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Covariant radiation hydrodynamics

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ABSTRACT. — Constitutive equations are determined for the radiation stresses in covariant radiation hydrodynamics within the general framework of extended thermodynamics. A unique expression is found for the variable Eddington factor.

RÉSUMÉ. — Nous déterminons des équations constitutives pour les efforts radiatifs en employant un formalisme covariant dans le cadre général de la thermodinamique étendue. Nous trouvons une expression unique pour le facteur d'Eddington variable.

1. INTRODUCTION

Radiation Hydrodynamics is a fundamental theory for Plasma Physics and Astrophysics. The physical model which it describes is a relativistic fluid coupled with a strong radiation field; the corresponding mathematical model is a set of equations describing the interaction between matter and radiation. The range of validity of this model, si restricted to those situations in which energy-momentum transfer is dominated by radiative processes.

The starting point for all treatments of radiative hydrodynamics is a transfer equation governing the evolution of the distribution function for photons. But in many situations solving such an equation can be either too expensive in terms of numerical cost or very unsatisfactory from a theoretical point of view, since usually one is interested mainly in the first moments of distribution function which are those which have a macroscopic interpretation. By taking the moments of the transfer equation one obtains an infinite set of equations for the moments of the distribution function; it is then necessary to adopt a closure approximation, linking the $(n+1)$-th moment to the lowest ones.

The simplest closure is the Eddington approximation, which assumes the radiation field to be in Local Thermodynamical Equilibrium (LTE). But this closure, although largely used in many practical situations, is unsatisfactory in all those cases in which dissipation is large.

In this article radiative hydrodynamics is treated within the general framework of extended thermodynamics. We seek for a closure at the second order, that is we want to find an expression for the radiative pressure tensor as a function of radiative energy and energy flux [*see* equation (10)]. As shown in [1], this amounts to finding a so called variable Eddington factor χ; its meaning, is that of an interpolating function between the collision dominated and collisionless regime; the values which χ must assume in the two opposite situations are respectivelly $\frac{1}{3}$ and 1. Many different variable Eddington factors have appeared in literature, obtained using different approaches.

Recently in [2] a new approach has been suggested: to seek for a variable Eddington factor and at the same time to impose the existence of a supplementary conservation law for the moment equations. The approach presented in [2] was successful: such an Eddington factor exists and, surprisingly, had already been found by Levermore supposing the radiation field isotropic in some inertial reference frame.

The above results were obtained within a special relativistic context assuming the underlying medium dynamically uncoupled with radiation; that is supposing that energy and momentum exchange between matter and radiation influences significantly only radiation.

In the present paper we extend the results of [2]; above all we use a covariant formalism; in all situations in which gravitational field is significant (early stages of universe or gravitational collapse) a general relativistic treatment is needed; moreover we assume the medium is dynamically coupled with radiation (the equation governing the motion of matter are taken into account).

We believe that the model we present here (the equations of motion for matter, the moment equations for radiation together with the closure and the supplementary conservation law) is conceptually rigorous and still sufficiently simple to be numerically praticable.

2. COVARIANT RADIATIVE TRANSFER EQUATION

In this article we deal with a radiation field which may be described by the photon distribution function f (number density of photons in phase space), depending on the coordinate x and the four-momentum p of photons; effects such as polarization, dispersion, coherence are neglected.

The transfer equation for such a distribution function is [3]

$$\frac{df}{d\lambda}=p^{\mu}\frac{\partial f}{\partial x^{\mu}}+\frac{dp^{\mu}}{d\lambda}\frac{\partial f}{\partial p^{\mu}}=n_0(\alpha-\beta f) \tag{1}$$

where λ is an affine parameter along the photon trajectories such that $p^{\mu}=\dfrac{dx^{\mu}}{d\lambda}$, α is the rate at which photons are injected (by emission or scattering) into the beam, βf is the rate at which photons are removed by scattering or absorption, and n_0 is the particle proper density of the medium through which radiation propagates.

If the change of momentum of photons (between two collisions with the medium particles) is due to the gravitational field we can write

$$\frac{dp^{\rho}}{d\lambda}=-\Gamma^{\rho}_{\mu\nu}p^{\mu}p^{\nu} \tag{2}$$

where $\Gamma^{\rho}_{\mu\nu}$ are the Christoffel symbols. Locally we can suppose $\Gamma^{\rho}_{\mu\nu}=0$.

Now let (n^{μ}) be the 4-velocity of an observer, normalized by $n^{\mu}n_{\mu}=-1$. In the reference frame of such an observer the decomposition

$$p^{\mu}=-\nu(n^{\mu}+l^{\mu}) \tag{3}$$

can be introduced, where l^{μ} is a 4-vector lying on the unit sphere of the 3-dimensional space orthogonal to (n^{μ}), while ν is the energy (which in our is also the frequency) of the photon as measured by the observer with 4-velocity (n^{μ}), that is

$$l^{\mu}n_{\mu}=0,\qquad l^{\mu}l_{\mu}=l,\qquad \nu=p^{\mu}n_{\mu}.$$

Usually the decomposition (3) is introduced by choosing as time congruence the one determined by the 4-velocity u^{μ} of the ambient medium;

$$p^{\mu}=-\nu_0(u^{\mu}+l^{\mu})$$

where ν_0 is the local rest frequency, while for describing the interaction between the radiation and the medium the quantities

$$\varepsilon(\nu_0)=\nu_0^2\,\alpha \qquad \text{and} \qquad \kappa(\nu_0)=\nu_0^{-1}\,\beta$$

are used which are respectively the emission and the absorption coefficient, as measured in the local rest frame.

Multiplying equation (1) by ν_0^2, using equation (2), introducing the decomposition (3) and finally integrating over ν_0, and invoking the equivalence principle, Anderson and Spiegel [3] obtained the transfer equation for the integrated proper intensity

$$\mathcal{I} = \int_0^\infty f \nu_0^3 \, d\nu_0,$$

which reads:

$$(u^\mu + l^\mu) \left\{ \nabla_\mu \mathcal{I} + 4 \mathcal{I} l^\sigma \nabla_\mu u_\sigma + l^\rho l^\sigma \frac{\partial \mathcal{I}}{\partial l^\rho} \nabla_\mu u_\sigma \right.$$
$$\left. + u^\rho l^\sigma \frac{\partial \mathcal{I}}{\partial l^\rho} \nabla_\mu u_\sigma - \frac{\partial \mathcal{I}}{\partial l^\rho} \nabla_\mu u^\rho \right\} = n_0 \varepsilon_0 - n_0 \langle \kappa \rangle \mathcal{I} \quad (4)$$

where

$$\varepsilon_0 = \int_0^\infty \varepsilon(\nu_0) \, d\nu_0$$

while $\langle \kappa \rangle$ is a sort of mean absorption coefficient

$$\langle \kappa \rangle = \mathcal{I}^{-1} \int_0^\infty \kappa(\nu_0) f \nu_0^3 \, d\nu_0.$$

The transfer equation (4) has the disadvantage of being too complicated to be used in many practical cases, having to be integrated over the coordinate space and the unit sphere. To circumvent this difficulty it is a common procedure to take the moments of this equation integrating over the unit sphere. In this way we obtain the zeroth order moment equation:

$$\nabla_\mu (J_0 u^\mu + H_0^\mu) + u^\nu H_0^\mu \nabla_\nu u_\mu + K_0^{\mu\nu} \nabla_\nu u_\mu = n_0 \langle \kappa \rangle (B - J_0), \quad (5)$$

where

$$\left. \begin{aligned} J_0 = \frac{1}{4\pi} \int_{4\pi} \mathcal{I} \, d\Omega, \qquad H_0^\mu = \frac{1}{4\pi} \int \mathcal{I} l^\mu \, d\Omega, \\ K_0^{\mu\nu} = \frac{1}{4\pi} \int \mathcal{I} l^\mu l^\nu \, d\Omega \end{aligned} \right\} \quad (6)$$

are, respectively, the radiation energy density, energy flux and stress tensor measured in the rest frame, $d\Omega$ being the element of solid angle over the unit sphere

$$d\Omega = \delta(u^\mu l_\mu) \, \delta(l^\nu l_\nu + 1) \, d^4 l,$$

while $B = \varepsilon_0 / \langle \kappa \rangle$ is the source function.

Multiplying equation (5) by l^μ, and again integrating, we obtain

$$\nabla_\nu T^{\mu\nu} = 4\pi n_0 \langle \kappa \rangle [u^\mu (B - J_0) - H_0^\mu], \tag{7}$$

where

$$T^{\mu\nu} = \int p^\mu p^\nu \nu_0 \, d\nu_0 \, d\Omega = 4\pi [J_0 u^\mu u^\nu + u^\mu H_0^\nu + u^\nu H_0^\mu + K_0^{\mu\nu}] \tag{8}$$

is the radiation stress-energy tensor.

One could continue this procedure by successively multiplying the transfer equation by $l^{\alpha_1} \dots l^{\alpha_h}$ and then integrating, thereby obtaining the hierarchy [3] of moment equations for the moments of the integrated proper intensity

$$M^{\alpha_1 \dots \alpha_h} = \frac{1}{4\pi} \int_{4\pi} \mathcal{I} \, l^{\alpha_1} \dots l^{\alpha_h} \, d\Omega.$$

The point is that if one wants to obtain a closed set of equations for $M^{\alpha_1 \dots \alpha_h}$ (number of equations equal to the number of unknowns), one must necessarly have an expression linking the n-th order moment to the lowest ones: the so called closure problem arises.

