this type, since this would preclude the ξ_l-dependence of H_l, which is our main interest. However, it is easy to find an analogue of equation (3), in which such a dependence is allowed:

$$N_l H_l(x_1,\ldots,x_{N_l})=\sum_{i=1}^{N_l}\hat{\eta}_i(\varepsilon(x_i))+\frac{1}{N_l-1}\sum_{i\neq 1}^{N_l}\hat{\eta}_{ij}(V(x_i, x_j)), \tag{4}$$

with continuous functions $\varepsilon: X\to\mathscr{A}$, and $V: X\times X\to\mathscr{A}\otimes\mathscr{A}$. We can look at equation (4) as a special case of equation (3), using the isomorphism $\mathscr{C}(X, \mathscr{A})\cong\mathscr{C}(X)\otimes\mathscr{A}$ [24]. If we take $\varepsilon\in\mathscr{C}(X, \mathscr{A})$ and

$V \in \mathscr{C}(X, \mathscr{A}) \otimes \mathscr{C}(X, \mathscr{A}) \cong \mathscr{C}(X^2, \mathscr{A} \otimes \mathscr{A})$ in equation (3), H_l becomes an element of $\mathscr{C}(X, \mathscr{A})^{N_l} \cong \mathscr{C}(X^{N_l}, \mathscr{A}^{N_l})$. Equation (4) is then nothing but the evaluation of equation (3) at a point $(x_1 \ldots x_{N_l}) \in X^{N_l}$. This suggests the definition ([17], [18]) of "inhomogeneous mean field systems" as systems, whose Hamiltonian densities are given by the evaluations of an approximately symmetric net $H_l \in \mathscr{C}(X, \mathscr{A})^{N_l}$. This definition is adequate for discussing the thermostatic properties of these systems. However, for dynamical problems more stringent assumptions are needed. The simplest of these is to impose strict symmetry, which still contains the case of general two-body interactions, *i.e.* the case of most immediate physical interest.

ASSUMPTION 1′. — $H \in \mathscr{Y}(\mathscr{C}(X, \mathscr{A}))$.

While the above condition is certainly the simplest assumption needed for our theory the following much weaker, but somewhat more technical assumption is sufficient. It was motivated by the mean field versions of lattice spin systems, which are not confined to n-body interactions with some fixed bounded n. It also turns out that this condition is a rather natural hypothesis in several of our results below.

ASSUMPTION 1. — *There is an index set* Γ, *for each* $\gamma \in \Gamma$ *an integer* $n(\gamma) \in \mathbb{N}$, *and for each* $\gamma \in \Gamma$ *and* $l \in I$ *permutation symmetric hermitian elements* $H_l^\gamma \in \mathscr{C}(X, \mathscr{A})^{n(\gamma)}$, $H^\gamma \in \mathscr{C}(X, \mathscr{A})^{n(\gamma)}$ *such that*

(*a*) $\lim_l \| H_l^\gamma - H^\gamma \| = 0$,

(*b*) $\sum_\gamma n(\gamma)^2 \sup_l \| H_l^\gamma \| < \infty$,

(*c*) *For each* $\gamma \in \Gamma$ *the set* $\{ H_l^\gamma \mid l \in I \}$ *is precompact in* $\mathscr{C}(X, \mathscr{A})^{n(\gamma)}$.

The Hamiltonians are constructed from these operators as

$$H_l = \sum_{\gamma \,:\, n(\gamma) \leq N_l} \mathrm{sym}_l(H_l^\gamma).$$

Assumption 1′ is trivially implied by this by taking a single γ with H_l^γ independent of l. However, even in the simplest examples Assumption 1 allows convenient additional flexibility in the definition of the models. For example, if the factor $(N_l - 1)^{-1}$ in equation (4) is replaced by N_l^{-1}, the resulting net of Hamiltonians no longer satisfies 1′, but Assumption 1, which depends only on the asymptotic behaviour of this factor, is obviously satisfied with Γ a one-point set. Part (*c*) of Assumption 1 is not needed when the systems are simply labelled by their size $N_l \equiv l$. In that case it follows from (*b*) and the observation that for each l the set $\{ l' \mid l' \not> l \}$ is finite. Part (*c*) is also easy to check for lattice models (*see* Example 5 below).

Note that by either of these assumptions each H_l is permutation symmetric. It is important to keep in mind that this does *not* mean that each

$H_l(\xi_l) \in \mathcal{A}^{N_l}$ is permutation symmetric: the symmetrization operation implicit in this Assumption refers to simultaneous operations on the locations and the site-labels. More formally, we have for any $F : X^k \to \mathcal{A}^k$, and any (k, l)-embedding η:

$$(\hat{\eta} F)(\xi_l) = \hat{\eta}(F(\xi_l \eta)), \tag{5}$$

where on the left hand side $\hat{\eta} : \mathcal{C}(X, \mathcal{A})^k \to \mathcal{C}(X, \mathcal{A})^{N_l}$, and on the right hand side $\hat{\eta} : \mathcal{A}^k \to \mathcal{A}^{N_l}$. We shall often have to pass from the level of the observable algebras $\mathcal{A}^{N_l}$ (or "$\mathcal{A}$-level") to the level of the function algebras $\mathcal{C}(X, \mathcal{A})^{N_l}$ (or "$\mathcal{C}$-level"). The basic operator for this is the symmetrized evaluation operator

$$R_l : \quad \mathcal{C}(X, \mathcal{A})^{N_l} \to \mathcal{A}^{N_l} : \qquad F \mapsto (\mathrm{sym}_l F)(\xi_l). \tag{6}$$

Thus the choice of location parameters ξ_l is implicit in this operator. The symmetrization (which is over permutations here) is redundant, when this R_l is applied to a symmetric element of $\mathcal{C}(X, \mathcal{A})^{N_l}$ like the Hamiltonian, *i.e.* we have $R_l(H_l) = H_l(\xi_l)$. In equation (2) we defined sym_l as an operator from $\mathcal{A}^k$ to $\mathcal{A}^{N_l}$. Therefore, $F \mapsto (\mathrm{sym}_l F)(\xi_l)$ also defines an operator from $\mathcal{C}(X, \mathcal{A})^k$ to $\mathcal{A}^{N_l}$. which we shall likewise denote by R_l. This operator satisfies the equation

$$R_l \hat{\eta} = R_l : \mathcal{C}(X, \mathcal{A})^k \to \mathcal{C}(X, \mathcal{A})^{N_l} \tag{7}$$

for all (k, l)-embeddings η.

Of course, in order to get a sensible limiting behaviour of these models, we also have to impose conditions on the location parameters ξ_l.

ASSUMPTION 2. – $(\xi_l \in X^{N_l})_{l \in I}$ *has a limiting density*, i.e. *there is a probability measure* μ *on* X, *such that for all* $f \in \mathcal{C}(X)$:

$$\lim_l \frac{1}{N_l} \sum_{i=1}^{N_l} f(\xi_{l,i}) = \int \mu(dx) f(x).$$

We do not assume that the support of the limiting measure μ, which we denote by $X' \subset X$, is the entire compact set X.

Some technical problems are greatly simplified, when the algebras involved do not become "too large". The following Assumption is of this kind. We shall indicate later, how it can be relaxed, and which of our results depend on it.

ASSUMPTION 3. – X *is metrizable and* $\mathcal{A}$ *is separable.*

Since $\mathcal{C}(X)$ is separable iff X is metrizable, we can say equivalently that $\mathcal{C}(X, \mathcal{A})$ is separable. This completes the definition of the class of models treated in this paper.

We close this section with some examples designed to point out possible physical interpretations of the mathematical structure defined by our assumptions, and, especially, of the space X, and the role of the "inhomogeneity" in our theory.

Example 1. – In the beginning of this section we have already mentioned the case of a sequence of finer and finer lattices fitted into a compact region $X \subset \mathbb{R}^d$. Here we take $l \in I = \mathbb{R}^+$ as the inverse lattice spacing, so that $\{\xi_{l,i}\} = X \cap l^{-1}\mathbb{Z}^d$. Evidently, the limiting density μ of these points is the normalized restriction of Lebesgue measure to X. In order to obtain a finite energy per particle, in spite of the unbounded number of particles in each finite volume, each two-body interaction term in equation (4) is multiplied by the inverse particle number. Thus the strength of the interaction between any two particles goes to zero in the thermodynamic limit.

Example 2. – There is a dual way of looking at the same systems, which is closer to the scheme described in the introduction: we then have a fixed lattice, say the cubic lattice $\mathbb{Z}^d \subset \mathbb{R}^d$. We now consider larger and larger regions, namely scaled copies of a fixed compact set X. In order to make these systems identical to those in example (1), however, we now have to scale the interaction to longer and longer range. If $z_1, \ldots, z_{N_l}$ are the lattice points in lX, we must set $\xi_{l,i} = l^{-1} z_i$, and the Hamiltonian in equation (4) becomes

$$N_l H_l(z_1, \ldots, z_{N_l}) = \sum_{i=1}^{N_l} \hat{\eta}_i(\varepsilon(lz_i)) + \frac{1}{N_l - 1} \sum_{i \neq 1}^{N_l} \hat{\eta}_{ij}(V(lz_i, lz_j)).$$

Note that the equivalence between these two ways of looking at the system reflects the coherence of the microscopic and the macroscopic views of the system, as set out in the introduction.

Example 3. – Let us take again a fixed lattice $\mathbb{Z}^d \subset \mathbb{R}^d$, with $\{\xi_{l,i}\}$ the lattice points in an increasing set of regions labelled by l. This time we shall not consider any rescaling of the arguments in the interaction terms, so it would seem that we just have a standard lattice model, which would be rendered trivial by multiplying the interaction with $1/N_l$. However, we shall consider two-body interactions, which not only have long range but infinite range, in the sense that the potential $V(x, y)$ does not go to zero as the points x, y approach infinity. Let us assume for simplicity that $\varepsilon(x)$ and $V(x, y)$ in equation (4) converge in norm, whenever x or y or both go to infinity in such a way that the unit vectors $x/|x|$, $y/|y|$ converge. Then we shall take X as the "directional" compactification of $\mathbb{R}^d$, *i.e.* equal to $\mathbb{R}^d$ with an added sphere at infinity. The limit conditions on ε and V are just equivalent to the existence of continuous extensions of these functions from $\mathbb{R}^d$ to X, resp. from $(\mathbb{R}^d)^2$ to X^2. Then $H_l \in \mathscr{C}(X, \mathscr{A})^{N_l}$ is defined by equation (4) in terms of these extensions. The limiting density in this example depends on the shape of the regions going to infinity, but it is always supported by the sphere $X' \equiv X \setminus \mathbb{R}^d$. For spheres around the origin of increasing radius the measure is just the surface measure of the sphere. Other conditions about the behaviour of the Hamiltonian at infinity can be accommodated by choosing different compactifications of $\mathbb{R}^d$. For example, if ε and V, and – if present – the higher order terms are almost periodic functions, the space X will be the Bohr-compactification [15] (or, more precisely, a separable quotient of it).

Example 4. – In the previous examples X was the configuration space, or some space closely related to it. However, this is by no means necessary. The simplest choice for X is a finite set. The resulting class of models might be called **multi-species** homogeneous mean field models. Assumption 2 then simply means that the relative particle numbers of the species converge. A study of the dynamics of Josephson junctions based on such a model can be found in [25].

Example 5. – There is a canonical way to obtain a homogeneous mean field model from an arbitrary quantum spin system on a lattice. Consider as in [20] an interaction potential Φ, which assigns to each finite subset $\Lambda_0 \subset \mathbb{Z}^d$ and operator Φ_{Λ_0} in the local algebra $\mathcal{A}(\Lambda_0) \equiv \bigotimes_{z \in \Lambda_0} \mathcal{A}_z$, where all $\mathcal{A}_z$, $z \in \mathbb{Z}$ are isomorphic copies of a fixed unital C*-algebra $\mathcal{A}$. The Hamiltonian of the system in a finite region Λ is then defined as

$$\mathcal{H}_\Lambda = \sum_{\Lambda_0 \subset \Lambda} \Phi_{\Lambda_0}.$$

Now let I be a net of finite regions $\Lambda_l \subset \mathbb{Z}^d$, going to $\mathbb{Z}^d$ in the sense of van Hove [20]. By $N_l = |\Lambda_l|$ we denote the number of points in Λ_l. Then, as shown in [8], the operators

$$H_l = \frac{1}{N_l} \operatorname{sym}_l (\mathcal{H}_{\Lambda_l})$$

satisfy Assumption 1, with the index set Γ chosen as the set of regions Λ_0 containing the origin, provided

$$\sum_{\Lambda_0 \ni 0} |\Lambda_0| \, \| \Phi_{\Lambda_0} \| \leqq \infty.$$

It is interesting to note that this condition is less stringent than the condition, under which the existence of the dynamics in the thermodynamic limit is proven in [20], which is of the same form as the above, with $|\Lambda_0|$ replaced by $\exp |\Lambda_0|$. Of course, this procedure generates a homogeneous mean field model. But applying the same method to a multi-species lattice system, where different species are assigned to the different particles in the elementary cell of a lattice, one obtains a multi-species model in the above sense. A discussion of some models generated in this way can be found in [9] and [12].

Example 6. — In some models X can be a part of momentum space. The most important example is the **BCS-model** without the "tight-binding" approximation. The equilibrium aspects of this model are discussed in [7]. For a study of some dynamical properties of the homogeneous version of the model, which is called the "tight binding" or "strong coupling" approximation, *see* [5].

Example 7. — If one thinks of the $\xi_{l,i}$ as "external parameters" it is natural sometimes to consider them to be given as random variables. Models of this kind are called **site-random models**, because there is one random variable for each of the N_l particles or "sites" [6]. The $\xi_{l,i}$ are called "quenched" random variables, because they are fixed once and for all, *i.e.* we are interested in the properties of each individual sample. In the simplest models all the $\xi_{l,i}$ are taken to be independent and distributed according to the same probability measure μ. Then by the law of large numbers Assumption 2 holds with probability one. Note that this Assumption is the only property of the sample, which enters our results. Once it is checked for a particular sample there will no further "almost never" occurring exceptional events to be taken into account. It is clear that the method in example 5 for constructing mean field models can also be applied to site-random spin systems on a lattice, yielding a rather general class of models satisfying our assumptions.

