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A. ECHEVERRÍA ENRÍQUEZ
M. C. MUÑOZ LECANDA

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Variational calculus in several variables: a hamiltonian approach

by

A. ECHEVERRÍA ENRÍQUEZ and M. C. MUÑOZ LECANDA

Dto. Matem. Applicada y Telemática,
Univers. Politècnica de Cataluña,
Aptdo. 30002, 08080-Barcelona, Spain

ABSTRACT. — Classical mechanics is formulated in two different ways, lagrangian mechanics and hamiltonian mechanics, and one of them is dual of the other. This paper studies the cases in which a hamiltonian formulation is possible for first order multiple integral variational calculus, that is classical field theory. In those cases we give a hamiltonian version of the theory and study the equivalence with the lagrangian formulation under suitable conditions of regularity.

RÉSUMÉ. — Il y a deux formulations standard de la mécanique classique, mécanique lagrangienne et mécanique hamiltonienne, et l'une est duale de l'autre. Cet article étudie les cas dans lesquels il est possible de faire une formulation hamiltonienne du calcul de variations multiples de premier ordre, c'est-à-dire de la théorie classique des champs. Dans ces cas nous donnons la version hamiltonienne de la théorie et l'équivalence entre toutes les deux formulations si nous avons des conditions suffisantes de régularité.

0. INTRODUCTION

As it is well known, the appropriate domain for the study of the lagrangian formulation of first order variational calculus problems, or classical field theory, is $J^1(E)$, the bundle of 1-jets of local sections of a differentiable fiber bundle $\pi: E \to X$. *See* references [1], [2] and [10].

In the one dimensional case we have that $X=\mathbb{R}$, $E=M\times\mathbb{R}$, where M is a differentiable manifold, and $J^1(E)=TM\times\mathbb{R}$ is a vector bundle. These properties of E and $J^1(E)$ allow the dualisation of the problem, via the Legendre transformation, and the construction of a hamiltonian formulation based on the existence of the 1-canonical form on T^*M.

The aim of this paper is to establish the conditions in which it is possible to formulate a hamiltonian theory, dual of the initial lagrangian one, and the study of the conditions for the equivalence between both theories.

Recently there have been several attempts to study this problem. *See* for example [4] and [8]. In these formulations, which use essentially the affine structure of $J^1(E)$, there is no way to define canonically the basic geometrical objects, like the hamiltonian function, the contact one form, etc., as in the analytical mechanics. In reference [10], the aim is directed towards local problems and their coordinate expressions.

It seems inescapable, in order to obtain a hamiltonian formulation of these problems, to add a geometrical condition. We think that this condition must be as simple as possible and general enough to cover the most classical problems in this theory. Moreover, the condition we have chosen, the existence of an horizontal subbundle of the tangent bundle of E, that is a connection on TE (*see* § 2.1), allows us to hold an strict parallelism with the geometrical formulation of the analytical mechanics.

On the basis of this condition the followings problems are studied and solved:

(1) To construct a natural dual vector bundle associated to $J^1(E)$.

(2) To obtain and characterize an analogous of the canonical Hamilton-Cartan form.

(3) To state the Hamilton-Jacobi variational problem and obtain the hamiltonian equations for a critical section.

(4) To construct a canonical Legendre transformation FL, such that the pull-back by FL of the Hamilton-Cartan form is the well known Poincaré-Cartan form, see references [5] and [6], and study a sufficient condition for the existence of a hamiltonian formulation associated to a Lagrangian.

(5) To study and characterize the ker FL_*. The case $X=\mathbb{R}$ is developed in references [2] and [3].

(6) To state the different variational problems related to a quasiregular Lagrangian and establish the conditions for their equivalence.

(7) To establish the equivalent to the classical Liouville theorem and study the hamiltonian version of the minimal surfaces problem.

This work is organized as follows:

- Point (1) can be found in Section 2.
- Points (2) and (3) are in Section 3.
- Point (4) is in Section 5.
- Points (5) and (6) are in Section 6.
- Point (7) is in Section 7.

In this last paragraph we also analyze the fundamental obstruction to the application of the theory to the electromagnetic field, that is the non regularity of the Legendre transformation or, equivalently, the existence of constraints.

This last example brings up the necessity of developing an adequate theory for the study of constrained systems in multiple dimensional variational problems, as has been recently done in the case of one dimensional problems. *See* for example [8].

1. 1-JETS BUNDLES OF SECTIONS. DUALITY

1.1. Affine bundle structure

Let $\pi: \mathrm{E} \rightarrow \mathrm{X}$ be a differentiable fibre bundle with fibre type F. Let $\dim \mathrm{X}=m$, $\dim \mathrm{F}=n$.

Let $e \in \mathrm{E}$ with $\pi(e)=x$ and U an open neihborhood of x in X. Let $s: \mathrm{U} \rightarrow \mathrm{E}$ be a local differentiable section with $s(x)=e$. Let $\Gamma(e)$ be the set of such sections when the open set U varies.

If $s, s' \in \Gamma(e)$ we say that $s \approx s'$ if and only if $\mathrm{T}_x s=\mathrm{T}_x s'$. Trivially, this is an equivalence relation and we call E_e the quotient set of $\Gamma(e)$ by this relation.

Put $\mathrm{J}^1(\mathrm{E})=\bigcup_{e \in \mathrm{E}} E_e$ and $p: \mathrm{J}^1(\mathrm{E}) \rightarrow \mathrm{E}$ for the natural projection. Then the set $\mathrm{J}^1(\mathrm{E})$ with its natural structure is a differentiable fibre bundle over E. The fibre has dimension $m \times n$. Moreover, the projection $\pi \circ p: \mathrm{J}^1(\mathrm{E}) \rightarrow \mathrm{X}$ makes $\mathrm{J}^1(\mathrm{E})$ a differentiable fibre bundle over X.

Let $\bar{e} \in \mathrm{E}_e$, so $p(\bar{e})=e$, with $\pi(e)=x$. We have that $\mathrm{E}_e \subset \operatorname{Hom}(\mathrm{T}_x \mathrm{X}, \mathrm{T}_e \mathrm{E})$, but if s is a representant of $\bar{e}$, then $\mathrm{T}_e \pi \circ \mathrm{T}_x s=\mathrm{id}_{\mathrm{T}_x \mathrm{X}}$, then E_e is the inverse image of the identity by the morphism:

$$\operatorname{Hom}(\mathrm{T}_x \mathrm{X}, \mathrm{T}_e \mathrm{E}) \xrightarrow{\eta} \operatorname{Hom}(\mathrm{T}_x \mathrm{X}, \mathrm{T}_x \mathrm{X})$$
$$\varphi \mapsto \mathrm{T}_e \pi \circ \varphi$$

We have proven that $\eta^{-1}(\mathrm{id}) \supset \mathrm{E}_e$ and the converse is trivial if you take local coordinates.

But $\ker \eta = \mathrm{Hom}(\mathrm{T}_x \mathrm{X}, \mathrm{V}_e \mathrm{E})$ where $\mathrm{V}_e \mathrm{E}$ is the set of tangent vectors of $\mathrm{T}_e \mathrm{E}$ vertical with respect to $\pi : \mathrm{E} \to \mathrm{X}$. Then if φ_e is an element of E_e we have:

$$\mathrm{E}_e = \varphi_e + \mathrm{Hom}(\mathrm{T}_x \mathrm{X}, \mathrm{V}_e \mathrm{E}) = \varphi_e + \mathrm{V}_e \mathrm{E} \otimes \mathrm{T}_x^* \mathrm{X}$$

Let $\bar{\mathrm{E}} = \bigcup_{e \in \mathrm{E}} \mathrm{V}_e \mathrm{E} \otimes \mathrm{T}_x^* \mathrm{X} = \mathrm{VE} \otimes_{\mathrm{E}} \pi^* \mathrm{T}^* \mathrm{X}$. We have proven that $\mathrm{J}^1(\mathrm{E})$ is an affine bundle on E and that the associated vector bundle is $\bar{\mathrm{E}} = \mathrm{VE} \otimes_{\mathrm{E}} \pi^* \mathrm{T}^* \mathrm{X}$, where $\pi^* \mathrm{T}^* \mathrm{X}$ is the pull-back of $\mathrm{T}^* \mathrm{X}$ to E. For more details *see* [7] and [11].

