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Constructing quantum dissipations and their reversible states from classical interacting spin systems

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ABSTRACT. — The relation between certain quantum systems and classical stochastic processes—e.g. in the method of functional integration—is formulated on the level of the dynamics for both quantum and classical dissipative time evolutions. An essentially unique quantum dissipation is constructed from a classical interacting spin system, preserving the notion of detailed balance. Translation invariant and reversible infinite volume quantum dynamics are found in this way and the Hamiltonian is recovered from the action of the generator in the GNS-representation of the corresponding groundstate for which a Feynmann-Kac formula holds. Local reversibility of quantum dissipations is shown to give rise to an almost classical characterization of the corresponding quantum states.

Key words : Quantum Dynamical Semigroups, Reversibility, Gibbs States, Ground States, Feynman-Kac Formula, Interacting Spin Systems.

RÉSUMÉ. — La relation entre certains systèmes quantiques et des processus stochastiques classiques — p.e. dans la méthode de l'intégration fonctionnelle — est formulée au niveau des dynamiques pour les évolutions

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temporelles dissipatives à la fois quantiques et classiques. Une dissipation essentiellement unique est construite à partir d'un système de spins classiques en interaction qui conserve la notion de bilan détaillé. De cette façon, on trouve une dynamique quantique à volume infinie, réversible et invariante sous les translations. L'hamiltonien est obtenu à partir de l'action du générateur dans la représentation GNS de l'état fondamental correspondant pour lequel une formule de Feynman-Kac est vérifiée. Nous montrons que réversibilité locale des dissipations quantiques donne lieu à une caractérisation presque classique des états quantiques correspondants.

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1. INTRODUCTION

At many occasions the flow of ideas from one branch of physics to another has passed via the mathematical structure of function space integration. The development of quantum mechanics – also including relativistic corrections, e. g. [1] – through the Feynman-Kac formula is widely used and successfully applied to a variety of problems in particle physics, condensed matter physics or statistical physics. The literature on the subject is vast and we only mention the books [2] and [3] as reference texts.

Feynman's formula gives an integral representation for the solutions of the equations of quantum mechanics. Due to mathematical difficulties in treating certain complex measures – *see* however [4] – rigourous work has primarily been concentrated on the Feynman-Kac formula which gives a path-space representation for the kernel of $e^{-\beta H}$, $\beta \geqq 0$, where H is a quantum Hamiltonian. A well known example is the harmonic oscillator with Hamiltonian $H = -\frac{1}{2}\Delta + \frac{1}{2}q^2 - \frac{1}{2}$ for which the associated classical (diffusion) process is the Ornstein-Uhlenbeck velocity process. One important consequence is the derivation of relations between ground state expectations for H and the expected value of functions of the configurations on which the appropriate classical stochastic process is living. These relations roughly look like

$$\left.\begin{array}{c}\displaystyle\int \prod_{i=1}^{n} F_i(q(t_i))\, dP = \langle \Omega, F_1 e^{-(t_2-t_1)H} F_2 \dots F_n \Omega \rangle, \\ t_1 \leqq t_2 \dots \leqq t_n \end{array}\right\} \qquad (1.1)$$

for $F_i(q)$ bounded functions of q, Ω the ground state of H, and dP the classical process or path-space measure. Many types of quantum systems can be treated in this way, also at non-zero temperatures. Interesting applications in the context of quantum statistical mechanics together with a clear exposition of the main ideas can e. g. be found in [5]. In most cases, these representations are inspired by the hope to learn something about the properties of quantum systems specified by a Hamiltonian from investigating an associated classical reversible process, and in this way the rigorous study of quantum ground states has profited from results of classical Gibbs measure-theory, e. g. recently in [6], [7] and [8].

At first sight however, it may appear strange to have relations between a static quantum system in its ground state and a classical stochastic dynamics. Is there an analogous bona fide quantum dynamics which intermediates in this connection, and, if yes, can one learn something about it and about its characterization of the quantum ground state? The present study addresses this problem and the paper therefore originates from two complementary questions:

1. Can these relations be more naturally formulated on the level of the generators (or dynamics) for *both* quantum and classical time evolutions, and, if yes, how to recover the quantum Hamiltonian appearing in (1.1)?

2. Can one learn something from these relations about quantum states solely characterized by reversibility properties for a certain quantum evolution?

The second question is (perhaps overly) ambitious, but it is in part motivated by the rather unsatisfactory state of affairs concerning quantum dissipative processes. Of course this problem has a long history with a number of fundamental results. Gorini *et al.* ([9], [10]) have studied the most general generator of a quantum dynamical semigroup for the case of finite dimensional Hilbert spaces, while independently Lindblad has generalized this to any separable Hilbert space [11]. Moreover, the notion of detailed balance and its connection with equilibrium states was developed in a series of papers, including ([12], [13], [14]). Still, we have not found in the literature any explicit constructions of infinite volume quantum detailed balance semigroups for even the simplest non-trivial spin 1/2 lattice systems. One of the results of our investigation is to present such a construction for ground states of certain quantum Hamiltonians. They will then be characterized by a local reversibility property. Let us add that characterizing quantum equilibrium states via (local) detailed balance properties is not new, *see* e. g. ([12], [13], [14]), but here this problem is analyzed for quantum ground states and – as we said already – we explicitly construct the underlying infinite volume dynamics. Furthermore, we investigate in detail the consequences of such local properties and we find that – unlike the classical situation – it has a rather drastic effect on the nature of the quantum state.

All this is however based on the answer we give to the first question. There, we show how, starting from a classical reversible interacting spin system, one can determine an essentially unique quantum dissipation which preserves the reversibility condition on the full quantum algebra. The quantum Hamiltonian is found after passing to the so called GNS-construction.

In the next Section we give the construction of this quantum dynamics. Section 3 is devoted to the study of the nature of the associated quantum states. We find the ground states of certain quantum Hamiltonians, and recover the usual Feynman-Kac formulation from their relation with classical Gibbs measures. The proofs of all results are postponed until Section 4.

2. FROM CLASSICAL INTERACTING SPIN SYSTEMS TO QUANTUM DISSIPATIONS

We consider the hypercubic lattice $\mathbb{Z}^d$ in d dimensions, to each of whose sites $x \in \mathbb{Z}^d$ we assign a spin variable with values $\sigma_x \in \{-1, +1\}$. The configuration space is $X = \{-1, +1\}^{\mathbb{Z}^d}$ while its restriction to some region $K \subset \mathbb{Z}^d$ is denoted by X_K. $C(K)$ denotes the set of continuous functions on K.

We choose a finite set $\Lambda \subset \mathbb{Z}^d$ containing the origin such that its translates $\Lambda_x \equiv \Lambda + x \equiv \{ y + x \in \mathbb{Z}^d \mid y \in \Lambda \}$ cover the whole lattice:

$$\bigcup_{x \in \mathbb{Z}^d} \Lambda_x = \mathbb{Z}^d.$$

Let Q_x be the set of all permutations of the configurations in Λ_x, *i.e.* $U \in Q_x$ if for each $\sigma \in X$

(i) $U \sigma \in X$ with $(U \sigma)_y = \sigma_y$, $\forall y \in \mathbb{Z}^d \setminus \Lambda_x$,

(ii) there is a unique $\eta \equiv U^{-1} \sigma \in X$ such that $U \eta = \sigma$ (defining $U^{-1} \in Q_x$).

The local dynamics of the classical system is described by a collection of transition rates $c_x(U, \sigma)$, $U \in Q_x$, $x \in \mathbb{Z}^d$. They are assumed to be non-negative, bounded, local and translation invariant functions of $\sigma \in X$:

$$\left.\begin{array}{ll} \text{(i)} & c_x(U, \sigma) \geqq 0 \\ \text{(ii)} & \sup_{\sigma \in X} \sum_{U \in Q_0} c_0(U, \sigma) < \infty, \end{array}\right\} \tag{2.1}$$

(iii) there is a finite $\bar{\Lambda} \subset \mathbb{Z}^d$ such that $c_0(U, \sigma)$ does not depend on any σ_y with $y \notin \bar{\Lambda}$, for all $U \in Q_0$;

(iv) for all $x \in \mathbb{Z}^d$, $U_x \in Q_x$, $\sigma \in X$, $c_x(U_x, \sigma) = c_0(U, \tau_x \sigma)$ with $U \eta \equiv U_x(\tau_{-x} \eta)$; $(\tau_x \eta)_y \equiv \eta_{x+y}$ for $\eta \in X$.