3. MEAN FIELD LIMIT OF STATES

Consider a net $(\omega_l)_{l\in I}$ of states on $\mathscr{A}^{N_l}$. Each of these states is defined on a different algebra, so in order to compare them, and define a notion of "thermodynamic limit" for such nets, we have to specify on which observables two states ω_l and $\omega_{l'}$ are to give similar expectation values. One set of observables, on which such comparison makes sense, is given in terms of (k, l)-embeddings, as defined in the previous section, *i.e.* one might call ω_l and $\omega_{l'}$ similar, if $\omega_l\hat{\eta}(A)\approx\omega_{l'}\hat{\eta}'(A)$ for all $A\in\mathscr{A}^k$ and certain pairs (η, η') of a (k, l)- and a (k, l')-embedding. The choice of a class of pairs for which the comparison is made, determines the nature of the comparison. The crudest choice, namely allowing *all* pairs leads to the following definition:

DEFINITION 3.1. — *A net of states* $(\omega_l)_{l\in I}$ *on* $\mathscr{A}^{N_l}$ *is said to have a* **homogeneous mean field limit**, *if for all* $k\in\mathbb{N}$, $A\in\mathscr{A}^k$, *and all nets* $(\eta_l)_{l\in I}$ *of* (k, l)*-embeddings the limit* $\lim\omega_l\hat{\eta}_l(A)$ *exists.*

Any permutation invariant state Ω on the C*-inductive limit algebra $\mathcal{A}^\infty = \overline{\bigcup_{n\in\mathbb{N}} \mathcal{A}^n}^{\|\cdot\|}$ determines such a net via $\omega_l = \Omega \,|\, \mathcal{A}^{N_l}$. In this case $\omega_l\,\hat{\eta}\,(A)$ is independent of the (k, l)-embedding η, and even independent of l, so the limit exists trivially. An important special case of this are the homogeneous product states $\omega_l = \rho^{N_l}$, where ρ is a fixed state on $\mathcal{A}$, and we use the notation ρ^N for the N-fold tensor product of the state ρ with itself. We shall denote the state space of the C*-algebra $\mathcal{A}$ by $K(\mathcal{A})$, and this space will be equipped with the weak* topology, unless otherwise stated. Since the "one-particle" algebra $\mathcal{A}$ is separable by Assumption 3, $K(\mathcal{A})$ is a compact metrizable space, so Baire and Borel measures on $K(\mathcal{A})$ coincide.

PROPOSITION 3.2. – *Suppose that* $(\omega_l)_{l\in I}$ *has a homogeneous mean field limit. Then there exists a unique probability measure* M_ω *on* $K(\mathcal{A})$ *such that for all* $A \in \mathcal{A}^k$ *and all nets* $(\eta_l)_{l\in I}$:

$$\lim_l (\omega_l\,\hat{\eta}_l)(A) = \int M_\omega(d\rho)\,\rho^k(A). \tag{8}$$

For all $(A_l)_{l\in I} \in \tilde{\mathcal{Y}}(\mathcal{A})$

$$\lim_l \omega_l(A_l) = \int M_\omega(d\rho)\,(jA)(\rho), \tag{9}$$

where $j : \tilde{\mathcal{Y}}(\mathcal{A}) \to \mathcal{C}(K(\mathcal{A}))$ *is defined by*

$$(jA)(\rho) = \lim_l \rho^{N_l}(A_l). \tag{10}$$

If $\omega_l\,\mathrm{sym}_l = \omega_l$ *for all* $l \in I$, *then the existence of the limit in equation* (9) *for all* $(A_l)_{l\in I} \in \mathcal{Y}(\mathcal{A})$ *is also sufficient for* $(\omega_l)_{l\in I}$ *to have a mean field limit.*

Proof. – We show first that $\lim_{l\in I} \omega_l\,\hat{\eta}_l(A)$ is independent of $(\eta_l)_{l\in I}$. This follows easily from the observation that all subnets of a convergent net converge to the same limit: let I_1, I_2 be disjoint subsets of I, both of which contain arbitrarily large elements. Then given any two nets η and η' we can produce a third net η'', such that $\eta''_l = \eta_l$ for $l \in I_1$, and $\eta''_l = \eta'_l$ for $l \in I_2$. Hence

$$\lim_{l\in I} \omega_l\,\hat{\eta}_l(A) = \lim_{l\in I_1} \omega_l\,\hat{\eta}''_l(A) = \lim_{l\in I_2} \omega_l\,\hat{\eta}''_l(A) = \lim_{l\in I} \omega_l\,\hat{\eta}'_l(A).$$

In particular, the equation $\Omega_k(A) = \lim_{l\in I} \omega_l\,\hat{\eta}_l(A)$ defines a permutation invariant state on $\mathcal{A}^k$. Since $\Omega_{k+1}(A\otimes\mathbf{1}) = \Omega_k(A)$, these states together define a permutation invariant state Ω on the C*-inductive limit of these algebras. Such a state has a unique integral decomposition

$\Omega = \int M(d\rho)\,\rho^\infty$ by Størmer's de Finetti Theorem [23], from which we get $\Omega_k = \int M(d\rho)\,\rho^k$ by restriction.

The limit $\lim_{l \in I} \omega_l \hat{\eta}_l(A)$ is even uniform with respect to η_l: for fixed hermitian $A \in \mathscr{A}^k$, and every l, let η_l^+ (resp. η_l^-) be a (k, l)-embedding for which $\omega_l \hat{\eta}(A)$ becomes maximal (resp. minimal) among all choices of such embeddings. Then both nets $\omega_l \hat{\eta}_l^\pm$ converge. Since the two limits have to be equal by the preceding argument, we can find for any $\varepsilon > 0$ an $l_\varepsilon \in I$ such that for all $l > l_\varepsilon$, and all (k, l)-embeddings $\eta \,|\, \omega_l \hat{\eta}(A) - \Omega_k(A) \,| \leqq \varepsilon$. Averaging over all (k, l)-embeddings, we obtain also that $|\, \omega_l \operatorname{sym}_l(A) - \Omega_k(A) \,| \leqq \varepsilon$.

For a strictly symmetric net $(A_l)_{l \in I}$ of degree k, we have for some $\tilde{A} \in \mathscr{A}^k$ that $\omega_l \operatorname{sym}_l(\tilde{A}) = \omega_l(A_l)$ for any (k, l)-embedding η, and $\rho^{N_l}(A_l) \equiv \rho^k(\tilde{A}) = (jA)(\rho)$ so that $\Omega_k(\tilde{A}) = \int M(d\rho)\,(jA)(\rho)$. Hence equation (9) holds for $(A_l)_{l \in I} \in \mathscr{Y}(A)$. Since each $\tilde{A} \in \mathscr{A}^k$ determines a strictly symmetric net $(A_l)_{l \in I}$, and for symmetric states $\omega_l \hat{\eta}_l(\tilde{A}) = \omega_l(A_l)$, equation (9) is indeed just a restatement of Definition 3.1. It is easy to see from the definition of approximate symmetry that the limit defining j, and the limit in equation (9) are uniform in ρ, and ω_l, respectively. From this one readily concludes equation (9) for $(A_l)_{l \in I} \in \tilde{\mathscr{Y}}$. ■

The following example shows how nets with a homogeneous mean field limit arise naturally in quantum spin systems on a lattice.

Example. — Let $\{\mathscr{A}(\Lambda) = \bigotimes_{i \in \Lambda} \mathscr{A} \,|\, \Lambda \subset \mathbb{Z}^d \text{ finite}\}$ be the net of local algebras of a quantum lattice system, and let ω be a translation invariant state on the quasi-local algebra $\mathscr{A}^\infty = \overline{\bigcup \mathscr{A}(\Lambda)}^{\|\cdot\|}$. Let $(\Lambda_l)_{l \in I}$ be a net of finite regions going to $\mathbb{Z}^d$ in the sense of van Hove. With some numbering of the N_l points in Λ_l chosen, we shall identify $\mathscr{A}^{N_l}$ and $\mathscr{A}(\Lambda)$. Then we claim that the net $(\omega_l)_{l \in I}$ with

$$\omega_l = (\omega \,|\, \mathscr{A}(\Lambda)) \circ \operatorname{sym}_l \tag{11}$$

has a homogeneous mean field limit. Because for $A \in \tilde{\mathscr{Y}}$

$$\| A_l - \operatorname{sym}_l(A_l) \| \to 0,$$

and because the states ω_l are symmetric by construction, our claim is equivalent to the existence of the limits $\lim_l \omega(A_l)$ for all $A \in \tilde{\mathscr{Y}}(\mathscr{A})$. Here we have considered $\mathscr{A}(\Lambda)$ as a subalgebra of $\mathscr{A}^\infty$, so the evaluation of ω makes sense. This has been shown by [21], [14] in the special case of sequences of the form

$$X_l = (\text{sym}_l(A_1)) . (\text{sym}_l(A_2)) \ldots (\text{sym}_l(A_r)),$$

where $\mathscr{A}_1, \ldots, \mathscr{A}_r \in \mathscr{A}$ are one-site observables. Linear combinations of such expressions were called "intensive polynomials" in [13]. Therefore it just remains to show that the intensive polynomials are dense in $\tilde{\mathscr{Y}}(\mathscr{A})$ in the sense that all $Y \in \tilde{\mathscr{Y}}$ can be approximated uniformly for large n by such sequences. Equivalently, we can show that the corresponding set $\{jX\} \subset \mathscr{C}(K(\mathscr{A}))$ is norm dense. This, however, is an immediate consequence of the Stone-Weierstraß theorem. Note that without the symmetrization in equation (11) the restrictions of the translation invariant state ω typically will not have a mean field limit. Suppose to the contrary that the net $(\omega | \mathscr{A}^{N_l})_{l \in I}$ has a mean field limit, and consider a net $(\eta_l)_{l \in I}$ of $(2, l)$-embeddings, which for every l embeds the two sites at the same distance $x \in \mathbb{Z}^d$ in the lattice. Then $\omega \hat{\eta}(A \otimes B)$ is the two-point correlation function of ω evaluated at x. By Proposition 3.2 the limit of this (l-independent) quantity must be independent of the net $(\eta_l)_{l \in I}$, hence the two-point function of ω must be constant, which is true for permutation symmetric states, but quite untypical for general translation invariant states.

We now take Definition 3.1 as a model for defining mean field limits also in the inhomogeneous case. The basic idea is to demand the existence of the limit $\omega_l \hat{\eta}_l(A)$ only if $\hat{\eta}_l$ maps the observable A into some predictable "location". The most straight-forward realization of this idea is the following.

DEFINITION 3.3. — *A net $(\omega_l)_{l \in I}$ is said to have a* **mean field limit** *with respect to $(\xi_l)_{l \in I}$, if for all k, $A \in \mathscr{A}^k$, and nets $(\eta_l)_{l \in I}$ of (k, l)-embeddings the convergence of $\xi_l \eta_l$ in X^k implies the convergence of $\omega_l \hat{\eta}_l(A)$.*

Basic properties of such limits are collected in the following result. Recall that X' denotes the support of the limiting density measure μ, which may be properly contained in X.

PROPOSITION 3.4. — *Suppose $(\omega_l)_{l \in I}$ has a mean field limit. Then for every $k \in \mathbb{N}$ there is a continuous function $\Omega : (X')^k \to K(\mathscr{A}^k)$,* such that *$\lim_l \xi_l \eta_l = x \in (X')^k$ implies*

$$\lim_l \omega_l \hat{\eta}_l(A) = \Omega_x(A).$$

Moreover, the convergence is uniform in the sense that for any $\varepsilon > 0$, and $A \in \mathscr{A}^k$, and any continuous metric d_k for X^k there is an $l_0 \in I$ and $\delta > 0$ such that for any (k, l)-embeddings η with $l > l_0$ and $d_k(\xi_l \eta, x) \leqq \delta$:

$$|\omega_l \hat{\eta}(A) - \Omega_x(A)| \leqq \varepsilon.$$

Proof. — Since X is metrizable, we can pick a metric d for X, and use in the first part of the proof the metric $d_k(x, y) = \sum_{i=1}^{k} d(x_i, y_i)$ for X^k. Our first aim is to show that if $x \in (X')^k$, there are sufficiently many nets $\xi_l \eta_l$ converging to x. More precisely, there is a net $(\varepsilon_l)_{l \in I}$, $\varepsilon_l > 0$ such that $\lim_l \varepsilon_l = 0$, and for all $x \in (X')^k$, $l \in I$ there is a (k, l)-embedding η such that $d_k(x, \xi_l \eta) \leqq \varepsilon_l$. Obviously, we can take $\varepsilon_l = \sup_x \min_\eta d_k(x, \xi_l \eta)$, and have to show that this $\varepsilon_l \to 0$. Suppose to the contrary that there is a subnet along which $\varepsilon_l \geqq \varepsilon_0 > 0$. Then there are x_l such that $\min_\eta d_k(x_l, \xi_l \eta) > \varepsilon_0/2$, and by compactness we may take the subnet (x_l) to be convergent to, say, $x \in (X')^k$. Thus for sufficiently large l we have $\min_\eta d_k(x, \xi_l \eta) > \varepsilon_0/4$. By going to a further subnet we can find an $i \leqq k$ such that

$$\inf_\eta d(x_i, \xi_{l,i}) \geqq \varepsilon_0/4k \equiv \varepsilon_1.$$

Consider the function $f \in \mathscr{C}(X)$ with $f(y) = \varepsilon_1 - d(x_i, y)$, where this quantity is positive, and $f(y) = 0$ otherwise. Then by the preceding we have $N_l^{-1} \Sigma f(\xi_{l,i}) \equiv 0$, and by Assumption 2 this quantity converges along any subnet to $\int \mu(dx) f(x) = 0$, which contradicts $x_i \in \operatorname{supp} \mu$.

We now define $\Omega_x(A) = \lim_l \omega_l \hat{\eta}_l(A)$ for $x = \lim_l \xi_l \eta_l \in (X')^k$, arguing as in the proof of Proposition 3.2 that this limit is independent of the choice of $(\eta_l)_{l \in I}$. Now let $x, x_n \in (X')^k$, for $n \in \mathbb{N}$ with $x_n \to x$. Then by the preceding paragraph we can find for each $n \in \mathbb{N}$ an l_n, and a (k, l_n)-embedding η_{l_n} such that

$$|\omega_{l_n} \hat{\eta}_{l_n}(A) - \Omega_{x_n}(A)| \leqq 2^{-n}.$$

We can define η_l on the remaining $l \in I$ such that $\xi_l \eta_l \to x$. Then in the above equation the first term in the absolute value converges to $\Omega_x(A)$. Hence $\lim_n \Omega_{x_n}(A) = \Omega_x(A)$, which in the metrizable space $(X')^k$ implies the continuity of Ω.