We denote by $\bar{p} : \bar{\mathrm{E}} \to \mathrm{E}$ the natural projection and write $\bar{\pi} = \pi \circ \bar{p}$.

Let $s : \mathrm{U} \to \mathrm{E}$ be a local section of π. If $x \in \mathrm{U}$ then the map $x \mapsto \mathrm{T}_x s$ gives us an element of $\mathrm{E}_{s(x)} \subset \mathrm{J}^1(\mathrm{E})$. Thus the section s allows us to construct a section $\bar{s}$ of the projection $\pi \circ p$. We call $\bar{s}$ the canonical lift, or the canonical extension, of s to $\mathrm{J}^1(\mathrm{E})$.

1.2. Duality

For any $e \in \mathrm{E}$, let W be an open neighborhood of e in E and $f : \mathrm{W} \to \mathrm{X}$ a differentiable function. Let $\mathrm{F}(\mathrm{W}; e)$ be the set of such functions that verify $f(e) = \pi(e) = x$. Let $\mathrm{F}(e)$ be the set of such functions when the open set W varies.

If $f, g \in \mathrm{F}(e)$ we put $f \approx g$ if and only if $\mathrm{T}_e f|_{\mathrm{V}_e \mathrm{E}} = T_e g|_{\mathrm{V}_e \mathrm{E}}$. This is an equivalence relation and we call $\tilde{\mathrm{E}}$ the quotient set of $\mathrm{F}(e)$ by that relation. We have $\tilde{\mathrm{E}}_e = \mathrm{Hom}(\mathrm{V}_e \mathrm{E}, \mathrm{T}_x \mathrm{X}) = \mathrm{V}_e^* \mathrm{E} \otimes \mathrm{T}_x \mathrm{X}$.

Now we put $\tilde{\mathrm{E}} = \mathrm{V}^* \mathrm{E} \otimes_{\mathrm{E}} \pi^* \mathrm{TX}$ and the natural projection $\tilde{p} : \tilde{\mathrm{E}} \to \mathrm{E}$ makes $\tilde{\mathrm{E}}$ a vector bundle on E. Its rank is $m \times n$. Let $\tilde{\pi} = \pi \circ \tilde{p}$.

The bundle $\tilde{\mathrm{E}}$ is the dual bundle of $\bar{\mathrm{E}}$ and both are the fundamental objects to construct our theory.

2. ADMISSIBLE BUNDLES AND CANONICAL LOCAL CHARTS

2.1. Definitions, examples and general properties

Let $\pi : \mathrm{E} \to \mathrm{X}$ be a differentiable fibre bundle with $\dim \mathrm{X} = m$ and $\dim \mathrm{E} = m + n$.

DEFINITION. — We say that $\pi : \mathrm{E} \to \mathrm{X}$ *is an admissible bundle if:*

(i) E *is a bundle with a connection, that is: There exists a vector subbundle* HE *of* TE *such that* $\mathrm{T}_e \mathrm{E} = \mathrm{V}_e \mathrm{E} \oplus \mathrm{H}_e \mathrm{E}$, *for all* $e \in \mathrm{E}$.

(ii) *The horizontal distribution on* E *is involutive.*

Examples:

(*a*) The trivial bundles $E = X \times F$ where F is a differentiable manifold.

(*b*) Let $\pi: E \to X$ be a vector bundle with a linear connections ∇. Associated to ∇ there exists a canonical decomposition $TE = VE \oplus HE$ where the horizontal vector fields form an involutive distribution if the curvature vanishes.

(*c*) Particular cases of (*b*) are the tensor bundles on a manifold X with a connection with null curvature.

(*d*) Let $\pi: E \to X$ be a principal bundle with a principal connection. If the curvature vanishes then the horizontal distribution is involutive.

Comment. — All this theory can be developed with condition (i) alone. Condition (ii) enables us to construct special coordinate systems as we will see in the following.

PROPOSITION. — *Let* $\pi: E \to X$ *be an admissible bundle and* $e \in E$. *There exists a local chart* (W, φ) *in* $e \in E$ *with coordinate functions* $x^1, \ldots, x^m$, $y^1, \ldots, y^n$ *such that:*

(i) $\{\partial/\partial y^i;\ i=1, \ldots, n\}$ *generate* $VE|_W$.

(ii) $\{\partial/\partial x^i;\ i=1, \ldots, m\}$ *generate* $HE|_W$.

(iii) *There exist functions* $\hat{x}^i$, *defined on* $U = \pi(W)$, *with* $x^i = \hat{x}^i \circ \pi$ *and such that* $(U, \hat{x})$ *is a local chart in* $\pi(e) \in X$.

Proof. — Let $\mathscr{D}(H)$ and $\mathscr{D}(V)$ be the horizontal and vertical distributions on E and $e \in E$. Both distributions are involutive. Hence there exists a local chart in e, (W, x^i, y^i), verifying conditions (i) and (ii).

Moreover, the functions x^i are constants along the fibres of π, then condition (iii) is true.

For more details *see* [13]. □

DEFINITION. — *If* $\pi: E \to X$ *is an admissible bundle and* (W, x^i, y^j) *is a local chart verifying the conditions of the proposition we say that it is an canonical chart.*

With these canonical charts we can obtain canonical systems of coordinates in $\bar{E}$ and $\tilde{E}$. Remember that $\bar{E} = VE \otimes_E \pi^* T^* X$ and $\tilde{E} = \bar{E}^*$.

So let $\bar{e}$ be a point in $\bar{E}$ with $e = \bar{p}(\bar{e})$ and (W, x^i, y^j) a canonical chart in $e \in E$. We can construct a canonical local chart in $\bar{E}$, $(U, \bar{x}^i, \bar{y}^j, v^j{}_i)$, in the following way:

$$U = \bar{p}^{-1}(W), \qquad \bar{x}^i = x^i \circ \bar{p},$$
$$\bar{y}^j = y^j \circ \bar{p}, \qquad v^j{}_i(\bar{e}) = \bar{e}(\partial/\partial x^i, dy^j)$$

In the same way, if $\tilde{e} \in \tilde{E}$ we have $(V, \tilde{x}^i, \tilde{y}^j, p^i{}_j)$ in $\tilde{E}$ as:

$$V = \tilde{p}^{-1}(W), \qquad \tilde{x}^i = x^i \circ \tilde{p},$$
$$\tilde{y}^j = y^j \circ \tilde{p}, \qquad p^i{}_j(\tilde{e}) = \tilde{e}(dx^i, \partial/\partial y^j)$$

For simplicity we put x, y instead of $\bar{x}$, $\bar{y}$ or $\tilde{x}$, $\tilde{y}$.

The importance of these canonical coordinate systems is that the critical sections equations are the usal ones, as we will see in the sequel.

From now on we will only speak about admissible bundles and we will only use canonical systems of coordinates. Observe that every vector field on E splits in one horizontal part and another vertical one. We put $v(\mathrm{Y})$ for the vertical component of $\mathrm{Y} \in \mathscr{X}(\mathrm{E})$.