The rate $c_x(U, \sigma)$ represents the weight which is given to the transition from σ to $U \sigma$, the new configuration which coincides with σ outside Λ_x, and it is therefore natural also to assume that

(v) $c_x(\mathrm{U}, \sigma) = c_x(\mathrm{U}', \sigma)$ whenever $\mathrm{U}\sigma = \mathrm{U}'\sigma$. (2.2)

Moreover we avoid certain degeneracies by requiring also that

(vi) $c_x(\mathrm{U}, \sigma) = c_x(\mathrm{U}', \sigma)$, $\forall \sigma \in \mathrm{X}$, implies $\mathrm{U} = \mathrm{U}' \in \mathrm{Q}_x$. (2.3)

Familiar examples include the case of spin flip systems where $\Lambda = \{0\}$ and spin exchange systems where in the simplest cases $\Lambda = \{0, e_1, \ldots, e_d\}$ for $\{e_\alpha\}_{\alpha=1}^d$ the unit vector in $\mathbb{Z}^d$ and where $c_0(\mathrm{U}, \sigma)$ is different from zero only if the permutation U exchanges the nearest neighbour spins σ_0 and σ_{e_α} for some $\alpha = 1, \ldots, d$.

Suppose there is given a Hamiltonian H for some finite range, translation invariant potential $\{\mathrm{J_R}\}$ where $\mathrm{R} \subset \mathbb{Z}^d$ runs over the finite subsets of the lattice, that is, formally

$$\mathrm{H}(\sigma) = \sum_{\mathrm{R}} \mathrm{J_R} \prod_{x \in \mathrm{R}} \sigma_x \tag{2.4}$$

with $\mathrm{J_R} = \mathrm{J}_{\mathrm{R}+x} \in \mathbb{R}$ and $\mathrm{J_R} = 0$ whenever the number of elements in R exceeds a given number. Then, the energy differences

$$\Delta_{\mathrm{U}} \mathrm{H}(\sigma) \equiv \mathrm{H}(\mathrm{U}\sigma) - \mathrm{H}(\sigma) \tag{2.5}$$

are well defined for all $\mathrm{U} \in \mathrm{Q}_x$, $x \in \mathbb{Z}^d$ and we assume that the transition from σ to $\mathrm{U}\sigma$ and vice versa are related via *the condition of detailed balance*, *i.e.* for all $\mathrm{U} \in \mathrm{Q}_x$, $x \in \mathbb{Z}^d$,

$$c_x(\mathrm{U}, \sigma) = c_x(\mathrm{U}^{-1}, \mathrm{U}\sigma)\, e^{-\Delta_{\mathrm{U}} \mathrm{H}(\sigma)}. \tag{2.6}$$

Let $\mathcal{G}(\mathrm{H})$ be the set of all Gibbs measures with respect to the Hamiltonian H of (2.4). Fix $\mu \in \mathcal{G}(\mathrm{H})$. By definition, μ is a probability measure on the Borel σ-algebra of X and satisfies the so-called DLR-equation, *see* for example [15], *i.e.*

$$\int f(\mathrm{U}^{-1}\sigma)\, d\mu(\sigma) \equiv \int f(\sigma)\, e^{-\Delta_{\mathrm{U}} \mathrm{U}(\sigma)}\, d\mu(\sigma) \tag{2.7}$$

for all $f \in \mathrm{C}(\mathrm{X})$ and $\mathrm{U} \in \mathrm{Q}_x$, $x \in \mathbb{Z}^d$.

The generator L of the corresponding interacting spin system is first defined on local functions f by

$$\mathrm{L} f(\sigma) \equiv \sum_{x \in \mathbb{Z}^d} \sum_{\mathrm{U} \in \mathrm{Q}_x} c_x(\mathrm{U}, \sigma)[f(\mathrm{U}\sigma) - f(\sigma)]. \tag{2.8}$$

Using standard theory, *see* for example [16], it can be shown from (2.6)-(2.8) that its closure in $\mathrm{L}^2(\mu)$, still denoted by L, is the generator of a Markov semigroup $\mathrm{S}(t)$, $t \geqq 0$, and is a self-adjoint operator on $\mathrm{L}^2(\mu)$. Moreover, defining for each $\mathrm{U} \in \mathrm{Q}_0$ the bounded operator $\mathrm{L_U}$ on $\mathrm{L}^2(\mu)$ by

$$\mathrm{L_U} f(\sigma) = c_0(\mathrm{U}, \sigma)[f(\mathrm{U}\sigma) - f(\sigma)], \tag{2.9}$$

we have that

$$\int g(\sigma)[L_U + L_{U^{-1}}] f(\sigma)\, d\mu(\sigma) = \int [L_U + L_{U^{-1}}] g(\sigma) f(\sigma)\, d\mu(\sigma) \quad (2.10)$$

for every f, $g \in L^2(\mu)$.

We connect this structure with the corresponding one for a quantum spin system where to each site $x \in \mathbb{Z}^d$ we now associate the vector

$$\left.\begin{aligned} |\sigma_x\rangle &= \begin{pmatrix}1\\0\end{pmatrix} \quad \text{if } \sigma_x = +1 \\ &= \begin{pmatrix}0\\1\end{pmatrix} \quad \text{if } \sigma_x = -1. \end{aligned}\right\} \quad (2.11)$$

In this way, the configurations $\{\sigma_y, y \in K\}$ in some finite region $K \subset \mathbb{Z}^d$ generate an orthonormal basis $\{|\sigma\rangle_K \equiv \bigotimes_{x \in K} |\sigma_x\rangle\}$ of the Hilbert space $\mathcal{H}_K \equiv \bigotimes_{x \in K} \mathbb{C}^2$. The Pauli matrices σ_x^1, σ_x^2, σ_x^3, $x \in \mathbb{Z}^d$, can be used to decompose any operator A on $\mathcal{H}_K$ as

$$A = \sum_{R \subset K} \sigma_R^1 f_R(\sigma^3) \quad (2.12)$$

where for each $R \subset K$,

$$\sigma_R^1 |\sigma\rangle_K \equiv |\sigma^R\rangle_K; \qquad \begin{aligned} (\sigma^R)_x &= -\sigma_x \quad \text{if } x \in R \\ &= \sigma_x \quad \text{if } x \notin R, \end{aligned}$$

and

$$f_R(\sigma^3)|\sigma\rangle_K \equiv f_R(\sigma)|\sigma\rangle_K$$

with $f_R \in C(K)$.

The set of all local operators (2.12) is denoted by $\mathcal{A}_{loc} \equiv \bigcup_{|K| < \infty} \mathcal{A}_K$, $\mathcal{A}_K \equiv \bigotimes_{x \in K} \mathcal{A}_x$ and $\mathcal{A}_x \equiv M_2(\mathbb{C})$, the two by two matrices. The infinite volume algebra $\mathcal{A}$ of the quantum system is the uniform closure of $\mathcal{A}_{loc}$. The subalgebra of classical operators, $\mathcal{A}_{cl}$, is defined by

$$\mathcal{A}_{cl} \equiv \{f(\sigma^3) \mid f \in C(X)\}. \quad (2.13)$$

For a permutation $U \in Q_0$ define the unitary operator $\tilde{U} \in \mathcal{A}_\Lambda$ by

$$\tilde{U}|\sigma\rangle_\Lambda = |U\sigma\rangle_\Lambda \quad (2.14)$$

for all $\sigma \in X_\Lambda$. Remark that $\tilde{U}$ has the decomposition (2.12) with $K = \Lambda$ and

$$f_R(\sigma) \equiv \delta_{U, R}(\sigma) = \delta_{U\sigma, \sigma^R} \quad (2.15)$$

where δ is the Kronecker delta function.

Given $\mu \in \mathscr{G}(H)$, define $\pi_\mu : \mathscr{A} \to \mathscr{B}(L^2(\mu))$, a map from the quantum algebra $\mathscr{A}$ to the bounded operators on $L^2(\mu)$, by the action on $g \in L^2(\mu)$:

$$[\pi_\mu(\sigma_R^1 F(\sigma^3))g](\sigma) \equiv e^{-(1/2)[H(\sigma^R)-H(\sigma)]} F(\sigma^R) g(\sigma^R), \qquad (2.16)$$

for all finite $R \subset \mathbb{Z}^d$ and local functions F on X.

The following Proposition can be found in [8].

Proposition 2.1. – *π_μ extends to a representation of $\mathscr{A}$. π_μ is irreducible iff μ is an extremal Gibbs measure.*

We obtain a (quantum) state ω_μ on $\mathscr{A}$ by defining

$$\omega_\mu(A) \equiv \int [\pi_\mu(A)\,1](\sigma)\, d\mu(\sigma). \qquad (2.17)$$

Equivalently, the state ω_μ is defined through its GNS-representation, *see* [17], and the GNS-triplet $(\mathscr{H}_\mu, \pi_\mu, \Omega_\mu)$ is given by $\mathscr{H}_\mu = L^2(\mu)$, $\Omega_\mu = 1$ and π_μ as in (2.16). Thus, for that choice,

$$\omega_\mu(A) = (\Omega_\mu, \pi_\mu(A)\Omega_\mu)_{\mathscr{H}_\mu}. \qquad (2.18)$$

Notice that for all local permutations U one has

$$[\pi_\mu(\tilde{U})h](\sigma) = e^{-(1/2)\Delta_{U^{-1}}H(\sigma)} h(U^{-1}\sigma) \qquad (2.19)$$

for all $h \in L^2(\mu)$. This is a direct consequence of (2.15) and (2.16).

