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Classical dynamics of non-holonomic systems: a geometric approach

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ABSTRACT. — A frame independent formulation of Classical Analytical Dynamics in the language of jet-bundle theory is developed. The geometrical environment is general enough to accomodate arbitrary "ideal" non-holonomic systems, independently of any assumption of linearity, through a suitable implementation of Gauss' principle of minimal constraint. The resulting dynamical scheme is analysed in detail. Comparison with other, more traditional formulations, is examined.

RÉSUMÉ. — On présente ici une formulation de la Mécanique Analytique en termes de jets, indépendante des choix du repère. La généralité de l'impostation géométrique permet de traiter des systèmes non holonômes « idéaux » arbitraires, sans aucune hypothèse de linéarité des contraintes, par une utilisation adéquate d'un principe de minimalité des contraintes dû à Gauss. On étudie de façon détaillée le schéma dynamique qui en résulte, en comparaison aussi d'autres formulations plus traditionnelles.

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INTRODUCTION

A major source of difficulty in the extension of Analytical Mechanics to arbitrary non-holonomic systems comes from pretending that a purely *statical* definition of the class of ideal constraints – the celebrated D'Alembert Principle ([1], [2]) – is powerful enough to cover all admissible cases, thus excluding at the outset any involvement of Kinematics at a *constitutive* level, even in the presence of arbitrary (possibly non-linear) kinetic constraints.

The implementation of this viewpoint meets unavoidably with the problem that – apart for the holonomic and linear non-holonomic cases – there is no "natural" algorithm allowing to translate the *kinematical* restrictions placed by the constraints into *geometrical* conditions on the virtual displacements of the system.

One is free, of course, to settle the question on *axiomatic* grounds, through an *a priori* ansatz, concerning the very *definition* of the concept of virtual displacement in the presence of kinetic constraints. Significant in this sense is the contribution of N. G. Chetaev ([3], [4]) (*see* also [5], [6] for more recent developments).

The resulting scheme, however, is not entirely satisfactory – at least from a foundational viewpoint – since, rather than moving from "first principles", it reflects essentially the consequences of the definitions adopted.

All drawbacks indicated above may be overcome at the outset, simply by *dropping* D'Alembert's principle as one of the cornerstones of Analytical Dynamics, and replacing it by the more powerful *principle of minimal constraint* of K. F. Gauss ([7], [3], [8], [2], [9]).

In this paper, we propose a detailed discussion of this point. The plan of presentation may be traced as follows:

In Section 1 we review the foundations of Classical Mechanics in the language of jet-bundle theory ([10], [11]), with special emphasis on the invariant, frame-independent aspects. The concept of *dynamical flow* will be singled out, as the natural object involved in the description of the interactions. Various aspects connected with the geometric representation of the constraints will be worked out in detail.

In Section 2 we discuss the implementation of the principle of determinism in the presence of constraints, through the introduction of the concept of *constitutive characterization* of the reactive forces. A general criterion of ideality, valid for arbitrary holonomic and non-holonomic systems, independently of any requirement of linearity, will be formulated, on the basis of a suitable revisitation of Gauss' principle of minimal constraints ([7], [3], [8], [2], [9]).

A comparison with D'Alembert's principle will show that the latter—completed with Chetaev' prescription on the definition of the virtual displacements in the non-linear, non-holonomic case—is automatically embodied into the newer scheme as an ordinary *theorem*.

The Section will end up with the indication of a possible "differential" formulation of Gauss' principle, similar in form, but intrinsically different and definitely more satisfactory than D'Alembert's one, based on the use of a new class of virtual objects, expressing the virtual variations of the *velocities* of the system.

Finally, in Section 3 we shall discuss the representation of the equations of motion, first in an intrinsic formulation, and then also in a more traditional *Lagrangian* language, the second alternative involving the description of the reactive forces due to the kinetic constraints, through the introduction of suitable Lagrange multipliers.

Various aspects of the resulting dynamical scheme will be briefly considered: among others, the *invariance* of the equations of motion under arbitrary changes in the *representation* of the constraints, the interplay between first integrals and kinetic constraints, etc.

Throughout the paper, special attention has been paid to the foundational aspects, mainly in connection with the construction of a geometrical set-up particularly suited to a frame independent formulation of Dynamics, an aspect that—in the authors' opinion—is seldom covered in the literature.

Non-holonomic constraints appear in the literature at the end of the XIX century, through the pioneering works of Ferrers [12], Neumann [13], Maggi [14], Appell ([15], [16], [17]), Voronets [18], Jourdain [19], Volterra [20], Chaplygin [21] and others. These Authors extended the holonomic Lagrangian formalism to linear non-holonomic systems, in order to embody, e. g., the rolling of rigid bodies. A systematic treatment of this subject may be found in several textbooks (*see*, among others, the classical treatises of Levi-Civita [2], Whittaker [9], Pars [22], Hamel [8], Gantmacher [23], Neimark and Fufaev [24]).