Usually one seeks for an expression for the second moment of the rest intensity, the stress tensor $K^{\mu\nu}$. The simplest closure at this level is the Eddington approximation, which assumes $K_0^{\mu\nu}$ isotropic

$$K_0^{\mu\nu} = \frac{1}{3} J h_0^{\mu\nu} \tag{9}$$

where $h_0^{\mu\nu} = u^\mu u^\nu + g^{\mu\nu}$ is the projection tensor along the temporal congruence determined by u^μ. But in [3] it has been shown that (9) does not describe viscous stresses adequately. In order to treat situations significatively far from Local Thermodynamical Equilibrium, one must have a non-isotropic expression for $K_0^{\mu\nu}$. Anderson and Spiegel [3] expanding around equilibrium (that is using as smallness parameter the mean free path) and taking only linear terms, obtained an expression for $K_0^{\mu\nu}$ in which the anisotropic part of $K_0^{\mu\nu}$ is proportional to the shear tensor of the underlying medium (*see* also Hsieh and Spiegel [4]). Here we shall follow a different approach wich is also largely followed in literature.

3. EDDINGTON FACTORS

Now we suppose that the stress tensor $K^{\mu\nu}$, as measured by an observer with 4-velcity n^μ, is a function of the lowest moments J and H^μ. The most general expression for such a $K^{\mu\nu}$ is:

$$K^{\mu\nu} = \frac{1}{3} J h^{\mu\nu} + q \left(H^\mu H^\nu - \frac{1}{3} H^2 h^{\mu\nu} \right) \tag{10}$$

where we have taken into account the condition $K^\mu_\mu = J$, while q is a function of the scalars J and $H = (H^\mu H_\mu)^{1/2}$. Then the determination of $K^{\mu\nu}$ is now reduced to finding a scalar function q. It is interesting to notice that our starting hypothesis is somewhat similar to a sort of unidirectionality of the problem. Levermore [1] has in fact shown, in a special relativistic context, that if the radiation intensity is symmetric about a preferred direction, then the stress tensor can be written as

$$K^{\mu\nu} = J \left(\frac{1-\chi}{2} h^{\mu\nu} + \frac{3\chi - 1}{2} \frac{H^\mu H^\nu}{H^2} \right); \tag{11}$$

this expression is equivalent to (10) if $q = \frac{J}{H^2} \frac{3\chi - 1}{2}$.

The function χ is called the variable Eddington factor; it is commonly supposed to depend on H and J through their ratio $f = H/J$, while in this paper we will obtain this result as consequence of universal principles such as the entropy principle and the relativity principle. How to construct a function χ is a question amply studied, and in fact many Eddington factors have already appeared in literature (*see* [1]). However in choosing such a function one must obey some constraints that we want here to briefly recall.

In the limit of Local Thermodynamical Equilibrium, (*i.e.* $f=0$), the pressure tensor must be isotropic:

$$\chi(0) = \frac{1}{3};$$

in the opposit limit, the free streaming limit, ($f=1$), all pressure is concentrated along the direction of radiation energy flux, that is

$$\chi(1) = 1.$$

Moreover one must have

$$f^2 \leqq \chi(f) \leqq 1 \tag{12}$$

because $K^{\mu\nu}$ is the second moment of the integrated proper intensity. Now, if we suppose the underlying medium is static in special relativity, so that the comoving reference frame where $u^\alpha = \delta^\alpha_0$ is inertial, and write the system constituted by equation (7) together with the closure (11), specializing it to a one dimensional spatial geometry (so that all vectors and tensors reduce to scalars) we obtain:

$$\left.\begin{aligned} \frac{\partial \mathrm{J}}{\partial x^0}+\frac{\partial \mathrm{H}}{\partial x^1}=n_0(\kappa)(\mathrm{B}_0-\mathrm{J}) \\ \frac{\partial \mathrm{H}_0}{\partial x^0}+\chi'\frac{\partial \mathrm{H}_0}{\partial x^1}+(\chi+\chi' f)\frac{\partial \mathrm{J}_0}{\partial x^1}+n\langle\kappa\rangle \mathrm{H}_0=0 \end{aligned}\right\} \tag{13}$$

where $'=\frac{d}{df}$. The quasilinear system (13), as it can be easily seen, is hyperbolic; then it admits discontinuous solutions in first derivatives; the propagation velocities of these discontinuities are:

$$\lambda_{\pm}=(\chi'\pm\sqrt{(\chi'^2)+4\chi' f+4\chi}).\frac{1}{2}.$$

It is natural to ask that these velocities grow with f. So one obtains that χ must be a convex function of f,

$$\chi''\geqq 0. \tag{14}$$

These are the main and more natural conditions to impose to variable Eddington factors. For a more detailed and extensive treatment *see* [5].

4. THE COMPLETE SYSTEM OF EQUATIONS

In this section we want to write the equations governing the motion of our radiative fluid. We shall suppose the medium dynamic, in the sense that there is a significant exchange of energy and momentum, between the medium and radiation. Moreover the medium is supposed to be perfect, that is the stress-energy tensor of the matter is

$$\mathrm{T}_m^{\mu\nu}=(e+p)\,u^\mu u^\nu+p g^{\mu\nu}, \tag{15}$$

with e total energy-density, p pressure, measured in the local rest-frame; all dissipative processes are then supposed to be due to the presence of radiation. This model is argued to correctly describe all situations in which heat conduction is dominated by transport of photons, such as early stages of Universe (radiation-dominated era) and certain phases of stellar evolution like gravitational collapse or supernovae explosion.

Now we decompose the stress-energy tensor of radiation

$$\mathrm{T}_r^{\mu\nu}=4\pi\,[\mathrm{J}\,n^\mu n^\nu+n^\mu \mathrm{H}^\nu+n^\nu \mathrm{H}^\mu+\mathrm{K}^{\mu\nu}] \tag{16}$$

where n^μ is the 4-velocity of a generic observer which measures radiation energy density J, radiation energy flux (H) and radiation momentum flux (K). Notice that the decomposition (8) is a particular case of (16). To close our system we suppose that in the reference frame n^μ a generalized

Eddigton closure holds:

$$K^{\mu\nu}=\frac{1}{3}\,J\,h^{\mu\nu}+q(J,H)\left(H^{\mu}H^{\nu}-\frac{1}{3}\,H^{2}h^{\mu\nu}\right) \tag{17}$$

where now $h^{\mu\nu}=n^{\mu}n^{\nu}+g^{\mu\nu}$.

We are now able to write the equations our system must obey. They are the balance law for the stress energy tensor

$$T^{\mu\nu}{}_{;\mu}=(T^{\mu\nu}_{m})_{;\mu}+(T^{\mu\nu}_{r})_{;\mu}=0, \tag{18}$$

the conservation of particle number

$$(n_0\,u^{\mu})_{;\mu}=0 \tag{19}$$

where n_0 is the number particle density measured by the comoving observer u^{μ}, and the equation

$$T^{\mu\nu}_{r\,;\mu}=4\,\pi\,n_0\,\langle\,\kappa\,\rangle\,[(B-J_0)\,u^{\mu}-H^{\mu}_{0}] \tag{20}$$

where, of course now for $T^{\mu\nu}_{r}$ we use the decomposition (16) instead of (8).

Now we want to find a closure for this set of equations, *i.e.* an expression for the function $q(J, H)$ [or equivalently $\chi(J, f)$] by imposing the entropy and the relativity principles.

5. THE ENTROPY AND RELATIVITY PRINCIPLES

Let us rewrite our system of balance equations as

$$\left.\begin{aligned}(n_0\,u^{\mu})_{,\mu}&=0\\ T^{\mu\nu}_{m,\,\mu}&=4\,\pi f^{\nu}\\ \bar{T}^{\mu\nu}_{r,\,\mu}&=-f^{\nu}\end{aligned}\right\} \tag{21}$$

where

$$T^{\mu\nu}_{m}=(e+p)\,u^{\mu}\,u^{\nu}+p g^{\mu\nu} \tag{22}$$

$$\bar{T}^{\mu\nu}_{r}=\frac{1}{4\pi}\,T^{\mu\nu}_{r}=J\,n^{\mu}\,n^{\nu}+2\,n^{(\mu}H^{\nu)}+\frac{1}{2}\,J\,h^{\mu\nu}+J\,\varphi(f,J)\left(\frac{H^{\mu}H^{\nu}}{H^{\alpha}H_{\alpha}}-\frac{1}{3}\,h^{\mu\nu}\right) \tag{23}$$

$$\varphi=q\,H^{2}\,J^{-1};\qquad f=J^{-1}\sqrt{H^{\alpha}H_{\alpha}}$$

$$f^{\nu}=-n_0\,\langle\,\kappa\,\rangle\,[(B-J_0)\,n^{\mu}-H^{\mu}_{0}]$$

and $p=p(n_0, e)$ is a function whose generality is restricted by the Gibbs relation, *i.e.* $\exists\, s(n_0, e)$ such that

$$T\,ds=d(e/n_0)+p\,d(1/n_0), \tag{24}$$

T being the absolute temperature.

Moreover the following conditions are verified

$$n_\mu n^\mu = -1; \qquad n_\mu H^\mu = 0. \tag{25}$$

Then the system (21) has nine equations in the nine unknowns n_0, e, J and the independent components of u^μ and H^μ. We now want to impose the entropy principle [6] for the system (21), *i.e.* that there are two function h^α, g such that the relation

$$h^\alpha_{,\alpha} = g \tag{26}$$

holds for every solution of the system (21).