Continuing now with the dynamics, we first say that an operator $\mathscr{L}$ from a dense *-subalgebra into $\mathscr{A}$ is a dissipation if

$$\left.\begin{array}{ll} \text{(i)} & \mathscr{L}(\mathbf{1}) = 0, \\ \text{(ii)} & [\mathscr{L}(A)]^* = \mathscr{L}(A^*), \\ \text{(iii)} & \mathscr{L}(A^*A) - \mathscr{L}(A^*)A - A^*\mathscr{L}(A) \geqq 0, \end{array}\right\} \qquad (2.20)$$

for all A in the domain $D(\mathscr{L})$ of $\mathscr{L}$. A^* is the adjoint of $A \in \mathscr{A}$.

The quantum version of the (classical) generators L_U, $U \in Q_0$, defined in (2.8) are the so-called Lindblad operators $\mathscr{L}_V$, $V \in \mathscr{A}_{\bar{\Lambda}}$ with $\bar{\Lambda}$ defined in (2.2), (iii) [11]. They have the form (Heisenberg picture):

$$\mathscr{L}_V(A) \equiv V^*AV - \frac{1}{2}\{V^*V, A\} \qquad (2.21)$$

where $\{A, B\} \equiv AB + BA$; $A, B \in \mathscr{A}$.

Clearly, $\mathscr{L}_V$ extends to a bounded dissipation on $\mathscr{A}$ and it is known that $\mathscr{L}_V$ is the generator of a completely positive semigroup on $\mathscr{A}$ ([11], [18]). We say that $\mathscr{L}_V$ is *reversible* for a state ω on $\mathscr{A}$ if

$$\omega(A\,\mathscr{L}_V(B)) = \omega(\mathscr{L}_V(A)\,B) \qquad (2.22)$$

for all $A, B \in \mathscr{A}$.

The extension from the classical process defined by (2.1)-(2.8) to a quasi-unique quantum evolution preserving the local reversibility as expressed in (2.10), is the subject of the next Propositions.

PROPOSITION 2.2. — *Suppose that* $U=U^{-1}\in Q_0$. *The following statements about the generator* $\mathcal{L}_V$, $V\in\mathcal{A}_{\bar\Lambda}$ *of* (2.21) *are equivalent.*

(i) $\mathcal{L}_V$ *satisfies*

(α) $\mathcal{L}_V\mathcal{A}_{cl}\subseteq\mathcal{A}_{cl}$ *and*

$$\mathcal{L}_V F(\sigma^3)=(L_U F)(\sigma^3) \quad \textit{for all } F\in\mathcal{A}_{cl}$$

(β) $\mathcal{L}_V$ *is reversible for* ω_μ.

(ii) *There exists a unitary operator* $W\in\mathcal{A}_{cl}\cap\mathcal{A}_{\bar\Lambda}$ *such that*

$$\mathcal{L}_V(A)=\mathcal{L}_{WV(U)}(A) \quad \textit{for all } A\in\mathcal{A}$$

where

$$V(U)=(\tilde U-\mathbf{1})\,c_0(U,\sigma^3)^{1/2}\in\mathcal{A}_{\bar\Lambda} \tag{2.23}$$

and $\tilde U$ *is defined in* (2.14).

Remarks:

(i) As W is a unitary operator, it is easy to see that for all $A\in\mathcal{A}$

$$\mathcal{L}_{WV(U)}(A)=\mathcal{L}_{V(U)}(A)+V(U)^*\,\mathcal{L}_W(A)\,V(U) \tag{2.24}$$

and because $W\in\mathcal{A}_{cl}$ it follows that

$$\mathcal{L}_{WV(U)}(F)=\mathcal{L}_{V(U)}(F)$$

for all $F\in\mathcal{A}_{cl}$.

(ii) From the expression (2.23) combined with (2.6) and (2.19) one computes:

$$\begin{aligned}[\pi_\mu(V(U))\Omega_\mu](\sigma)&=e^{-(1/2)\Delta_{U^{-1}}H(\sigma)}c_0(U,U^{-1}\sigma)^{1/2}-c_0(U,\sigma)^{1/2}\\&=0\end{aligned}$$

and thus

$$\omega_\mu(V(U)^*V(U))=0. \tag{2.25}$$

Therefore by Schwarz inequality one has for all $B\in\mathcal{A}$,

$$|\omega_\mu(BV(U))|^2\leq\omega_\mu(B^*B)\,\omega_\mu(V(U)^*V(U))=0. \tag{2.26}$$

Combining (2.24) and (2.26) yields for $\mathcal{L}_V$ of Proposition (2.2)

$$\omega_\mu(A\,\mathcal{L}_V(B))=\omega_\mu(A\,\mathcal{L}_{V(U)}(B)) \tag{2.27}$$

for all $A, B\in\mathcal{A}$.

Now we handle the case $U\neq U^{-1}$. In general $\mathcal{L}_{V(U)}\mathcal{A}_{cl}\not\subseteq\mathcal{A}_{cl}$ for $V(U)$ as in (2.23). For general $U\in Q_x$ we define instead

$$V_x(U)\equiv\tilde U f_x(\sigma^3)-g_x(\sigma^3) \tag{2.28}$$

with

$$\begin{aligned}f_x(\sigma^3)&=c_x(U,\sigma^3)^{1/2}\\g_x(\sigma^3)&=c_x(U^{-1},\sigma^3)^{1/2},\end{aligned}$$

$x\in\mathbb{Z}^d$, and simply write $\mathcal{L}_U\equiv\mathcal{L}_{V_x(U)}$.

That this is the good choice follows from Proposition 2.2 and:

PROPOSITION 2.3. – *For all* $U \in Q_x$, $x \in \mathbb{Z}^d$, *one has*

(i) $(\mathcal{L}_U + \mathcal{L}_{U^{-1}})(\mathcal{A}_{cl}) \subset \mathcal{A}_{cl}$ *and* $(\mathcal{L}_U + \mathcal{L}_{U^{-1}})[F(\sigma^3)] = (L_U F + L_{U^{-1}} F)(\sigma^3)$ *for all* $F \in C(X)$;

(ii) $\mathcal{L}_U + \mathcal{L}_{U^{-1}}$ *is reversible for* ω_μ.

Note the correspondence of (ii) with (2.10). Thus far however we have only considered strictly local dynamics with bounded generator. To construct the associated infinite volume quantum dynamics coinciding on the classical algebra with the process defined by (2.8), we define the operator $\mathcal{L}$ on $\mathcal{A}$ with domain $D(\mathcal{L}) = \mathcal{A}_{loc}$ and action

$$\mathcal{L}(A) \equiv \sum_{x \in \mathbb{Z}^d} \sum_{U \in Q_x} \mathcal{L}_U, \qquad A \in \mathcal{A}_{loc} \tag{2.29}$$

We say that $\mathcal{L}$ is locally reversible for a state ω on $\mathcal{A}$ if for all $U \in Q_x$, $x \in \mathbb{Z}^d$,

$$\omega(A(\mathcal{L}_U + L_{U^{-1}})(B)) = \omega((\mathcal{L}_U + \mathcal{L}_{U^{-1}})(A)B) \tag{2.30}$$

for all $A, B \in \mathcal{A}_{loc}$. Note that even for classical systems local reversibility [as in (2.10)] is not equivalent with (global) reversibility, except for example for spin flip systems where under the condition of positivity of the rates, both are equivalent with (2.6) which then becomes the definition of the stochastic Ising model or Glauber type dynamics [16].

THEOREM 2.4. – *The operator* $\mathcal{L}$ *defined by* (2.29) *is a dissipation on* $\mathcal{A}$. $\mathcal{A}_{loc}$ *is a core for* $\mathcal{L}$ *and its closure is the generator of a spatially homogeneous strongly continuous semi-group* γ_t, $t \geqq 0$, *of completely positive unity preserving contractions on* $\mathcal{A}$. *Furthermore,* $\gamma_t F(\sigma^3) = S(t) F(\sigma^3)$, $F \in C(X)$, *with* $S(t)$ *the Markov semigroup obtained from* (2.8) *and* $\mathcal{L}$ *is locally reversible for the state* ω_μ.