For any $x \in (X')^k$ and $\varepsilon > 0$, and an arbitrary metric d_k we can find $\delta(\varepsilon, x)$ and $l_0(\varepsilon, x)$ such that for any (k, l)-embeddings η with $l > l_0$ and $d_k(\xi_l \eta, x) \leqq \delta$ we have $|\omega_l \hat{\eta}(A) - \Omega_x(A)| \leqq \varepsilon$. [Otherwise we could easily construct a net $(\eta_l)_{l \in I}$ such that $\xi_l \eta_l \to x$, but $\omega_l \hat{\eta}_l(A) \nrightarrow \Omega_x(A)$.] It remains to be shown that we can take $\delta(\varepsilon, x)$ and $l(\varepsilon, x)$ to be independent of x. Now $x \mapsto \Omega_x(A)$ is uniformly continuous, so we can find for all $\varepsilon > 0$ a δ_1 such that $d_k(x, y) \leqq \delta_1$ implies $|\Omega_x(A) - \Omega_y(A)| \leqq \varepsilon$. Consider a covering of $(X')^k$ by open balls at each x with radius less than $\delta(\varepsilon, x)/2$

and less than δ_1. Let $\{x_\alpha\}$ be the centers of the balls in a finite subcover. Let δ be the smallest of the radii in this subcover, and let l_0 be larger than all $l(\varepsilon, x_\alpha)$. Then if $l>l_0$ and $d(x, \xi_l\eta)\leqq\delta$ for some (k, l)-embedding η, then there is an α such that $d(x, x_\alpha)\leqq\delta\leqq\delta_1$, hence $d(\xi_l\eta, x_\alpha)\leqq 2\delta\leqq\delta(\varepsilon, x_\alpha)$. Consequently,

$$|\omega_l\hat{\eta}(A)-\Omega_x(A)|\leqq|\omega_l\hat{\eta}(A)-\Omega_{x_\alpha}(A)|+|\Omega_{x_\alpha}(A)-\Omega_x(A)|\leqq 2\varepsilon. \quad \blacksquare$$

Another notion of inhomogeneous mean field limits can be obtained by the same device that we used in section 2 to define the Hamiltonians of inhomogeneous mean field systems, *i.e.* passing from the "$\mathscr{A}$-level" to the "$\mathscr{C}$-level", and applying the "homogeneous" version of the definition to $\mathscr{C}(\mathscr{X}, \mathscr{A})$.

DEFINITION 3.5. — *A net* $(\omega_l)_{l\in I}$, $\omega_l\in K(\mathscr{A}^{N_l}$ *is said to have a* **weak mean field limit**, *if the net* $\omega_l R_l\in K(\mathscr{C}(X, \mathscr{A})^{N_l})$ *has a homogeneous mean field limit.*

In view of equation (7) this is equivalent to the existence of $\lim_l \omega_l R_l(F)$ for all $F\in\mathscr{C}(X, \mathscr{A})^k$ and all k.

The following Proposition describes the limiting object of a weak field limit. In its proof we need for the first time the important direct integral decomposition

$$\Phi=\int^\oplus \mu_\Phi(dx)\,\varphi_x \tag{12}$$

of an arbitrary state $\Phi\in K(\mathscr{C}(X, \mathscr{A}))$, which is shorthand for

$$\Phi(F)=\int\mu_\Phi(dx)\,\varphi_x(F(x)), \tag{13}$$

where $\varphi_x\in K(\mathscr{A})$, and μ_Φ is a probability measure on X. Inserting functions of the form $F(x)=f(x)\mathbf{1}_{\mathscr{A}}$ into this equation we see that μ_Φ is the Radon measure representing the restriction of Φ to the first factor of $\mathscr{C}(X, \mathscr{A})\cong\mathscr{C}(X)\otimes\mathscr{A}$. The function $x\mapsto\varphi_x(A)$ must be measurable for all $A\in\mathscr{A}$, and since $\mathscr{A}$ is separable (*cf.* Assumption 3) the states φ_x are uniquely determined by Φ up to a set of μ_Φ-measure zero. We shall denote by $K^\mu(\mathscr{C}(X, \mathscr{A}))\subset K(\mathscr{C}(X, \mathscr{A}))$ the set of states. whose restriction to $\mathscr{C}(X)$ is given by the limiting density measure μ, *i.e.* the set of states for which $\mu_\Phi=\mu$.

PROPOSITION 3.6. — *Every net* $(\omega_l)_{l\in I}$ *has a subnet, along which it has a weak mean field limit. Suppose the net* $(\omega_l)_{l\in I}$ *has a weak mean field limit. Then for each* $k\in\mathbb{N}$ *there is a function* $x\in X^k\mapsto\Omega_x\in K(\mathscr{A}^k)$, *which is measurable in the sense that for all* $A\in\mathscr{A}^k$, $x\mapsto\Omega_x(A)$ *is measurable, and* *has the property that for all* $F \in \mathscr{C}(X, \mathscr{A})^k$

$$\lim_l \omega_l R_l(F) = \int \mu(dx_1) \ldots \mu(dx_k) \Omega_{x_1, \ldots, x_k}(F(x_1, \ldots, x_k)). \quad (14)$$

Ω_x *is uniquely determined by this equation for* μ^k*-almost all* $x \in X^k$.

Proof. – The existence of a convergent subnet is trivial: the states $\omega_l R_l$ can be extended to states $\tilde{\omega}_l$ on $\mathscr{C}(X, \mathscr{A})^\infty$, and since $K(\mathscr{C}(X, \mathscr{A})^\infty)$ is weak* compact, the extensions $\tilde{\omega}_l$ converges along a subnet. Along this subnet the limits of $\omega_l R_l \hat{\eta}_l(F) = \tilde{\omega}_l(F)$ exist for all $F \in \mathscr{C}(X, \mathscr{A})^k$ and nets $(\eta_l)_{l \in I}$, so $(\omega_l R_l)_{l \in I}$ has a homogeneous mean field limit.

Suppose now that $(\omega_l)_{l \in I}$ has a weak mean field limit. Then from Proposition 3.2 we get a probability measure $M_{\omega R}$ on the state space of $\mathscr{C}(X, \mathscr{A})$ such that for $F \in \mathscr{C}(X, \mathscr{A})^k$ $\omega_l R_l(F)$ converges to $\int M_{\omega R}(d\Phi) \Phi^k(F)$.

In the limit the fact that the restriction of $\omega_l R_l$ to

$$\mathscr{C}(X^{N_l}) \subset \mathscr{C}(X^{N_l}) \otimes \mathscr{A}^{N_l} \equiv \mathscr{C}(X, \mathscr{A})^{N_l}$$

depends only on ξ_l, forces the measure $M_{\omega R}$ to be supported by $K^\mu(\mathscr{C}(X, \mathscr{A}))$ ([17], [18]). Therefore, the formula of the Proposition holds with

$$\Omega_{x_1, \ldots x_k} = \int M_{\omega R}(d\Phi) \varphi_{x_1} \otimes \ldots \varphi_{x_k}, \quad (15)$$

where in the integral on the right the connection between the integration variable Φ and the φ_x is given by equation (12). ■

PROPOSITION 3.7. – *If a net* $(\omega_l)_{l \in I}$ *has a mean field limit, then it also has a weak mean field limit. Moreover, the functions* $x \mapsto \Omega_x$ *of Propositions* 3.4 *and* 3.6 *coincide* μ^k*-almost everywhere.*

Proof. – We have to show that for every $F \in \mathscr{C}(X, \mathscr{A})^k$ the limit of $\omega_l R_l(F)$ exists, and is equal to $\int \mu^k(dx) \Omega_x(F(x))$. Since $\omega_l R_l$ is by definition a symmetric state on $\mathscr{C}(X, \mathscr{A})^k$, this expression is independent of η_l, and by definition of R and sym [equations (6, 2, 5)] it is equal to the mean over all (k, l)-embeddings η of

$$\omega_l((\hat{\eta} F)(\xi_l)) = \omega_l \hat{\eta}(F(\xi_l \eta)) \quad (16)$$

Since $F: X^k \to \mathscr{A}^k$ is continuous, and X^k is compact, we can find for every $\varepsilon_1 > 0$ a $\delta_1 > 0$ such that $d_k(x, y) \leqq \delta_1$ implies $\| F(x) - F(y) \| \leqq \varepsilon_1$. Moreover, the range of F is norm compact, and can thus be covered by finitely many balls of any given radius. Applying Proposition 3.4 to the centers of the balls in a suitable covering, we obtain for every $\varepsilon_2>0$ a $\delta_2>0$ and an l_2 such that for all $y\in X^k$, $x\in(X')^k$, $l>l_2$ and (k, l)-embeddings η with $d_k(\xi_l\eta, x)\leqq\delta_2$ we have $|\omega_l\hat{\eta}(F(y))-\Omega_x(F(y))|\leqq\varepsilon_2$.

Cover X^k by finitely many, say M, balls B_α of diameter less than either δ_1 or δ_2, and pick positive functions $f_\alpha\in\mathscr{C}(X^k)$ with support in B_α with $\sum_\alpha f_\alpha\equiv 1$. Furthermore, pick for each α a point $x_\alpha\in B_\alpha$ such that $x_\alpha\in(X')^k$ if $B_\alpha\cap(X')^k\neq\varnothing$. Then if $l>l_2$ we have for each (k, l)-embedding η

$$\begin{aligned}\omega_l\hat{\eta}(F(\xi_l\eta)) &= \sum_\alpha f_\alpha(\xi_l\eta)\,\omega_l\hat{\eta}(F(\xi_l\eta))\\ &= \sum_\alpha f_\alpha(\xi_l\eta)(\omega_l\hat{\eta}(F(x_\alpha))+u_\alpha\varepsilon_1)\\ &= u_1\varepsilon_1+\sum_{x_\alpha\in(X')^k} f_\alpha(\xi_l\eta)(\Omega_{x_\alpha}(F(x_\alpha))+u_\alpha\varepsilon_2)\\ &\quad+\|F\|\sum_{x_\alpha\notin(X')^k} f_\alpha(\xi_l\eta)\,u_\alpha,\end{aligned}$$

where in each expression u_i stands for "some number with modulus $\leqq 1$". In the last expression η or ξ_l appear only in the argument of the f_α's. Therefore we need to compute the average of $f(\xi_l\eta)$ over all (k, l)-embeddings η. This is the average of $(\xi_l\eta)$ over all *injective* maps η, which differs only by corrections of order k/N_l from the average over *all* maps η, which by Assumption 2 converges to the integral with respect to μ^k. Hence we can find for any ε_3 an l_3 such that for all α and all $1>l_3$

$$\left|\operatorname*{Mean}_\eta\{f_\alpha(\xi_l\eta)\}-\int\mu(dx_1)\ldots\mu(dx_k)\,f_\alpha(x_1,\ldots x_k)\right|\leqq\varepsilon_3.$$

Combining these estimates we obtain

$$\begin{aligned}\omega_l R_l\hat{\eta}_l(F) &= u_1\varepsilon_1+u_2\varepsilon_2+M u_3\varepsilon_3+\sum_\alpha\int\mu^k(dx)\,f_\alpha(x)\,\Omega_{x_\alpha}(F(x_\alpha))\\ &= u_1\varepsilon_1+u_2\varepsilon_2+M u_3\varepsilon_3\\ &\quad+\sum_\alpha\int\mu^k(dx)\,f_\alpha(x)(\Omega_x(F(x))+2u_\alpha\varepsilon_2+u'_\alpha\varepsilon_1)\\ &= \int\mu^k(dx)\,\Omega_x(F(x))+3u_1\varepsilon_1+3u_2\varepsilon_2+M u_3\varepsilon_3,\end{aligned}$$

where we used ε_2 to estimate the modulus of continuity of $x\mapsto\Omega_x(F(x_\alpha))$ for each α. ■

This Proposition prompts the following definition:

DEFINITION 3.8. — *A* **macroscopic state** *is given by a family of measurable functions* $x\in X^k\mapsto\Omega_x\in K(\mathscr{A}^k)$, *called its* **correlation functions,** *such that*

$$\Omega_{x_1,\ldots x_{k+1}}(A_1\otimes\ldots A_k\otimes\mathbf{1}_{\mathscr{A}})=\Omega_{x_1,\ldots,x_k}(A_1\otimes\ldots A_k)$$

and

$$\Omega_{x_{\pi 1}, \ldots, x_{\pi k}}(A_{\pi 1} \otimes \ldots A_{\pi k}) = \Omega_{x_1, \ldots, x_k}(A_1 \otimes \ldots A_k)$$

for any permutation of $\{1, \ldots, k\}$. *The* **Størmer measure** *of a macroscopic state is the unique measure on* $K^{\mu}(\mathscr{C}(X, \mathscr{A}))$ *such that*

$$\Omega_{x_1, \ldots, x_k} = \int M(d\Phi)\, \varphi_{x_1} \otimes \ldots \varphi_{x_k}, \tag{17}$$

where in the integral $\Phi = \int^{\oplus} \mu(dx)\, \varphi_x$.

Note that this definition is not identical with that given in the introduction. The two definitions are, however, closely related. The limit $\Omega^x(A) = \lim_l \omega_l(\sigma_{z_l}(A))$ discussed in the introduction defines a state on the quasi-local algebra $\mathscr{B} \equiv \mathscr{A}^{\infty}$ for every $x \in X$ and $l^{-1} z_l \to x$. In contrast, the correlation functions defined above are defined for tuples $x \in X^k$, and are states on $\mathscr{A}^k$. Suppose that in the setting of the introduction A is strictly local, say $A \in \mathscr{D}(\Lambda_0) \equiv \mathscr{A}^k$, where $\Lambda_0 \subseteq \mathbb{Z}^d$ comprises k sites. Then if $l^{-1} z_l \to x$, we also have $l^{-1}(z_l + \zeta_i) \to x$ for every $\zeta_i \in \Lambda_0$. Since the $\xi_{l,j}$ label the points of $X \cap (l^{-1} \mathbb{Z}^d)$, there is a (k, l)-embedding η_l with $\xi_{l,j} = l^{-1}(z_l + \zeta_i)$ for $j = \eta(i)$. Obviously, $\xi_l \eta_l \to (x, \ldots, x) \in X^k$. Hence, if the mean field limit of $(\omega_l)_{l \in I}$ exists Ω^x is well defined and by equation (17)

$$\left.\begin{aligned} \Omega^x(A) = \Omega_{x, \ldots, x}(A) &= \int M(d\Phi)\, \varphi_x^k(A) \\ \text{or} \qquad & \\ \Omega^x &= \int M(d\Phi)\, \varphi_x^{\infty} \end{aligned}\right\} \tag{18}$$

for all $x \in X$. Thus the macrostate $\Omega^x \in K(\mathscr{A}^{\infty})$ in the sense of the introduction is automatically permutation invariant, and its Størmer measure can be obtained immediately from the Størmer measure of the macroscopic state in the sense of the above definition.

Since we want to investigate the behaviour of nets $(\omega_l)_{l \in I}$ under time evolutions, it is crucial to establish that the properties investigated are preserved under the transformation $\omega_l \mapsto \omega_l \alpha_l^t$. Definition 3.5 happily enjoys that property, but we were not able to show the same of 3.3. Therefore it is of interest to find further conditions, which are more easily seen to be time invariant, and which together with the existence of a weak mean field limit imply the existence of a mean field limit in the sense of 3.3. The following condition has this property, but we believe that it is not optimal, *i.e.* that much weaker conditions with this property can be found. We shall show in section 4, however, that the net of equilibrium states for any Hamiltonian satisfying Assumption 1 satisfies this condition. Since equilibrium states for a perturbed Hamiltonian are often natural candidates for initial states in a model, the condition does include many cases of interest.

