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On long time behaviour of a class of reaction-diffusion models

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ABSTRACT. — This is a short account of some results on long time behaviour of a class of interacting particle systems obtained by the superposition of Glauber and Kawasaki dynamics. Under some macroscopic limit they are described by reaction-diffusion equations, and the problem treated here refers to longer times, when propagation of chaos does not hold anymore. Precisely, one is interested in the escape from unstable equilibrium points and the onset of phase separation. The results reported here have been proven in a series of papers by P. Calderoni, A. De Masi, A. Pellegrinotti, E. Presutti and M. E. Vares.

Key words : Reaction-diffusion equations, Glauber and Kawasaki dynamics, propagation of chaos, escape from unstable equilibrium points.

RÉSUMÉ. — Cet article est une brève description de quelques résultats sur le comportement à long terme d'une classe de systèmes de particules en interaction obtenus par superposition des dynamiques de Glauber et de Kawasaki. Dans une certaine limite macroscopique ces systèmes sont décrits par des équations de réaction-diffusion et nous considérons ici des temps très longs où la propagation du chaos n'a plus lieu. Plus précisement, nous nous intéressons à l'éloignement des points d'équilibre instable et à l'apparition de la séparation de phase. Les résultats exposés ici ont été démontrés dans une série d'articles dus à P. Calderoni, A. De Masi, A. Pellegrinotti, E. Presutti et M. E. Vares.

1. INTRODUCTION

Many highly important achievements in the theory of stochastic processes during the last two decades were related to the study of interacting Markovian systems with infinitely many components. Among many problems and motivations concurring for such big development one certainly finds those connected with non-equilibrium Statistical Mechanics. In this scenary it appears one basic and deep problem: better understanding and justification of the hydrodynamical description for physical fluids, where macroscopic properties should be described by the equations of Euler and Navier-Stokes, while the molecules follow hamiltonian laws.

As it is well known, the appearance of such type of collective behaviour in large systems with local interactions should be a quite general phenomenon and its rigourous derivation is a fascinating problem. Well, for stochastic dynamics this transition between microscopic and macroscopic levels has been the object of rigourous analysis, and this has definitely contributed to a better understanding of the underlying phenomena.

In the rigourous derivations of hydrodynamics for stochastic particle systems many results have been obtained so far. In the diffusive case a large class of "gradient type" systems has been studied in the last decade. Interesting examples of "non-gradient" models have also been studied in this period. Also asymmetric systems (in a Euler-type scaling) leading to hyperbolic conservation laws, which exhibit shock phenomena, have been treated by using coupling techniques and attractiveness properties. In more particular cases even the microscopic structure of the shock front has been characterized. The very recent developments of the techniques initiated by Guo, Papanicoloau and Varadhan based on entropy inequalities, and closely related to large deviations has opened a new and exciting road to the derivation of hydrodynamics.

In this note we shall focus on a quite particular type of problem in the frame of reaction-diffusion models giving a short account of a series of results (some of which already published and others still in progress). Before going into this set up it is convenient to provide some general references, so that a reader not familiar with Makovian particle systems, could find his way for a deeper study. Concerning general aspects of interacting particle systems, basic references are the books of Liggett (1985) and Durrett (1988). On the other side, there are several surveys on the macroscopic description and collective behaviour of stochastic particle systems. Among those we indicate: the survey on hydrodynamics by De Masi, Ianiro, Pellegrinotti and Presutti (1984), the reviews by Presutti (1986) and Presutti (1987), and the lecture notes on propagation of chaos, by Sznitman (1989). An extensive and very updated survey is contained in the recent book by Spohn (1989). In these surveys the reader will find a very good description of what we have sketched above, with detailed references. At several points, I will refer to the lecture notes by De Masi and Presutti (1989), which contain a detailed exposition of another technique based on the study of correlations functions.