We shall prove that this condition is satisfied iff $\varphi(f, J)$ depends only on f and moreover

$$\varphi(f) = 2 \pm \sqrt{4-3f^2} \tag{27}$$

that is the same closure we found in the special relativistic and static case [2]!!

To obtain such result let us first remember that the condition (26) is equivalent to imposing that there exist the function λ, λ_β, ψ_β, called Lagrange Multipliers, such that the relation

$$h^\alpha_{,\alpha} - g + \lambda (n_0 u^\alpha)_{,\alpha} + \lambda_\beta (T^{\beta\alpha}_m - 4\pi f^\beta) + \psi_\beta (T^{\beta\alpha}_{r,\alpha} + f^\beta) = 0 \tag{28}$$

holds for every value of the independent variables (the proof of this property may be found in papers such as [7], [8], [9]).

We adopt now an idea developed in papers such as [10] (concerning the classical case), [11] and [12] (concerning the relativistic case) to define

$$h'^\alpha = h^\alpha + \lambda n_0 u^\alpha + \lambda_\beta T^{\beta\alpha}_m + \psi_\beta \bar{T}^{\beta\alpha}_r \tag{29}$$

and to take the Lagrange Multipliers as independent variables; they are also called the "mean field".

As consequence, the relation (28) becomes

$$-h'^\alpha_{,\alpha} + n_0 u^\alpha \lambda_{,\alpha} + T^{\beta\alpha}_m \lambda_{\beta,\alpha} + \bar{T}^{\beta\alpha}_r \psi_{\beta,\alpha} + g + (4\pi\lambda_\beta - \psi_\beta) f^\beta = 0$$

that must hold for every value of λ, λ_β, ψ_β from which fact we have that

$$\left.\begin{aligned} n_0 u^\alpha &= \frac{\partial h'^\alpha}{\partial \lambda} \\ T^{\beta\alpha}_m &= \frac{\partial h'^\alpha}{\partial \lambda_\beta} \\ \bar{T}^{\beta\alpha}_r &= \frac{\partial h'^\alpha}{\partial \psi_\beta} \end{aligned}\right\} \tag{30}$$

and

$$g = (\psi_\beta - 4\pi\lambda_\beta) f^\beta.$$

Now in the appendix it will be proved that the first members of equations. $(30)_{1,2}$ do not depend on ψ_β; consequently, from $(30)_{1,2}$ we obtain that h'^{α} can be written as sum of a function depending only on λ, λ_β and of a function depending only on ψ_β, *i.e.*,

$$h'^{\alpha}=h_0^{\alpha}(\lambda, \lambda_\beta)+h_1^{\alpha}(\psi_\beta) \tag{31}$$

As consequence the system (30) splits up into two parts:

$$\left.\begin{aligned} n_0 u^{\alpha} &= \frac{\partial h_0^{\alpha}}{\partial \lambda} \\ T_m^{\beta\alpha} &= \frac{\partial h_0^{\alpha}}{\partial \lambda_\beta} \end{aligned}\right\} \tag{32}$$

$$\bar{T}_r^{\beta\alpha}=\frac{\partial h_1^{\alpha}}{\partial \psi\beta} \tag{33}$$

Now in equation $(32)_2$ we must impose the expression (22) for $T_m^{\beta\alpha}$; its symmetry (as shown in reference [13]) is equivalent to assuming the existence of a scalar-valued function $h_0(\lambda, \lambda_\alpha)$ such that

$$h_0^{\alpha}=\frac{\partial h_0}{\partial \lambda_\alpha};$$

moreover the relativity principle imposes that h_0 can be written as $h_0(\lambda, G_1)$ where $G_1=\lambda_\alpha \lambda^{\alpha}$ (*see* reference [14], [15]), from which

$$h_0^{\alpha}=2\frac{\partial h_0}{\partial G_1}\lambda^{\alpha};$$

consequently the system (32) becomes:

$$n_0 u^{\alpha}=2\frac{\partial^2 h_0}{\partial G_1\,\partial \lambda}\lambda^{\alpha}$$

$$(e+p)\,u^{\alpha}u^{\beta}+pg^{\alpha\beta}=4\frac{\partial^2 h_0}{\partial G_1^2}\lambda^{\alpha}\lambda^{\beta}+2\frac{\partial h_0}{\partial G_1}g^{\alpha\beta}$$

from which $-(n_0)^2=4\left(\dfrac{\partial^2 h_0}{\partial G_1\,\partial \lambda}\right)^2 G_1\,(\Rightarrow G_1<0)$ and then

$$n_0=2\left|\frac{\partial^2 h_0}{\partial G_1\,\partial \lambda}\right|(-G_1)^{1/2}$$

$$u^{\alpha}=\frac{|\partial^2 h_0/\partial G_1\,\partial \lambda|}{\partial^2 h_0/\partial G_1\,\partial \lambda}(-G_1)^{-1/2}\lambda^{\alpha}$$

$$p=2\frac{\partial h_0}{\partial G_1}; \qquad e=-4\,G_1\frac{\partial^2 h_0}{\partial G_1^2}-2\frac{\partial h_0}{\partial G_1}.$$

From this relations we can see that it is sufficient to know the function $p(\lambda, G_1)$ because consequently they give

$$p=p(\lambda, G_1); \qquad n_0=(-G_2)^{1/2}\left|\frac{\partial p}{\partial \lambda}\right|; \tag{34}$$

$$e=-2\mathrm{G}_1\frac{\partial p}{\partial \mathrm{G}_1}-p;\qquad h_0^\alpha=p\lambda^\alpha;$$
$$u^\alpha=\frac{|\partial p/\partial\lambda|}{\partial p/\partial\lambda}(-\mathrm{G}_1)^{-1/2}\lambda^\alpha.$$

These relations may be used to obtain

$$d(e/n_0)+p\,d(1/n_0)=\frac{|\partial p/\partial\lambda|}{\partial p/\partial\lambda}(-\mathrm{G}_1)^{-1/2}d\left[-\lambda-2\mathrm{G}_1\frac{\partial p}{\partial \mathrm{G}_1}\left(\frac{\partial p}{\partial\lambda}\right)^{-1}\right],$$

so that the Gibbs relation (24) is satisfied iff:

$$\mathrm{T}=(-\mathrm{G}_1)^{-1/2}$$

and

$$s=\left|\frac{\partial p}{\partial\lambda}\right|\left(\frac{\partial p}{\partial\lambda}\right)^{-1}\left[-\lambda-2\mathrm{G}_1\frac{\partial p}{\partial \mathrm{G}_1}\left(\frac{\partial p}{\partial\lambda}\right)^{-1}\right].\tag{35}$$

We have then obtained two solutions according to the sign of $\frac{\partial p}{\partial\lambda}$. The relations (34), (35) prove also the invertibility of the functions $\lambda=\lambda(e, n_0, u^\alpha)$, $\lambda^\alpha(e, n_0, u^\alpha)$, a property that we had assumed when we took the Lagrange Multipliers as variables; in fact they give

$$\mathrm{G}_1=-\frac{1}{\mathrm{T}^2};\qquad \lambda^\alpha=\left|\frac{\partial p}{\partial\lambda}\right|\left(\frac{\partial p}{\partial\lambda}\right)^{-1}\mathrm{T}^{-1}u^\alpha;\tag{36}$$
$$\lambda=\left|\frac{\partial p}{\partial\lambda}\right|\left(\frac{\partial p}{\partial\lambda}\right)^{-1}\left(\frac{e+p}{n_0\mathrm{T}}-s\right)=-\left|\frac{\partial p}{\partial\lambda}\right|\left(\frac{\partial p}{\partial\lambda}\right)^{-1}\frac{\partial(n_0 s)}{\partial n_0}$$

It remains now to impose the expression (23) for $\bar{\mathrm{T}}_r^{\mu\nu}$ in the condition (33); as seen for $\mathrm{T}_m^{\alpha\beta}$, also the symmetry of $\bar{\mathrm{T}}_r^{\beta\alpha}$ shows that there is a scalar-valued functions $h_1(\mathrm{G}_2)$ where $\mathrm{G}_2=\psi_\beta\psi^\beta$, such that

$$h_1^\alpha=\frac{\partial h_1}{\partial\psi_\alpha}=2\frac{\partial h_1}{\partial \mathrm{G}_2}\psi^\alpha\tag{37}$$

so that equation (33) becomes

$$\bar{\mathrm{T}}_r^{\alpha\beta}=4\frac{\partial^2 h_1}{\partial(\mathrm{G}_2)^2}\psi^\alpha\psi^\beta+2\frac{\partial h_1}{\partial \mathrm{G}_2}g^{\alpha\beta}\tag{38}$$

which must be compared with the expression (23).

For this purpose let us first introduce the representations for n^μ and H^μ, *i.e.*,

$$n^\mu=a\lambda^\mu+b\psi^\mu;\qquad \mathrm{H}^\mu=c\lambda^\mu+d\psi^\mu\tag{39}$$

where a, b, c, d are related by

$$a^2\mathrm{G}_1+b^2\mathrm{G}_2+2ab\mathrm{G}+1=0\tag{40}$$

$$c(a\,G_1+b\,G)+d(b\,G_2+a\,G)=0 \tag{41}$$
$$J^2 f^2 = H^\mu H_\mu = c^2 G_1 + d^2 G_2 + 2\,c\,dG \tag{42}$$

in order to satisfy the conditions $n_\mu n^\mu = -1$; $n_\mu H^\mu = 0$ and the definition $J^2 f^2 = H^\mu H_\mu$.