In this way we have constructed a dynamical quantum system $(\mathcal{A}, \omega_\mu, \gamma_t)$ which is an extension of the classical system $(\mathcal{A}_{cl} = C(X), \mu, S(t))$ such that the property of local reversibility is preserved. Moreover, we know from Proposition 2.2 that there is essentially a unique way of doing this. In the next Section we investigate what kind of quantum states we obtain if a local property as (2.30) is imposed.

3. LOCAL REVERSIBILITY AND ITS CONSEQUENCES FOR THE QUANTUM STATE

Another way of defining Gibbs states for classical Hamiltonians as in (2.5) would be to write instead of (2.7) a local Markov property for the probability measure μ: its conditional distributions inside a volume K given the outside are local functions of the configurations in the boundary $\partial K \subset \mathbb{Z}^d \setminus K$. This is in fact a direct consequence of the local reversibility (2.10) (plus positivity of the rate functions) with U a spin flip operation. While the construction of the previous Section naturally extends the local reversibility to quantum dynamics, we expect that in fact states describing quantum equilibrium generically do not have such local properties as for example the Markov property discussed above. The first question is therefore what the consequences of (2.30) are.

Let $\mathscr{B}$ be a C*-algebra of observables (like the $\mathscr{A}$ of the previous Section), ω a state on $\mathscr{B}$ and δ the generator of a quantum Hamiltonian dynamics α_t ($\alpha_t = e^{t\delta}$ is a one-parameter group of *-automorphisms of $\mathscr{B}$). ω is a *ground state* for α_t if

$$-i\,\omega(A^* \delta(A)) \geqq 0 \tag{3.1}$$

for all A in the domain $D(\delta)$ of δ. The physical significance of (3.1) can be understood in the context of correlation inequalities as clarified for the classical situation in [19]. It follows that ω is α_t-invariant [17], theorem 5.3.19. Moreover, with $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$ the GNS triplet of the ground state ω, there exists a non-negative operator H_ω on $\mathscr{H}_\omega$ such that

$$\left.\begin{aligned} &\pi_\omega(\alpha_t(A)) = e^{itH_\omega}\pi_\omega(A)\,e^{-itH_\omega} \\ &H_\omega \Omega_\omega = 0. \end{aligned}\right\} \tag{3.2}$$

PROPOSITION 3.1. – *Let $\mathscr{H}$ be a Hilbert space and ω a state on $\mathscr{B}(\mathscr{H})$, the bounded operators on $\mathscr{H}$. For $V \in \mathscr{B}(\mathscr{H})$, define the bounded dissipation $\mathscr{L}_V$ by*

$$\mathscr{L}_V(A) \equiv V^* AV - \frac{1}{2}\{V^* V, A\}, \qquad A \in \mathscr{B}(\mathscr{H}).$$

Then, the following conditions are equivalent

(i) *$\mathscr{L}_V$ is reversible for ω.*

(ii) *either* (α) *or* (β) *are satisfied:*

(α) $\quad V = V^*$ *(up to a phase) and* $\omega([A, V]) = 0,\ \forall A \in \mathscr{B}(\mathscr{H})$ $\quad$ (3.3)

(β) $\quad \omega(V^* V) = 0$ $\quad$ (3.4)

Remarks:

ad(α) Suppose that ω is an α_t-KMS state on $\mathscr{B}(\mathscr{H})$, (3.3) then implies that V is α_t-invariant [17], Theorem 5.3.28.

ad(β) Let ω be a state such that (3.4) holds.

From Schwarz inequality,

$$|\omega(AV)|^2 \leqq \omega(A^* A)\,\omega(V^* V) = 0 \tag{3.5}$$

for all $A \in \mathscr{B}(\mathscr{H})$. Let

$$\delta(A) \equiv i[V^* V, A]. \tag{3.6}$$

Then, from (3.5) we get for all $A \in \mathscr{B}(\mathscr{H})$

$$-i\omega(A^* \delta(A)) = \omega((VA)^* VA) \geqq 0 \tag{3.7}$$

and ω is a ground state for the Hamiltonian

$$\tilde{\mathscr{H}} \equiv V^* V. \tag{3.8}$$

Furthermore, again from (3.5), for all A, $B \in \mathscr{B}(\mathscr{H})$,

$$\frac{1}{2} i\omega(A\,\delta(B)) = \omega(A\,\mathscr{L}_V(B)). \tag{3.9}$$

Thus, the GNS Hamiltonian H_ω, defined in (3.2), is completely specified by

$$H_\omega(\pi_\omega(A)\Omega_\omega) = -2\pi_\omega(\mathscr{L}_V A)\Omega_\omega \tag{3.10}$$

for all $A \in \mathscr{B}(\mathscr{H})$. This relation is the key-step together with Propositions 2.2 and 2.3 in understanding the usual Feynman-Kac formula on the level of the dynamics for both the quantum and the classical system.

Clearly, the next is the application of Proposition 3.1 to the system of Section 2 with $V = V_x(U)$, *see* (2.28).

First observe that

$$V_x(U^{-1}) = -\tilde{U}^{-1} V_x(U) \tag{3.11}$$

and therefore

$$V_x(U)^* V_x(U) = V_x(U^{-1})^* V_x(U^{-1}) \tag{3.12}$$

for all $U \in Q_x$, $x \in \mathbb{Z}^d$.

Furthermore, by (2.16), (2.19) and (2.6)

$$\begin{aligned}[\pi_\mu(V_x(U))\Omega_\mu](\sigma) &= e^{-1/2\Delta_{U^{-1}}H(\sigma)} c_x(U, U^{-1}\sigma)^{1/2} - c(U^{-1}, \sigma) \\ &= 0\end{aligned}$$

and thus

$$\omega_\mu(V_x(U)^* V_x(U)) = 0. \tag{3.13}$$

By Proposition 3.1 ω_μ is reversible for $\mathscr{L}_U$ and $\mathscr{L}_{U^{-1}}$ separately and ω_μ is a ground state for the (formal) quantum Hamiltonian.

$$\tilde{H} \equiv \sum_{x \in \mathbb{Z}^d} \sum_{U \in Q_x} V_x(U)^* V_x(U) \tag{3.14}$$

$\tilde{H}$ is translation invariant by construction. The derivation δ corresponding to (3.14) is densily defined on the local algebra $\mathscr{A}_{loc}$ and given by

$$\delta(A) \equiv i \sum_{x \in \mathbb{Z}^d} \sum_{U \in Q_x} [V_x(U)^* V_x(U), A]. \tag{3.15}$$

Furthermore, it is well known [17], theorem 6.2.4, that δ generates a quantum Hamiltonian dynamics α_t, $t \in \mathbb{R}$, on $\mathscr{A}$:

$$\alpha_t(\mathrm{A}) = \lim_{\mathrm{K} \to \infty} e^{it\tilde{\mathrm{H}}_{\mathrm{K}}} \mathrm{A}\, e^{-it\tilde{\mathrm{H}}_{\mathrm{K}}} \in \mathscr{A} \tag{3.16}$$

where

$$\tilde{\mathrm{H}}_{\mathrm{K}} = \sum_{x \in \mathrm{K}} \sum_{\mathrm{U} \in \mathrm{Q}_x} \mathrm{V}_x(\mathrm{U})^* \mathrm{V}_x(\mathrm{U}). \tag{3.17}$$

In the following we study the set of quantum states for which the quantum dissipation $\mathscr{L}$ (2.29), Theorem 2.4 is locally reversible. Proposition 3.1 is a key result for that. However we will exclude case (α) of this Proposition. If for example $\mathrm{U} = \mathrm{U}^{-1}$, then it is easy to see that:

$$\mathrm{V}_x(\mathrm{U})^* = \mathrm{V}_x(\mathrm{U}) \quad \text{iff} \quad \Delta_{\mathrm{U}}\mathrm{H} = 0 \tag{3.18}$$

PROPOSITION 3.2. — *Let* ω *be a state on* $\mathscr{A}$. *Take* $\mathrm{U} \in \mathrm{Q}_0$ *and suppose that* $c_0(\mathrm{U}, \sigma) > 0$ *and* $\Delta_{\mathrm{U}}\mathrm{H} \neq 0$.

If ω *is reversible for* $\mathscr{L}_{\mathrm{U}} + \mathscr{L}_{\mathrm{U}^{-1}}$, *then*

$$\omega(\mathrm{V}_0(\mathrm{U})^* \mathrm{V}_0(\mathrm{U})) = 0. \tag{3.19}$$

As a consequence of Proposition 3.1 ω is reversible for $\mathscr{L}_{\mathrm{U}}$ and $\mathscr{L}_{\mathrm{U}^{-1}}$ separately. Therefore the statement of Proposition 3.2 is closely related with the observation

$$\mathscr{L}_{\mathrm{U}} \mathscr{A}_{\mathrm{cl}} \not\subseteq \mathscr{A}_{\mathrm{cl}}$$

which was already mentioned in Section 2.