We shall now concentrate on a class of stochastic dynamics $\sigma^{\varepsilon}(., t)$ with values on $X=\{-1, +1\}^{\mathbb{Z}}$, which under certain macroscopic limit are described by a reaction-diffusion equation

$$\left.\begin{aligned} \partial_t m &= \frac{1}{2}\partial_r^2 m + F(m) \\ m(., 0) &= m_0(.). \end{aligned}\right\} \tag{1.1}$$

To many others (*see* references below) we may add a simple mathematical reason for the choice of models of this type: despite its simplicity, these reaction-diffusion equations have a very rich structure. The main questions we shall be concerned here refer to typical (long time) behaviour of the particle system as it escapes from unstable equilibrium points of (1.1). To make this more precise we first recall in section 2 that the derivation of (1.1) from $\sigma^{\varepsilon}(.)$ involves a limiting procedure which requires a fixed time interval, where propagation of chaos in proven. We are then concerned with what happens at longer times, *i.e.* just after it escapes from the behaviour predicted by (1.1). For example: "when" and "how" do the random fluctuations with respect to (1.1) become macroscopic? What does it happen "at" this time? The main motivation is in fact, to understand the onset of "phase separation".

For this type of study we were initially motivated by several articles in the physics literature, as De Pasquale, Tartaglia and Tombesi (1985), Broggi, Lugiato and Colombo (1985), Baras, Nicolis, Malek-Mansour and Turner (1983) and Meyer, Ahlers and Cannell (1987). In these papers similar transient behaviour for other stochastic dynamics has been considered, and experimental results are also discussed.

2. DESCRIPTION OF THE MODEL

We will be considering a family $\sigma^{\varepsilon}(., t)$ with $0<\varepsilon\leq 1$ of Markov processes, taking values on $X=\{-1, +1\}^{\mathbb{Z}}$. They are obtained by a superposition of two dynamics: one is of Glauber (spin flip) type and the other is a Kawasaki (stirring) dynamics. Moreover, the stirring dynamics is speeded up. More precisely, the generator L_{ε} of our Markov process $\sigma^{\varepsilon}(., t)$, when applied to cylinder functions on X, is written as:

$$L_{\varepsilon} f(\sigma) = \varepsilon^{-2} L_0 f(\sigma) + L_G f(\sigma) \tag{2.1 a}$$

where

$$L_0 f(\sigma)=\frac{1}{2}\sum_x (f(\sigma^{x,\,x+1})-f(\sigma)), \tag{2.1 b}$$

$$L_G f(\sigma)=\sum_x c(x,\sigma)(f(\sigma^x)-f(\sigma)) \tag{2.1 c}$$

and

$$\sigma^{x,\,x+1}(y)=\begin{cases}\sigma(y) & \text{if } y\notin\{x, x+1\}\\ \sigma(x) & \text{if } y=x+1\\ \sigma(x+1) & \text{if } y=x\end{cases}$$

$$\sigma^x(z)=\begin{cases}\sigma(z) & \text{if } z\neq x\\ -\sigma(z) & \text{if } z=x\end{cases}$$

and $c(x,\sigma)=c(0,\tau_x\sigma)$, with $c(0,.)$ being a positive cylinder function on X. [Here we are using the notation $\tau_x\sigma(y)=\sigma(x+y)$ for all $y\in\mathbb{Z}$.]

This model was introduced by De Masi, Ferrari and Lebowitz (1986) who proved the following strong form of propagation of chaos, given in Theorem (2.2) below.

NOTATION. — (*a*) If μ is a probability on X, then P_μ^ε represents the law of the process $\sigma^\varepsilon(.,t)$ (on its path space) when $\sigma^\varepsilon(.,0)$ has distribution μ.

(*b*) We shall use the same symbol to denote both a probability measure and the expectation with respect to it.

(*c*) If $-1\leqq m\leqq 1$, ν_m denotes the product Bernoulli probability on X with $\nu_m(\sigma(x))=m$ for all $x\in X$.