By substituting the expressions (39) in (23) and comparing with (38) one obtains

$$4\frac{\partial^2 h_1}{\partial G_2^2} = \frac{1}{3}(4-\varphi)\,J\,b^2 + 2\,b\,d + \varphi\frac{1}{Jf^2}d^2 \tag{43}$$
$$0 = \frac{1}{3}(4-\varphi)\,J\,a^2 + 2\,ac + \varphi\frac{1}{Jf^2}c^2 \tag{44}$$
$$0 = \frac{1}{3}(4-\varphi)\,J\,ab + ad + bc + \varphi\frac{1}{Jf^2}cd \tag{45}$$
$$2\frac{\partial h_1}{\partial G_2} = \frac{1}{3}J(1-\varphi) \tag{46}$$

Now the sum of relation (43) multiplied by G_2, relation (44) by G_1, (45) by $2\,G$ and (46) by 4 gives

$$4\,G_2\frac{\partial^2 h_1}{\partial G_2^2} + 8\frac{\partial h_1}{\partial G_2} = 0,$$

[where relations (40), (41) have been also used]; this is a differential equation for the unknown function h_1 whose solution is

$$h_1 = -\gamma\,G_2^{-1} + \bar{\gamma} \tag{47}$$

with γ, $\bar{\gamma}$ constants arising from the integration. The sum of equations (45) and (44) multiplied times $-\frac{b}{a}$ gives

$$0 = (ad - bc)\left(1 + \frac{\varphi}{Jf^2}\,\frac{c}{a}\right)$$

from which

$$c = -a\,J f^2\,\varphi^{-1} \tag{48}$$

which substituted in equation (44) gives $\varphi^2 - 4\,\varphi + 3f^2 = 0$ from which the aforesaid expression (27) directly follows.

After that the relations (40) – (42), (46), (48) give a, c, d, J, f as functions of G_1, G_2, G, *i.e.*,

$$a = G_1^{-1}\left(-b\,G \pm \sqrt{b^2(G^2 - G_1 G_2) - G_1}\right) \tag{49}$$
$$c = -6\,a\gamma f^2 G_2^{-2}[\varphi(1-\varphi)]^{-1} \tag{50}$$
$$d = 6\,a\gamma f^2 G_2^{-2}[\varphi(1-\varphi)(b\,G_2 + a\,G)]^{-1}(a\,G_1 + b\,G) \tag{51}$$
$$J = 6\,\gamma\,G_2^{-2}(1-\varphi)^{-1} \tag{52}$$

$$f^2=16a^2(G^2-G_1G_2)(bG_2+aG)^2 \times[(G^2-G_1G_2)a^2+3(bG_2+aG)^2]^{-2} \quad (53)$$

while b remains arbitrary. [It is obvious that $G^2-G_1G_2>0$; in fact from $\lambda^\mu\lambda_\mu=-\frac{1}{T^2}$ we have that λ^μ is time-like and then its direction can be taken as o-axis of a reference frame, while the 1-axis can be chosen such that $\lambda^\mu=\left(\frac{1}{T}, 0, 0, 0\right)$; $\psi^\mu=(-GT, \psi^1, 0, 0)$ from which $G^2-G_1G_2=\frac{1}{T^2}(\psi^1)^2>0.$]

It could be proved that (52), (53) are invertible and give G, G_2 as functions of J and f; after that from (39), (36) one could obtain ψ^μ, λ^μ, λ as functions of the variables e, n_0, J, u^μ, H^μ. We omit the proof because such invertibility is also a consequence of the hyperbolicity that will be proved in the next section. We conclude by observing that the function h'^α in relation (31), by means of the results $(34)_4$, (37), (47) assumes the form

$$h'^\alpha=p(\lambda, G_1)\lambda^\alpha+2\gamma G_2^{-2}\psi^\alpha \quad (54)$$

while the field equations (21), by means of the system (30), become

$$\frac{\partial^2 h'^\alpha}{\partial F_A\,\partial F_B}F_{B,\alpha}=g^A \quad (55)$$

where

$A, B=0, \ldots, 8$; $F_8=\lambda$, $F_\mu=\lambda_\mu$; $F_{4+\mu}=\psi_\mu g^8=0$; $g^\mu=4\pi f^\mu$; $g^{4+\mu}=-f^\mu$

for $\mu=0, 1, 2, 3$; that is a symmetric system of partial differential equations.

In the next section we shall prove that it is hyperbolic in the time direction of u^α [it is hyperbolic in every time direction if a further condition is verified, *i.e.* condition (60)].

Therefore the use of the elements of the mean field as variables has the advantage of giving a symmetric hyperbolic system.

Moreover, if we use equation (31) such system splits up into two independent systems (except for the second members) *i.e.*, into the five equations

$$\frac{\partial^2 h_0^\alpha}{\partial\lambda^2}\gamma_{,\alpha}+\frac{\partial^2 h_0^\alpha}{\partial\lambda_\mu\,\partial\lambda}\lambda_{\mu,\alpha}=0$$

$$\frac{\partial^2 h_0^\alpha}{\partial\lambda\,\partial\lambda_\beta}\lambda_{,\alpha}+\frac{\partial^2 h_0^\alpha}{\partial\lambda_\mu\,\partial\lambda_\beta}\lambda_{\mu,\alpha}=4\pi f^\beta$$

for the five unknowns λ, λ_μ and into the four equations

$$\frac{\partial^2 h_1^\alpha}{\partial\psi_\mu\,\partial\psi_\beta}\lambda_{\mu,\,\alpha} = -f^\beta$$

for the four unknowns ψ_μ.

Obviously in these systems we have

$$h_0^\alpha = p(\lambda, G_1)\lambda^\alpha \tag{56}$$

$$h_1^\alpha = 2\gamma\, G_2^{-2}\psi^\alpha \tag{57}$$

6. THE HYPERBOLICITY REQUIREMENT

So far we have obtained the closure (27) for the system (21) such that this system can be put in the symmetric form (55) if the Lagrange Multipliers are taken as variables.

Now symmetric systems are very nice to treat because with only one further assumption, *i.e.* that the function $h'^\alpha\xi_\alpha$ is convex for a time-like 4-vector ξ_α, they give as results that

1) all the eigenvalues are real;

2) there is a basis of $\mathbb{R}^9$ (in this case) constituted by corresponding eigenvectors.

In other words this means that the system is hyperbolic.

We shall prove now that the convexity of $h'^\alpha\xi_\alpha$ is verified only by one of the closure conditions (27), *i.e.*, by

$$\varphi(f) = 2 - \sqrt{4 - 3f^2} \tag{58}$$

and moreover only for $0 < f < 1$.

To this end let us firstly prove that $G_2 < 0$ is a necessary condition for the convexity of $h'^\alpha\xi_\alpha$; in fact let us consider the quadratic form

$$Q = \xi_\alpha \frac{\partial^2 h'^\alpha}{\partial F_A\,\partial F_B}\,\delta F_A\,\delta F_B. \tag{59}$$

For a variation in which $\delta\lambda = 0$; $\delta\lambda_\mu = 0$, Q becomes

$$\bar{Q} = \xi_\alpha \frac{\partial^2 h'^\alpha}{\partial\psi_\mu\,\partial\psi_\nu}\,\delta\psi_\mu\,\delta\psi_\nu.$$

We can evaluate $\bar{Q}$ in the reference frame where $\xi^\mu \equiv (\xi^0, 0, 0, 0)$; $\psi^\mu \equiv (\psi^0, \psi^1, 0, 0)$ and consider the particular variations with $\delta\psi^1 = \delta\psi^3 = 0$; in this case it becomes

$$\bar{Q} = 8\gamma\, G_2^{-3}\xi_0\psi^0\{(\delta\psi_0)^2[-6\,G_2^{-1}(\psi^0)^2 - 3] + (\delta\psi_2)^2\}.$$

But the convexity requirement is satisfied only if the coefficients of $(\delta\psi_0)^2$ and $(\delta\psi_2)^2$ in $\bar{Q}$ have the same sign.

Consequently condition $G_2<0$ is necessary for convexity. Now if we substitute c, d, J from (50), (51), (52) into (42) we obtain

$$(b\,G_2+a\,G)^2\,\varphi^2=(G^2-G_1\,G_2)\,a^2\,f^2.$$

By using this relation and (27), (40) we obtain

$$\begin{aligned}\frac{4}{3}\,\frac{\varphi(1-\varphi)}{f^2}\,(b\,G_2+a\,G)^2 &= \frac{4\,\varphi-4\,\varphi^2}{3\,f^2}\,(b\,G_2+a\,G)^2=\frac{3\,f^2-3\,\varphi^2}{3\,f^2}.(b\,G_2+a\,G)^2\\ &=f^{-2}\,[f^2\,(b\,G_2+a\,G)^2-(G^2-G_1\,G_2)\,a^2\,f^2]=-G_2>0.\end{aligned}$$

As consequence of this relation we obtain $\varphi(1-\varphi)>0$; this condition is not satisfied by $\varphi(f)=2+\sqrt{4-3\,f^2}$ and thus from the two closure conditions (27) only one can be taken; moreover, for such an expression of $\varphi(f)$, the condition $\varphi(1-\varphi)>0$ is satisfied iff $0<f<1$.