The following Proposition states that (3.19) is equivalent with a "DLR-equation" for ω:

PROPOSITION 3.3. — *Let* ω *be a state on* $\mathscr{A}$ *and take* $\mathrm{U} \in \mathrm{Q}_0$ *with* $c_0(\mathrm{U}, \sigma) > 0$. *Then the following statements are equivalent:*

(i) $\omega(\mathrm{V}_0(\mathrm{U})^* \mathrm{V}_0(\mathrm{U})) = 0$.

(ii) ω *satisfies the "DLR-equation":*

$$\omega(\tilde{\mathrm{U}}\mathrm{A}) = \omega(e^{-1/2\Delta_{\mathrm{U}}\mathrm{H}(\sigma^3)}\mathrm{A}) \tag{3.20}$$

for all $\mathrm{A} \in \mathscr{A}$.

Notice that from (3.20) one gets

$$\begin{aligned} \omega(\tilde{\mathrm{U}}\mathrm{A}\tilde{\mathrm{U}}^{-1}) &= \omega(e^{-1/2\Delta_{\mathrm{U}}\mathrm{H}(\sigma^3)}\mathrm{A}\tilde{\mathrm{U}}^{-1}) \\ &= \overline{\omega(\tilde{\mathrm{U}}\mathrm{A}^* e^{-1/2\Delta_{\mathrm{U}}\mathrm{H}(\sigma^3)})} \\ &= \omega(e^{-1/2\Delta_{\mathrm{U}}\mathrm{H}(\sigma^3)}\mathrm{A}\, e^{-1/2\Delta_{\mathrm{U}}\mathrm{H}(\sigma^3)}) \end{aligned} \tag{3.21}$$

for all $\mathrm{A} \in \mathscr{A}$.

For $\mathrm{A} = \mathrm{F}(\sigma^3) \in \mathscr{A}_{\mathrm{cl}}$ one gets from (3.21)

$$\omega(\mathrm{F}_{\mathrm{U}}(\sigma^3)) = \omega(e^{-1/2\Delta_{\mathrm{U}}\mathrm{H}(\sigma^3)}\mathrm{F}(\sigma^3)) \tag{3.22}$$

where $F_U(\sigma^3) \in \mathscr{A}_{cl}$ and defined by

$$F_U(\sigma^3)|\sigma\rangle = F(U^{-1}\sigma)|\sigma\rangle$$

Remark the correspondence of (3.22) with (2.7).

Let $\tilde{Q}_0$ be defined by

$$\tilde{Q}_0 \equiv \{ U \in Q_0 \,|\, c_0(U, .) > 0 \quad \text{and} \quad \Delta_U H \neq 0 \}. \tag{3.23}$$

We call the process (2.8) *irreducible* if for all $U \in Q_0$ there exists a sequence $U_1, \ldots, U_n \in \tilde{Q}_0$, $n \in \mathbb{N}$, such that

$$U = U_1 \ldots U_n \tag{3.24}$$

THEOREM 3.4. – *Let* ω *be a state on* $\mathscr{A}$ *and suppose the classical process* (2.8) *is irreducible.*

The following conditions are equivalent:

(i) *The quantum process* $\mathscr{L} = \sum_{x \in \mathbb{Z}^d} \sum_{U \in Q_x} \mathscr{L}_U$ *is locally reversible* (2.30) *for* ω.

(ii) *There exists a Gibbs measure* $\mu \in \mathscr{G}(H)$ *such that* $\omega = \omega_\mu$ (2.17).

If these conditions hold, then ω *is a ground state for the quantum Hamiltonian* (3.14), *and satisfies the DLR-equation* (3.20).

For classical systems it is well known that the states characterized by the local reversibility condition are exactly the Gibbs measures $\mu \in \mathscr{G}(H)$. The above Theorem is the analogous (extended) result for the quantum process $\mathscr{L}$ of Theorem 2.4. A state satisfying (2.30) must be a ground state and restricted to the classical algebra $\mathscr{A}_{cl}$ the state is a Gibbs measure $\mu \in \mathscr{G}(H)$.

We end this Section with some remarks about the Feynman-Kac formula in this context. Let $(\mathscr{H}_\mu, \pi_\mu, \Omega_\mu)$ be the GNS-triplet of ω_μ : $\mathscr{H}_\mu = L^2(\mu)$, $\Omega_\mu = \mathbf{1}$ and π_μ is defined in (2.16). Define the operator H_μ with domain $D(H_\mu) = \pi_\mu(\mathscr{A}_{loc})\Omega_\mu \subset \mathscr{H}_\mu$ by:

$$H_\mu(\pi_\mu(A)\Omega_\mu) = -i\pi_\mu(\delta(A))\Omega_\mu, \qquad A \in \mathscr{A}_{loc} \tag{3.25}$$

where δ is the derivation defined in (3.15). H_μ is densily defined and because ω_μ is a ground state for δ (or $\tilde{H}$), and thus time-invariant, one has that H_μ is symmetric. Furthermore, as δ is of finite range, $\delta(A) \in \mathscr{A}_{loc}$ if $A \in \mathscr{A}_{loc}$, one has that the set of local elements is a set of analytic vectors for δ [17], Theorem 6.2.4. Hence $\pi_\mu(\mathscr{A}_{loc})\Omega_\mu$ is a set of analytic vectors for H_μ. By the theorem of Nelson [20], Theorem X.39, H_μ is essentially self-adjoint and thus there exists a unique self-adjoint extension, again denoted by H_μ.

As $\pi_\mu(V_x(U))\Omega_\mu = 0$, $U \in Q_x$, $x \in \mathbb{Z}^d$, we get from (2.29)

$$H_\mu(\pi_\mu(A)\Omega_\mu) = -2\pi_\mu(\mathscr{L} A)\Omega_\mu \tag{3.26}$$

for all $A \in \mathscr{A}_{loc}$. Because $\mathscr{L}$ is a dissipation, Theorem 2.4 and $\mathscr{L}$ is reversible for ω_μ one has that H_μ is a positive operator. In fact H_μ is the GNS-Hamiltonian of ω_μ [*see* (3.2)].

Moreover if F is local, then from (3.26) and Theorem 2.4,

$$H_\mu(\pi_\mu(F(\sigma^3))\Omega_\mu) = -2\pi_\mu(LF(\sigma^3))\Omega_\mu.$$

where L is the classical generator (2.8). Identifying

$$\pi_\mu(F(\sigma_3))\Omega_\mu \equiv F \in L^2(\mu)$$

one gets

$$H_\mu F = -2LF.$$

Hence for all $F \in L^2(\mu)$, $t \geqq 0$

$$e^{-tH_\mu} F = S(2t)F$$

where $S(t)$ is the Markov semigroup of Theorem 2.4.

Finally for $F_1(\sigma^3), F_2(\sigma^3) \in \mathscr{A}_{cl}$ and $t_2 \geqq t_1$ one computes

$$\begin{aligned}\omega_\mu(\alpha_{it_1}(F_1(\sigma^3))\alpha_{it_2}(F_2(\sigma^3))) &= (\Omega_\mu, \pi_\mu(F_1(\sigma^3))e^{-(t_2-t_1)H_\mu}\pi_\mu(F_2(\sigma^3))\Omega_\mu) \\ &= \int d\mu(\sigma)F_1(\sigma)(S(2(t_2-t_1))F_2)(\sigma) \\ &= \int dP_\mu(\sigma)F_1(\sigma(2t_1))F_2(\sigma(2t_2))\end{aligned}$$

where dP_μ is the path-space measure of the process $\sigma(t)$ started from μ. By induction one gets, for $t_1 \leqq t_2 \leqq \ldots \leqq t_n$ and $F_1(\sigma^3), \ldots, F_n(\sigma^3) \in \mathscr{A}_{cl}$

$$\omega_\mu(\alpha_{it_1}(F_1(\sigma^3))\ldots\alpha_{it_n}(F_n(\sigma^3))) = \int dP_\mu(\sigma)F_1(\sigma(2t_1))\ldots F_n(\sigma(2t_n)).$$

It is this formula which goes under the name of Feynman and Kac.

4. PROOF OF THE RESULTS

We begin with the proof of Proposition 3.1, which is independent of the other results.

Proof of Proposition 3.1.