THEOREM (2.2) [De Masi, Ferrari, Lebowitz (1986)]. — *Let* m_0: $\mathbb{R}\to[-1,1]$ *be a function of class* C^3, *and let* μ^ε *be the product probability on* X *with* $\mu^\varepsilon(\sigma(x))=m_0(\varepsilon x)$, *for all* $x\in\mathbb{Z}$. *Then:*

$$\limsup_{\varepsilon\to 0}\sup_{t\leqq T}\sup_{\substack{x_1,\ldots,x_n\in\mathbb{Z}\\ \text{distinct}\\ |x_i|\leqq\varepsilon^{-1}r,\,\forall i}}\left|P_{\mu^\varepsilon}^\varepsilon\left(\prod_{i=1}^n\sigma(x_i\,t)\right)-\prod_{i=1}^n m(\varepsilon x_i,t)\right|=0 \tag{2.3}$$

for all $n\geqq 1$, $0<r<+\infty$ *and* $0<T<+\infty$, *where* $m(r,t)$ *is the solution of* (1.1) *with*

$$F(m)=-2\nu_m(\sigma(0)c(0,\sigma)). \tag{2.4}$$

Remarks. — (*a*) In [7] the above theorem has in fact been proven for higher dimensional processes, but the questions we shall be considering have been answered so far only in the one-dimensional case.

(*b*) Theorem (2.2) is saying that if we look, at time t, around $[\varepsilon^{-1}r]$ we shall see approximately $\nu_{m(r,t)}$. In particular, it implies the validity of a law of large numbers: if φ is a test function, $T \geqq 0$, and $\delta > 0$

$$\lim_{\varepsilon \downarrow 0} P^{\varepsilon}_{\mu^{\varepsilon}}\left[\left|\varepsilon \sum \varphi(\varepsilon x)\sigma(x, T) - \int \varphi(r) m(r, T)\, dr\right| \geqq \delta\right] = 0. \qquad (2.5)$$

That is, if measured on the spatial scale $[\varepsilon^{-1} r]$, the magnetization tends to a deterministic limit whose evolution is given by equation (1.1).

(*c*) A simple calculation gives that

$$L_{\varepsilon}\sigma(0) = \frac{\varepsilon^{-2}}{2}(\sigma(1) + \sigma(-1) - 2\sigma(0)) - 2\sigma(0)\, c(0, \sigma).$$

and from this the reader immediately sees that if propagation of chaos holds, the evolution of local magnetization must be given by (1.1).

(*d*) In [7] the authors also study the fluctuations of the above magnetization field. We shall be back to this at Section 3.

Note that in (2.3) or (2.5) the limit is obtained for a fixed T. It is natural to try to investigate the behaviour of $\sigma^{\varepsilon}(., t_{\varepsilon})$ when $t_{\varepsilon} \to +\infty$ in a suitable way. As previously discussed, this is precisely the type of questions we shall consider. For this, we shall look at cases where $F(m) = -V'(m)$ with $V(.)$ a double-well potential and our initial profile $m_0(r)$ is constant, the value being the saddle point of $V(.)$.

As we may expect, this behaviour changes drastically according to the shape of $V(.)$ near the saddle. The hyperbolic case is quite different from degenerated ones. To see this we shall fix our attention into two examples.

Example 1 (quadratic case):

$$c(0, \sigma) = 1 - \gamma\sigma(0)(\sigma(1) + \sigma(-1)) + \gamma^2 \sigma(-1)\sigma(1)$$

where $1/2 < \gamma \leqq 1$.

In this case $F(m) = -V'(m)$, where

$$V(m) = \frac{\beta m^4}{4} - \frac{\alpha m^2}{2}$$

with $\alpha = 2(2\gamma - 1)$ and $\beta = 2\gamma^2$.

Example 2 (quartic case):

$$c(0, \sigma) = \left(1 - \frac{1}{2}\sigma(0)\sigma(1)\right)\left(1 - \frac{1}{2}\sigma(0)\sigma(-1)\right)(1 - c\,\sigma(0)\sigma(2)\sigma(3)\sigma(4))$$

with $1/4 < c < 1$.

In this case $F(m) = -V'(m)$, where

$$V(m) = \frac{\beta m^6}{6} - \frac{\alpha m^4}{4}$$

with $\alpha = 2(c - 1/4)$ and $\beta = 3c/2$.

Thus in both examples $m=0$ is an unstable stationary solution of equation (1.1). We want to study how does $\sigma^{\varepsilon}(., t)$ escape from ν_0. In other words, how does it start the "phase separation" between $+m^*$ and $-m^*$? Here $\pm m^*$ denote the points of minimum of $V(.)$.