We have until now proved that the closure condition (58) for $0<f<1$ is a necessary condition for the convexity requirement. Let us now look for a sufficient condition. To this end let us observe that $Q=Q_1+Q_2$ where

$$Q_1=\xi_\alpha\left[\frac{\partial^2 h_0^\alpha}{\partial\lambda^2}\,(\delta\lambda)^2+2\,\frac{\partial^2 h_0^\alpha}{\partial\lambda\,\partial\lambda_\mu}\,\delta\lambda\,\delta\lambda_\mu+\frac{\partial^2 h_0^\alpha}{\partial\lambda_\mu\,\partial\lambda_\nu}\,\delta\lambda_\mu\,\delta\lambda_\nu\right];$$

$$Q_2=\xi_\alpha\,\frac{\partial^2 h_1^\alpha}{\partial\psi_\mu\,\partial\psi_\nu}\,\delta\psi_\mu\,\delta\psi_\nu$$

and then Q is positive (or negative) definite if so are both Q_1 and Q_2. Now, by using the relations (32) we obtain

$$Q_1=\xi_\alpha\,[\delta(n_0\,u^\alpha)\,\delta\lambda+\delta T_m^{\beta\alpha}\,\delta\lambda_\beta]$$

which by using (22), (36) and (24) becomes

$$\begin{aligned}Q_1=\left|\frac{\partial p}{\partial\lambda}\right|\left(\frac{\partial p}{\partial\lambda}\right)^{-1}\Bigg\{\xi_\alpha\,u^\alpha\Bigg[e_T\,T^{-2}\,(\delta T)^2+\frac{p_{n_0}}{n_0\,T}\,(\delta n_0)^2\\ +\frac{e+p}{T}\,\delta u^\beta\,\delta u_\beta\Bigg]+2\,T^{-1}\,p_T\,\xi_\alpha\,\delta u^\alpha\,\delta T+2\,P_{n_0}\,T^{-1}\,\xi_\alpha\,\delta u^\alpha\,\delta n_0\Bigg\}\end{aligned}$$

where we have taken into account that, from (24) it follows that

$$\frac{\partial s}{\partial n_0}=\frac{1}{n_0\,T}\,e_{n_0}-\frac{e+p}{n_0^2\,T};\qquad \frac{\partial s}{\partial T}=\frac{1}{n_0\,T}\,e_T,$$

whose symmetry condition gives $-n_0\,e_{n_0}-T\,p_T+e+p=0$. In the reference frame in which $u^\alpha\equiv(1,0,0,0)$ (and consequently $\delta u^0=0$ because $u_\mu\,\delta u^\mu=0$) the above expression for Q_1 becomes

$$Q_1=\left|\frac{\partial p}{\partial\lambda}\right|\left(\frac{\partial p}{\partial\lambda}\right)^{-1}\frac{\xi_0}{T}\,M_{AB}\,\delta X^A\,\delta X^B$$

where $\delta X^A = (\delta u^1, \delta u^2, \delta u^3, \delta T, \delta n_0)$ and M_{AB} is the matrix

$$M_{AB} = \begin{pmatrix} e+p & 0 & 0 & P_T \xi_1/\xi_0 & P_{n_0} \xi_1/\xi_0 \\ 0 & e+p & 0 & P_T \xi_2/\xi_0 & P_{n_0} \xi_2/\xi_0 \\ 0 & 0 & e+p & P_T \xi_3/\xi_0 & P_{n_0} \xi_3/\xi_0 \\ P_T \xi_1/\xi_0 & P_T \xi_2/\xi_0 & P_T \xi_3/\xi_0 & e_T T^{-1} & 0 \\ P_{n_0} \xi_1/\xi_0 & P_{n_0} \xi_2/\xi_0 & P_{n_0} \xi_3/\xi_0 & 0 & P_{n_0}/n_0 \end{pmatrix}$$

Let M_i be the determinant of the matrix obtained taking the first i rows and the first i columns of M_{AB}; we have

$$M_1 = (e+p); \qquad M_2 = (e+p)^2; \qquad M_3 = (e+p)^3;$$
$$M_4 = (e+p)^2 [T^{-1} (e+p) e_T - (p_T)^2 + (\xi_0)^{-2} (p_T)^2]$$
$$M_5 = (n_0 T)^{-1} p_{n_0} (e+p)^2 \{(e+p) e_T - T (p_T)^2 - n_0 p_{n_0} e_T + [T (p_T)^2 + n_0 p_{n_0} e_T] (\xi_0)^{-2}\}$$

and consequently the quadratic form $Q_1\left(\dfrac{\partial p}{\partial \lambda}\right)(\xi_0)$ is certainly positive definite for $\xi_\alpha = u_\alpha$ (because we have $p_{n_0} > 0$, $e_T > 0$, the classical stability conditions on compressibility and specific heat; moreover in this case $\xi^0 = 1$); if instead this result is desired for every time-like 4-vector ξ_α the following condition must hold:

$$(e+p) e_T - T (p_T)^2 \geqq n_0 p_{n_0} e_T. \tag{60}$$

This condition is more important than the mere convexity along the time direction of u_α; in fact, as shown by Strumia, [16] it assures that the speeds of the shocks can not exceed the speed of light.

From the same paper [16] we learn that the equation (60) would be surely satisfied if another requirement is imposed; *i.e.* that the characteristic velocities corresponding to u^α do not exceed that of light. We can verify this statement also in this case; in fact the characteristic velocities are the roots of the following equation in the unknown μ:

$$0 = \det \begin{pmatrix} \mu T^{-1} (e+p) & 0 & 0 & T^{-1} p_T \eta_1 & T^{-1} p_{n_0} \eta_1 \\ 0 & \mu T^{-1} (e+p) & 0 & T^{-1} p_T \eta_2 & T^{-1} p_{n_0} \eta_2 \\ 0 & 0 & \mu T^{-1} (e+p) & T^{-1} p_T \eta_3 & T^{-1} p_{n_0} \eta_3 \\ T^{-1} p_T \eta_1 & T^{-1} p_T \eta_2 & T^{-1} p_T \eta_3 & T^{-2} \mu e_T & 0 \\ T^{-1} p_{n_0} \eta_1 & T^{-1} p_{n_0} \eta_2 & T^{-1} p_{n_0} \eta_3 & 0 & \mu p_{n_0} (n_0 T)^{-1} \end{pmatrix}$$

where η_α is such that $u_\alpha \eta^\alpha = 0$; $\eta^\alpha \eta_\alpha = 1$.

The solutions are $\mu = 0$ with multiplicity 3 and

$$\mu^2 = [e_T (e+p)]^{-1} [T (p_T)^2 + n_0 p_{n_0} e_T],$$

and thus by imposing that this value does not exceed the speed of light $c = 1$, we obtain again the condition (60).

It remains now to investigate if Q_2 is definite positive (or negative) with φ given by (58) and for $0<f<1$ (and consequently $G_2<0$). By using the expression (57) we obtain

$$Q_2 = \xi_\alpha \psi^\alpha [48\gamma G_2^{-4} (\psi^\mu \delta\psi_\mu)^2 - 8\gamma G_2^{-3} \delta\psi_\mu \delta\psi^\mu] - 16\gamma G_2^{-3} \psi^\mu \delta\psi_\mu \xi^\nu \delta\psi_\nu.$$

Now from $G_2<0$ we see that ψ^μ is time-like and then a reference frame can be chosen where $\psi^\mu \equiv ((-G_2)^{1/2}, 0, 0, 0)$. There we have

$$\begin{aligned} Q_2 &= 8\gamma(-G_2)^{-5/2} \xi_0 [3(\delta\psi_0)^2 + 2\delta\psi_0 \delta\psi_1 (\xi_1/\xi_0) + 2\delta\psi_0 \delta\psi_2 (\xi_2/\xi_0) \\ &\quad + 2\delta\psi_0 \delta\psi_3 (\xi_3/\xi_0) + (\delta\psi_1)^2 + (\delta\psi_2)^2 + (\delta\psi_3)^2] \\ &= 8\gamma(-G_2)^{-5/2} \xi_0^{-1} \{(\xi_0 \delta\psi_1 + \xi_1 \delta\psi_0)^2 + (\xi_0 \delta\psi_2 + \xi_2 \delta\psi_0)^2 \\ &\quad + (\xi_0 \delta\psi_3 + \xi_3 \delta\psi_0)^2 + [1 + 2(\xi_0)^2](\delta\psi_0)^2\} \end{aligned}$$

and then $Q_2 \gamma \xi_0$ is positive definite for every time-like ξ_α; consequently also the speeds of the shocks and the characteristic speeds do not exceed the speed of light.

Finally $Q = Q_1 + Q_2$ is positive (or negative) definite iff $\gamma \dfrac{\partial p}{\partial \lambda} > 0$, which is a restriction on the arbitrariness of the constant γ and of the sign of $\dfrac{\partial p}{\partial \lambda}$.

7. CONCLUSIONS

The results of this paper have been obtained only by imposing the entropy and relativity principles and the hyperbolicity requirement along every time direction ξ_α. They are very satisfactory; in fact we have obtained the expression (58) for the function $\varphi(f, J)$ so that the system of partial differential equations (21), is closed; moreover it is hyperbolic and can be put in the form of a symmetric hyperbolic system (30) if the Lagrange Multipliers are taken as independent variables.

This fact assures nice properties such as the well posedness of the Cauchy problem for smooth initial data, *i.e.* existence, uniqueness and continuous dependence in a neighborhood of the initial manifold [17].

We notice that the entropy and relativity principle have not univocally selected the 4-velocity n^μ of the general relativistic reference frame in which the closure condition (10) holds. This means that the choice of the reference frame in which to impose the closure condition (10) can be made only on physical ground.