Suppose that $\mathscr{L}_V$ is reversible for a given state ω on $\mathscr{B}(\mathscr{H})$. Let e, f, e', f' be unit vectors of $\mathscr{H}$, $\langle\,.\,,\,.\,\rangle$ and consider the rank-one operators $B = |e\rangle\langle f|$, $C = |e'\rangle\langle f'|$, *i.e.* for $\varphi \in \mathscr{H}$.

$$B\varphi = \langle f, \varphi\rangle e.$$

Then $\omega(B\mathscr{L}_V(C))=\omega(\mathscr{L}_V(B)C)$ implies

$$\langle f|\{\omega(|e\rangle\langle f'|V)V^*-\frac{1}{2}\omega(|e\rangle\langle f'|V^*V)\mathbf{1}\}|e'\rangle$$
$$=\langle f|\{\omega(V^*|e\rangle\langle f'|)V-\frac{1}{2}\omega(V^*V|e\rangle\langle f')\mathbf{1}\}|e'\rangle$$

Since this holds for all unit vectors $e, f, e', f'\in\mathscr{H}$ one has

$$\omega(AV)V^*-\omega(V^*A)V=\frac{1}{2}\omega([A, V^*V])\mathbf{1} \tag{4.1}$$

for all $A\in\mathscr{B}(\mathscr{H})$.

If $V^*=e^{i\varphi}V$, $\varphi\in\mathbb{R}$, then, from (4.1),

$$\omega([A, V])V=\frac{1}{2}\omega([A, V^*V])\mathbf{1} \tag{4.2}$$

and thus $\omega([A, V])=0$ for all A.

If $V^*\neq e^{i\varphi}V$, for all $\varphi\in\mathbb{R}$, then [taking $A=\mathbf{1}$ in (4.1)] necessarily $\omega(V)=0$. By taking the expectation value of (4.1) in the state ω one gets $\omega([A, V^*V])=0$ and thus $\omega(AV)V^*=\omega(V^*A)V$. Because $V^*\neq e^{i\varphi}V$ it follows that $\omega(V^*V)=0$.

Rest to prove that (α) or (β) of condition (ii) are sufficient conditions for the reversibility of $\mathscr{L}_V$ w.r.t. ω. That (ii, α) implies (i) is an easy computation and (i) follows from (ii, β) because $\omega(V^*V)=0$ implies $\omega(AV)=0$ for all A [*see* (3.5)] and hence

$$\omega(A\mathscr{L}_V(B))=-\frac{1}{2}\omega(AV^*VB)=\omega(\mathscr{L}_V(A)B). \quad \blacksquare \tag{4.3}$$

Now we prove the results of Section 2.

Proof of Proposition 2.2.

We first prove that condition (i) of the Proposition implies (ii). Let V be an operator on $\mathscr{H}_{\bar{\Lambda}}$, then V can be written as

$$V=\sum_{R\subset\bar{\Lambda}}\sigma_R^1 f_R(\sigma^3) \tag{4.4}$$

with $f_R\in C(\bar{\Lambda})$ [*see* (2.12)].

Input of the first condition (i, α)

(i, α) implies that

$${}_{\bar{\Lambda}}\langle\sigma|\mathscr{L}_V F(\sigma^3)|\sigma\rangle_{\bar{\Lambda}}=(L_U F)(\sigma)=c(U, \sigma)(F(U\sigma)-F(\sigma)) \tag{4.5}$$

for all $F\in C(\bar{\Lambda})$ and $\sigma\in X_{\bar{\Lambda}}$.

Using (4.4) one computes

$$_{\bar{\Lambda}}\langle \sigma | \mathscr{L}_V F(\sigma^3) | \sigma \rangle_{\bar{\Lambda}} = \sum_{R \subset \bar{\Lambda}} |f_R(\sigma)|^2 (F(\sigma^R) - F(\sigma)). \tag{4.6}$$

Writing $F(U\sigma) = \sum_{R \subset \bar{\Lambda}} \delta_{U_{\sigma,\sigma} R} F(\sigma^R)$ and combining (4.5) and (4.6) one gets

$$F(\sigma)(c(U, \sigma)(1 - \delta_{U_{\sigma,\sigma}}) - \sum_{R \neq \Phi} |f_R(\sigma)|^2) + \sum_{R \neq \Phi} F(\sigma^R)(|f_R(\sigma)|^2 - \delta_{U_{\sigma,\sigma} R}\, c(U, \sigma)) = 0$$

for all $F \in C(\bar{\Lambda})$ and $\sigma \in X_{\bar{\Lambda}}$. Hence for all $R \neq \Phi$ (the empty set)

$$|f_R(\sigma)| = c(U, \sigma)^{1/2} \delta_{U_{\sigma,\sigma} R}$$

and V must have the form

$$V = f_\Phi(\sigma^3) + \sum_{R \neq \Phi} \sigma_R^1 \delta_{U, R}(\sigma^3) e^{i\varphi_R(\sigma^3)} c(U, \sigma^3)^{1/2} \tag{4.7}$$

where φ_R is a real function on $X_{\bar{\Lambda}}$ and $\delta_{U, R}(\sigma^3)$ is defined in (2.15).

From (4.7) one easily finds

$$V | \sigma \rangle_{\bar{\Lambda}} = X_1(\sigma) | \sigma \rangle_{\bar{\Lambda}} + X_2(\sigma) | U\sigma \rangle_{\bar{\Lambda}}$$

with

$$X_1(\sigma) = f_\Phi(\sigma) - \delta_{U_{\sigma,\sigma}} c(U, \sigma)^{1/2} e^{i\varphi_\Phi(\sigma)}$$
$$X_2(\sigma) = c(U, \sigma)^{1/2} . \exp i \sum_{R \subset \bar{\Lambda}} \varphi_R(\sigma) \delta_{U_{\sigma,\sigma} R}.$$

Thus V must have the form

$$V = \tilde{U} f(\sigma^3) + g(\sigma^3) \tag{4.8}$$

with $f, g \in C(\bar{\Lambda})$ and $|f(\sigma)| = c(U, \sigma)^{1/2}$.

Now we proceed with this form.

Take $\sigma, \eta \in X_{\bar{\Lambda}}$ and consider the rank one operators

$$P_{\sigma,\eta} = | \sigma \rangle_{\bar{\Lambda}}\, {}_{\bar{\Lambda}}\langle \eta | \in \mathscr{B}(\mathscr{H}_{\bar{\Lambda}}).$$

Using (4.8) one computes

$$\begin{aligned} \mathscr{L}_V(P_{\sigma,\eta}) = P_{\sigma,\eta}&\left(\bar{g}(\sigma) g(\eta) - \frac{1}{2}|g(\sigma)|^2 - \frac{1}{2}|g(\eta)|^2 - \frac{1}{2}|f(\sigma)|^2 - \frac{1}{2}|f(\eta)|^2\right) \\ &+ P_{U\sigma, U\eta} \bar{f}(U\sigma) f(U\eta) \\ &+ P_{U\sigma, \eta}\left(\bar{f}(U\sigma) g(\eta) - \frac{1}{2} g(\sigma) \bar{f}(U\sigma) - \frac{1}{2} f(\sigma) \bar{g}(U\sigma)\right) \\ &+ P_{\sigma, U\eta}\left(\bar{g}(\sigma) f(U\eta) - \frac{1}{2}\bar{g}(\eta) f(U\eta) - \frac{1}{2}\bar{f}(\eta) g(U\eta)\right). \end{aligned} \tag{4.9}$$

Taking $\sigma=\eta$ in (4.9), then condition (i, α), $\mathscr{L}_V \mathscr{A}_{cl} \subset \mathscr{A}_{cl}$ implies

$$\bar{f}(U\sigma)\, g(\sigma) = f(\sigma)\, \bar{g}(U\sigma). \tag{4.10}$$

Combining (4.8)-(4.10) we obtain

$$\begin{aligned}\mathscr{L}_V F(\sigma^3) &= \sum_{\sigma \in \bar{\Lambda}} F(\sigma)\, \mathscr{L}_V (P_{\sigma,\sigma}) \\ &= \sum_{\sigma \in \bar{\Lambda}} c(U, \sigma)(F(U\sigma) - F(\sigma))\, P_{\sigma,\sigma} \\ &= (L_U F)(\sigma^3)\end{aligned}$$

and we have that the form of V, (4.8) and (4.10) is necessary and sufficient for condition (i, α) of the Proposition.

Input of condition (i, β)

Here we apply Proposition 3.1 to the case $\mathscr{H} = \mathscr{H}_{\bar{\Lambda}}$, $\omega = \omega_\mu$ and $\mathscr{L}_V$ with V of the form (4.8), (4.10).

There are two cases.

Case 1 : input of $\omega_\mu (V^* V) = 0$.

Write

$$f(\sigma) = c(U, \sigma)^{1/2} e^{i\psi(\sigma)} \tag{4.11}$$

with ψ a real function on $X_{\bar{\Lambda}}$.