3. ESCAPE FROM UNSTABLE EQUILIBRIUM POINTS

The first results on this problem were obtained for a bounded macroscopic volume, *i. e.*, when we change $\mathbb{Z}$ to $\mathbb{Z}_{\varepsilon}=\mathbb{Z}/[\varepsilon^{-1} L]$, where $1 \leqq L < +\infty$ is fixed, and consider the process on the torus, so that (1.1) becomes an equation with $r \in [0, L]$ and periodic boundary conditions. In this case, the spatial structure will "disappear", and the model describes the escape for a system with one degree of freedom; the problem becomes simpler, but still meaningful and non-trivial. In Theorems (3.1) and (3.2) below we let ν_m^{ε} be the Bernoulli measure on $X_{\varepsilon}=\{-1, +1\}^{\mathbb{Z}_{\varepsilon}}$ with $\nu_m^{\varepsilon}(\sigma(x))=m$ for all x.

NOTATION. – μ_t^{ε} will denote the law of $\sigma^{\varepsilon}(., t)$.

THEOREM (3.1). – *Under the above conditions, if we consider the model of Example* 1, *and let* $\mu_0^{\varepsilon}=\nu_0^{\varepsilon}$, *then, as* $\varepsilon \to 0$*:*

$$\mu^{\varepsilon}_{\tau |\ln \varepsilon|} \overset{\omega^*}{\to} \begin{cases} \nu_0 & \text{if } \tau < 1/2\alpha \\ (\nu_{m^*}+\nu_{-m^*})/2 & \text{if } \tau > 1/2\alpha \end{cases}$$

and

$$\mu^{\varepsilon}_{|\ln \varepsilon|/2\alpha+t} \overset{\omega^*}{\to} \int \nu_m \lambda_t(dm)$$

where the λ_t *are probabilities on* $[-1, 1]$, *absolutely continuous with respect to Lebesgue measure, and as* $t \to +\infty$, *they converge weakly to* $(\delta_{m^*}+\delta_{-m^*})/2$.

Remark. – In the above theorem we in fact have that the expectation of the product of a fixed number of spins converges to the proper limit uniformly on their localization.

It is quite reasonable to expect the magnetization becoming finite ($\neq 0$) in the time scale $|\ln \varepsilon|$, due to the linear instability of $m \equiv 0$ in (1.1), and due to the initial fluctuations. Nevertheless, the interesting point it that despite of being produced by stochastic fluctuations the transition happens in a deterministic time (in the scale $|\ln \varepsilon|$), precisely $1/2\alpha$. This happens due to a very special adjustement between the magnitude of stochastic fluctuations and the deterministic part. If we modify the potential in a neighbourhood of the origin, this will not happen anymore. For this, let us recall the following.

THEOREM (3.2) [Calderoni, Pellegrinotti, Presutti, Vares (1989)]. – *Under the above conditions, take* $\mu_0^\varepsilon = \nu_0^\varepsilon$ *in Example* 2. *Then, as* $\varepsilon \to 0$*:*

$$\mu^\varepsilon_{\tau\varepsilon^{-1/2}} \overset{\omega^*}{\to} a(\tau)\,\nu_0 + (1-a(\tau))\,\frac{(\nu_{m^*}+\nu_{-m^*})}{2}$$

where $a(\tau) = P(S > \tau)$, S *being the explosion time of the diffusion* $dZ_t = \alpha Z_t^3\,dt + dW_t$, *with* $Z_0 = 0$ *and* W_t *a standard Brownian motion* (i.e. $S = \inf\{t > 0 : |Z_t| = +\infty\}$).

So, in this case the behaviour is characterized by what can be called "transient bimodality". In this situation the time of escape from $m \equiv 0$ is a stochastic variable. This is expected for any higher degree of degeneracy at zero, and even for "flat" potentials. But, as one can expect it becomes "infinitely" complicated to do this for particle systems. In Vares (1990) this was done for a one-dimensional diffusion obtained by the addition of a small white noise to a deterministic system.