2.2. Canonical diffeomorphisms

If $\pi : \mathrm{E} \to \mathrm{X}$ is an admissible bundle, consider the map $\mathrm{T}\pi : \mathrm{TE} \to \mathrm{TX}$. If we take $e \in \mathrm{E}$, then the restriction $\mathrm{T}\pi|_{\mathrm{H}_e \mathrm{E}} : \mathrm{H}_e \mathrm{E} \to \mathrm{T}_{\pi(e)}\mathrm{X}$ is an isomorphism. Hence we have the map

$$\sigma_c(e) : \mathrm{T}_{\pi(e)}\mathrm{X} \to \mathrm{T}_e \mathrm{E}, \qquad \sigma_c(e) = i_e \circ (\mathrm{T}\pi|_{\mathrm{H}_e \mathrm{E})})^{-1},$$

where $i_e : \mathrm{H}_e \mathrm{E} \hookrightarrow \mathrm{T}_e \mathrm{E}$ is the natural injection. Now the map

$$\begin{aligned} \sigma_c : \quad & \mathrm{E} \to \mathrm{Hom}(\pi^* \mathrm{TX}, \mathrm{TE}) \\ & e \to \sigma_c(e) \end{aligned}$$

is a differentiable section of the bundle $k : \mathrm{Hom}\,(\pi^* \mathrm{TX}, \mathrm{TE}) \to \mathrm{E}$.

PROPOSITION. — *If* $e \in \mathrm{E}$ *then* $\sigma_c(e) \in \mathrm{E}_e$, *the fibre of the bundle* $\mathrm{J}^1(\mathrm{E})$ *over the point* e.

Proof. — Take a canonical chart on E and the corresponding chart on X. Now you can construct a local section s of $\pi : \mathrm{E} \to \mathrm{X}$ such that $T_{\pi(e)} s = (\mathrm{T}_e \pi|_{\mathrm{H}_e \mathrm{E}})^{-1}$. □

Now consider the map:

$$\begin{aligned} \varphi : \quad & \mathrm{J}^1(\mathrm{E}) \to \bar{\mathrm{E}} \\ & \bar{e} \mapsto \bar{e} - \sigma_c(p(e)) \end{aligned}$$

This map is a bundle diffeomorphism because it is no more that the difference to one point in every fibre, that point is given by the section σ_c. Observe that the difference is made in the vector bundle $k : \mathrm{Hom}(\pi^* \mathrm{TX}, \mathrm{TE}) \to \mathrm{E}$, which contains $\mathrm{J}^1(\mathrm{E})$ as a subbundle (but not as a vector subbundle).

We call $\varphi : \mathrm{J}^1(\mathrm{E}) \to \bar{\mathrm{E}}$ the *canonical diffeomorphism* and it allows us to translate everything between $\bar{\mathrm{E}}$ and $\mathrm{J}^1(\mathrm{E})$.

In particular from a canonical chart on $\bar{\mathrm{E}}$, $(\mathrm{U}, x^i, y^j, v^j_i)$, we obtain one canonical chart on $\mathrm{J}^1(\mathrm{E})$, $(\varphi^{-1}(\mathrm{W}), x^i, y^j, v^j_i)$, just composing with φ. Observe that in such systems the mapping φ is the identity.

3. HAMILTONIAN FORMULATION

3.1. Structure forms and Hamilton-Cartan form

We can construct an element $\tilde{\theta} \in A^1(\tilde{E}, \pi^* TX)$, that is a 1-form on $\tilde{E}$ with values in $\pi^* TX$, in the following way:

Let $\tilde{e} \in \tilde{E} = V^* E \otimes_E \pi^* TX$ and $D \in \mathscr{X}(\tilde{E})$, a vector field tangent to $\tilde{E}$, then:

$$\tilde{\theta}(\tilde{e}; D) = \tilde{e}(v_e(T_{\tilde{e}} \tilde{p}(D)))$$

where v_e is the canonical projection $v_e : T_e E \to T_e^v E$.

DEFINITION. — *The* 1*-form* $\tilde{\theta}$ *is called the* canonical 1-form *on* $\tilde{E}$.

PROPOSITION. — *The canonical* 1*-form on* $\tilde{E}$ *is differentiable.*

Proof. — We are going to compute the coordinates of $\tilde{\theta}$ in a canonical system of coordinates (V, x^i, y^j, p^i_j). Let $\tilde{e}$ be a point of $\tilde{E}$.

Put $\tilde{e} = (e, \sum \lambda^i_j \, dy^j \otimes \partial/\partial x^i)$ with $\lambda^i_j = p^i_j(\tilde{e})$ and

$$D = \sum \alpha^i \, \partial/\partial x^i + \sum \beta^j \, \partial/\partial y^j + \sum \gamma^i_j \, \partial/\partial p^i_j \in \mathscr{X}(\tilde{E})$$

then

$$\begin{aligned} \tilde{\theta}(\tilde{e}; D) &= \tilde{e}(v(T_{\tilde{e}} \tilde{p}(D))) = \tilde{e}(v(\textstyle\sum \alpha^i \, \partial/\partial x^i + \sum \beta^j \, \partial/\partial y^j)) \\ &= \tilde{e}(\textstyle\sum \beta^j \, \partial/\partial y^j) = (\sum \lambda^i_j \, \beta^j \, \pi^* \, \partial/\partial x^i)(\tilde{e}) \\ &= (\textstyle\sum p^i_j \, \beta^j \, \pi^* \, \partial/\partial x^i)(\tilde{e}) = (\sum p^i_j \, dy^j \otimes \pi^* \, \partial/\partial x^i)(\tilde{e}; D) \end{aligned}$$

Hence: $\tilde{\theta} = \sum p^i_j \, dy^j \otimes \pi^* \, \partial/\partial x^i$ in the open set V.

So the coordinates of $\tilde{\theta}$ are differentiable functions of the coordinates. □

From now on we will write: $\tilde{\theta} = \sum p^i_j \, dy^j \otimes \partial/\partial x^i$. The following proposition is a characterization of $\tilde{\theta}$:

PROPOSITION. — $\tilde{\theta}$ *is the only element of* $A^1(\tilde{E}, \pi^* TX)$ *which verifies the following conditions:*

(i) $\tilde{\theta}$ *is zero on the tangent vector fields on* $\tilde{E}$ *which are vertical for* $\tilde{p} : \tilde{E} \to E$.

(ii) *For every section* $\gamma : E \to \tilde{E}$ *of* $\tilde{p} : \tilde{E} \to E$ *we have*

$$\gamma^* \tilde{\theta} = \gamma \circ v$$

where v *is to take the vertical part on* TE.

Comment. — Observe that $\gamma(e)$ can be understood as a linear form on $V_e E$ but $(\gamma^* \tilde{\theta})_e$ is a linear form on $T_e E$, both valued in $(\pi^* TX)_e$.

Proof. — Trivially $\tilde{\theta}$ verifies condition (i). For the condition (ii), if $e \in E$ and $D \in \mathscr{X}(E)$ we have:

$$(\gamma^* \tilde{\theta})(e;\, D) = \tilde{\theta}(\gamma(e);\, T_e \gamma(D)) = \gamma(e)(v \circ T_{\gamma(e)} \tilde{p})(T_e \gamma(D))$$
$$= \gamma(e)(v(D)) = (\gamma \circ v)(e;\, D)$$

because $\tilde{p} \circ \gamma = id$.

Uniqueness. — Suppose $\alpha \in A^1(\tilde{E}, \pi^* TX)$ verifies (i) and (ii), we will see that α is determined in every point of $\tilde{E}$.

Let $\tilde{e} \in \tilde{E}$, $\tilde{D} \in \mathscr{X}(\tilde{E})$ with $e = \tilde{p}(\tilde{e})$. There exist a local section $\gamma : E \to \tilde{E}$ and a vector field D on E such that:

(*a*) $\gamma(e) = \tilde{e}$.