From (2.16), (2.19) and (2.6) one obtains

$$\begin{aligned}[\pi_\mu (V)\, \Omega_\mu](\sigma) &= e^{-(1/2)\,\Delta_U H(\sigma)} f(U\sigma) + g(\sigma) \\ &= c(U, \sigma)^{1/2} e^{i\psi(U\sigma)} + g(\sigma).\end{aligned}$$

Hence $\omega_\mu (V^* V) = 0$ implies

$$g(\sigma) = -c(U, \sigma)^{1/2} e^{i\psi(U\sigma)}. \tag{4.12}$$

The functions f and g, (4.11) and (4.12) satisfy the conditions (4.10) and together with (4.8) we get that V has the form

$$V = W(\tilde{U} - \mathbf{1})\, c(U, \sigma^3)^{1/2} = WV(U)$$

where W is a unitary operator, $W \in \mathscr{A}_{cl} \cap \mathscr{A}_{\bar{\Lambda}}$, given by

$$W\,|\,\sigma\rangle_{\bar{\Lambda}} = e^{i\psi(U\sigma)}\,|\,\sigma\rangle_{\bar{\Lambda}}.$$

Case 2 : input of $V^* = e^{i\varphi} V\ (\varphi \in \mathbb{R})$ and $\omega_\mu([V, A]) = 0$ for all $A \in \mathscr{B}(\mathscr{H}_{\bar{\Lambda}})$.

First note that we may suppose that $V^* = V\ (\varphi = 0)$ because

$$\mathscr{L}_{e^{i\varphi} V} \equiv \mathscr{L}_V \qquad \text{for all} \quad \varphi \in \mathbb{R}.$$

By applying $V^* = V$ on the vector $|\,\sigma\rangle_{\bar{\Lambda}}$, $\sigma \in X_{\bar{\Lambda}}$ we get

$$(\bar{f}(U\sigma) - f(\sigma))\,|\,U\sigma\rangle_{\bar{\Lambda}} = (\bar{g}(\sigma) - g(\sigma))\,|\,\sigma\rangle_{\bar{\Lambda}}. \tag{4.13}$$

If $U\sigma \neq \sigma$, (4.13) gives

$$\bar{f}(U\sigma) = f(\sigma)$$

and hence from (4.8)

$$c(U, U\sigma) = c(U, \sigma). \tag{4.14}$$

This last equality holds for all $\sigma \in X_{\bar{\Lambda}}$, and therefore from (2.6) we get

$$\Delta_U H(\sigma) = 0 \tag{4.15}$$

for all $\sigma \in X_{\bar{\Lambda}}$.

On the other hand (4.13) also implies

$$\bar{f}(U\sigma) + \bar{g}(\sigma) = f(\sigma) + g(\sigma) \tag{4.16}$$

for all $\sigma \in X_{\bar{\Lambda}}$. Let

$$\gamma(\sigma) \equiv f(U\sigma) + g(\sigma) \in C(\bar{\Lambda}).$$

From the explicit form of V, (4.8) and (2.19), (4.15) we have

$$[\pi_\mu(V)\Omega_\mu](\sigma) = \gamma(\sigma). \tag{4.17}$$

Now we apply the condition

$$\omega_\mu([V, A]) = 0. \tag{4.18}$$

Choosing $A = F(\sigma^3)\sigma_R^1 \in \mathscr{A}_{\bar{\Lambda}}$, $F \in C(\bar{\Lambda})$, $R \subset \bar{\Lambda}$, we get from (4.17), (4.18) and (2.16)

$$\int d\mu(\sigma)\,\bar{\gamma}(\sigma)\,F(\sigma)\,e^{-1/2\Delta_R H(\sigma)} = \int d\mu(\sigma)\,F(\sigma)\,e^{-1/2\Delta_R H(\sigma)}\,\gamma(\sigma^R) \tag{4.19}$$

where $\Delta_R H(\sigma) = H(\sigma^R) - H(\sigma)$.

Since μ is a Gibbs measure, $\gamma \in C(\bar{\Lambda})$ is a local function and (4.19) holds for all $F \in C(\bar{\Lambda})$, it must be that

$$\bar{\gamma}(\sigma) = \gamma(\sigma^R) = \gamma(\eta) \tag{4.20}$$

for all $\sigma, \eta \in X_{\bar{\Lambda}}$.

Combining (4.16) and (4.20) yields for all $\sigma \in X_{\bar{\Lambda}}$,

$$\left.\begin{aligned} f(U\sigma) &= f(\sigma) \\ f(\sigma) + g(\sigma) &= \gamma(\sigma) = \lambda \end{aligned}\right\} \tag{4.21}$$

where $\lambda \in \mathbb{R}$ is a constant.

Notice that this constant is given, from (4.17), by

$$\omega(V^* V) = \int d\mu(\sigma)\,|\gamma(\sigma)|^2 = \lambda^2 \tag{4.22}$$

Combining (4.10) with (4.21) implies

$$\lambda(f(\sigma)-\bar{f}(\sigma))=0.$$

The case $\lambda=0=\omega(V^* V)$ was already handled in Case 1 above.

If f is a real function, then we obtain from (4.8) and (4.21) that V must have the form

$$\begin{aligned} V &= (\tilde{U}-\mathbf{1})\, c(U, \sigma^3)^{1/2}+\lambda \\ &= V(U)+\lambda \end{aligned}$$

with the restriction (4.14), *i.e.* $V^*=V$.

The proof is completed by observing that

$$\mathscr{L}_{V+\lambda}\equiv\mathscr{L}_V$$

if $V=V^*$ and $\lambda\in\mathbb{R}$.

Finally that (ii) of the proposition implies (i), follows from the above construction. ■

Proof of Proposition 2.3

(i) follows from an easy computation and (ii) is a consequence of $\omega_\mu(V_0(U)^* V_0(U))=\omega_\mu(V_0(U^{-1})^* V_0(U^{-1})=0$ [*see* (3.12), 3.13)] and Proposition 3.1. ■

Proof of Theorem 2.4

From the definition of $\mathscr{L}$ (2.29) it follows that $\mathscr{L}$ is a dissipation on $\mathscr{A}$. The statements of the Theorem now follow from [17], Theorem 3.1.34, [21], Theorem 2.1 and Proposition (2.3). ■

Proof of Proposition 3.2

If $U=U^{-1}$, then $\mathscr{L}_U+\mathscr{L}_{U^{-1}}=2\mathscr{L}_U$. Because $\Delta_U H\neq 0$ one has that $V_0(U)^*\neq V_0(U)$ and the statement of the Proposition follows from Proposition 3.1.

Now take $U\neq U^{-1}$ and suppose that the statement of the Proposition is false:

$$\omega(V_0(U)^* V_0(U))>0 \tag{4.23}$$

Because $\Delta_U H\neq 0$ there exists $\tilde{\eta}\in X_{\tilde{\Lambda}}$ such that

$$c_0(U^{-1}, \tilde{\eta})\neq c_0(U, U^{-1}\tilde{\eta})$$

or

$$g_0(\tilde{\eta}) \neq f_0(U^{-1}\tilde{\eta}) \tag{4.24}$$

[for the definitions of f_0, g_0 *see* (2.28)].

Using the same methods of Proposition 3.1 one finds that (2.30) implies that

$$\omega(AV)V^* - \omega(V^*A)V + \omega(A\tilde{U}^{-1}V)(\tilde{U}^{-1}V)^* - \omega((\tilde{U}^{-1}V)^*A)\tilde{U}^{-1}V = \omega([A, V^*V]) \tag{4.25}$$

for all $A \in \mathscr{A}_{\bar{\Lambda}}$ and where $V \equiv V_0(U) = \tilde{U} f_0(\sigma^3) - g_0(\sigma^3)$ [*see* (2.28)].

Take any $\sigma \in X_{\bar{\Lambda}}$, then by applying (4.25) to the vector $|\sigma\rangle_{\bar{\Lambda}} \in \mathscr{H}_{\bar{\Lambda}}$ one finds

$$\begin{aligned}&[\omega([A, V^*V]) + (\omega(AV) - \omega(V^*A))g_0(\sigma) \\ &\quad + (\omega((\tilde{U}^{-1}V)^*A) - \omega(A\tilde{U}^{-1}V))f_0(\sigma)]\,|\sigma\rangle_{\bar{\Lambda}} \\ &= [\omega(AV)f_0(U^{-1}\sigma) + \omega((\tilde{U}^{-1}V)^*A)g_0(\sigma)]\,|U^{-1}\sigma\rangle_{\bar{\Lambda}} \\ &\quad - [\omega(V^*A)f_0(\sigma) + \omega(A\tilde{U}^{-1}V)g_0(U\sigma)]\,|U\sigma\rangle_{\bar{\Lambda}}\end{aligned} \tag{4.26}$$

As $U \neq U^{-1}$ there exists $\eta \in X_{\bar{\Lambda}}$ such that $U\eta \neq U^{-1}\eta$, and thus also $U\eta \neq \eta \neq U^{-1}\eta$.