We refer to Calderoni, Pellegrinotti, Presutti, and Vares (1989) for a proof of Theorem (3.2). There the reader will also find a more detailed discussion of the physical motivations, as well as more references.

Concerning Theorem (3.1), it has been presented firstly in De Masi, Presutti and Vares (1986). A gap in their proof has been corrected in De Masi, Pellegrinotti, Presutti and Vares (1990) where an unbounded volume is considered (*see* § 4). For the bounded volume case a corrected proof can be found in De Masi and Presutti (1989).

Going back to Theorem (2.2) it is very natural to ask about the behaviour of the fluctuations around the deterministic limit in (2.5). This has been studied by De Masi, Ferrari and Lebowitz (1986). Defining the fluctuation field through

$$Y_t^\varepsilon(\varphi) = \sqrt{\varepsilon} \sum_{x \in \mathbb{Z}} \varphi(\varepsilon x)\,(\sigma^\varepsilon(x, t) - P^\varepsilon_{\mu^\varepsilon}(\sigma^\varepsilon(x, t))),$$

where φ is a test function, they prove that, as $\varepsilon \to 0$, the processes Y^ε converge in law to a Generalized Ornstein-Uhlenbeck process. Examining the covariance function of this limiting process it may be convenient to write it as sum of two terms: a "regular" part, which in accordance with the Fluctuation-dissipation theorem solves a linearized version of (1.1), and a "singular" part. These correspond to the two noise processes in the generalized Ornstein-Uhlenbeck process: a "white noise" term and a "derivative of white noise" term. In particular we observe in Example 1 that starting from ν_0^ε the covariance of the limiting fluctuation field diverges exponentially as $t \to +\infty$, which in some sense "suggests" the escape in the logarithmic time scale. Of course this is just "informal" since the above limit of the fluctuation field is taken after fixing a time interval [0, T] and letting $\varepsilon \to 0$.

On the other side it is interesting to comment that the behaviour found in Theorem (2.3) is exactly analogous to that we would observe for a process $u^{\varepsilon}(., t)$ obtained through a stochastic pertubation of equation (1.1) by adding a small noise $\sqrt{\varepsilon}\,\alpha$, where α is a white noise in space and time (and we take a fixed volume [0, L] for the r variable with proper boundary conditions).

4. UNBOUNDED VOLUME

We now consider the model described in Example 1 of section 2, but on a larger spatial domain. One would like to consider the whole $\mathbb{Z}$, but for technical reasons, the results so far refer only to a microscopic domain of size $l_{\varepsilon}=[\varepsilon^{-1}|\ln\varepsilon|+1]$, hereafter denoted by $\mathbb{Z}_{\varepsilon}$, the integers $\mathbb{Z}$ module l_{ε}. This corresponds to a macroscopic domain of size $|\ln\varepsilon|$.

The physical picture we have when studying our system is that of "phase separation phenomena". Assume that we have a spin system at high temperature and with 0 magnetization. At such a temperature this magnetization value will be stable, but imagine the temperature is very fastly reduced below the critical one and that in this new situation the stable phases have magnetization $\pm m^*$. The evolution of the system will then describe phase separation phenomena; phases segregate, and clusters of each phase will appear in the space. The phase boundaries *i.e.* the transition regions from one phase to the other will then evolve in some complex space of patterns. There are several phenomenological equations which have been used to describe these phenomena, as for instance the Cahn-Hillard equations. However, also stochastic versions of our reaction-diffusion equation, as that mentioned earlier, are often considered as good models for these phenomena. Evidently, if the space regions where the phenomenon occurs are too small, as in the case considered before, the whole spatial structure is lost. Hence, the interest for considering larger sizes.

With this in mind we have enlarged the space to $\mathbb{Z}_{\varepsilon}$ and consider the interaction as given in Example 1. The difficulty is not just that the problem becomes really infinite-dimensional, but as the reader may imagine we must have a good control on the influence of the regions where the magnetization is small ("phase boundaries"). As in the other cases the proof of the theorem below relies on (i) sufficient sharp estimates of correlation functions and (ii) separation of several time scales. The result we have so far is the following.