The intuition leading to this condition is drawn from analogies with the convergence of nets of functions. In weak mean field limits the function Ω appears only in integrands together with continuous "test functions", which makes this analogous to weak convergence of functions. In view of Proposition 3.4 the mean field limits in the sense of Definition 3.3 correspond to uniform convergence of functions. In this analogy the following condition is a condition of "equicontinuity". It demands that the state is not too sensitive to permutations of the sites $\{1, \ldots, N_l\}$, which move the N_l points $\xi_{l,i} \in X$ only by a small total distance.

DEFINITION 3.9. – *A net* $(\omega_l)_{l \in I}$ *is called* **uniformly locally symmetric**, *if there is a continuous pseudo-metric d on* X *such that for sufficiently large l, and all permutations* π *of* $\{1, \ldots, N_l\}$:

$$\|\omega_l \hat{\pi} - \omega_l\| \leqq \sum_{i=1}^{N_l} d(\xi_{l,i}, \xi_{l,\pi(i)}). \tag{19}$$

This notion does not require the direct comparison of states ω_l for different l, nor does it require the existence of a weak mean field limit of the given net, as the example of the net of equilibrium states shows (*see* section 4). It also does not refer to the special observables in $\mathcal{A}^{N_l}$ of the form $\hat{\eta}(A)$ for a (k, l)-embedding η. This will be important in the proof of time invariance of this property, since the time evolutes $\alpha_l^t \hat{\eta}(A)$ will not also be of the form $\hat{\eta}(A_t)$.

For proving the announced consequence of uniform local symmetry we need the following result comparing the distance between k-tuples $\xi_l \eta$ and $\xi_l \eta'$ with the minimal "work" done by a permutation transforming η to η'.

LEMMA 3.10. – *Let d be a pseudo-metric on* $\{1, \ldots, N\}$. *For any permutation* π *of* $\{1, \ldots, N\}$ *set* $D(\pi) = \sum_{j=1}^{N} d(j, \pi(j))$. *Then for any injective maps* $\eta, \eta' : \{1, \ldots, k\} \to \{1, \ldots, N\}$ *we have*

$$\sum_{i=1}^{k} d(\eta(i), \eta'(i)) \leqq \inf\{D(\pi) \mid \eta' = \pi \circ \eta\} \leqq 2 \sum_{i=1}^{k} d(\eta(i), \eta'(i)).$$

Proof. – For any π with $\eta' = \pi \circ \eta$ the sum on the left hand side contains only those terms in the sum defining $D(\pi)$ with $j = \eta(i)$ for some i. This proves the first inequality. To see the second, note that the condition

$\eta'=\pi\circ\eta$ determines the images of all $\eta(i)$, and hence forces all possible chains of the form

$$\eta(i_1),\ \eta'(i_1)=\eta(i_2),\qquad \eta'(i_2)=\eta(i_3),\ \ldots,\ \eta'(i_{L-1})=\eta(i_L),\ \eta'(i_L)$$

to appear in the cycle decomposition of π. Maximal chains of this kind are disjoint, because η and η' are injective. Consider a permutation π, which leaves all points invariant, which do not appear in any chain (*i.e.* the union of the ranges of η, η'). We are free to define $\pi(\eta'(i_L))=\eta(i_1)$ for any maximal chain. Then the sum for $D(\pi)$ contains all terms $d(\eta(i),\eta'(i))$ plus the additional terms $d(\eta'(i_L),\eta(i_1))$ from each chain. But by the triangle inequality these terms are bounded by the sum of the lengths of the links of the chain. This gives an upper bound for $D(\pi)$ in which each $d(\eta(i),\eta'(i))$ is counted exactly twice. ■

PROPOSITION 3.11. – *Suppose* $(\omega_l)_{l\in I}$ *is uniformly locally symmetric, and has a weak mean field limit. Then* $\omega_l\hat{\eta}_l(A)$ *is convergent, whenever* $\xi_l\eta_l$ *converges to a point in* $(X')^k$.

Proof. – Let d be the pseudo-metric in Definition 3.9. Applying the preceding Lemma to the metric $\tilde{d}(i,j)=d(\xi_{l,i},\xi_{l,j})$, we get with the permutation π minimizing $\tilde{D}(\pi)$:

$$\begin{aligned}|\omega_l\hat{\eta}(A)-\omega_l\hat{\eta}'(A)| = |(\omega_l-\omega_l\hat{\pi})(\hat{\eta}(A))| &\leqq \|\omega_l-\omega_l\hat{\pi}\|\,\|A\|\\ &\leqq\tilde{D}(\pi)\|A\|\leqq 2\,d_k(\xi_l\eta,\xi_l\eta')\,\|A\|,\end{aligned}\qquad(20)$$

where $d_k(x,y)=\sum_{i=1}^{k} d(x_i,y_i)$.

Now let $A\in\mathscr{A}^k$, and let $(\eta_l)_{l\in I}$ be an arbitrary net of (k,l)-embeddings such that $\xi_l\eta_l\to x\in X^k$. We have to construct for every ε an l_ε such that $|\omega_l\hat{\eta}_l(A)-\omega_{l'}\hat{\eta}_{l'}(A)|\leqq\varepsilon$ for $l,\ l'>l_\varepsilon$. Pick a function $X\in\mathscr{C}(X^k)$ with support in an ε-neighbourhood of $x\in(X')^k$ such that $\int\mu^k(dy)\chi(y)=1$. Let $F\in\mathscr{C}(X,\mathscr{A})^k$ be given by $F(y)=\chi(y).A\in\mathscr{A}^k$.

Then by assumption $\omega_l R_l(F)$ is convergent. On the other hand, from the proof of 3.7 we get for sufficently large l:

$$\begin{aligned}\omega_l R_l(F)&=\operatorname*{Mean}_{\eta}\{\xi_l\eta)\,\omega_l\hat{\eta}(A)\}\\ &=\omega_l\hat{\eta}_l(A)\operatorname*{Mean}_{\eta}\{\chi(\xi_l\eta)\}+2u\varepsilon\|A\|\\ &=\omega_l\hat{\eta}_l(A)\left(u\varepsilon+\int\mu^k(dy)\chi(y)\right)+2u\varepsilon\|A\|\\ &=\omega_l\hat{\eta}_l(A)+3u\varepsilon\|A\|,\end{aligned}$$

where in each expression u stands for some other number with $|u|\leqq 1$. At the second inequality we use that for large $l\,d_k(\xi_l\eta_l,x)\leqq\varepsilon$, and we can use inequality (20) for η_l and all η in the support of χ. At the second equality we use that $x \mapsto \chi(x) . \mathbf{1}$ is also in $\mathscr{C}(X, \mathscr{A})^k$, so that $\underset{\eta}{\text{Mean}} \{\chi(\xi_l \eta_l)\}$ approximates the integral 1 for large l. Since $\omega_l R_l(F)$ converges, $|\omega_l \hat{\eta}_l(A) - \omega_{l'} \hat{\eta}_{l'}(A)| \leqq A \varepsilon \|A\|$ for sufficiently large l. ■

An important special case of nets with either kind of mean field limit are those, for which Ω_x is almost everywhere a product state, *i.e.* the correlation functions of the macroscopic state factorize. These states have a special meaning in the general scheme outlined in the introduction. For thermodynamic problems the quantum states in the microscopic description of the system can rarely be taken to be pure states. In keeping with this, the notions of mean field limit introduced in this section are all consistent with statistical mixtures (*i.e.* convex combinations $\omega_l = \lambda \omega_l' + (1-\lambda) \omega_l''$ of convergent nets ω_l', ω_l'' are again convergent). The set of "macroscopic limit states" is hence also closed under statistical mixtures. On the other hand, the "state" of a system is considered as a non-statistical entity in most macroscopic theories, and mixed states are only introduced artificially as probability measures over such deterministic states. A notion of thermodynamic limit for a net of microscopic states will naturally yield mixed as well as pure states of the macroscopic theory. Indeed, one expects that the limit states form a simplex, whose extreme points could be called "macroscopically pure" or "dispersion free". For the mean field limits studied in this paper, the simplex of macroscopic states consists of the Størmer measures $M_{\omega R}$ arising in the weak mean field limit. The relevant definition of purity is recorded in the following definition for later reference.

DEFINITION 3.12. — *A macroscopic state is said to be* **macroscopically pure**, *if its Størmer measure is concentrated on one point*

$$\Phi = \int^{\oplus} \mu(dx)\, \varphi_x \in K^{\mu}(\mathscr{C}(X, \mathscr{A})),$$

or, equivalenty, if its correlation functions are of the form

$$\Omega_{x_1, \ldots, x_k} = \varphi_{x_1} \otimes \ldots \otimes \varphi_{x_k},$$

for all $k \in \mathbb{N}$ *and for* μ^k*-almost all* $(x_1, \ldots, x_k) \in X^k$, *with a measurable function* $x \in X \mapsto \varphi_x \in K(\mathscr{A})$.

If a state on a quantum lattice system is ergodic, *i.e.* extremal in the set of translation invariant states, then the net derived from it by symmetrizing its local restrictions (*see* the example at the beginning of this section) is macroscopically pure [21].

It is easy to construct examples of nets converging to macroscopically pure states: let $x \in X \mapsto \varphi_x \in K(\mathscr{A})$ be any function, and define

$$\omega_l = \varphi_{\xi_{l,1}} \otimes \ldots \otimes \varphi_{\xi_{l, N_l}}. \tag{21}$$

Then with

$$\Omega_{x_1, \ldots x_k} = \varphi_{x_1} \otimes \ldots \otimes \varphi_{x_k}$$

we get $\omega_l \hat{\eta}_l = \Omega_{\xi_l \eta}$. If $x \mapsto \varphi_x$ is continuous in the weak* topology, then since the tensor product of states on C*-algebras is jointly weak* continuous, $x \in X^k \mapsto \Omega_x$ is also weak* continuous. Hence the convergence of $\xi_l \eta_l \to x$ implies the weak* convergence of $\omega_l \hat{\eta}_l$ to Ω_x, and the net $(\omega_l)_{l \in I}$ defined by equation (21) has a mean field limit.

If $x \mapsto \varphi_x$ is even norm continuous, $d(x, y) = \| \varphi_x - \varphi_y \|$ defines a continuous pseudo-metric on X, and it is easy to verify that the inequality in Definition 3.9 is satisfied for this d and the net in equation (21). Hence $(\omega_l)_{l \in I}$ is uniformly locally symmetric. It is easy to see that conversely, if ω_l is uniformly locally symmetric, and of the form (21), then $x \mapsto \varphi_x$ must be norm continuous.

For the net $(\omega_l)_{l \in I}$ to have a weak mean field limit, it is clearly necessary that Ω_x, and hence the function φ be measurable. However, this condition is not sufficient, and the existence of the weak mean field limit depends on finer properties of the net $(\xi_l)_{l \in I}$ than those postulated in Assumption 2.

4. EQUILIBRIUM STATES

In this section we shall discuss the mean field limits of a distinguished net of states, namely the thermal equilibrium states of the system. The equilibrium statistical mechanics of inhomogeneous mean field systems was studied in [18]. We review here only those results, which are of relevance in the present context, and establish some tools necessary for the study of the dynamics. The main result in this section not contained in [18] is that under Assumption 1 the net of equilibrium states is always uniformly locally symmetric.

Since the Hamiltonians H_l are all assumed to be bounded, KMS states for α_l^t exist only if $\mathscr{A}$ has a faithful finite trace "tr". For example, if $\mathscr{A}$ is a finite dimensional matrix algebra $\mathscr{A}$ carries a natural trace, giving weight one to each minimal projection. (The theory in [18] also allows an unbounded one-particle term in the Hamiltonian, so the existence of a trace is not essential.) Thus we can define the equilibrium state $\rho_l^\beta \in K(\mathscr{A}^{N_l})$ of the l-th system by its density matrix:

$$\rho_l^\beta(A) = \frac{\operatorname{tr}(A\, e^{-\beta N_l H_l(\xi)})}{\operatorname{tr}(e^{-\beta N_l H_l(\xi)})}. \tag{22}$$

Does this net of states have a mean field limit? It is easy to construct examples, in which even the weak mean field does not exist. However, by

Proposition 3.6 the weak mean field limit must always exist along suitable subnets. Therefore it is of interest to find the possible limit points of such subnets. A useful characterization of the possible limits is given by the following Proposition, the Gibbs variational principle [18]. In order to state it we need the notion of entropy of states on $\mathscr{C}(X, \mathscr{A})$. For a state on $\mathscr{A}$ of the form $\rho(A)=\operatorname{tr}(D_\rho A)$ (*i.e.* a state with density matrix D_ρ) the entropy is defined as usual by $s(\rho)=-\operatorname{tr}(D_\rho \ln(D_\rho))$. For $\Phi \in K^\mu(\mathscr{C}(X, \mathscr{A}))$, as defined before Proposition 3.6, we set

$$S(\Phi)=S\left(\int^{\oplus} \mu(dx)\,\varphi_x\right)=\int \mu(dx)\,s(\varphi_x). \tag{23}$$

For the statement of the following result recall also the definition of $j\mathrm{H}$ in equation (10), which we use here for $\tilde{\mathscr{Y}}(\mathscr{C}(X, \mathscr{A}))$ rather $\tilde{\mathscr{Y}}(\mathscr{A})$.

PROPOSITION 4.1. – *Let* $(H_l)_{l \in I}$ *and* $(\xi_l)_{l \in I}$ *satisfy Assumptions* 1, 2 *and* 3 *of section* 2 *and define* $(\rho_l^\beta)_{l \in I}$ *by equation* (22). *Suppose that* M^β *is the Størmer measure of the weak mean field limit of* $(\rho_l^\beta)_{l \in I}$ *along a suitable subnet. Then* M^β *is supported by the non-empty compact subset of* $K^\mu(\mathscr{C}(X, \mathscr{A}))$ *on which the functional*

$$f^\beta(\Phi)=(j\mathrm{H})(\Phi)+\frac{1}{\beta}S(\Phi) \tag{24}$$

attains its absolute minimum. The value of the minimum is equal to

$$\lim_l \frac{-1}{\beta N_l} \ln \operatorname{tr}\left(e^{-\beta N_l H_l(\xi_l)}\right).$$

The functional f^β is the limiting free energy per particle. Clearly, when it has a unique minimum all subnets of $(\rho_l^\beta)_{l \in I}$ must converge to the same limit, and hence this net itself has a limit, whose Størmer measure is the point measure on the minimizer. Non-uniqueness of the minimizer is related to the presence of phase transitions, and it is easy to see that in such circumstances the weak mean field limit need not exist.

It is rather surprising that the condition of uniform local symmetry holds for the net $(\rho_l^\beta)_{l \in I}$ even if the limit does not exist. This tells us that the mean field limit in the sense of Definition 3.3 always exists along suitable subnets, and that in the absence of phase transitions this limit exists without qualifications. For establishing this result we need a preparatory Lemma, which will also play an important role in proving the time invariance of uniform local symmetry.