Now take $\sigma = \eta$ in (4.26), then one gets

$$\omega(A\tilde{U}^{-1}V) = -\frac{f_0(\eta)}{g_0(U\eta)}\omega(V^*A), \qquad A \in \mathscr{A}_{\bar{\Lambda}}. \tag{4.27}$$

Since $\tilde{U} \in \mathscr{A}_{\bar{\Lambda}}$ is a unitary operator one has

$$\sup_{A \in \mathscr{A}_{\bar{\Lambda}}} \frac{|\omega(A\tilde{U}^{-1}V)|}{\|A\|} = \sup_{A \in \mathscr{A}_{\bar{\Lambda}}} \frac{|\omega(V^*A)|}{\|A\|}.$$

Hence from (4.23) and (4.27) we obtain

$$f_0(\eta) = g_0(U_\eta) \qquad \text{for all } \eta \text{ s.t. } U\eta \neq U^{-1}\eta \tag{4.28}$$

$$\omega(A\tilde{U}^{-1}V) = -\omega(V^*A) \qquad \text{for all } A \in \mathscr{A}_{\bar{\Lambda}}. \tag{4.29}$$

From (4.28) one has

$$c_0(U^{-1}, \eta) = c_0(U, U^{-1}\eta) \tag{4.30}$$

for all η s.t. $U\eta \neq U^{-1}\eta$.

Remark that (4.30) also holds for the configurations $\sigma \in X_{\bar{\Lambda}}$ satisfying $U\sigma = U^{-1}\sigma = \sigma$.

Therefore one has that the configuration $\tilde{\eta}$ (4.24) satisfies

$$U\tilde{\eta} = U^{-1}\tilde{\eta} \neq \tilde{\eta}. \tag{4.31}$$

Taking $\sigma=\tilde{\eta}$ in (4.26) and using (4.29) one gets

$$\omega([A, V^* V])=0 \tag{4.32}$$

$$(\omega(AV)+\omega(V^* A))(f(U\tilde{\eta})-f(\tilde{\eta}))=0 \tag{4.33}$$

for all $A\in\mathscr{A}_{\bar{\Lambda}}$.

From (2.2), (4.24) and (4.31) it follows that

$$\begin{aligned} f_0(U\tilde{\eta}) &= c_0(U, U\tilde{\eta}) \\ &= c_o(U, U^{-1}\tilde{\eta}) \\ &\neq c_0(U^{-1}, \tilde{\eta}) \\ &= c_0(U, \tilde{\eta}) \\ &= f_0(\tilde{\eta}). \end{aligned}$$

Hence from (4.33)

$$\omega(AV)=-\omega(V^* A), \qquad A\in\mathscr{A}_{\bar{\Lambda}}. \tag{4.34}$$

Now take again a configuration η with $U\eta\neq U^{-1}\eta$. Then using (4.28), (4.29) and (4.34) one gets from (4.26)

$$\omega(AV)(f_0(\eta)-g_0(\eta))=0$$

and because $\omega(V^* V)>0$ it follows that $f_0(\eta)=g_0(\eta)$ for all η such that $U\eta\neq U^{-1}\eta$.

But this holds also for the configurations σ satisfying $U\sigma=U^{-1}\sigma$. Hence $f_0=g_0$. But this is a contradiction with $U\neq U^{-1}$ (2.3) and the Proposition is proven. ■

Proof of Proposition 3.3

Let $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS-triplet of ω. Then (i) implies

$$\pi_\omega(V_0(U))\Omega_\omega=0$$

and thus

$$\pi_\omega(\tilde{U})\tau_\omega(f_0(\sigma^3))\Omega_\omega=\pi_\omega(g_0(\sigma^3))\Omega_\omega. \tag{4.35}$$

Remark that for all $F\in\mathscr{A}_{cl}$

$$\tilde{U}F(\sigma^3)=F_U(\sigma^3)\tilde{U} \tag{4.36}$$

where $F_U(\sigma^3)|\sigma\rangle_{\bar{\Lambda}}=F(U^{-1}\sigma)|\sigma\rangle_{\bar{\Lambda}}$.

Combining (4.35), (4.36) with $f>0$ $(U\in\tilde{Q})$ and (2.6) we get

$$\begin{aligned} \pi_\omega(\tilde{U})\Omega_\omega &= \pi_\omega([f(U^{-1}\sigma^3)]^{-1} g(\sigma^3))\Omega_\omega \\ &= \pi_\omega(e^{-1/2\Delta_{U^{-1}} H(\sigma^3)})\Omega_\omega. \end{aligned} \tag{4.37}$$

Hence

$$\begin{aligned} \omega(\tilde{U}^{-1}A) &= (\pi_\omega(\tilde{U})\Omega_\omega, \pi_\omega(A)\Omega_\omega) \\ &= (\pi_\omega(e^{-1/2\Delta_{U^{-1}} H(\sigma^3)})\Omega_\omega, \pi_\omega(A)\Omega_\omega) \\ &= \omega(e^{-1/2\Delta_{U^{-1}} H(\sigma^3)} A). \end{aligned}$$

Thus we have proven that (i) implies (ii). That (ii) implies (i) follows from the same arguments. ■

Proof of Theorem 3.4

That the states ω_μ, $\mu \in \mathscr{G}(H)$ are locally reversible for $\mathscr{L}$ follows from (3.13) and Proposition (3.1). One also immediately has that ω_μ is a ground state for $\tilde{H}$ (3.14) and satisfies the DLR-equation (Proposition 3.3).

Rest to prove that (i) implies (ii).

Let ν be the restriction of ω to the classical algebra $\mathscr{A}_{cl} = C(X)$: ν is a measure on $X = \mathbb{Z}^d$ and

$$\omega(F(\sigma^3)) = \int_X d\nu(\sigma) F(\sigma). \tag{4.38}$$

As ω is locally reversible for $\mathscr{L}$ it follows from proposition (3.2), (3.3) and formula (3.22) that

$$\int_X d\nu(\sigma) F(U^{-1}\sigma) = \int_X d\nu(\sigma) e^{-\Delta_U H(\sigma)} F(\sigma) \tag{4.39}$$

for all $U \in \tilde{Q}_0$ and $F(\sigma^3) \varepsilon \mathscr{A}_{cl}$.

From the translation invariance of the process (2.2), (iv), one has that (4.39) holds for all $U_x \in \tilde{Q}_x$, $x \in \mathbb{Z}^d$, where U_x is the translation of $U_0 \in Q_0$.

As the process is irreducible an easy computation yields that (4.39) holds for all local permutations U and hence $\nu \in \mathscr{G}(H)$.

As in the proof of Proposition 3.3 let $(\mathscr{H}_\omega, \pi_\omega, \Omega_\omega)$ be the GNS triplet of ω. Take $U \in \tilde{Q}_0$.

From Proposition (3.2) we have that

$$\pi_\omega(V_0(U))\Omega_\omega = 0.$$

From (4.36) and (4.37) we obtain

$$\pi_\omega(\tilde{U}F(\sigma^3))\Omega_\omega = \pi_\omega(F_U(\sigma^3) e^{-1/2\Delta_{U^{-1}} H(\sigma^3)})\Omega_\omega. \tag{4.40}$$

As the process is irreducible an induction argument gives that (4.40) holds for all $F(\sigma^3) \in \mathscr{A}_{cl}$ and $U = U_1 \ldots U_n \in Q_0$; where $U_1, \ldots, U_n \in \tilde{Q}_0$.

Again by the translation invariance of the process (4.40) holds for all local permutations U. In particular this holds for the transformation U_R, $U_R \sigma = \sigma^R$, defined in (2.12), R a finite subset of $\mathbb{Z}^d$. Hence from (4.38) and (4.40), $\sigma_R^1 \equiv \tilde{U}_R$,

$$\begin{aligned}\omega(\sigma_R^1 F(\sigma^3)) &= (\Omega_\omega, \pi_\omega(\sigma_R^1 F(\sigma^3))\Omega_\omega) \\ &= \int_X d\nu(\sigma) F(\sigma^R) e^{-1/2\{H(\sigma^R)-H(\sigma)\}} \\ &= \int_X d\nu(\sigma)[\pi_\nu(\sigma_R^1 F(\sigma^3))\mathbf{1}](\sigma) \\ &= \omega_\nu(\sigma_R^1 F(\sigma^3))\end{aligned}$$

with $\nu \in \mathscr{G}(H)$.

This proves the Theorem. ■

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