(*b*) $T_e \gamma(D) - \tilde{D}(\tilde{e})$ is vertical with respect to $\tilde{p}$.

Observe that γ and D can be computed in a local coordinate system $(x^i, y^j, p^i{}_j)$ in the following way: Take $\gamma = (x^i, y^j, f^i{}_j)$ with $f^i{}_j(e) = p^i{}_j(\tilde{e})$ and D with:

$$D(e;\, x^i) = \tilde{D}(\tilde{e};\, x^i), \qquad D(e;\, y^j) = \tilde{D}(\tilde{e};\, y^j)$$

Now:

$$\alpha(\tilde{e}; \tilde{D}) = \alpha(\gamma(e); T_e \gamma(D)) = (\gamma^* \alpha)(e; D) = \gamma(e; v(D)) = \tilde{\theta}(\gamma(e); \tilde{D}) = \tilde{\theta}(\tilde{e}; \tilde{D})$$

Hence α is completely determined. □

From now on we will suppose that the manifold X has a volume form ω. This form is pulled-back to E, $\bar{E}$ and $\tilde{E}$ with the same notation.

DEFINITION. — *The form* $\tilde{\theta} \wedge \omega$ *will be called the* canonical m-form *associated to the volume form* ω, *where* $\wedge$ *is defined by the bilinear map:*

$$A^1(\tilde{E}, \pi^* TX) \times A^m(\tilde{E}) \to A^m(\tilde{E})$$
$$(\alpha \otimes u,\, \omega) \mapsto \alpha \wedge i_u \omega$$

Observe that $\tilde{\theta} \wedge \omega$ is an ordinary m-form on $\tilde{E}$.

If we take a canonical system of coordinates the expression of the canonical m-form is

$$\tilde{\theta} \wedge \omega = J \sum (-1)^{i-1} p^i{}_j\, dy^j \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^m$$

if the volume form is $\omega = J\, dx^1 \wedge \ldots \wedge dx^m$.

Suppose we have a Hamiltonian on $\tilde{E}$, that is a chosen function $H : \tilde{E} \to \mathbb{R}$.

DEFINITION. — *We call* Hamilton-Carton form *associated to the Hamiltonian* H, *the form*

$$\tilde{\Theta} = \tilde{\theta} \wedge \omega - H\, \omega$$

Remember that we put ω for the pull-back of the volume element on X to E, $\bar{E}$ and $\tilde{E}$. The expression of $\tilde{\Theta}$ in a canonical local chart comes trivially from the expressions given above

3.2. Hamilton-Jacobi variational problem and critical sections

Let $\sigma : X \to \tilde{E}$ be a section of $\tilde{\pi} : \tilde{E} \to X$. Consider the integral:

$$I(\sigma) = \int_X \sigma^* \tilde{\Theta}$$

This functional I is defined on the set $\mathrm{Secc}\,\tilde{E} \subset \Gamma(\tilde{E})$ of those sections such that the above integral exists. Observe that if X is compact then $\mathrm{Secc}\,\tilde{E} = \Gamma(\tilde{E})$.

The Hamilton-Jacobi variational problem associated to the function H is to find the critical points of the functional I in the following sense:

DEFINITION. — *We say that the section $\sigma : X \to \tilde{E}$ is a critical point of the functional* I *if*

$$\int_X \sigma^* L_D \tilde{\Theta} = 0$$

for every vector field $D \in \mathscr{X}(\tilde{E})$ *with compact support.*

THEOREM (First characterization of critical sections). — *A section* $\sigma : X \to \tilde{E}$ *is critical if and only if*

$$\sigma^* (i_D d\tilde{\Theta}) = 0$$

for every vector field $D \in \mathscr{X}(\tilde{E})$ *with compact support.*

Proof :

$$0 = \int_X \sigma^* L_D \tilde{\Theta} = \int_X \sigma^* i_D d\tilde{\Theta} + \int_X \sigma^* d i_D \tilde{\Theta} = \int_X \sigma^* i_D d\tilde{\Theta}$$

because D has compact support.

Now the result follows by the Fundamental Lemma of the Calculus of Variations. □

THEOREM (Second characterization of critical sections). — *Let* $\gamma \in \mathrm{Secc}(\Lambda^m TX)$ *be the only m-vector field such that* $\gamma(\omega) = 1$. *Then a section* $\sigma : X \to \tilde{E}$ *is critical if and only if*

$$i_{\sigma_* \gamma} d\tilde{\Theta} = 0.$$

Proof :

$\Rightarrow$

If $i_{\sigma_* \gamma} d\tilde{\Theta} \neq 0$ then there exists D such that $i_{\sigma_* \gamma} i_D d\tilde{\Theta} \neq 0$ hence the section is not critical because $\sigma^* (i_D d\tilde{\Theta}) \neq 0$.

$\Leftarrow$

If $\sigma^* (i_D d\tilde{\Theta}) \neq 0$ for some D, then $(i_D d\tilde{\Theta})(\sigma_* \gamma) \neq 0$. □

4. LAGRANGIAN PROBLEMS AND CRITICAL SECTIONS

In this paragraph we summarize some results of [5] that are necessary for the complete understanding of the rest of this paper. For the details see the just mentioned paper.

A Lagrangian for the bundle $\pi: E \to X$ is a function $L: \bar{E} \to \mathbb{R}$ and if $s: X \to E$ is a section of $\pi: E \to X$ we consider the functional:

$$\mathscr{L}(s) = \int_X \bar{s}^* L\, \omega$$

where $\bar{s}: X \to \bar{E}$ is the canonical lift of s to $\bar{E}$ via the canonical diffeomorphism $\varphi: J^1(E) \to \bar{E}$. We must restraint the problem to the set $\mathrm{Secc}(E) \subset \Gamma(E)$ of sections such that the integral exists.

The Hamilton variational problems associated to the Lagrangian L is to find the critical points of the functional $\mathscr{L}$, that is the sections s such that:

$$\int_X \bar{s}^* L_D L\, \omega = 0$$

for every vector field D on $\bar{E}$ with compact support.

THEOREM [5]. — *A section $s: X \to E$ is critical if and only if*

$$\bar{s}^* i_D\, d\Theta = 0$$

where Θ is the Poincaré-Cartan form on $\bar{E}$ defined by:

$$\Theta = \theta \wedge \Omega_L - L\,\omega$$

θ being the canonical 1-form on $\bar{E}$ and Ω_L the Legendre form transformation associated to the given variational problem.

Comment. — The expressions of the before mentioned forms in a canonical local chart, are the following:

$$\theta = \sum (dy^j - \sum v^j{}_i\, dx^i) \otimes \partial/\partial y^j$$
$$\omega = J\, dx^1 \wedge \ldots \wedge dx^m$$
$$\Omega_L = J \sum (-1)^i (\partial L/\partial v^j{}_i)\, dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^m \otimes \pi^* dy^j$$

our notation is slightly different because we put $v^j{}_i$ instead of p_{ij} as it is used in [5].

In fact, in the original paper [5], the above theorem is stated on $J^1(E)$. We have translated the problem to $\bar{E}$ via the canonical diffeomorphism. On the other side in that reference there is a local construction of the Poincaré-Cartan form Θ. We now offer another, which is related only with the geometrical objects associated to the problem.