LEMMA 4.2. – *Let* H *satisfy Assumption* 1. *Then there exists a continuous pseudo-metric* $d: X \times X \to \mathbb{R}$ *such that for all* $l \in I$ *and* $x, y \in X^{N_l}$:

$$\| H_l(x)-H_l(y) \| \leqq \frac{1}{N_l} \sum_{i=1}^{N_l} d(x_i, y_i)$$

Proof. — For each $\gamma \in \Gamma$ we define a pseudo-metric d_γ on X by

$$d_\gamma(x, y) = \sup_l \sup_{z_2, \ldots z_{N_l} \in X} n(\gamma) \| H_l^\gamma(x, z_2, \ldots z_{N_l}) - H_l^\gamma(y, z_2, \ldots z_{N_l}) \|.$$

It is clear that d_γ satisfies the triangle inequality. We have to prove that it is a continuous function of both arguments. Let $\varepsilon > 0$, and $\bar{x}, \bar{y} \in X$. By part (*c*) of Assumption 1 we can find finitely many $l_1, \ldots l_r \in I$ such that for every l there is one α such that $\| H_l^\gamma - H_{l_\alpha}^\gamma \| \leqq \varepsilon/4$. We need to find a neighbourhood of $\bar{x}, \bar{y}$, on which the supremum only over $(z_2, \ldots, z_{N_l})$ is less than $\varepsilon/2$, for each of the r indices l_α. Note that due to the continuity of H_l^γ the norm in the definition of d_γ is continuous in all its arguments. The existence of the desired neighbourhood, and hence the continuity of d_γ therefore follows from the observation that for any continuous function $f: X \times Y \to \mathbb{R}$, defined on the product of two compact sets, the function $x \mapsto \sup_y f(x, y)$ is continuous.

Then for $x, y \in X^{n(\gamma)}$, and arbitrary $l \in I$ we get

$$\| H_l^\gamma(x) - H_l^\gamma(y) \| \leqq \frac{1}{n(\gamma)} \sum_{i=1}^{n(\gamma)} d_\gamma(x_i, y_i)$$

by splitting the difference on the left hand side into $n(\gamma)$ terms, in each of which only one coordinate changes, and using the symmetry of H_l^γ. By equation (2) we get for all $x, y \in X^{N_l}$

$$\begin{aligned} &\| \mathrm{sym}_l(H_l^\gamma(x)) - \mathrm{sym}_l(H_l^\gamma(y)) \| \\ &\qquad \leqq \frac{(N_l - n(\gamma))!}{N_l!} \sum_\eta \| H_l^\gamma(x\eta) - H_l^\gamma(y\eta) \| \\ &\qquad \leqq \frac{(N_l - n(\gamma))!}{n(\gamma) N_l!} \sum_\eta \sum_{r=1}^{n(\gamma)} d_\gamma((x\eta)_r, (y\eta)_r) = \frac{1}{N_l} \sum_{i=1}^{N_l} d_\gamma(x_i, y_i), \end{aligned}$$

where the sum in η is over all (k, l)-embeddings. The normalization factor $1/N_l$ is determined by observing that each term $d_\gamma(x_i, y_i)$ appears in the double sum with the same frequency, and that in all three sums the number of terms is equal to the inverse normalization factor.

We now define $d(x, y) = \sum_\gamma d_\gamma(x, y)$. This sum exists, since

$$d_\gamma(x, y) \leqq 2 n(\gamma) \sup_l \| H_l^\gamma \|,$$

and due to (*a*) of Assumption 1. Since this sum converges absolutely, it is also clear that d is continuous. Moreover,

$$\begin{aligned} \| H_l(x) - H_l(y) \| &\leqq \sum_\gamma \| \mathrm{sym}_l(H_l^\gamma(x)) - \mathrm{sym}_l(H_l^\gamma(y)) \| \\ &\leqq \frac{1}{N_l} \sum_\gamma \sum_{i=1}^{N_l} d_\gamma(x_i, y_i) \leqq \frac{1}{N_l} \sum_{i=1}^{N_l} d(x_i, y_i). \quad \blacksquare \end{aligned}$$

PROPOSITION 4.3. – *Let $\beta>0$, and suppose that $(H_l)_{l\in I}$ satisfies Assumption* 1. *Then the net $(\rho_l^\beta)_{l\in I}$ of equilibrium states as defined by equation* (22) *is uniformly locally symmetric.*

Proof. – We have to make use of the perturbation theory of Gibbs states developed by Araki [2]. A good expository account is to be found in [4]. Let $\mathscr{A}$ be any C*-algebra with a strongly continuous one-parameter automorphism group $t\mapsto\alpha^t\in\mathrm{Aut}(\mathscr{A})$. Let $H=H^*\in\mathscr{A}$. Then a perturbed automorphism group $\tilde{\alpha}^t$ is defined by the integral equation

$$\tilde{\alpha}^t(A)=\alpha^t(A)+\int_0^t ds\,\tilde{\alpha}^{t-s}(i[H,\alpha^s(A)]).$$

If α^t happens to be generated by a Hamiltonian H_0, the Hamiltonian generating $\tilde{\alpha}^t$ will be H_0+H. This is the reason for calling H the "relative Hamiltonian" of the perturbation. Now Araki constructs for any a state ρ, which is β-KMS for α^t, a perturbed β-KMS state for $\tilde{\alpha}^t$, denoted ρ^H. (Actually, Araki uses this notation for a certain non-normalized linear functional on $\mathscr{A}$, but we follow [4] here and take ρ^H to be normalized as a state.) In the case of a matrix algebra and ρ a state with density matrix $D_\rho=\exp(H_0)$ one obtains the state with density matrix proportional to $\exp(H_0+H)$. Putting together the estimates in Theorem 5.4.4 of [4], we find that $\|\rho^H-\rho\|\leqq 2\|H\|(1-2)\|H\|)^{-1}$, whenever $2\|H\|<1$. Using the fact that $(\rho^H)^K=\rho^{H+K}$ we thus obtain for sufficiently small $\|H-K\|$ the bound $\|\rho^H-\rho^K\|\leqq(2+\varepsilon)\|H-K\|$. This can be iterated with the triangle inequality to remove the restriction on $\|H-K\|$, and letting $\varepsilon\to 0$ we find

$$\|\rho^H-\rho^K\|\leqq 2\|H-K\|. \tag{25}$$

The equilibrium states ρ_l^β are simply the perturbations of the normalized trace on $\mathscr{A}^{N_l}$ by the relative Hamiltonian $-\beta N_l H_l(\xi_l)$. Therefore, by the estimate (25) we have for any permutation of $\{1,\ldots,N_l\}$:

$$\begin{aligned}\|\rho_l^\beta\hat{\pi}-\rho_l^\beta\| &\leqq 2\|\hat{\pi}^{-1}(\beta N_l H_l(\xi_l))-\beta N_l H_l(\xi_l)\|\\ &\leqq 2\beta N_l\|H_l(\xi_l\pi)-H_l(\xi)\|\\ &\leqq 2\beta\sum_{i=1}^{N_l} d(\xi_{l,i},\xi_{l,\pi(i)}),\end{aligned}$$

where at the last step we have used Lemma 4.2. Hence Definition 3.9 is satisfied with a metric proportional to that appearing in the Lemma. ■

The Euler-Lagrange equations of the variational principle of Proposition 4.1 turn out to be of interest for the study of the limiting dynamics. In the general setting of [18] they do not always make sense, since f^β need not be differentiable. However, since we assumed $\dim\mathscr{A}<\infty$, the entropy term in f^β is differentiable, and as a consequence of Assumption 1 the energy density jH is twice differentiable. The following Lemma gives the form of the derivative of jH, or "effective Hamiltonian".

LEMMA 4.4. — *Let* H *satisfy Assumption* 1, *and let* $\Phi \in K(\mathscr{C}(X, \mathscr{A}))$. *Then there is a unique element* $H_\Phi \in \mathscr{C}(X, \mathscr{A})$, *called the* **effective Hamiltonian** *of* H *in the state* Φ, *such that for all* $\Psi \in K(\mathscr{C}(X, \mathscr{A}))$

$$\Psi(H_\Phi) = \frac{d}{dt}(jH)((1-t)\Phi + t\Psi)\big|_{t=0+} = \sum_\gamma n(\gamma)(\Psi - \Phi)\otimes\Phi^{n(\gamma)-1}(H^\gamma). \quad (26)$$

Proof. — We first compute jH in terms of the H^γ of Assumption 1:

$$\begin{aligned}(jH)(\Phi) &= \lim_l \Phi^{N_l}(H_l) \\ &= \lim_l \sum_\gamma \Phi^{n(\gamma)}(H_l^\gamma) = \sum_\gamma \lim_l \Phi^{n(\gamma)}(H_l^\gamma) \\ &= \sum_\gamma \Phi^{n(\gamma)}(H^\gamma). \end{aligned} \quad (27)$$

The interchange of limit and sum is justified, because by part (*b*) of Assumption 1 the sum converges absolutely and uniformly in l. Due to the permutation symmetry of each H^γ the derivative of one term in this sum is

$$\frac{d}{dt}((1-t)\Phi + t\Psi)^{n(\gamma)}(H^\gamma)\big|_{t=0} = n(\gamma)(\Psi - \Phi)\otimes\Phi^{n(\gamma)-1}(H^\gamma).$$

This expression is bounded by $c_\gamma \equiv 2n(\gamma)\sup_l \|H_l^\gamma\|$, and hence the sum of the derivatives has a convergent majorant by part (*b*) of Assumption 1. This shows that the second and the third expression in the Lemma are equal.

It remains to show an element H_Φ with the stated properties exists. Again we consider at first only one term in the sum. By [24], Cor. IV.4.25 there is a completely positive map $\theta: \mathscr{C}(X, \mathscr{A})^{n(\gamma)} \to \mathscr{C}(X, \mathscr{A})$, called the conditional expectation with respect to $\Phi^{n(\gamma)-1}$, such that $\theta(F\otimes G) = F\,\Phi^{n(\gamma)-1}(G)$ for $F \in \mathscr{C}(X, \mathscr{A})$, and $G \in \mathscr{C}(X, \mathscr{A})^{n(\gamma)-1}$. Clearly, we have $\Psi\otimes\Phi^{n(\gamma)-1}(F\otimes G) = \Psi(F)\,\Phi^{n(\gamma)-1}(G) = \Psi(\theta(F\otimes G))$, and hence $\Psi\otimes\Phi^{n(\gamma)-1} = \Psi\circ\theta$. We now set

$$H_\Phi^\gamma = n(\gamma)(\theta(H^\gamma) - \mathbf{1}_{\mathscr{A}}\Phi^{n(\gamma)}(H^\gamma)) \in \mathscr{C}(X, \mathscr{A}).$$

Hence

$$\Psi(H_\Phi^\gamma) = n(\gamma)(\Psi - \Phi)\otimes\Phi^{n(\gamma)-1}(H^\gamma).$$

Since $\|H_\Phi^\gamma\| \leqq 2n(\gamma)\|H^\gamma\| \leqq c_\gamma$, the sum $H_\Phi = \sum_\gamma H_\Phi^\gamma$ converges absolutely in $\mathscr{C}(X, \mathscr{A})$, and has the desired properties. The uniqueness of H_Φ is trivial since equation (26) holds for all states Ψ. ■

Since the directionnal derivative in the Lemma is taken only along the state space, the value of the derivative would appear to be defined up to a multiple of the identity. The definition in the Lemma fixes this constant term so that $\Phi(\mathrm{H}_\Phi)=0$. With a different convention $\tilde{\mathrm{H}}_\Phi=\mathrm{H}_\Phi+h(\Phi)\mathbf{1}$ the equation $\frac{d}{dt}(j\mathrm{H})((1-t)\Phi+t\Psi)=(\Psi-\Phi)(\tilde{\mathrm{H}}_\Phi)$ still holds. The applications of effective Hamiltonians to either thermostatics or dynamics are not affected by this choice.

Since $\mathrm{H}_\Phi\in\mathscr{C}(\mathrm{X},\mathscr{A})$, it is by definition a continuous $\mathscr{A}$-valued function on X. The values $\mathrm{H}_\Phi(x)\in\mathscr{A}$ of this function can be characterized in a way analogous to equation (26). Inserting the decomposition $\Phi=\int^\oplus \mu(dx)\varphi_x$ into that equation we obtain for all $\psi\in\mathrm{K}(\mathscr{A})$, $\Phi\in\mathrm{K}^\mu(\mathscr{C}(\mathrm{X},\mathscr{A}))$, and $x\in\mathrm{X}$:

$$\psi(\mathrm{H}_\Phi(x))=\sum_\gamma n(\gamma)\int\mu(dx_2)\ldots\mu(dx_{n(\gamma)})\\ \psi\otimes\varphi_{x_2}\otimes\ldots\varphi_{x_{n(\gamma)}}(\mathrm{H}^\gamma(x,x_2,\ldots,x_{n(\gamma)}))-\psi(\mathbf{1})\sum_\gamma\Phi^{n(\gamma)}(\mathrm{H}^\gamma). \quad (28)$$

It is easy to obtain from this a Lipshitz bound on H_Φ, which is uniform in Φ. Using the pseudo-metrics d_γ introduced in the proof of Lemma 4.2 and summing over γ, we obtain

$$\|\mathrm{H}_\Phi(x)-\mathrm{H}_\Phi(y)\|\leqq d(x,y), \quad (29)$$

where d is the pseudo-metric constructed in Lemma 4.2.

We can now state the Euler-Lagrange equations for the Gibbs variational principle:

$$0=\delta f^\beta(\Phi)=\int\mu(dx)\,\delta\varphi_x(\mathrm{H}_\Phi(x)+\beta^{-1}(\ln(\mathrm{D}_x)-\mathbf{1})),$$

where D_x denotes the density matrix of $\varphi_x\in\mathrm{K}(\mathscr{A})$. Since we are varying only over states, we have $\delta\varphi_x(\mathbf{1})=0$, and the argument of $\delta\varphi_x$ must be a multiple of $\mathbf{1}$. Hence the Euler-Lagrange equations are equivalent to the so-called gap-equation [11]:

$$\varphi_x(\mathrm{A})=\frac{\mathrm{tr}(\mathrm{A}\,e^{-\beta\mathrm{H}_\Phi(x)})}{\mathrm{tr}(e^{-\beta\mathrm{H}_\Phi(x)})}, \quad (30)$$

i.e. φ_x is the β-equilibrium state for the "local Hamiltonian" $\mathrm{H}_\Phi(x)$.

Note that the Lipshitz bound (29) and the estimate (25) imply that for any solution of the gap-equation we have $\|\varphi_x-\varphi_y\|\leqq 2\beta\,d(x,y)$. This not only applies to the absolute minima of f^β, *i.e.* the possible equilibrium states, but also to maxima and saddles, or local minima, which would correspond to metastable states. In all such states discontinuities are thus ruled our, *i.e.* there cannot be sharp phase separation in the models studied here.