THEOREM (4.1) [De Masi, Pellegrinotti, Presutti, Vares (1990)]. — *Under the above conditions, if* $\mu_0^\varepsilon = \nu_0^\varepsilon$ *then:*

$$(a)\quad \lim_{\varepsilon \to 0} \sup_{\substack{x_1, \ldots, x_n \in \mathbb{Z}_\varepsilon \\ \text{distinct}}} \left| \mu^\varepsilon_{\tau |\ln \varepsilon|} \left(\prod_{i=1}^n \sigma(x_i) \right) \right| = 0 \qquad \text{if} \quad \tau \leqq 1/2\alpha$$

$$(b)\quad \lim_{\varepsilon \to 0} \sup_{\substack{x_1, \ldots, x_n \in \mathbb{Z}_\varepsilon \\ \text{distinct}}} \left| \mu^\varepsilon_{|\ln \varepsilon|/2\alpha + |\ln \varepsilon|^{1/3}} \left(\prod_{i=1}^n \sigma(x_i) \right) - \tilde{\mathrm{E}} \prod_{i=1}^n \tilde{\rho}(\varepsilon |\ln \varepsilon|^{-1/2} x_i) \right| = 0$$

where $\tilde{\rho}(r) = m^* . \operatorname{sign}(\tilde{X}(r))$, *and* $\tilde{X}(.)$ *is a Gaussian process with* $\tilde{\mathrm{E}}\tilde{X}(r) = 0$, $\tilde{\mathrm{E}}(\tilde{X}(r)\tilde{X}(r')) = \exp(-\alpha(r-r')^2/2)$.

Remarks. — (1) The reader will notice at once that now the escape does not happen at time $\tau = 1/2\alpha$ in the scale $\tau|\ln \varepsilon|$, but "just after". (2) According to (*b*) of Theorem (4.1) at time $|\ln \varepsilon|/2\alpha + |\ln \varepsilon|^{1/3}$ we have clusters of $\pm m^*$ whose size is of the order $\varepsilon^{-1}|\ln \varepsilon|^{1/2}$. We conjecture that this picture is still true at later times as $\tau|\ln \varepsilon|$ with $1/2\alpha < \tau$, but we don't know how to prove this. Larger clusters of $\pm m^*$ should be formed at longer times. Of course, there is a all series of interesting questions connected to further evolution which involves multi-scale phenomena until at very long times a new quite different phenomenon becomes important; namely, the appearance of the different phase inside the region where the other one is present, also called "tunneling events". For its rigourous analysis it is necessary to develop more deeply the techniques on large deviations. In the setup of stochastic partial differential equations Faris and J. Lasinio (1982) have studied these problems. The results have then been extended by Cassandro, Olivieri and Picco (1986). We refer also to Brassesco (1989), and Martinelli, Olivieri and Scoppola (1989) for further results, and the connection with the so-called "pathwise approach to metastability", introduced by Cassandro, Galves, Olivieri, and Vares (1984).

5. BASIC IDEA OF THE PROOFS

In what follows we first briefly discuss the basic ideas behind the proofs in the quadratic case, so that we can understand the origin of the Gaussian process in (*b*) of Theorem (4.1). For simplicity, let us start recalling the basic scheme of the proof in the bounded volume case. The known result on the magnetization fluctuation field tells us that under $P^\varepsilon_{\nu_0^\varepsilon}$ and for fixed

t, the random variables describing the average magnetization

$$m^{\varepsilon}(t)=\varepsilon \sum_{x=0}^{[\varepsilon^{-1} L]} \sigma^{\varepsilon}(x, t)$$

should be approximately ($\varepsilon>0$, but small) normal with average 0 and standard deviation $c\sqrt{\varepsilon}\, e^{\alpha t}$ where c is some constant. (Precisely:

$$\varepsilon^{-1/2} m^{\varepsilon}(t) \underset{\varepsilon \to 0}{\overset{\omega^*}{\to}} \mathrm{N}(0, c^2 e^{2\alpha t}),\ cf.\ [7].)$$