With the same notation as in the beginning we have

$$J^1(E) \xrightarrow{p} E \xrightarrow{\pi} X$$

But $J^1(E)$ is an affine bundle on E, then the vertical tangent space to $J^1(E)$ at the point $\bar{e}$ is canonically isomorphic to $V_e E \otimes T_x^* X$ where $e = p(\bar{e})$ and $x = \pi(e)$. Hence we have a canonical ismorphism V between $\Gamma(Vp)$, the sections of the vertical bundle of $J^1(E)$ over E, and $\Gamma(p^*(\pi^* T^* X \otimes_E VE))$. This canonical isomorphism is an element of

$$\Gamma(Vp) \otimes \Gamma(p^*(\pi^* T^* X \otimes_E \Gamma VE))^*$$
$$\Gamma(Vp) \otimes \Gamma(p^*(\pi^* T^* X))^* \otimes \Gamma(p^* VE)^*$$
$$= \Gamma(Vp) \otimes \Gamma(p^*(\pi^* TX)) \otimes \Gamma(p^* VE)^*$$

Consider now the canonical structure form θ of $J^1(E)$. The form θ is an element of $\Gamma(T^* J^1(E)) \otimes_{\bar{E}} \Gamma(p^* VE)$. The bilinear product defined by the duality between $\Gamma(p^* VE)$ and $\Gamma(p^* VE)^*$ allows us to define the natural contraction

$$\bar{S} = i(V \otimes \theta) \in \Gamma(Vp) \otimes \Gamma(T^* J^1(E)) \otimes \Gamma(p^* \pi^* TX)$$

The form ω on $J^1(E)$, the pull-back of the volume form on X, is an element of $\Gamma(p^* \pi^* \Lambda^m T^* X)$, so taking into account the contraction between $\Gamma(p^* \pi^* TX)$ and $\Gamma(p^* \pi^* \Lambda^m T^* X)$ and the exterior product between $\Gamma(T^* J^1(E))$ and $\Gamma(\Lambda^{m-1} T^* J^1(E))$ we obtain the contraction:

$$S = i(\bar{S} \wedge \omega) \in \Gamma(Vp) \otimes \Gamma(p^* \pi^* \Lambda^m T^* X)$$

Now, as in analytical mechanics, we have that:

$$\Theta = -i(dL \otimes S) - L\,\omega \in \Gamma(\Lambda^m T^* J^1(E))$$

if we use the natural contraction between $\Gamma(Vp)$ and dL which is an element of $\Gamma(T^* J^1(E))$.

5. LEGENDRE TRANSFORMATION AND LAGRANGIAN FORMS

In this paragraph we study the relation between our hamiltonian formulation and the lagrangian formulation we have summarized in the last paragraph. The situation is very similar to the one we find in analytical mechanics.

5.1. Legendre transformation

Let $L : \bar{E} \to \mathbb{R}$ be a Lagrangian.

DEFINITION. – The map

$$FL : \quad \bar{E} \to \tilde{E}$$

defined by

$$\bar{e} \mapsto \mathrm{D}(\mathrm{L}|_{\bar{\mathrm{E}}_{\bar{p}(\bar{e})}})(\bar{e})$$

where D is the ordinary differential in a vector space, is called the *Legendre Transformation* associated to the Lagrangian L.

Obviously the expression of FL in canonical local charts is:

$$x^i = x^i, \qquad y^j = y^j, \qquad p^i_j = \partial \mathrm{L}/\partial v^j_i$$

DEFINITION. — The function

$$\begin{aligned} \mathrm{A}: \ & \bar{\mathrm{E}} \to \mathbb{R} \\ & \bar{e} \mapsto (\mathrm{FL}(\bar{e}))(\bar{e}) \end{aligned}$$

is called the *action* of the lagrangian L.

The function $\mathscr{E} = \mathrm{A} - \mathrm{L}$ is called the *energy* associated of the Lagrangian L.

The expressions of A and $\mathscr{E}$ in a canonical local chart are:

$$\mathrm{A} = \sum (\partial \mathrm{L}/\partial v^j_i)\, v^j_i, \qquad \mathscr{E} = \sum (\partial \mathrm{L}/\partial v^j_i)\, v^j_i - \mathrm{L}$$

5.2. Quasiregular and hyperregular Lagrangians

DEFINITION. — We say that L is a quasiregular Lagrangian if:

(i) The image of FL is a submanifold of $\tilde{\mathrm{E}}$ and the natural injection is an embedding.

(ii) The fibres of FL are connected submanifolds of $\bar{\mathrm{E}}$.

PROPOSITION. — *If* L *is quasiregular then* $\mathscr{E}$ *is* FL*-projectable.*

Proof. — We must prove that $\mathscr{E}$ is constant along the fibres of FL. This fibres are contained in the fibres of $\bar{p} : \bar{\mathrm{E}} \to \mathrm{X}$, hence we can limit our study to a canonical local chart (x^i, y^j, v^j_i).

If

$$\mathrm{D} = \sum \alpha^i\, \partial/\partial x^i + \sum \beta^j\, \partial/\partial y^j + \sum \gamma^j_i\, \partial/\partial v^j_i$$

is tangent to the fibres of FL then $\mathrm{FL}_* \mathrm{D} = 0$ and it verifies:

$$\begin{gathered} \alpha^i = 0, \qquad \beta^j = 0, \qquad \forall\, i, j; \\ \sum \gamma^j_i (\partial^2 \mathrm{L}/\partial v^k_h\, \partial v^j_i) = 0, \qquad \forall\, h, k \end{gathered}$$

Then we have:

$$\mathrm{D}\mathscr{E} = (\sum \gamma^j_i\, \partial/\partial v^j_i)(\sum (\partial \mathrm{L}/\partial v^h_k)\, v^h_k - \mathrm{L}) = \sum_{h,k} (\sum_{i,j} \gamma^j_i (\partial^2 \mathrm{L}/\partial v^k_h\, \partial v^j_i)) = 0 \quad \square$$

DEFINITION. — If L is quasiregular then there exist functions $\mathrm{H} : \tilde{\mathrm{E}} \to \mathbb{R}$ such that $\mathrm{FL}^* \mathrm{H} = \mathscr{E}$. Any one of such functions is called a Hamiltonian associated to the Lagrangian L.

Comments. — If FL is a diffeomorphism then we say that L is hyperregular. In this case there exists only one Hamiltonian H associated to the Lagrangian L.

If FL is locally a diffeomorphism then we say that the Lagrangian is locally regular. In this case there isn't any global Hamiltonian but locally there exist hamiltonian functions.

If FL is quasiregular but not a diffeomorphism then the Hamiltonian is not univocally defined but all of them coincide on the image of FL.

Now we are going to see how we can obtain the lagrangian formulation from the hamiltonian one, via the Legendre transformation.

5.3. Lagrangian form

Let L be a quasiregular Lagrangian and H a Hamiltonian associated to L.

DEFINITION. — We call *lagrangian form*, $\tilde{\Theta}_L$, the pull-back by FL of the Hamilton-Cartan form, that is: $\tilde{\Theta}_L = FL^* \tilde{\Theta}$.

The expression of $\tilde{\Theta}_L$ in a canonical local chart is:

$$\tilde{\Theta}_L = J \sum (-1)^{i-1} (\partial L/\partial v^j_i)\, dy^j \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^m$$
$$- J \sum v^j_i (\partial L/\partial v^j_i)\, dx^1 \wedge \ldots \wedge dx^m + JL\, dx^1 \wedge \ldots \wedge dx^m$$

where $\omega = J\, dx^1 \wedge \ldots \wedge dx^m$ is the volume form on X.

PROPOSITION. — *Let* $s : X \to E$ *be a section of* $\pi : E \to X$ *and* $\bar{s} : X \to \bar{E}$ *its canonical lift. Then:*

$$\bar{s}^* \tilde{\Theta}_L = \bar{s}^* L\, \omega$$

Proof. — We can use a canonical local chart (x^i, y^j, v^j_i). *Let* $s(x^i) = (x^h, f^j(x^i))$ be a section, then $\bar{s}(x^i) = (x^h, f^j(x^i), (\partial f^j/\partial x^h))$ and a little computation gives us the result. □

Comment. — Observe that the Lagrangian form $\tilde{\Theta}_L$ corresponds to the Poincaré-Cartan form in the above mentioned work [5] but with opposite sign.