5. MEAN FIELD LIMIT OF THE DYNAMICS

We turn our attention to the dynamics of the models in the thermodynamic limit. Our purpose is twofold. First, we establish conditions on the net $(H_l)_{l\in I}$ under which the existence of a weak mean field limit (Definition 3.5) or uniform local symmetry (Definition 3.9) for a net of states $(\omega_l)_{l\in I}$ are preserved by the automorphisms $(\alpha_l^t)_{l\in I}$. In both cases it will turn out to be sufficient that Assumption 1 is satisfied. Secondly, we obtain the exact form for the limiting dynamics.

The basic strategy of this section is to reduce all statements about inhomogeneous mean field dynamics to the corresponding statements about the homogeneous case, by passing from the "$\mathscr{A}$-level" to the "$\mathscr{C}$-level" of function algebras. The connection is made in the following Lemma. Define the net $(\tilde{\alpha}_l^t)_{l\in I}$ with $\tilde{\alpha}_l^t \in \mathrm{Aut}(\mathscr{C}(X, \mathscr{A})^{N_l})$ and

$$\tilde{\alpha}_l^t(A) = \exp(itN_l H_l)\, A \exp(-itN_l H_l). \tag{31}$$

Note that $\tilde{\alpha}_l^t$ does not depend on the choice of ξ_l. In fact, the following Lemma shows that it contains the full information about the observable algebra evolutions $(\alpha_l^t)_{l\in I}$ defined by equation (1) (which are the primary objects of interest), for all such choices of ξ_l. Note that the subalgebra $\mathscr{C}(X^{N_l})\,\mathbf{1}$ of $\mathscr{C}(X, \mathscr{A})^{N_l}$ belongs to the center, so it is left pointwise invariant under $\tilde{\alpha}^t$. This property, which amounts to the statement that the location parameters ξ_l are themselves not subject to a time evolution, is crucial for this Lemma.

LEMMA 5.1:

$$R_l\,\tilde{\alpha}_l^t = \alpha_l^t\, R_l. \tag{32}$$

Proof. – Since H_l, and hence $\tilde{\alpha}_l^t$, is permutation symmetric, $\mathrm{sym}_l\,\tilde{\alpha}_l^t = \tilde{\alpha}_l^t\,\mathrm{sym}_l$. The product in the C*-algebra $\mathscr{C}(X, \mathscr{A}) \cong \mathscr{C}(X^{N_l}, \mathscr{A}^{N_l})$ is defined as pointwise multiplication. Therefore, by expanding $\exp(itN_l H_l)$ into a power series, we find for every $F \in \mathscr{C}(X^{N_l}, \mathscr{A}^{N_l})$, and $x \in X^{N_l}$:

$$(\tilde{\alpha}_l^t F)(x) = \exp(itN_l H_l(x))\, F(x) \exp(-itN_l H_l(x)).$$

The result follows by putting $x = \xi_l$ in this equation, and applying the operator sym_l. ■

We begin the analysis of the $(\tilde{\alpha}_l^t)_{l\in I}$ with the following definition. Here and below $\mathscr{B}(\mathscr{A})$ denotes the space of bounded operators on the Banach space $\mathscr{A}$.

DEFINITION 5.2. — *Let* $\mathscr{A}$ *be a C*-algebra with unit* **1** *and let* $(N_l)_{l\in I}$ *be a net of natural numbers diverging to* ∞. *Then a net* $(\alpha_l)_{l\in I}: \alpha_l \in \mathscr{B}(\mathscr{A}^{N_l})$ *is called* **approximate symmetry preserving** *if for all approximately symmetric nets* $(A_l)_{l\in I}$ *with* $A_l\in\mathscr{A}^{N_l}$, *the net* $(\alpha_l A_l)_{l\in I}$ *is approximately symmetric.*

An exactly symmetric net $(A_l)_{l\in I}$ is viewed as the averages over N_l sites of some observable $\tilde{A}\in\mathscr{A}^k$ for some k, *i.e.* as a family of intensive observables. For a corresponding family of Hamiltonians of mean field type [*i.e.* of the form $(N_l H_l)_{l\in I}$ with $H_l=\mathrm{sym}_l\tilde{H}$ for some $\tilde{H}\in\mathscr{A}^{k'}$] one expects that the intensive character of the observable is preserved by the time evolution. This can be shown by essentially combinatorial arguments (*see* e.g. [13], [1]) using the series expansion of the time evolution. We appeal to a more general result in [8], where also some similar results are obtained under weaker assumptions.

PROPOSITION 5.3 [8]. — *Let* H *satisfy Assumption* 1. *Then for all* $t\in\mathbb{R}$ *the net of maps* $(\tilde{\alpha}_l^t)_{l\in I}$ *with* $\tilde{\alpha}_l^t\in\mathrm{Aut}(\mathscr{C}(X,\mathscr{A})^{N_l})$ *and* $\tilde{\alpha}_l^t(A)=\exp(itN_lH_l)A\exp(-itN_lH_l)$ *is approximate symmetry preserving.*

Approximate symmetry preservation is a sufficient condition for the propagation of weak mean field limits:

PROPOSITION 5.4. — *Let the net of states* $(\omega_l)_{l\in I}$ *on* $\mathscr{A}^{N_l}$ *have a weak mean field limit, and let* H *be such that for all* $t\in\mathbb{R}$, $(\tilde{\alpha}_l^t)_{l\in I}$ *is approximate symmetry preserving. Then for all* $t\in\mathbb{R}$ *the net of states* $(\omega_l\alpha_l^t)_{l\in I}$ *has a weak mean field limit.*

Proof. — By Lemma 5.1, for all $(F_l)_{l\in I}$ in $\tilde{\mathscr{Y}}(\mathscr{C}(X,\mathscr{A}))$

$$\lim_l \omega_l\alpha_l^t R_l(F_l)=\lim_l \omega_l R_l\tilde{\alpha}_l^t(F_l),$$

the right hand limit existing because $(\tilde{\alpha}_l^t)_{l\in I}$ is approximate symmetry preserving. Since the $\omega_l\tilde{\alpha}_l^t R_l$ are symmetric, then by Proposition 3.2, they have a homogeneous mean field limit and so $(\omega_l\alpha_l^t)_{l\in I}$ has a weak mean field limit. ■

We now turn to uniform local symmetry (*see* Definition 3.9). An essential tool for proving that this property is propagated under the time evolution is the uniform Lipshitz bound on the Hamiltonians shown in Lemma 4.2.

PROPOSITION 5.5. — *Let* H *satisfy Assumption* 1, *and let* d *be a pseudometric satisfying the conclusion of Lemma* 4.2. *Then for all* $l\in I$, $t\in\mathbb{R}$ *and all permutations* π *of* $\{1,\ldots,N\}$:

$$\|\alpha_l^t\hat{\pi}-\hat{\pi}\alpha_l^t\|\leqq 2|t|\sum_{i=1}^{N_l} d(\xi_{l,i},\xi_{l,\pi(i)}).$$

Moreover, for any uniformly local symmetric net $(\omega_l)_{l \in I}$ *the net* $(\omega_l \alpha_l^t)_{l \in I}$ *is also uniformly locally symmetric.*

Proof:

$$\| \alpha_l^t \hat{\pi} - \hat{\pi}\alpha_l^t \| = \left\| \int_0^t ds \frac{d}{ds} \alpha_l^s \hat{\pi} \alpha_l^{t-s} \right\|$$
$$\leqq |t| N_l \| [\mathrm{ad}\, H_l(\xi_l), \hat{\pi}] \|_{\mathscr{B}(\mathscr{A}^{N_l})}$$
$$\leqq 2 |t| N_l \| H_l(\xi_l) - \hat{\pi}(H_l(\xi_l)) \|_{\mathscr{A}^{N_l}} \leqq 2 |t| \sum_{i=1}^{N_l} d(\xi_{l,i}, \xi_{l,\pi(i)}).$$

To prove the second part of the proposition, suppose that $(\omega_l)_{l \in I}$ is uniformly locally symmetric and so satisfies equation (19) for some continuous pseudo-metric d_0 on X.

Then

$$\| \omega_l \alpha_l^t \hat{\pi} - \omega_l \alpha_l^t \| \leqq \| (\omega_l \hat{\pi} - \omega_l) \alpha_l^t \| + \| \omega_l \| \; \| \alpha_l^t \hat{\pi} - \hat{\pi} \alpha_l^t \| \leqq \sum_{i=1}^{N_l} d_t(\xi_{l,i}, \xi_{l,\pi(i)}),$$

where $d_t(x, y) = d_0(x, y) + 2 |t| d(x, y)$. ■

Finally, we give the form of the limiting dynamics. In a nutshell, the result is that the dynamics in the limit is generated by the same state dependent effective Hamiltonians which determine the equilibrium states via the gap equation.

THEOREM 5.6. – *Let* H *satisfy Assumption* 1, *and let* H_Φ *denote the effective Hamiltonian in the state* $\Phi \in K^\mu(\mathscr{C}(X, \mathscr{A}))$ *as defined in Lemma* 4.4. *Let* $(\omega_l)_{l \in I}$ *have a weak mean field limit,* i.e.

$$w - \lim_l \omega_l R_l = \int_{K^\mu(\mathscr{C}(X, \mathscr{A}))} M_{\omega R}(d\Phi) \Phi^\infty,$$

where $M_{\omega R}$ *is the Størmer measure of the macroscopic limit state. Then for all* $t \in \mathbb{R}$:

$$w - \lim_l \omega_l \alpha_l^t R_l = \int_{K^\mu(\mathscr{C}(X, \mathscr{A}))} M_{\omega R}(d\Phi) (\mathscr{F}_t \Phi)^\infty, \tag{33}$$

where $\mathscr{F}_t$ *is a one-parameter group of continuous transformations of* $K(\mathscr{C}(X, \mathscr{A}))$, *leaving* $K^\mu(\mathscr{C}(X, \mathscr{A}))$ *invariant. Moreover,* $t \mapsto \mathscr{F}_t \Phi$ *satisfies the differential equation*

$$\frac{d}{dt}(\mathscr{F}_t \Phi)(F) = (\mathscr{F}_t \Phi)(i [H_{\mathscr{F}_t \Phi}, F]) \tag{34}$$

for all $F \in \mathscr{C}(X, \mathscr{A})$.

Proof. – Since by Lemma 5.1 $R_l \tilde{\alpha}_l^t = \alpha_l^t R_l$, equation (33) concerns the same net $(\omega_l R_l)_{l \in I}$ which is assumed to have a homogeneous mean field limit. Therefore, the proof reduces essentially to a corresponding result about homogeneous mean field systems, proven in [8]. This states that for Hamiltonians $(\mathrm{H}_l)_{l \in \mathrm{I}}$ satisfying Assumption 1,

$$(j\tilde{\alpha}^t \mathrm{F})(\Phi) \equiv \lim_l \Phi^{\mathrm{N}_l}(\tilde{\alpha}_l^t \mathrm{F}_l) = \lim_l (\mathscr{F}_t \Phi)^{\mathrm{N}_l}(\mathrm{F}_l) \equiv (j\mathrm{F})(\mathscr{F}_t \Phi) \qquad (35)$$

for any $\mathrm{F} \in \tilde{\mathscr{Y}}(\mathscr{C}(\mathrm{X}, \mathscr{A}))$, the limit being uniform in $\Phi \in \mathrm{K}(\mathscr{C}(\mathrm{X}, \mathscr{A}))$, and with $(\mathscr{F}_t)_{t \in \mathbb{R}}$ a flow on $\mathrm{K}(\mathscr{C}(\mathrm{X}, \mathscr{A}))$. This flow is determined by the differential equation

$$\frac{d}{dt}(\mathscr{F}_t \Phi)(\mathrm{F}) = \sum_\gamma n(\gamma)(\mathscr{F}_t \Phi)^{n(\gamma)}(i[\mathrm{H}^\gamma, \mathrm{F} \otimes \mathbf{1}_{n(\gamma)-1}]).$$

Before converting this differential equation to equation (34) we give a brief explanation of equation (35). The existence of the left hand limit is just a restatement of Proposition 5.3: since $(\tilde{\alpha}_l^t)_{l \in \mathrm{I}}$ is approximate symmetry preserving, the limit must exist. That the evolution is implemented by the flow $\mathscr{F}_t$ is equivalent to saying that the "mean field limit" of $\tilde{\alpha}^t$ [8], *i.e.* the operators $\hat{\alpha}^t$ on $\mathscr{C}(\mathrm{K}(\mathscr{C}(\mathrm{X}, \mathscr{A})))$ with $\hat{\alpha}^t(j\mathrm{F}) = j(\tilde{\alpha}^t \mathrm{F})$ are homomorphisms. This follows immediately from the observation that each $\tilde{\alpha}^t$ is a homomorphism, and that $j : \tilde{\mathscr{Y}} \to \mathscr{C}(\mathrm{K}(\mathscr{C}(\mathrm{X}, \mathscr{A})))$ takes n-wise products into products of functions. Hence

$$\hat{\alpha}^t((j\mathrm{F})(j\mathrm{G})) = j\tilde{\alpha}^t(\mathrm{FG}) = j((\tilde{\alpha}^t \mathrm{F})(\tilde{\alpha}^t \mathrm{G})) = (j\tilde{\alpha}^t \mathrm{F})(j\tilde{\alpha}^t \mathrm{G}).$$

We return to the proof. Since the limit in equation (35) is uniform, we can integrate with respect to $\mathrm{M}_{\omega\mathrm{R}}(d\Phi)$. Since by hypothesis $(\omega_l \mathrm{R}_l)_{l \in \mathrm{I}}$ has a homogeneous mean field limit described by the measure $\mathrm{M}_{\omega\mathrm{R}}$, as in equation (9), we have for any $\mathrm{F} \in \tilde{\mathscr{Y}}(\mathscr{C}(\mathrm{X}, \mathscr{A}))$:

$$\begin{aligned}\lim_l \omega_l \alpha_l^t \mathrm{R}_l(\mathrm{F}_l) &= \lim_l \omega_l \mathrm{R}_l(\tilde{\alpha}_l^t \mathrm{F}_l) = \int \mathrm{M}_{\omega\mathrm{R}}(d\Phi)(j\tilde{\alpha}^t \mathrm{F})(\Phi) \\ &= \int \mathrm{M}_{\omega\mathrm{R}}(d\Phi)(j\mathrm{F})(\mathscr{F}_t \Phi) = \lim_l \int \mathrm{M}_{\omega\mathrm{R}}(d\Phi)(\mathscr{F}_t \Phi)^\infty(\mathrm{F}_l),\end{aligned}$$

which proves equation (33).