The proof of Theorem (3.1) involves an analysis of two different stages: in the first we study how $m^{\varepsilon}(.)$ changes from typical values of order $\sqrt{\varepsilon}$ to order ε^a, where $0<a<1/2$ and prove that the above gaussian approximation still holds. More precisely, if $0<\tau<1/2\alpha$ and if we let

$$Z_{\tau}^{\varepsilon}=e^{-\alpha\tau\,|\ln \varepsilon|}\, \varepsilon^{-1/2} m^{\varepsilon}(\tau\,|\ln \varepsilon|)$$

then, after a careful analysis of the correlation functions (*cf.* [8], [9]) we are able to prove that all the moments of Z_{τ}^{ε} converge to those of a normal random variable with average 0 and variance c^2. The true reason is that in this stage (from $\varepsilon^{1/2}$ to ε^a with $0<a<1/2$) $m^{\varepsilon}(.)$ evolves essentially as a linear stochastic differential equations, obtained by the addition of a white noise ($\sqrt{\varepsilon}\,\dot{w}$) to the linearized "macroscopic equation" ($\dot{m}=\alpha m$). But in fact, the noise can be switched off almost instantaneously (in $|\ln \varepsilon|$ scale), because its effect becomes negligible. The second stage involves the transition from ε^a with $0<a<1/2$ to finite values, and we prove this is essentially deterministic according to the macroscopic equation $\dot{m}=\alpha m-\beta m^3$.

The reader is probably wondering where does the space variable enters, since clearly $m^{\varepsilon}(.)$ is *not* a Markov process and *does not* obey a closed stochastic differential equation. Yes! But this enters only in the very beginning, *i.e.* in time intervals $[0, t_{\varepsilon}]$ where $t_{\varepsilon} \ll |\ln \varepsilon|$. At the second stage, that is, after time $\tau\,|\ln \varepsilon|$ with $\tau>0$ the configurations become sufficiently flat so that the spatial structure is lost, due to the bounded volume assumption. To make this more precise, it ammounts also to look at

$$e^{-\alpha t}\, \varepsilon^{-1/2} m^{\varepsilon}(t)$$

for $t \leqq t_{\varepsilon}$ with $t_{\varepsilon} \ll |\ln \varepsilon|$, proving it behaves as purely noise, and then to look at the previously defined process Z_{τ}^{ε}. When $\tau \in [\tau_0, \tau_1]$ with $0<\tau_0<\tau_1<1/2\alpha$, $Z_{\tau}^{\varepsilon}-Z_{\tau_0}^{\varepsilon}$ converges to zero in probability. A "$2\,\delta$-argument" is then needed – in the proper order – to conclude the correct approximation for Z_{τ}^{ε}.

This last discussion has been so presented because this is, somehow, the intuitive reasoning, but mainly because we can really extend it – after inserting the spatial dependence — to treat the unbounded volume. As already mentioned the proof for the Gaussian approximation of $Z_\tau^\varepsilon (0<\tau<1/2\alpha)$ in the bounded volume case is more direct. We refer to [9] for a complete proof.

As for Theorem (4.1), an analogous procedure has to be followed. Now there is a spatial structure and instead of the previous Z_τ^ε we are led to consider the field

$$Y_\tau^\varepsilon(\varphi)=\tilde{\varepsilon}^{1/2}\, e^{-\alpha\tau\,|\ln\varepsilon|}\sum_x \varphi(\tilde{\varepsilon}\, x)\,\sigma^\varepsilon(x,\, \tau\,|\ln\varepsilon\,|)$$

where $\tilde{\varepsilon}=\varepsilon/|\ln\varepsilon|^{1/2}$, and φ is a test function. Note that $\tilde{\varepsilon}^{-1}$ is the typical size of the clusters in Theorem (4.1) and we prove that for $0<\tau<1/2\alpha$ this is the length on which we have "flatness". Moreover, a simple computation with the pair correlation functions tells us that at time $|\ln\varepsilon|/2\alpha$ the typical local magnetization on the scale $\tilde{\varepsilon}^{-1}$ is still of order $|\ln\varepsilon|^{-1/4}$. So, up to this time we may use the linearized stochastic equation (analogous to the previous discussion) and at the last and final stage $(|\ln\varepsilon|/2\alpha\leqq t\leqq|\ln\varepsilon|/2\alpha+|\ln\varepsilon|^{1/3})$ the full non linear macroscopic evolution, given by (1.1), comes in, with negligeable noise. This proof of Theorem (4.1) has been provided by De Masi, Pellegrinotti, Presutti and Vares (1990).