A comment on variational principles

Usually the Hamilton variational problem is the following: Is there any section $s : X \to E$ with compact support and critical for the functional

$$s \mapsto \int_{\bar{s}} L\, \omega?$$

On the other side the variational principle of Hamilton-Jacobi asks for the critial sections of the functional

$$s \mapsto \int_{\bar{s}} \mathrm{FL}^* (\tilde{\theta} \wedge \omega) - \mathscr{E}\,\omega$$

Observe that, from the definition, $\mathrm{FL}^* \tilde{\Theta} = \tilde{\Theta}_{\mathrm{L}}$, and the above Proposition, $\bar{s}^* \tilde{\Theta}_{\mathrm{L}} = \bar{s}^* \mathrm{L}\,\omega$, we deduce:

$$\int_{\bar{s}} \mathrm{FL}^* (\tilde{\theta} \wedge \omega) - \mathscr{E}\,\omega = \int_{\bar{s}} \mathrm{FL}^* (\tilde{\Theta}) = \int_{\bar{s}} \mathrm{L}\,\omega$$

that is, the equivalence between both variational principles.

Notice that the energy $\mathscr{E}$ verifies

$$\bar{s}^* \mathrm{FL}^* (\tilde{\theta} \wedge \omega) - \bar{s}^* (\mathscr{E}\,\omega) = \bar{s}^* (\mathrm{L}\,\omega)$$

for any section $s : \mathrm{X} \to \mathrm{E}$ and its canonical lift. It is not difficult to prove that the energy is univocally determined by this property, so we can conclude that the energy is the only function that achieves the equivalence between both variational principles.

6. ASSOCIATED PROBLEMS AND EQUIVALENCIES

Let $\mathrm{L} : \bar{\mathrm{E}} \to \mathbb{R}$ be a quasiregular Lagrangian. Consider the following sets:

$$\mathrm{V}_1 = \{ s : \mathrm{X} \to \mathrm{E};\ \bar{s}^* (i_{\bar{\mathrm{D}}}\, d\tilde{\Theta}_{\mathrm{L}}) = 0,\ \forall\, \mathrm{D} \in \mathscr{X}^c (\bar{\mathrm{E}}) \}$$
$$\mathrm{V}_2 = \{ s : \mathrm{X} \to \bar{\mathrm{E}};\ s^* (i_{\bar{\mathrm{D}}}\, d\tilde{\Theta}_{\mathrm{L}}) = 0,\ \forall\, \mathrm{D} \in \mathscr{X}^c (\bar{\mathrm{E}}) \}$$
$$\mathrm{V}_3 = \{ \sigma : \mathrm{X} \to \tilde{\mathrm{E}};\ \sigma^* (i_{\mathrm{D}} d\tilde{\Theta}) = 0,\ \forall\, \mathrm{D} \in \mathscr{X}^c (\tilde{\mathrm{E}}) \}$$

Where $\mathscr{X}^c$ means "with compact support".

V_1 is the set of solutions of the Hamilton variational problem associated to the Lagrangian L. If we take sections $s : \mathrm{X} \to \bar{\mathrm{E}}$ instead of $s : \mathrm{X} \to \mathrm{E}$ and their canonical lifts, then the set of solutions is V_2.

V_3 is the set of solutions of the Hamiltonian variational problem for the Hamiltonian H associated to the quasiregular Lagrangian L.

The canonical lift gives us a natural injection of V_1 into V_2. We will study the other relations between V_1, V_2 and V_3. We call these relations Equivalence Theorems. The equivalence between V_1 and V_2 is usually called Regularity Problem.

6.1. Characterization of ker FL_*

THEOREM:

$$\ker FL_* = \operatorname{rad} d\tilde{\Theta}_L \cap \mathscr{X}^v_p(\bar{E})$$

where $\mathscr{X}^v_p(\bar{E})$ *are the vector fields on* $\bar{E}$ *that are vertical with respect to the projection* $\bar{p} : \bar{E} \to E$ *and* $\operatorname{rad} d\tilde{\Theta}_L$ *is the set of vector fields* D *such that* $i_{\bar{D}}\, d\Theta_L = 0$.

Proof. – This is a local result, then we can use a canonical system of coordinates (x^i, y^j, v^j_i).

Let $D = \sum \alpha^i \partial/\partial x^i + \sum \beta^j \partial/\partial y^j + \sum \gamma^j_i \partial/\partial v^j_i$ a vector field. If D is in $\mathscr{X}^v_p(\bar{E})$ then $\alpha^i = 0$, $\beta^j = 0$ for all i, j.

If D is in ker FL_* then:

$$\alpha^i = 0, \qquad \beta^j = 0, \qquad \sum_{h,k} (\partial^2 L/\partial v^k_h \partial v^j_i)\, \gamma^k_h = 0 \quad \text{for all } i, j$$

Conversely, if $D \in \operatorname{rad} d\tilde{\Theta}_L \cap \mathscr{X}^v_p(\bar{E})$ then $\alpha^i = 0$ and $\beta^j = 0$, for all i, j, because D is vertical with respect to p. Hence the expression of D is:

$$D = \sum \gamma^j_i\, \partial/\partial v^j_i$$

On the other side from the expression of $\tilde{\Theta}_L$ in a canonical coordinate system, we have that:

$$\begin{aligned} d\tilde{\Theta}_L = {} & J \sum (-1)^{i-1} (\partial^2 L/\partial v^k_h \partial v^j_i)\, dv^k_h \wedge dy^j \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^m \\ & - J \left(\sum (\partial L/\partial v^j_i)\, dv^j_i + \sum v^j_i (\partial L/\partial v^k_h \partial v^j_i)\, dv^k_h\right) \wedge dx^1 \wedge \ldots \wedge dx^m \\ & + J \sum (\partial L/\partial v^k_h)\, dv^k_h \wedge dx^1 \wedge \ldots \wedge dx^m \\ & + \text{other terms without } dv^j_i \end{aligned}$$

Then, if $D = \sum \gamma^j_i\, \partial/\partial v^j_i$ verifies $i_{\bar{D}}\, d\Theta_L = 0$, we have:

$$\sum_{h,k} (\partial^2 L/\partial v^k_h \partial v^j_i)\, \gamma^k_h = 0 \quad \text{for all } i, j$$

And the result follows. □

6.2. Equivalence theorems

THEOREM. – *The natural injection* $i : V_1 \hookrightarrow V_2$ *is onto if and only if* $\ker FL_* = \{0\}$.

Proof:

$\Rightarrow$

Suppose that $\ker FL_* \neq 0$. Let $D \neq 0$ be a vector field in $\ker FL_*$ and τ a local one-parameter group associated to D. We have that $p \circ \tau = p$ because $D \in \mathscr{X}^v_p(\bar{E})$. Moreover, as $D \in \operatorname{rad} d\tilde{\Theta}_L$, then $i_{\bar{D}}\, d\tilde{\Theta}_L = 0$, hence

$L_D d\tilde{\Theta} = 0$, where L_D is the Lie derivative with respect to D. Then we have $\tau^* d\tilde{\Theta}_L = d\tilde{\Theta}_L$, so $d\tilde{\Theta}_L$ is invariant under the action of τ.

Let $s \in V_1$, then $\bar{s} \in V_2$. We are going to obtain an element $\bar{\sigma} \in V_2$ with $\bar{\sigma} \notin i(V_1)$.