We believe that the choice of the reference frame in which to write the equations governing the motion of a radiative fluid is a fundamental questions for radiation hydrodynamics. In our case this amounts to finding the reference system which describes the radiation field by a distribution function whose multipole series can be reasonably truncated at second level [18].

Although the comoving reference frame can be a good choice for many situations, we believe that in general the optimal choice is the "thermal rest-frame" as determined by the entropy current [19].

Finally we want to stress that our system of nine equations in nine unknowns, written using as variables the Lagrange Multipliers, splits in two systems of five and four equations respectively [*see* equations (32) and (33)] whose differential parts are independent.

This can significatively simplify the numerical resolution of our system.

APPENDIX

Here we have to prove the property of system (30) in section V, *i.e.* that the first members of $(30)_{1,2}$ do not depend of ψ_β. Now the symmetry of $T_m^{\alpha\beta}$ proves that there is a function W such that $h'^\alpha = \dfrac{\partial W}{\partial \lambda^\alpha}$ as we can see from of reference [13]. Moreover W is a scalar-valued function of λ, λ^α, ψ^α and consequently W can be expressed as a function of λ, G_1, G_2 and $G = \lambda^\alpha \psi_\alpha$ (*see* reference [14] for a proof of this statement); from this fact it follows that

$$h'^\alpha = \frac{\partial W}{\partial \lambda^\alpha} = 2\,\frac{\partial W}{\partial G_1}\,\lambda^\alpha + \frac{\partial W}{\partial G}\,\psi^\alpha.$$

Thus the system (30) becomes

$$\left.\begin{aligned}
n_0\,u^\alpha &= 2\,\frac{\partial^2 W}{\partial G_1\,\partial\lambda}\,\lambda^\alpha + \frac{\partial^2 W}{\partial\lambda\,\partial G}\,\psi^\alpha \\
T_m^{\alpha\beta} &= 4\,\frac{\partial^2 W}{\partial G_1^2}\,\lambda^\alpha\lambda^\beta + 4\,\frac{\partial^2 W}{\partial G_1\,\partial G}\,\lambda^{(\alpha}\psi^{\beta)} \\
&\quad + 2\,\frac{\partial W}{\partial G_1}\,g^{\alpha\beta} + \frac{\partial^2 W}{\partial G^2}\,\psi^\alpha\psi^\beta \\
\bar{T}_r^{\alpha\beta} &= 2\,\frac{\partial^2 W}{\partial G_1\,\partial G}\,\lambda^\alpha\lambda^\beta + 4\,\frac{\partial^2 W}{\partial G_1\,\partial G_2}\,\lambda^\alpha\psi^\beta \\
&\quad + \frac{\partial^2 W}{\partial G^2}\,\lambda^\beta\psi^\alpha + 2\,\frac{\partial^2 W}{\partial G\,\partial G_2}\,\psi^\alpha\psi^\beta + \frac{\partial W}{\partial G}\,g^{\alpha\beta}
\end{aligned}\right\} \tag{A.1}$$

where the symmetry of $\bar{T}_r^{\beta\alpha}$ implies that

$$\frac{\partial^2 W}{\partial G^2} = 4\,\frac{\partial^2 W}{\partial G_1\,\partial G_2}. \tag{A.2}$$

From $(A.1)_1$ we obtain

$$\left.\begin{aligned} n_0 &= \left[-4\left(\frac{\partial^2 W}{\partial G_1\,\partial\lambda}\right)^2 G_1 - \left(\frac{\partial^2 W}{\partial\lambda\,\partial G}\right)^2 G_2 - 4\,\frac{\partial^2 W}{\partial\lambda\,\partial G_1}\,\frac{\partial^2 W}{\partial\lambda\,\partial G}\,G\right]^{1/2} \\ u^\alpha &= n_0^{-1}\left(2\,\frac{\partial^2 W}{\partial G_1\,\partial\lambda}\,\lambda^\alpha + \frac{\partial^2 W}{\partial\lambda\,\partial G}\,\psi^\alpha\right) \end{aligned}\right\} \quad (A.3)$$

and from $(A.1)_2$ by using the expression (22) for $T_m^{\beta\alpha}$ we obtain:

$$\left.\begin{aligned} &p = 2\,\frac{W}{\partial G_1}; \qquad (e+p)\left(\frac{\partial^2 W}{\partial G_1\,\partial\lambda}\right)^2 = n_0^2\,\frac{\partial^2 W}{\partial G_1^2}; \\ &(e+p)\,\frac{\partial^2 W}{\partial G_1\,\partial\lambda}\cdot\frac{\partial^2 W}{\partial G\,\partial\lambda} = n_0^2\,\frac{\partial^2 W}{\partial G_1\,\partial G}; \\ &(e+p)\left(\frac{\partial^2 W}{\partial\lambda\,\partial G}\right)^2 = n_0^2\,\frac{\partial^2 W}{\partial G^2} \end{aligned}\right\} \quad (A.4)$$

Let us consider firstly the case $\dfrac{\partial^2 W}{\partial G^2}=0$; from (A.4) we obtain then $\dfrac{\partial^2 W}{\partial\lambda\,\partial G} = \dfrac{\partial^2 W}{\partial G_1\,\partial G} = 0$ *i.e.* the function $\dfrac{\partial W}{\partial G}$ does not depend on G or λ or G_1; then it is a function of G_2 that integrated with respect to G gives $W = G f(G_2) + f_1(G_1, G_2, \lambda)$ where f_1 is the constant (with respect to G) arising from the integration.

Substituting this expression in (A.2) we obtain $\dfrac{\partial^2 f_1}{\partial G_1\,\partial G_2} = 0$ and then

$$W = G f(G_2) + f_2(G_1, \lambda) + f_3(G_2, \lambda)$$

which, substituted in $(A.1)_{1,2}$ gives $n_0 u^\alpha$ and $T_m^{\alpha\beta}$ as functions depending only on λ, λ_β as we desired to prove. $\Bigg($Moreover we have $h'^\alpha = 2\,\dfrac{\partial f_2(G_1, \lambda)}{\partial G_1}\,\lambda^\alpha + f(G_2)\,\psi^\alpha$, *i.e.* h'^α is sum of a function depending only on λ, λ^α and of a function depending only on $\psi^\alpha\Bigg)$.

No other cases are possible because we shall see now that if we suppose $\dfrac{\partial^2 W}{\partial G^2} \neq 0$, we obtain an absurd.

In fact from (23) we have $\bar{T}_r^{\mu\nu} g_{\mu\nu} = 0$, which imposed on $(A.1)_3$ gives

$$\frac{\partial^2 W}{\partial G_2\,\partial G}\,G_2 + \frac{\partial^2 W}{\partial G_1\,\partial G}\,G_1 + \frac{\partial^2 W}{\partial G^2}\,G + 2\,\frac{\partial W}{\partial G} = 0,$$

where (A.2) has been used; but we can express W as a composite function by means of another function $\tilde{W}(x, y, z, \lambda)$, *i. e.* $W = G_2^{-1} \tilde{W}(GG_2^{-1}, G_1 G_2^{-1}, G_2, \lambda)$ which, substituted in the above relation expresses it in the form $\frac{\partial^2 \tilde{W}}{\partial z \, \partial x} = o$. We then have

$$W = G_2^{-1} W_1 (GG_2^{-1}, G_1 G_2^{-1}, \lambda) + G_2^{-1} W_2 (G_1 G_2^{-1}, G_2, \lambda) \quad (A.5)$$

where the arbitrary functions $W_1(x, y, \lambda)$ and $W_2(y, z, \lambda)$ have been used. Now in $(A.4)_4$ we have $\frac{\partial^2 W}{\partial \lambda \, \partial G} \neq 0$ $\left(\text{because } \frac{\partial^2 W}{\partial G^2} \neq 0\right)$ and then it gives the function $(e+p)$ which substituted in $(A.4)_{2,3}$ gives

$$\begin{gathered} \frac{\partial^2 W}{\partial G^2}\left(\frac{\partial^2 W}{\partial G_1 \, \partial \lambda}\right)^2 = \frac{\partial^2 W}{\partial G_1^2}\left(\frac{\partial^2 W}{\partial \lambda \, \partial G}\right)^2 \\ \frac{\partial^2 W}{\partial G_1 \, \partial \lambda} \frac{\partial^2 W}{\partial G^2} = \frac{\partial^2 W}{\partial G_1 \, \partial G} \frac{\partial^2 W}{\partial \lambda \, \partial G} \end{gathered} \quad (A.6)$$

which multipled times $-\frac{\partial^2 W}{\partial G_1 \, \partial \lambda}$ transforms the expression of the precedent relation in

$$\frac{\partial^2 W}{\partial G_1 \, \partial G} \frac{\partial^2 W}{\partial G_1 \, \partial \lambda} = \frac{\partial^2 W}{\partial G_1^2} \frac{\partial^2 W}{\partial \lambda \, \partial G}$$

$\left(\text{where } \frac{\partial^2 W}{\partial \lambda \, \partial G} \neq 0 \text{ has again been used}\right)$; in this relation we may substitute $\frac{\partial^2 W}{\partial G_1 \, \partial \lambda}$ from (A.6) obtaining

$$\left(\frac{\partial^2 W}{\partial G_1 \, \partial G}\right)^2 = \frac{\partial^2 W}{\partial G_1^2} \frac{\partial^2 W}{\partial G^2}. \quad (A.7)$$