Finally we briefly discuss the basic scheme of the proof of Theorem (3.2). It again strongly relies on the so-called "v-function" technique, which here consists in obtaining estimates for some truncated correlation functions by exploiting the self-duality (*cf.* [13]) of the Kawaski dynamics. (*See* [9] for an exposition of this.) The result is now qualitatively different from the quadratic case: the escape time is random, that is, we may always see ν_0 with positive probability, and this is the reason for the name *bimodality*. Concerning the proof, there is an obvious difference: since the linear term in (1.1) vanishes we cannot use Gaussian approximations until the magnetizations is "almost" finite. The proof involves a separation of several scales in order to understand how typical values of $m^\varepsilon(.)$ change from $\sqrt{\varepsilon}$ to finite values. The first, and "decisive" step is when $m^\varepsilon(.)$ changes from typical values of order $\sqrt{\varepsilon}$ (finite time) to order $\varepsilon^{1/4-\delta}$ where δ is positive but arbitrarily small. This is where the really stochastic event is contained. Afterwards it is an approximately deterministic evolution and it takes a much shorter time (negligible in the $\varepsilon^{-1/2}$ scale). For a precise statement of this we refer to Theorem (3.1) in Calderoni *et al.* (1989). But to get the flavour of the proof, and why one should expect the validity of Theorem (3.2), we first recall that the finite time fluctuation field $\varepsilon^{-1/2}\, m^\varepsilon(t)$ has a distribution which converges, as $\varepsilon\to 0$, to a Gaussian with average 0 and variance ct (for suitable c). Thus we are led to make a comparison between $m^\varepsilon(.)$ which, as before, is *not* a Markov process, with the one dimensional diffusions $\tilde{m}_\varepsilon(.)$, solution of

$$d\tilde{m}_\varepsilon(t)=(\alpha\,\tilde{m}_\varepsilon(t)^3-\beta\,\tilde{m}_\varepsilon(t)^5)\,dt+\sqrt{\varepsilon}\,d\mathrm{W}_t$$
$$\tilde{m}_\varepsilon(0)=0$$

where $\mathrm{W}(.)$ is a Brownian motion on $\mathbb{R}$. The process defined by

$$\tilde{\mathrm{Z}}_\varepsilon(t)=\varepsilon^{-1/4}\,\tilde{m}_\varepsilon(\varepsilon^{-1/2}\,t)$$

is easily seen to converge in law to the diffusion $\mathrm{Z}(.)$ of Theorem (3.2) and, moreover, if we take $\delta>0$ small and:

$$\tilde{\mathrm{S}}_\varepsilon=\inf\{\,t>0:|\tilde{\mathrm{Z}}_\varepsilon(t)|=\varepsilon^{-\delta}\}$$

then it can be proven that $\tilde{\tau}_\varepsilon$ converges in law to the random time S in Theorem (3.2). Well, we must prove that for $\delta>0$ sufficiently small we can really treat the process

$$\mathrm{Z}_\varepsilon(t)=\varepsilon^{-1/4}\,m^\varepsilon(\varepsilon^{-1/2}\,t)$$

as the above $\tilde{\mathrm{Z}}_\varepsilon$, up to $\tilde{\mathrm{S}}_\varepsilon$, and this is non trivial. Again, if we look at $\tilde{m}_\varepsilon$ (properly rescaled) we easily see that the further evolution up to the time it becomes finite is essentially deterministic and takes a time $\ll \varepsilon^{-1/2}$. As one can imagine, the procedure for the process which really interests us, that is $m^\varepsilon(.)$, is much more involved and it has to be done in several steps, to take care of the Glauber interactions and the non-Markovian character in each of them. See [4] for the details.

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