Let $\bar{\sigma} = \tau \circ \bar{s}$. If $\bar{D} \in \mathscr{X}_c(\bar{E})$ we have:

$$\bar{\sigma}^* (i_{\bar{D}} d\tilde{\Theta}_L) = (\tau . \bar{s})^* (i_{\bar{D}} d\tilde{\Theta}_L) = \bar{s}^* (\tau^* (i_{\bar{D}} d\tilde{\Theta}_L))$$
$$= \bar{s}^* (i_{\tau_*^{-1}(\bar{D})} \tau^* d\tilde{\Theta}_L) = \bar{s}^* (i_{\tau_*^{-1}(\bar{D})} d\tilde{\Theta}_L) = 0$$

because $s \in V_1$. Then $\bar{\sigma} \in V_2$.

Now we are going to prove that $\bar{\sigma} \notin i(V_1)$. Suppose that $\bar{\sigma} \in i(V_1)$, then there locally exists $s'(x^i) = (x^i, h^j)$ such that $\bar{\sigma} = \bar{s'}$, that is $\bar{\sigma}(x) = (x^i, h^j, \partial h^j/\partial x^i)$. But observe that if $s(x^i) = (x^i, f^j)$ and

$$\tau(x^i, y^j, v^j_i) = (x^i, y^j, \tau^j_i(x^j, y^j, v^j_i))$$

then we have:

$$(x^i, h^j, \partial h^j/\partial x^i) = (\tau . \bar{s})(x^i) = \tau(x^i, f^j, \partial f^j/\partial x^i) = (x^i, f^j, \tau^j_i(x^i, f^j, \partial f^j/\partial x^i))$$

then $h^j = f^j$, $\partial f^j/\partial x^i = \tau^j_i(x^i, f^j, \partial f^j/\partial x^i)$ and we have that $\tau = \mathrm{id}$, hence $D = 0$, against the hypothesis.

$\Leftarrow$

$\Rightarrow$ Suppose there exists $\bar{s} \in V_2$, $\bar{s} \notin i(V_1)$, then locally

$$s(x^i) = (x^i, f^j, h^j_i) \qquad \text{with} \quad h^j_i \neq \partial f^j/\partial x^i \quad \text{for some } i, j$$

Let us consider the vector field:

$$D = \sum (\pi \circ p)^* (h^j_i - \partial f^j/\partial x^i)\, \partial/\partial v^j_i \in \mathscr{X}^v_p(\bar{E})$$

Then D is different from zero. We are going to see that $D \in \mathrm{rad}\, d\tilde{\Theta}_L$ that is:

$$\sum (\partial^2 L/\partial v^k_h \partial v^j_i)(h^j_i - \partial f^j/\partial x^i) = 0 \quad \text{for all } h, k$$

We know that $\bar{s} \in V_2$, then $\bar{s}^* i_{\bar{D}} d\tilde{\Theta}_L = 0$ for all $\bar{D} \in \mathscr{X}_c(\bar{E})$. If we take vector fields $\bar{D} = \sum \gamma^k_h \partial/\partial v^k_h$ we have:

$$i_{\bar{D}} d\tilde{\Theta}_L = J \sum (-1)^{i-1} (\partial^2 L/\partial v^k_h \partial v^j_i) \gamma^k_h dy^j \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^m$$
$$- J \sum (\partial^2 L/\partial v^k_h \partial v^j_i) v^j_i \gamma^k_h dx^1 \wedge \ldots \wedge dx^m$$

then:

$$0 = \bar{s}^* i_{\bar{D}} d\tilde{\Theta}_L = [\sum (\partial^2 L/\partial v^k_h \partial v^j_i) \gamma^k_h (\partial f^j/\partial x^i - h^j_i)]\, \omega$$

hence the condition is verified, because γ^k_h are arbitrary coefficients, and $\ker FL_* \neq \{0\}$. $\square$

THEOREM. — *Consider the mappings:*

$$\varphi : V_1 \to V_3 \quad \text{defined by } \varphi(s) = FL \circ \bar{s}$$
$$\psi : V_2 \to V_3 \quad \text{defined by } \psi(\bar{s}) = FL \circ \bar{s}$$
$$\eta : V_3 \to V_1 \quad \text{defined by } \eta(\sigma) = \tilde{p} \circ \sigma$$

If FL *is a diffeomorphism, then they are correctely defined and they are bijections so the three different variational problems are equivalent.*

Proof. – If $\bar{s} \in V_2$ then $\psi(\bar{s}) \in V_3$ for if D has compact support on $\tilde{E}$ then:

$$\psi(\bar{s})^* i_D d\tilde{\Theta}_L = \bar{s}^* i_{FL_*^{-1} D} d\tilde{\Theta}_L = 0$$

because $\bar{s} \in V_2$.

Observe that $\psi^{-1}(\sigma) = FL^{-1} \circ \sigma$.

Now if $\sigma \in V_3$ then $\sigma^* i_D d\tilde{\Theta} = 0$, but $\tilde{p} \circ \sigma = \bar{p} \circ FL^{-1} \circ \sigma$. because FL is a fibre morphism, and $FL^{-1} \circ \sigma \in V_2$, so by the above theorem we have that $V_2 = V_1$ and the result follows. □

7. APPLICATIONS AND EXAMPLES

7.1. Liouville theorem

Let $\pi : E \to X$ be an admissible bundle, $\tilde{E} \to \mathbb{R}$ a hamiltonian function and ω one volume element on X.

PROPOSITION. – *Let* $D \in \mathcal{X}(\tilde{E})$. *If the integral curves of* D *are contained in the image of critical sections and the section* $\sigma : X \to \tilde{E}$ *is critical for the form* $d\tilde{\Theta}$, *then it is also critical for the form* $L_D d\tilde{\Theta}$.

Proof. – Suppose that σ is critical for $d\tilde{\Theta}$, then $\sigma^*(i_X d\tilde{\Theta}) = 0$, for any $X \in \mathcal{X}(\tilde{E})$. We must deduce this implies that for any $X_0 \in \mathcal{X}(\tilde{E})$ we have:

$$\sigma^*(i_{X_0} L_D d\tilde{\Theta}) = 0$$

The problem is equivalent to see that:

$$i_{X_1} \dots i_{X_m} i_{X_0} L_D d\tilde{\Theta} = 0$$

for any $X_1, X_2, \dots, X_m$ vector fields tangent to the image of the section σ.

We have:

$$L_D d\tilde{\Theta} = i_D dd\tilde{\Theta} + d(i_D d\tilde{\Theta}) = d(i_D d\tilde{\Theta})$$

then we must show that:

$$d(i_D d\tilde{\Theta})(X_0, X_1, \dots, X_m) = 0$$

for any $X_0 \in \mathcal{X}(\tilde{E})$ and any $X_1, X_2, \dots, X_m$ vector fields tangent to the image of the section σ. We have:

$$d(i_D d\tilde{\Theta})(X_0, X_1, \dots, X_m)$$
$$= \sum_{0 \leq j \leq m} (-1)^j X_j((i_D d\tilde{\Theta})(X_0, X_1, \dots, \hat{X}_j, \dots, X_m)) +$$

$$+\sum_{0\leq i\leq j\leq m}(-1)^{i+j}((i_D d\tilde{\Theta})([X_i, X_j], X_0, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_m))$$
$$=\sum_j(-1)^j X_j(-(i_{X_0} d\tilde{\Theta})(D, X_1, \ldots, \hat{X}_j, \ldots, X_m))$$
$$+\sum_{0\leq i<j\leq m}(-1)^{i+j}((i_D d\tilde{\Theta})([X_i, X_j], X_0, X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_m))$$

but, as $\sigma^*(i_X d\tilde{\Theta})=0$ for all $X\in\mathscr{X}(\tilde{E})$, then the first summation is zero because of the conditions on the vector field D.