Finally, we express the above differential equation for $\mathscr{F}_t \Phi$ in terms of the effective Hamiltonian. Recall the definition

$$\mathrm{H}_\Phi^\gamma = n(\gamma)(\theta(\mathrm{H}^\gamma) - \mathbf{1}_{\mathscr{A}} \Phi^{n(\gamma)}(\mathrm{H}^\gamma))$$

of the contribution of a single γ to the effective Hamiltonian, where $\theta : \mathscr{C}(\mathrm{X}, \mathscr{A}) \otimes \mathscr{C}(\mathrm{X}, \mathscr{A})^{n(\gamma)-1} \to \mathscr{C}(\mathrm{X}, \mathscr{A})$ is the conditional expectation with respect to $\Phi^{n(\gamma)-1}$. A basic property [24] of such conditional expectations is that $\theta((\mathrm{F} \otimes \mathbf{1})\mathrm{H}(\mathrm{G} \otimes \mathbf{1})) = \mathrm{F}\theta(\mathrm{H})\mathrm{G}$ for $\mathrm{F}, \mathrm{G} \in \mathscr{C}(\mathrm{X}, \mathscr{A})$ and

$H \in \mathscr{C}(X, \mathscr{A})^{n(\gamma)}$. This allows us to compute

$$\begin{aligned}\Phi(i[H^\gamma_\Phi, F]) &= in(\gamma)\,\Phi(\theta(H^\gamma)F - F\,\theta(H^\gamma))\\ &= n(\gamma)\,\Phi(\theta(i[H^\gamma, F\otimes \mathbf{1}_{n(\gamma)-1}]))\\ &= n(\gamma)\,\Phi^{n(\gamma)}(i[H^\gamma, F\otimes \mathbf{1}_{n(\gamma)-1}]).\end{aligned}$$

Summing this over γ we obtain the equality of the two forms of the generator. ■

6. DYNAMICS OF MACROSCOPIC STATES

In section 3 we defined various notions of mean field limits for a net $(\omega_l)_{l\in I}$ of states on $\mathscr{A}^{N_l}$. The limit of such a net is a macroscopic state in the sense of Definition 3.8. It is described completely by the family of correlation functions $\Omega: (X')^k \to K(\mathscr{A}^k)$ or, equivalently, by the Størmer measure $M_{\omega R}$ on $K^\mu(\mathscr{C}(X, \mathscr{A}))$.

In the last section we saw that the existence of weak mean field limits (Definition 3.5), and under additional assumptions also the existence of mean field limits (Definition 3.3), is preserved under the time evolutions described in section 2. Therefore there must be a well-defined time evolution for the limit states. In this section we shall describe this time evolution of macroscopic states, and some of its basic properties.

In Theorem 5.6 the dynamical evolution in the limit is described on the $\mathscr{C}$-level, *i.e.* in terms of the algebra $\mathscr{C}(X, \mathscr{A})$. It is clear from that Theorem that the measure $M_{\omega R}$ will be transformed into $M_{\omega R}\circ \mathscr{F}_t^{-1}$, where $\mathscr{F}_t$ is the flow on $K(\mathscr{C}(X, \mathscr{A}))$ described there. This flow just describes the evolution of the point measures, which correspond to the macroscopically pure states (*see* Definition 3.12). Clearly, macroscopically pure states remain macroscopically pure under time evolution, and more general states are transformed by decomposing the initial state into a mixture of macroscopically pure states, letting these evolve independently under $\mathscr{F}_t$, and recomposing the mixture with the same weights. It therefore suffices to study the flow $\mathscr{F}_t$ on $K^\mu(\mathscr{C}(X, \mathscr{A}))$.

For all times $\mathscr{F}_t\Phi$ will have a decomposition

$$\mathscr{F}_t \Phi = \int^{\oplus} \mu(dx)\,\varphi_{x,t}. \tag{36}$$

The basic evolution equation for $\varphi_{x,t}$ follows from eauation (34) by substituting the above decomposition, and using the effective local Hamiltonian $H_\Phi(x)$. For all $A\in\mathscr{A}$, and μ-almost all x we have:

$$\frac{d}{dt}\varphi_{x,t}(A) = \varphi_{x,t}(i[H_{\mathscr{F}_t\Phi}(x), A]). \tag{37}$$

When Φ arises from a mean field limit, not just a weak mean field limit, we know that $\varphi_{x,t}$ is even a continuous function, so the above equation holds pointwise on $X' = \operatorname{supp} \mu$. It is important to note that through the dependence of $H_{\mathcal{F}_t \Phi}$ on Φ, and therefore on all $\varphi_{y,t}$ with $y \neq x$, the equations for different x are coupled.

The local Hamiltonian $H_\Phi(x)$ is determined from the Hamiltonians as given in Assumption 1 by equation (28). As the simplest, and physically most important case consider H to be strictly symmetric of degree 2, and therefore of the form given in equation (4). Thus the stronger Assumption 1′ is satisfied and in Assumption 1 we can take $\Gamma = \{\gamma\}$ with $n(\gamma) = 2$, and $H_l^\gamma \equiv \tilde{H} \in \mathcal{C}(X, \mathcal{A})^2$. In order to get the net in equation (4) we have to set

$$\tilde{H}(x_1, x_2) = \frac{1}{2}(\varepsilon(x_1) \otimes \mathbf{1} + \mathbf{1} \otimes \varepsilon(x_2)) + V(x_1, x_2). \tag{38}$$

Inserting equation (28) into equation (37) we get

$$\begin{aligned}\frac{d}{dt}\varphi_{x,t}(A) &= 2\int \mu(dy)\, \varphi_{x,t} \otimes \varphi_{y,t}(i[\tilde{H}(x, y), A \otimes \mathbf{1}]) \\ &= \varphi_{x,t}(i[\varepsilon(x), A]) + 2\int \mu(dy)\, \varphi_{x,t} \otimes \varphi_{y,t}(i[V(x, y), A \otimes \mathbf{1}]).\end{aligned} \tag{39}$$

For the rest of this section we shall study the basic properties of the solutions of this equation [resp. equation (37)]. Since the flow $\mathcal{F}_t$ derives from a net of Hamiltonians, it is natural to ask about the conservation of energy. In the models we consider the natural energy function on $K(\mathcal{C}(X, \mathcal{A}))$ is $\Phi \mapsto (j H)(\Phi)$, which we already encountered in section 4 as the internal energy per particle in the thermodynamic limit. Using the statement in Lemma 4.4 that the derivative of jH in the direction of Ψ at the point Φ is given by $\Psi(H_\Phi)$, and equation (34) we compute

$$\frac{d}{dt}(j H)(\mathcal{F}_t \Phi) = \left(\frac{d}{dt}\mathcal{F}_t \Phi\right)(H_{\mathcal{F}_t \Phi}) = \Phi(i[H_{\mathcal{F}_t \Phi}, H_{\mathcal{F}_t \Phi}]) = 0.$$

Hence energy is conserved by the flow $\mathcal{F}_t$.

Usually, one expects a macroscopic system to exhibit irreversible dynamical behaviour. Since the flow is also defined for negative times, and $\mathcal{F}_t \circ \mathcal{F}_{-t} = \mathrm{id}$, this is, however, not true for mean field dynamical systems. For example, we cannot expect to prove a general H-Theorem, since nothing in our theory would distinguish the nets of Hamiltonians $(H_l)_{l \in I}$ and $(-H_l)_{l \in I}$, so both $\mathcal{F}_t$ and $\mathcal{F}_{-t}$ would have to decrease the entropy. Therefore it is natural to expect that entropy is also conserved under the flow $\mathcal{F}_t$. A conceptual problem with this is that in a general C*-algebra $\mathcal{A}$ there is no natural definition of (absolute) entropy of a state. We shall therefore demonstrate a stronger statement, which trivially implies the conservation of entropy, whenever this term makes sense (*e.g.* in a finite dimensional matrix algebra).

PROPOSITION 6.1. – *Let $\mathscr{F}_t$ be as in Theorem* 5.6, *and let*

$$\Phi = \int^{\oplus} \mu_\Phi(dx)\, \varphi_x \in \mathrm{K}(\mathscr{C}(\mathrm{X}, \mathscr{A})).$$

Then there is a norm-continuous family of unitaries $t \mapsto \mathrm{U}_t \in \mathscr{C}(\mathrm{X}, \mathscr{A})$ depending on Φ) such that for $\mathrm{F} \in \mathscr{C}(\mathrm{X}, \mathscr{A})$, $\mathrm{A} \in \mathscr{A}$:

$$\left.\begin{aligned} &(\mathscr{F}_t \Phi)(\mathrm{F}) = \Phi(\mathrm{U}_t \mathrm{F} \mathrm{U}_t^*) \\ &\textit{resp. for } \mu\textit{-almost all } x: \\ &\varphi_{x,t}(\mathrm{A}) = \varphi_x(\mathrm{U}_t(x)\, \mathrm{A}\, \mathrm{U}_t(x)). \end{aligned}\right\} \tag{40}$$

Proof. – The equivalence of the two versions of equation (40) follows from the decomposition $\mathscr{F}_t \Phi = \int^{\oplus} \mu(dx)\, \varphi_{x,t}$ by taking $\mathrm{F} \in \mathscr{C}(\mathrm{X}, \mathscr{A})$ of the form $\mathrm{F}(x) = f(x)\,\mathrm{A}$ with an arbitrary continuous scalar function f. Consider now the differential equation

$$\frac{d}{dt}\mathrm{U}_t = i\,\mathrm{U}_t \mathrm{H}_{\mathscr{F}_t \Phi}, \qquad \mathrm{U}_0 = \mathbf{1}, \tag{41}$$

i.e. the Schrödinger equation with time dependent Hamiltonian $\mathrm{H}_{\mathscr{F}_t \Phi}$. Since this Hamiltonian is uniformly bounded (by $\sum_\gamma n(\gamma)\, \| \mathrm{H}^\gamma \|$) we can solve this equation by iteration of the corresponding integral equation. It is straightforward to check from equation (41) that $\mathrm{U}_t^* \mathrm{U}_t \equiv \mathrm{U}_t \mathrm{U}_t^* \equiv \mathbf{1}$. Moreover, the states $\Phi_t(\mathrm{F}) \equiv \Phi(\mathrm{U}_t \mathrm{F} \mathrm{U}_t^*)$ satisfy the differential equation

$$\begin{aligned} \frac{d}{dt}\Phi_t(\mathrm{F}) &= \Phi(\mathrm{U}_t(i\mathrm{H}_{\mathscr{F}_t \Phi})\mathrm{F}\mathrm{U}_t^*) + \Phi(\mathrm{U}_t \mathrm{F}(-i\mathrm{H}_{\mathscr{F}_t \Phi})\mathrm{U}_t^*) \\ &= \Phi_t(i[\mathrm{H}_{\mathscr{F}_t \Phi}, \mathrm{F}]), \end{aligned}$$

with initial condition $\Phi_0 = \Phi$. This is precisely the equation determining $\mathscr{F}_t \Phi$, so we must have $\mathscr{F}_t \Phi = \Phi_t$. ■

Hence for all times $\varphi_{x,t}$ and $\varphi_{x,0}$ are equivalent by transformation with a unitary in $\mathscr{A}$. In particular, these states will be normal in precisely the same set of representations of $\mathscr{A}$.

Another corollary of this proposition is that the thermodynamic equilibrium states discussed in section 4 for finite dimensional $\mathscr{A}$ are also dynamically stable. Since the entropy of a state on a finite dimensional algebra is invariant under unitary transformations, we have that $s(\varphi_{x,t}) = s(\varphi_{x,0})$ for all t, and, consequently the entropy defined in equation (23) is constant under the flow. Taking this together with the conservation of internal energy we get $f^\beta(\mathscr{F}_t \Phi) \equiv f^\beta(\Phi)$ for all t. Consequently, if

Φ is a local extremum for one of the functions f^{β}, it is stable for the flow $\mathscr{F}_t$ in the sense that if Φ' is close to Φ, $\mathscr{F}_t \Phi'$, being confined to a level set of f^{β} which is contained in a neighbourhood of Φ, will remain close to Φ for all $t \in \mathbb{R}$. In particular, the equilibrium states, which are absolute minima of f^{β} are stable, as are "metastable states" defined as local minima of f^{β}, and even maxima. These states are not asymptotically stable, however, in the sense that no matter how close Φ' is to ρ, $\mathscr{F}_t \Phi'$ will not converge to Φ. Other stationary points, *i.e.* saddle points of f^{β} are also stationary points for the flow: since they satisfy the gap equation (30), we have $\varphi_x(i[H_{\Phi}(x), A]=0$ for all A, and hence $\dfrac{d}{dt}\Phi=0$. But these points will usually be unstable, in the sense that no matter how close Φ' is to Φ, it cannot be guaranteed that $\mathscr{F}_t \Phi'$ will stay close to Φ.

The dynamical equation (37) is particularly easy to solve, when the Hamiltonian is that of a homogeneous mean field system, *i.e.* $H_l(x) \in \mathscr{A}^{N_l}$ is independent of $x \in X^{N_l}$ for all l. Of course, this does not imply that $\varphi_{x,t}$ is independent of $x \in X$, so the states may still be "inhomogeneous". One then sees from equation (28) that $H_{\Phi}(x) \equiv H_{\bar{\varphi}}$ is independent of x, and, moreover, only depends on the "average" state $\bar{\varphi} \equiv \int \mu(dx)\, \varphi_x \in K(\mathscr{A})$. Integrating equation (37) with respect to $\mu(dx)$ we obtain a closed evolution equation for $\bar{\varphi}$, namely

$$\frac{d}{dt}\bar{\varphi}_t(A)\big|_{t=0} = \bar{\varphi}_t(i[H_{\bar{\varphi}_t}, A]). \tag{42}$$

This is precisely the equation we would have had to solve for the homogeneous mean field system with Hamiltonians $H_l \equiv H_l(x) \in \mathscr{A}^{N_l}$. Since $H_{\Phi}(x)$ depends only on $\bar{\varphi}$. The solution of equation (42) determines $H_{\mathscr{F}_t \Phi}(x) = H_{\bar{\varphi}_t} \equiv h_t$, so that equation (37) simply becomes a Schrödinger equation

$$\frac{d}{dt}\varphi_t(A) = \varphi_t(i[h_t, A])$$

with time dependent Hamiltonian h_t for each of the states $\varphi_{x,t}$. This is in full agreement with the intuition about mean field systems, which has the individual systems move in an effective external field determined by averaging over all subsystems.