By using (A.5) the relations (A.2), (A.7) become

$$\frac{\partial^2 W_1}{\partial x^2} + 4x \frac{\partial^2 W_1}{\partial x \, \partial y} + 4y \frac{\partial^2 W_1}{\partial y^2} + 8 \frac{\partial W_1}{\partial y} = -8 \frac{\partial W_2}{\partial y} - 4y \frac{\partial^2 W_2}{\partial y^2} + 4z \frac{\partial W_2}{\partial z} \quad (A.8)$$

$$\left(\frac{\partial^2 W_1}{\partial x \, \partial y}\right)^2 - \frac{\partial^2 W_1}{\partial x^2} \frac{\partial^2 W_1}{\partial y^2} = \frac{\partial^2 W_1}{\partial x^2} \frac{\partial^2 W_2}{\partial y^2} \quad (A.9)$$

where $\frac{\partial^2 W_1}{\partial x^2} \neq 0$ because $0 \neq \frac{\partial^2 W}{\partial G^2} = G_2^{-3} \frac{\partial^2 W_1}{\partial x^2}$; now $\frac{\partial^2 W_2}{\partial y^2}$ depends only on y, z, λ; but from (A.9) we learn that it does not depend on z and then is a function only of y, λ; it can be integrated and then gives that

W_2 can be expressed as

$$W_2 = f_0(y, \lambda) + z^2 f_1(z, \lambda) y + f_2(z, \lambda) \tag{A.10}$$

where the coefficient z^2 before f_1 has been introduced for later convenience. Substituting W_2 from (A.10) in (A.8) we find

$$\frac{\partial^2 W_1}{\partial x^2} + 4x \frac{\partial^2 W_1}{\partial x \partial y} + 4y \frac{\partial^2 W_1}{\partial y^2} + 8 \frac{\partial W_1}{\partial y} + 8 \frac{\partial f_0(y, \lambda)}{\partial y} + 4y \frac{\partial f_0(y, \lambda)}{\partial y^2} = 4z^3 \frac{\partial f_1(z, \lambda)}{\partial z}$$

whose first member does not depend on z and whose second member is consequently a function only of λ, *i. e.*,

$$z^3 \frac{\partial f_1(z, \lambda)}{\partial z} = 2 f_3(\lambda),$$

from which $f_1(z, \lambda) = -f_3(\lambda) z^{-2} + f_4(\lambda)$ which substituted in (A.10) and then in (A.5) gives

$$W = z^{-1} [W_1(x, y, \lambda) + f_0(y, \lambda) - f_3(\lambda)] + z^{-1} [f_2(z, \lambda) + z^2 f_4(\lambda)]. \tag{A.11}$$

But $W_1(x, y, \lambda)$ is an arbitrary function of x, y, λ; then the same thing can be said for $W_1^*(x, y, \lambda) = W_1(x, y, \lambda) + f_0(y, \lambda) - f_3(\lambda)$. Moreover the term $z^{-1}[f_2(z, \lambda) + z^2 f_4(\lambda)]$ does not give any contribution to h'^{α} and then it does not play any role in the subsequent equations we have obtained from it (as could be easily verified); consequently, without loss of generality, we can take $f_2 = f_4 = 0$.

After that, equation (A.11) becomes $W = z^{-1} W_1^*(x, y, \lambda)$ *i. e.*, the equation (A.5) is satisfied with $W_2 = 0$ and with W_1^* instead of W_1; but it is useless to replace an arbitrary function as W_1 by another arbitrary function W_1^* of the same variables; then we can simply assume equation (A.5) with

$$W_2 = 0. \tag{A.12}$$

After that equations (A.8) and (A.9) become

$$\frac{\partial^2 W_1}{\partial x^2} + 4x \frac{\partial^2 W_1}{\partial x \partial y} + 4y \frac{\partial^2 W_1}{\partial y^2} + 8 \frac{\partial W_1}{\partial y} = 0 \tag{A.13}$$

$$\left(\frac{\partial^2 W_1}{\partial x \partial y} \right)^2 - \frac{\partial^2 W_1}{\partial x^2} \frac{\partial^2 W_1}{\partial y^2} = 0. \tag{A.14}$$

Moreover equation (A.6) becomes

$$\frac{\partial^2 W_1}{\partial y \partial \lambda} \frac{\partial^2 W_1}{\partial x^2} - \frac{\partial^2 W_1}{\partial x \partial y} \frac{\partial^2 W_1}{\partial x \partial \lambda} = 0. \tag{A.15}$$

Defining F (x, y, λ) from

$$\frac{\partial^2 W_1}{\partial x\,\partial y}=F\frac{\partial^2 W_1}{\partial x^2} \tag{A.16}$$

equations (A.14), (A.13) and (A.15) become respectively

$$\frac{\partial^2 W_1}{\partial y^2}=F^2\frac{\partial^2 W_1}{\partial x^2} \tag{A.17}$$

$$\frac{\partial^2 W_1}{\partial x^2}(1+4xF+4yF^2)+8\frac{\partial W_1}{\partial y}=0 \tag{A.18}$$

$$\frac{\partial^2 W_1}{\partial y\,\partial \lambda}=F\frac{\partial^2 W_1}{\partial x\,\partial \lambda}. \tag{A.19}$$

Now if we take the derivative of equation (A.16) with respect to x and after that with respect to y, we have respectively

$$\frac{\partial^3 W_1}{\partial x^2\,\partial y}=\frac{\partial F}{\partial x}\,\frac{\partial^2 W_1}{\partial x^2}+F\frac{\partial^3 W_1}{\partial x^3} \tag{A.20}$$

$$\frac{\partial^3 W_1}{\partial x\,\partial y^2}=\left(\frac{\partial F}{\partial y}+p\frac{\partial F}{\partial x}\right)\frac{\partial^2 W_1}{\partial x^2}+F^2\frac{\partial^3 W_1}{\partial x^3} \tag{A.21}$$

where (A.20) has been used; if we take the derivative of equation (A.17) with respect to x and comparing with (A.21) we obtain

$$\frac{\partial F}{\partial y}=F\frac{\partial F}{\partial x}. \tag{A.22}$$

By taking the derivative of equation (A.18) with respect to x and using (A.16) we obtain

$$\frac{\partial^3 W_1}{\partial x^3}(1+4Fx+4F^2y)=-4\left(3F+x\frac{\partial F}{\partial x}+2yF\frac{\partial F}{\partial x}\right)\frac{\partial^2 W_1}{\partial x^2}. \tag{A.23}$$

By taking the derivative of equation (A.18) with respect to y, substituting in the relation so obtained the expressions of $\frac{\partial^3 W_1}{\partial x^2\,\partial y}$ and $\frac{\partial^2 W_1}{\partial y^2}$ from (A.20) and (A.17) respectively and using (A.23) we obtain

$$\frac{\partial^2 W_1}{\partial x^2}\,\frac{\partial F}{\partial x}(1+4xF+4yF^2)=0; \tag{A.24}$$

if $1+4xF+4yF^2=0$ from (A.18) it would follow that $\frac{\partial W_1}{\partial y}=0$ which substituted in (A.5) and $(A.4)_1$ would give the absurd result $p=0$.

Then (A.24) necessarily gives $\frac{\partial F}{\partial x}=0$; consequently we have $\frac{\partial F}{\partial y}=0$ from (A.22); moreover we can take the derivative of (A.19) with respect to x

and use the derivative of equation (A.16) with respect to λ, obtaining $\frac{\partial F}{\partial \lambda}=0$; we have then that F is a constant. This fact permits integration of equation (A.16) to give

$$\frac{\partial W_1}{\partial y}=F\frac{\partial W_1}{\partial x}+q(y,\lambda). \qquad (A.25)$$

Substituting (A.25) into (A.17) and (A.19) we obtain that q is a constant. We can now express W_1 as a composite function by means of another function $W_3(\omega, y, \lambda)$ where $\omega=x+Fy$, *i. e.*

$$W_1=W_3(x+Fy, y, \lambda)+qy$$

so that equation (A.25) becomes $\frac{\partial W_3}{\partial y}=0$ so that

$$W_1=W_3(x+Fy, \lambda)+qy.$$

Substituting this expression in (A.18) we obtain

$$\frac{\partial^2 W_3}{\partial \omega^2}(1+4F\omega)+8F\frac{\partial W_3}{\partial \omega}+8q=0. \qquad (A.26)$$

If $F=0$ this expression can be integrated and gives

$$W_3=-4q\omega^2+\delta(\lambda)\omega+\nu(\lambda);$$

but in this case we have also $x=\omega$,

$$W_1=-4qx^2+\delta(\lambda)x+\nu(\lambda)+qy;$$
$$W=-4qG^2G_2^{-3}+\delta(\lambda)GG_2^{-2}+\nu(\lambda)G_2^{-1}+qG_1G_2^{-2}$$

and then $\frac{\partial^2 h'^\alpha}{\partial\lambda_\beta\,\partial\lambda_\gamma}=0$ against the requested convexity of $h'^\alpha\xi_\alpha$.

Therefore the case $F=0$ must not be considered. The case $F\neq 0$ remains; in this case (A.26) can be integrated and give

$$W_3=F^{-1}[-(1+4F\omega)^{-1}\delta(\lambda)/4-q\omega+\nu(\lambda)];$$

From this relation and (A.5), $(A.3)_1$, $(A.4)_{1,4}$ we find

$$n_0=|\delta'(\lambda)|\,\eta^{-3/2} \qquad \text{and} \qquad p=\frac{1}{3}e=2F\delta(\lambda)\eta^{-2} \qquad (A.27)$$

(from which $T^\alpha_m\alpha=0$), where $\eta=-G_2(1+4FGG_2^{-1}+4F^2G_1G_2^{-1})$.