For the second summation, all the items with $i>0$ are zero for the reason just mentioned above. For $i=0$ we have:

$$\sum_j(-1)^j((i_D d\tilde{\Theta})([X_0, X_j], X_1, \ldots, \hat{X}_j, \ldots, X_m))$$
$$=-\sum_j(-1)^j((i_{[X_0, X_j]} d\tilde{\Theta})(D, X_1, \ldots, \hat{X}_j, \ldots, X_m))=0$$

for the same reason. □

Commentary. — If $m=1$, that is in the case of analytical mechanics, this result reduces to $L_D d\tilde{\Theta}=0$, that is the classical Liouville theorem, because if D is tangent to the critical sections we have:

$$i_D i_X d\tilde{\Theta}=0 \quad \text{for any } X\in\mathscr{X}(\tilde{E})$$

then $i_D d\tilde{\Theta}=0$, hence $L_D d\tilde{\Theta}=0$ because $d\tilde{\Theta}$ is closed.

7.2. Minimal surfaces in $\mathbb{R}^3$

We look for differentiable mappings $\varphi : \mathbb{R}^2 \to \mathbb{R}$ such that their graphic have minimal area as sets of $\mathbb{R}^3$.

Then $X=\mathbb{R}^2$, $E=\mathbb{R}^2\times\mathbb{R}$ and:

$$J^1(E)=\bar{E}=VE\otimes_E\pi^* T^* X=T\,\mathbb{R}\otimes_E\pi^* T^*\mathbb{R}^2$$
$$\tilde{E}=V^* E\otimes_E\pi^* TX=T^*\mathbb{R}\otimes_E\pi^* T\,\mathbb{R}^2$$

hence: $\bar{E}=\mathbb{R}\otimes_E\mathbb{R}^2$, $\tilde{E}=\mathbb{R}\otimes_E\mathbb{R}^2$.

Let x^1, x^2 be the coordinates in the first $\mathbb{R}^2$, and y the coordinate of $\mathbb{R}$, then v^1 and v^2 are the coordinates of the fibres of $\bar{E}$ and p_1, p_2 the corresponding coordinates for the fibres of $\tilde{E}$.

A section $s : \mathbb{R}^2\to\mathbb{R}^2\times\mathbb{R}$ is a function $y=f(x_1, x_2)$ and its 1-jet prolongation is the mapping:

$$(x^1, x^2)\mapsto(x^1, x^2, f(x^1, x^2), \partial f/\partial x^1, \partial f/\partial x^2)$$

The Lagrangian is the function:

$$L(x^1, x^2, y, v^1, v^2)=(1+(v^1)^2+(v^2)^2)^{1/2}$$

The Legendre transformation is given by:

$$FL(x^1, x^2, y, v^1, v^2) = (x^1, x^2, y, v^1/L, v^2/L)$$

then L is hyperregular.

Now the action and the energy are:

$$A = L - 1/L, \qquad \mathscr{E} = -1/L$$

hence the Hamiltonian is:

$$H = -(1 - p_1^2 - p_2^2)^{1/2}$$

If, as usually, $\omega = dx^1 \wedge dx^2$ then:

$$\tilde{\Theta} = p_1\, dy \wedge dx^2 - p_2\, dy \wedge dx^1 - H\, dx^1 \wedge dx^2$$

and:

$$d\tilde{\Theta} = dp_1 \wedge dy \wedge dx^2 - dp_2 \wedge dy \wedge dx^1 - dH \wedge dx^1 \wedge dx^2$$

A section $\sigma : \mathbb{R}^2 \to \tilde{E}$ is given by:

$$\sigma(x^1, x^2) = (x^1, x^2, f(x^1, x^2), g_1(x^1, x^2), g_2(x^1, x^2))$$

and if σ is critical then:

$$\partial f/\partial x^1 = -p_1/H, \qquad \partial f/\partial x^2 = -p_2/H$$
$$\partial g_1/\partial x^1 = -\partial g_2/\partial x^2$$

are the Hamilton equations of the problem.

These equations are completely equivalent to the classical one since, as we said, the Lagrangian is hyperregular.

7.3. The electromagnetic field

In this case the base space X is the Minkowsky real space $\mathbb{R}^4$, the manifold E is $T^*\mathbb{R}^4$ and $\pi : E \to X$ is the canonical projection. Thus we have a vector bundle, hence $J^1(E)$ is canonically isomorphic to

$$\bar{E} = VE \otimes \pi^* T^* X = \pi^* T^* X \otimes \pi^* T^* X.$$

The dual bundle is $\tilde{E} = \pi^* TX \otimes \pi^* TX$.

The Lagrangian is $L(\beta) = (1/4)\,\|\mathscr{A}(\beta)\|$, where $\mathscr{A}$ is the alternating operator and we use the induced metric on $\bar{E}$ by the metric on X.

If $\bar{\alpha}$ is the 1-jet prolongation of the section α then $L(\bar{\alpha}) = (1/4)\,\|\mathscr{A}(\bar{\alpha})\| = (1/4)\,\|d\alpha\|$. Observe that $L(\bar{\alpha}) = L(\overline{\alpha + df})$, where f is a differentiable function on X, and this is one of the sources of the gauge problems in electromagnetic theory.

If we call x^i the coordinates on X, y^j the coordinates on the fibres of E and v^{ij} the induced coordinates on the fibres of $\bar{E}$, then

$$L(x, y, v) = (1/4)\Big(\sum_{i<j<4} (v^{ij} - v^{ji})^2 - \sum_{i<4} (v^{i4} - v^{4i})^2\Big)$$

For a section $\alpha = \sum f_i \, dx^i$ we have:

$$L(\bar{\alpha}) = (1/4)\Big(\sum_{i<j<4} (\partial f_j/\partial x^i - \partial f_i/\partial x^j)^2 - \sum_{i<4} (\partial f_4/\partial x^i - \partial f_i/\partial x^4)^2 \Big) = (1/4)(\|\vec{H}\|^2 - \|\vec{E}\|^2)$$

where $\vec{H}$ and $\vec{E}$ are the classical magnetic and electric fields.

The lagrangian equations are the well known Maxwell equations.

The Legendre transformation, FL, is given by:

$$\begin{aligned} x &= x \\ y &= y \\ p_{ij} &= (1/2)(v^{ij} - v^{ji}), && i<j<4 \\ p_{ij} &= -(1/2)(v^{ij} - v^{ji}), && j<i<4 \\ p_{i4} &= -(1/2)(v^{i4} - v^{4i}), && i<4 \\ p_{4i} &= (1/2)(v^{i4} - v^{4i}), && i<4 \\ p_{ii} &= 0 \end{aligned}$$

Obviously this transformation is not a diffeomorphism between $\bar{E}$ and $\tilde{E}$. Observe that the image of FL is the vector bundle $\Lambda^2 TX$, subbundle of $\tilde{E}$.

Following the usual steps we find that the corresponding Hamiltonian is

$$H(x, y, p) = \sum_{i<j<4} p_{ij}^2 - \sum_{i<4} p_{i4}^2$$

and in the same way we can obtain the associated Hamilton equations.

In this case, as a consequence of the non regularity of the Legendre transformation, we cannot apply the equivalence theorems. In fact, it can be suspected that the equivalence, if it exists, must be between the solutions of the Hamilton equations and certain classes of solutions of the Maxwell equations.

The origin of this non regularity is the inadequacy of the space $J^1(E)$ to the problem of the electromagnetic field. In order to solve this kind of problems it is necessary to develop the theory of degenerate variational problems with several variables.

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