Some of the simplification in the case of a homogeneous mean field Hamiltonians is still possible, when the Hamiltonian does not depend on the full set X of variables. More formally, let Y be another compact space, and suppose that $K_l \in \mathscr{C}(Y, \mathscr{A})^{N_l}$ satisfies Assumption 1. Let $\gamma: X \to Y$ be a continuous map, and let $\gamma^{N_l}: X^{N_l} \to Y^{N_l}$ denote the componentwise application of γ. Then $H_l(x) = K_l(\gamma^{N_l}(x))$ satisfies Assumption 1. We denote by $\gamma_*: \mathscr{C}(Y, \mathscr{A}) \to \mathscr{C}(X, \mathscr{A})$ the map defined by $\gamma_*(F) = F \circ \gamma$, and by $\gamma^*: K(\mathscr{C}(X, \mathscr{A})) \to K(\mathscr{C}(Y, \mathscr{A}))$ the adjoint of γ_*. Note that Y carries a natural measure ν such that $\int \nu(dy) f(y) = \int \mu(dx) f(\gamma x)$, and $\gamma^*(K^\mu(\mathscr{C}(X, \mathscr{A}))) \subset K^\nu(\mathscr{C}(Y, \mathscr{A}))$. We can then solve the equation (37) in two steps. We first map the initial state $\Phi = \int^\oplus \mu(dx) \varphi_x \in K^\mu(\mathscr{C}(X, \mathscr{A}))$ into an initial state $\gamma^* \Phi \in K(\mathscr{C}(Y, \mathscr{A}))$. Just like the net $(H_l)_{l \in I}$ the net $(K_l)_{l \in I}$ defines a flow $\hat{\mathscr{F}}_t$ on $K(\mathscr{C}(Y, \mathscr{A}))$, determined by the analogue of the dynamical equation (37). We next solve this equation for $\hat{\mathscr{F}}_t \gamma^* \Phi$. A short computation, using equation (35) shows that $\hat{\mathscr{F}}_t \gamma^* = \gamma^* \mathscr{F}_t$. The solution $\hat{\mathscr{F}}_t \gamma^* \Phi = \gamma^* \mathscr{F}_t \Phi$ then determines $H_{\mathscr{F}_t \Phi}(x) = K_{\gamma^* \mathscr{F}_t \Phi}(\gamma x)$ in the original equation (37), and the equations for different x can be solved independently. Moreover, these equations are identical for x_1 and x_2 (apart from initial conditions) if $\gamma x_1 = \gamma x_2$.

When $\dim \mathscr{A} < \infty$, equation (37) can be viewed as a system of equations for a finite number (namely, $\dim \mathscr{A}$) of x, t-dependent fields. When the Hamiltonian has a special form the number of relevant fields can sometimes be reduced significantly. As an example consider the case of spin systems with pair interactions. Then $\mathscr{A}$ is the algebra of $(2s+1) \times (2s+1)$-matrices, and there are three spin operators $S_\alpha \in \mathscr{A}$, $\alpha = 1, 2, 3$ generating a representation of SU(2). In terms of these operators, let the one-particle energy $\varepsilon: X \to \mathscr{A}$ and the interaction $V: X \times X \to \mathscr{A}^2$ be given by expressions of the form

$$\varepsilon(x) = \sum_\alpha \varepsilon_\alpha(x) S_\alpha \quad \text{and} \quad V(x, y) = \sum_{\alpha, \beta} V_{\alpha\beta}(x, y) S_\alpha \otimes S_\beta. \tag{43}$$

We can then introduce a three-component field $F_\alpha: X \times \mathbb{R} \to \mathbb{R}$ by $F_\alpha(x, t) = \varphi_{x, t}(S_\alpha)$.

Then equation (39) becomes

$$\left.\begin{aligned} &\frac{d}{dt} F_\alpha(x, t) = \sum_{\beta, \gamma} \varepsilon_{\alpha\beta\gamma} (\mathscr{I} F)_\beta(x) F_\gamma(x, t) \\ &\textit{with} \\ &(\mathscr{I} F)_\beta(x) = \frac{1}{2} \varepsilon_\beta(x) + \int \mu(dy) \sum_\delta V_{\beta\delta}(x, y) F_\delta(y, t), \end{aligned}\right\} \tag{44}$$

where $\varepsilon_{\alpha\beta\gamma}$ denotes the completely antisymmetric tensor. Note that this equation is independent from the spin s, which characterizes the representation of SU(2). Given the solution of equation (44) we find $H_{\mathscr{F}_t \Phi}(x) = \sum_\alpha (\mathscr{I} F)_\alpha(x, t) S_\alpha(x)$, and we can solve equation (37) separately for each x. It is clear that $\varphi_{x, t}$ will differ from $\varphi_{x, 0}$ only by a rotation in the given representation of SU (2). A similar reduction to a small number of fields is always possible when the Hamiltonian depends linearly on tensor products of the generators of the representation of some Lie group.

7. DISCUSSION

What can be learn from the models treated in this paper for the fundamental problem of local equilibrium? As in the introduction let us consider a fixed compact region $X \subset \mathbb{R}^d$, given on the macroscopic length scale, whose multiples contain more and more (namely N_l) points $\Lambda_l = (lX) \cap \mathbb{Z}^d$ of the lattice with microscopic lattice constant. In the general scheme as in the mean field case we consider the net of local algebras $\mathscr{D}(\Lambda) \simeq \bigotimes_{z \in \Lambda} \mathscr{A} \simeq \mathscr{A}^N$. The connection between the macroscopic and the microscopic point of view hinges on the dual role of this net, which in the inductive limit approximates both the quasi-local algebra $\mathscr{D}$ of the whole system and the algebra $\mathscr{B}_x$ "at the point x". Explicitly, this connection is made by identifying a fixed configuration Λ_0 of finitely many sites with suitable sites in Λ_l all of which are near lx on the macroscopic scale. In the general scheme we insist that the geometric interrelations between the sites of Λ_0 are respected by this embedding, *i. e.* the identification is made by a translation automorphism σ_{z_l}. In the mean field case we were much more liberal by admitting *any* embedding taking the points close to lx.

The crucial step in both cases is to define a notion of thermodynamic limit (or mean field limit) for nets $(\omega_l)_{l \in I}$ of states on these algebras. The main requirement is that for all acceptable nets of embeddings, the expectation values for the embedded observables converge. Since this is postulated for all finite Λ_0, the limits define a state on the inductive limit $\mathscr{B}_x \cong \mathscr{D} \cong \overline{\bigcup \mathscr{D}_{\Lambda_0}(\Lambda_0)}$. In the mean field case we thus obtain the inductive limit algebra $\mathscr{B}_x \equiv \mathscr{A}^\infty = \overline{\bigcup \mathscr{A}^n}$. The mean field condition for $(\omega_l)_{l \in I}$ then forces the macroscopic state Ω^x at x to be permutation symmetric. Similarly, we obtain translation invariant states Ω^x in the general scheme. It is clear that since the conditions for thermodynamic limits of states are less stringent in the general scheme, every net $(\omega_l)_{l \in I}$ with a mean field limit also has a thermodynamic limit, and the corresponding macroscopic local states Ω^x will be the same. More precisely, the correlation functions of a macroscopic limit state in the sense of Definition 3.8, which arises as the mean field limit of a net $(\omega_l)_{l \in I}$ determine the macroscopic local states Ω^x via equation (18).

The converse is false in general, but if the macroscopic state is macroscopically pure, *i. e.* if its Størmer measure is concentrated on a single point $\Phi = \int^{\oplus} \mu(dx)\,\varphi_x$, the states $\Omega^x = (\varphi_x)^\infty$ in turn determine the correlation functions. Since the case of macroscopically pure states is certainly the most relevant, and since the mixed case can be obtained from it by trivial convex combinations, we felt justified to use the term "macroscopic state" for both the function $x \mapsto \Omega^x$ and for the collection of correlation functions. However, it is also possible to incorporate macroscopically mixed states into the general scheme. We then have to postulate the existence of $\Omega^{a_l \dots z_l}(\mathrm{A}, \dots, \mathrm{Z}) = \lim_l \omega_l(\sigma_{a_l}(\mathrm{A}) \dots \sigma_{z_l}(\mathrm{Z}))$, where A, ..., Z are strictly local and $l^{-1} a_l, \dots, l^{-1} z_l$ converge to different points $\hat{a}, \dots, \hat{z}$ in X. The pure case is again characterized by the property that this limit is equal to $\Omega^{\hat{a}}(\mathrm{A}) \dots \Omega^{\hat{z}}(\mathrm{Z})$.

The central result of our paper is that in the mean case it is possible to impose simple convergence conditions on the net $(\omega_l)_{l \in \mathrm{I}}$, which are preserved under the time evolution. The art of defining the right notion of thermodynamic limit for $(\omega_l)_{l \in \mathrm{I}}$ in the general scheme lies in finding conditions with just this property. It is unlikely that the existence of the limits defining Ω^x is by itself a time invariant condition. Even if it is, however, the local states would be described by arbitrary translation invariant states on a lattice system, which is a large set without any useful characterizations. In contrast, in the mean field case we ended up with permutation symmetric states, which are easily parametrized by means of Størmer's theorem. It is here that the physical intuition of local equilibrium becomes vital: in addition to the existence of the Ω^x one expects that there is a large class of "physically interesting" nets $(\omega_l)_{l \in \mathrm{I}}$, for which the Ω^x turn out to be KMS-states for a suitably defined local time evolution in $\mathrm{K}(\mathscr{B}_x)$. (The KMS property can be defined in the Schrödinger picture, too [19].)

Unfortunately, our present study does not suggest any definition of a suitable class of physically interesting nets, since the KMS-property of Ω^x is in conflict with the mean field nature of our system in a sense we shall now describe.

There is no difficulty in defining the macroscopic time evolution by $(\Omega_t)^x = \lim_l \omega_l \circ \alpha^l_{\varkappa_l t} \circ \sigma_{lx}$ as in the introduction. Since the existence of a mean field limit implies the existence of Ω^x by equation (18), and since the former property is preserved in time, the evolution of the macrostates $t \mapsto \Omega_t$ makes sense. Thus, choosing $\varkappa_{l=1}$ and setting $\Omega^x = \int \mathrm{M}(d\Phi)\,\varphi_x^\infty$ we have

$$(\Omega_t)^x = \lim_l (\omega_l \circ \alpha^l_t)\,\hat{\eta}^x_l = \int \mathrm{M}(d\Phi)\,(\varphi_{x,t})^\infty, \tag{45}$$

where $\varphi_{x,t}$ is given by equation (40), and $\hat{\eta}_l^x$ denotes the (k, l)-embedding (k chosen appropriately) with $\xi_l \eta_l^x$ converging to $(x, \ldots, x)$. Physically, the choice $\kappa_l = 1$ means that disturbances travel macroscopic distances in unscaled (microscopic) times due to the macroscopic interaction length of the mean field Hamiltonians. In contrast, the general scheme suggests a rescaling with $\kappa_l \to \infty$, since any local observable evolved by a fixed microscopic time will still be well localized, and will only affect regions at macroscopic distances after infinite time. This would be required in order to define a *local* time evolution α_t^x in the first place. But this is precisely what breaks down in the mean field case: any attempt to define a time evolution for observables near a point x runs into the difficulty that due to the extremely long range of the interaction the state of any distant region will have a non-negligible effect on the state near x. In this sense the macroscopic range of the interaction contradicts the idea which is at the root of the concept of local equilibrium, namely that the approach to equilibrium happens in parallel on a short time scale within many macroscopically small regions, which have pratically no interaction on that scale.

This does not exclude the possibility that the Ω^x might be KMS states for some time evolution α_t^x on $\mathscr{B}_x$ which is related to the Hamiltonian in a different way. While this is in itself a major departure from the general scheme, even this weakened form of a local KMS condition is not feasible in general. To see this let us restrict to the most relevant special case, namely that of macroscopically pure states, and for simplicity we take $\mathscr{A}$ to be a matrix algebra. Then $(\Omega_t)^x = (\varphi_{x,t})^\infty$ for some solution of equation (37). We know from Proposition 6.1 that $\varphi_{x,t}$ is unitarily equivalent to $\varphi_{x,0}$ for all t. Can the infinite product states formed from these one-particle states be KMS-states for the same time evolution α_t^x, with possibly different temperatures $\beta_t(x)$? A time evolution from a standard lattice interaction will hardly satisfy this property, since its KMS states will in general not be product states, and not even permutation symmetric. However, we have seen that homogeneous mean field systems have equilibrium states which are homogeneous product states. Therefore it is tempting to look for an approximately symmetric net $(K_l)_{l \in I}$, $K_l \in \mathscr{A}^{N_l}$ of Hamiltonian densities of a homogeneous mean field system [and a corresponding net $(\alpha_t^{x,l})_{l \in I}$ of automorphisms in analogy to equation (1)] such that $\varphi_{x,t}$ determines a β_t-equilibrium state. [Indeed, we have a natural approximately symmetric net of Hamiltonians associated with each point $x \in X$, namely the operators $(H_l(x, \ldots, x))_{l \in I}$]. By the homogeneous version [16] of Proposition 4.1 the net K_l is such that $\varphi_{x,t}$ minimizes the functional

$$\varphi \mapsto (j\mathrm{K})(\varphi) + \frac{1}{\beta_t(x)} s(\varphi).$$

Assume that φ_1 and φ_2 minimize the free energy functional at inverse temperatures $\beta_1 < \beta_2$, and set $s_i = s(\varphi_i)$ and $u_i = (j\mathrm{K})(\varphi_i)$. Then the variational principle implies $u_2 + s_2/\beta_1 \geqq u_1 + s_1/\beta_1$, and a similar relation with 1 and 2 interchanged. But if φ_1 and φ_2 are unitarily equivalent, then $s_1 = s_2$, so we also have $u_1 = u_2$, and both states minimize both free energy functionals. But then the gap equation (30) must hold for φ_i at two different temperatures with the same effective Hamiltonian. This is only possible when the effective Hamiltonian vanishes, *i.e.* when both φ_1 and φ_2 are equal to the trace. Applying this reasoning to $\varphi_{x,t}$ we see that there is no non-trivial time evolution after all.

To conclude the discussion we would like to point out some possible extensions of our work to larger classes of models. The separability Assumption 3 is the easiest to drop. In fact, we might say that it holds without loss of generality, if we keep X and $\mathscr{A}$ to the sizes required by the Hamiltonian. For this we replace X by its quotient with respect to the pseudo-metric introduced in Lemma 4.2, and replace $\mathscr{A}$ by the C*-algebra generated by all effective Hamiltonians (and their analogues for $\mathrm{H}_l \in \mathscr{A}^{\mathrm{N}_l}$). We can then use the methods of section 6, to first solve the dynamical equation (37) in this separable model, and to obtain the time dependence of the effective Hamiltonians, in which the states move independently. Our technical reason for introducing Assumption 3 was to be able to conclude continuity from the convergence of sequences in Proposition 3.4, and to make the states φ_x in the integral decomposition $\Phi = \int^{\oplus} \mu(dx)\,\varphi_x$ well-defined μ-almost everywhere.

The most severe technical restriction is the boundedness of the Hamiltonians, which is part of Assumption 1. We remark that the general definition of mean field dynamical semigroups in [8] does not require bounded generators. Both in equilibrium [18], and in-non-equilibrium it is relatively easy to allow an unbounded one-particle term in the Hamiltonian, and to reproduce all the results in this paper. However, a general theory which also admitted unbounded interactions would be riddled with domain problems. For unbounded interactions it would not even be clear whether the dynamics is well-defined for each l separately, and we do not expect this problem to become simpler in the mean field limit. Naturally, these questions can be answered in some models, but the presentation of techniques for doing this would have distracted too much from the main objective of the paper. For the study of equilibrium states in homogeneous models one only needs lower bounds on the Hamiltonian, and one can even allow the Hamiltonians to increase with n. This does not hold in the inhomogeneous case, since Assumption 2 is not strong enough to control upper bounds on expectation values of functions with singularities in X-space.

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