Consequently we have

$$d(e/n_0)+pd(1/n_0)=2F\frac{|\delta'|}{\delta'}\eta^{-1/2}d\{-\lambda+4\delta(\lambda)[\delta'(\lambda)]^{-1}\}$$

and then $T=2|F|\eta^{-1/2}$;

$$S=\frac{|\delta' F|}{\delta' F}\{-\lambda+4\delta(\lambda)[\delta'(\lambda)]^{-1}\}$$

from which $\eta=\eta(T)$; $\lambda=\lambda[S(n_0, T)]$ can be obtained. Moreover we find

$$h'^{\alpha}=\delta(\lambda)\eta^{-2}(2F\lambda^{\alpha}+\psi^{\alpha})-qG_2^{-2}F^{-1}\psi^{\alpha}$$

from which

$$\bar{T}_r^{\alpha\beta}=\frac{\partial h'^{\alpha}}{\partial\psi_{\beta}}=4\delta\eta^{-3}(2F\lambda^{\alpha}+\psi^{\alpha})(2F\lambda^{\beta}+\psi^{\beta})+\delta(\lambda)\eta^{-2}g^{\alpha\beta}$$
$$+4qF^{-1}G_2^{-3}\psi^{\alpha}\psi^{\beta}-qF^{-1}G_2^{-2}g^{\alpha\beta}$$

which compared with expression (23) and by using (39)-(42) (which hold also in this case) gives

$$4qF^{-1}G_2^{-3}+4\delta\eta^{-3}=\frac{1}{3}(4-\varphi)Jb^2+2bd+\varphi\frac{1}{Jf^2}d^2 \qquad (A.28)$$

$$16\delta F^2\eta^{-3}=\frac{1}{3}(4-\varphi)Ja^2+2ac+\varphi\frac{1}{Jf^2}c^2 \qquad (A.29)$$

$$8\delta F\eta^{-3}=\frac{1}{3}(4-\varphi)Jab+ad+bc+\varphi\frac{1}{Jf^2}cd \qquad (A.30)$$

$$\delta\eta^{-2}-qF^{-1}G_2^{-2}=\frac{1}{3}J(1-\varphi) \qquad (A.31)$$

which are to be substituted for the corresponding expressions (43)-(46). If we add to equation (A.28) the equations (A.29), (A.30), (A.31) multiplied respectively by $G_2^{-1}G_1$, $2G_2^{-1}G$, $4G_2^{-1}$ and we take into account equations (40), (41) we find an identity; consequently equation (A.28) imposes no restrictions.

The sum of equation (A.29) multiplied times (aG_1+bG) and of (A.30) times (bG_2+aG) gives

$$8\delta F\eta^{-3}[2F(aG_1+bG)+bG_2+aG]$$
$$=\frac{1}{3}(4-\varphi)Ja[a(aG_1+bG)+b(bG_2+aG)]$$
$$+2ac(aG_1+bG)+(ad+bc)(bG_2+aG)$$

that by using (40), (41) gives

$$c=-\frac{1}{3}(4-\varphi)Ja-8\delta F\eta^{-3}[2F(aG_1+bG)+bG_2+aG]$$

that substituted in equation (A.29) gives the following relation:

$$0=\frac{a^2J}{9f^2}(4-\varphi)(-\varphi^2+4\varphi-3f^2)+\frac{16}{3}\delta\eta^{-3}F[f^{-2}\varphi(4-\varphi)(2FG_1+G)$$
$$+abf^{-2}(-\varphi^2+4\varphi-3f^2)(2FG+G_2)-3F-3a^2(2FG_1+G)]$$

$$+64\,\delta^2\,\eta^{-6}\frac{\varphi}{J\,f^2}F^2\,[a(2\,FG_1+G)+b(2\,FG+G_2)]^2. \quad (A.32)$$

Now we have that from (A.27) we can obtain η and λ as functions of n_0 and e; but $\varphi(f, J)$ does not depend on n_0, e and so it cannot depend on λ and η; moreover we have from $(A.27)_1$ that $\delta'(\lambda)\neq 0$ and then $\delta(\lambda)$ is an invertible function of λ; consequently $\varphi(f, J)$ does not depend on η and δ.

Equation (A.32) can be considered as a third order algebraic equation in the unknown φ; its solutions must remain the same if we put $\delta=0$ in this equation, because φ does not depend on δ; in this way we obtain

$$\varphi=4 \qquad \text{or} \qquad \varphi=2\pm\sqrt{4-3\,f^2}.$$

Similarly the solutions of (A.32) must remain the same in the limit $\delta^{-1}\to 0$; in this way we obtain $\varphi=0$ (that we cannot accept because it contradicts the precedent result), and $a(2\,FG_1+G)=-b(2\,FG+G_2)$.

But in this case equation (40) multiplied times $(2\,FG_1+G)^2$ gives $(2\,FG_1+G)^2+b^2(G^2-G_1\,G_2)\,\eta=0$ that must be verified also for $2\,FG_1+G\neq 0$ because G_1, G and λ are indipendent variables. Then the above relation cannot hold because $G^2-G_1\,G_2\geqq 0$ [In fact if $G_1\geqq 0$ we have this result from the identity $G^2-G_1\,G_2=G_1\,\eta+(G+2\,FG_1)^2>0$, while if $G_1<0$ we may consider the reference frame in which $\lambda^\alpha\equiv(\sqrt{-G_1},0,0,0)$; $\psi^\alpha\equiv(\psi^0,\psi^1,0,0)$ and there we have $G^2-G_1\,G_2=-G_1\,(\psi^1)^2\geqq 0$.]

Then we have obtained an absurd; this fact proves that the only acceptable solution is that we found in section V.

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REFERENCES

[1] C. D. LEVERMORE, Relating Eddington Factors to Flux Limiters, *J.Q.S.R.T.*, Vol. **31**, 1984, p. 149.

[2] A. M. ANILE, S. PENNISI and M. SAMMARTINO, A Thermodynamical Approach to Eddington Factors, *J. Math. Phys.* (submitted).

[3] J. L. ANDERSON and E. A. SPIEGEL, The moment Method in Relativistic Radiative Transfer, *Ap. J.*, Vol. **171**, 1972, p. 127.

[4] S. H. HSIEH and E. A. SPIEGEL, The Equations of Photohydrodynamics, *Ap. J.*, Vol. **207**, 1976, p. 244.

[5] C. D. LEVERMORE, *Bounds on Eddington Factors*, Lawrence Livermore National Laboratory, Livermore, 1982.

[6] A. M. ANILE, *Relativistic Fluids and Magnetofluids*, Cambridge University Press, 1989.

[7] I.-SHIH LIU, Method of Lagrange Multipliers for Exploitation of the Entropy Principle, *Arch. Rat. Mech. Anal.*, Vol. **46**, 1972, p. 131.
[8] K. O. FRIEDRICHS, Symmetric Hyperbolic Linear Differential Equations, *Comm. Pure Appl. Math.*, Vol. **7**, 1954, p. 345.
[9] K. O. FRIEDRICHS and P. D. LAX, Systems of Conservation Equations with a Convex Extension, *Proc. Nat. Acad. Sci. U.S.A.*, Vol. **68**, 1971, p. 1686.
[10] G. BOILLAT, Sur l'existence et la Recherche d'Equation de Conservation Supplémentaires pour les Systèmes Hyperboliques, *C. R. Acad. Sci. Paris*, T. **278**, Series A, 1974, p. 909.
[11] T. RUGGERI, Struttura dei Sistemi alle Derivate Parziali compatibili con un Principio di Entropia, Suppl. *B.U.M.I del G.N.F.M.*, *Fisica Matematica*, Vol. **4**, 1985, p. 5.
[12] T. RUGGERI and A. STRUMIA, Main Field and Convex Covariant Density for Quasi-Linear Hyperbolic Systems; Relativistic Fluid Dynamics, *Ann. Inst. H. Poincaré, Phys. Théor.*, Vol. **34**, 1981, p. 165.
[13] S. PENNISI, Some Consideration on a Non-Linear Approach to Extended Thermodynamics, I.S.I.M.M., *Symposium of Kinetic Theory and Extended Thermodynamics*, Bologna, May 18-20, Pitagora Editrice Bologna, 1987, p. 259.
[14] S. PENNISI, Some Representation Theorems in a 4-Dimensional Inner Product Space, Suppl. *B.U.M.I. Fisica Matematica*, Vol. **5**, 1986, p. 191.
[15] S. PENNISI and M. TROVATO, Mathematical Characterization of Functions Underlying the Principle of Relaticity, *Le Matematiche* (accepted).
[16] A. STRUMIA, Wave Propagation and Symmetric Hyperbolic Systems of Conservation Laws with Constrained Field Variables. II. Symmetric Hyperbolic Systems with Constrained Fields, *Il Nuovo Cimento*, Vol. **101**, B, 1988, p. 19.
[17] A. FISHER and D. P. MARSDEN, The Einstein Evolution Equations as a First Order Quasilinear Symmetric Hyperbolic System, *Comm. Math. Phys.*, Vol. **28**, 1972, p. 1.
[18] K. S. THORNE, Relativistic Radiative Transfer: Moment Formalism, *Mon. Not. R. Astron. Soc.*, Vol. **194**, 1981, p. 439.
[19] B. CARTER, *Conductivity with Causality in Relativistic Hydrodynamics: the Regular Solution to Eckart's Problem*, Preprint, 1987.

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