The representation of the equations of motion for linear non-holonomic systems in Lagrangian form, and the related problem of the existence of a variational formulation has been treated by Chaplygin [21], Khmelevskii [25], and Iliev [26]; the same problem has been dealt with by Novoselov ([27] to [31]) and Sumbatov ([32], [33]), making use of the so-called Helmholtz conditions [9].

Finally, the argument has been analysed by Moshchuk ([34], [35]), within the context of Hamiltonian systems, and by Rumiantsev ([36], [37], [38], [5], [39]), Khmelevskii [40], Naziev [41] and Pignedoli [42], also in connection with the extension of Hamilton-Jacobi theory to the non-holonomic case.

More geometrically-oriented contributions to the theory of linear kinetic constraints have been given by Cattaneo [43], Clauser [44], Ferrarese [45]. Further references on these topics may be found in [42].

Levi-Civita [2], Appell ([15], [16], [17]), Delassus ([46], [47], [48]), Hamel [8], Neimark and Fufaev [24], have discussed the problem of motion in the presence of non-linear kinetic constraints; other Authors, such as Valcovici ([49], [50], [51]), Ghori [52], Duo San [53], Kirgetov ([54], [55]), Kuznetsov [56] have extended the analysis to the case of constraints involving higher order derivatives.

All these approaches rely more or less directly on Gauss' principle of minimal constraint ([7], [8], [9]). In this respect, they are most closely related to the line of thought followed in the present paper.

Alternative approaches to the equations of motion, based on Chetaev' definition of the virtual displacements in the presence of non-linear non-holonomic constraints, may be found in Kirgetov ([54], [55]), Pozharitskii [57], Pironneau [6] and Rumiantsev [39].

Among other topics, we mention the problem of stability for the equilibrium positions and for the orbits of non-holonomic systems. A possible approach, outlined e. g. in [24], is based on the linearization of the equations of motion and on the subsequent determination of the characteristic exponents. Other results, mainly for merostatic and periodic motions, have been obtained by Semenova [58], Salvadori ([59], [60]), Rumiantsev [61], Karapetyan ([62] to [65]), Fusco and Oliva [66].

In spite of its relatively old origin, Mechanics of non-holonomic systems is still an open field of research. Among the most recent contributions, in addition to those already mentioned in the text, we recall the works of Polyakhov [67], Vershik [68], Weber [69], Fufaev ([70], [71]), Fam Guen [72], Benenti [73], Virga [74], Cardin and Zanzotto [75].

1. FOUNDATIONS

1.1. Free systems

(*a*) In Classical Physics, a *frame-independent* description of the mechanical behaviour of a material point **P** may be based on the introduction of a 4-dimensional affine bundle $\mathscr{V}_4$, called the *space-time*, fibered over the real line $\mathbb{R}$ (the "time-axis"), with projection $t : \mathscr{V}_4 \to \mathbb{R}$ yielding the usual *absolute time* function.

The choice of an arbitrary frame of reference $\mathscr{I}$ determines a corresponding representation of $\mathscr{V}_4$ as a cartesian product $\mathscr{V}_4 = \mathbb{R} \times \mathscr{E}_3$, $\mathscr{E}_3$ denoting the *reference space* associated with $\mathscr{I}$. The resulting projection $\mathbf{x} : \mathscr{V}_4 \to \mathscr{E}_3$ will be called the *relativization* of $\mathscr{V}_4$ *induced* by $\mathscr{I}$.

Through the previous construction, each fiber $t=\text{const.}$ in $\mathscr{V}_4$ can be made into an euclidean three-space, *isometric*, via $\mathbf{x}(.)$, to the reference space $\mathscr{E}_3$. According to the Axiom of *absolute space*, the resulting geometry is in fact *independent* of the specific choice of $\mathscr{I}$, and is therefore an *intrinsic* property of the space-time manifold $\mathscr{V}_4$.

For dynamical purposes, it is convenient to embody the inertial mass of $\mathbf{P}$ into the fiber geometry, by *re-scaling* all distances by a common factor m, according to the convention

$$(d(\mathscr{P}, \mathscr{Q}))^2 := m\,|\mathbf{x}(\mathscr{P}) - \mathbf{x}(\mathscr{Q})|^2$$

$(\mathscr{P}, \mathscr{Q} \in \mathscr{V}_4,\ t(\mathscr{P}) = t(\mathscr{Q}))$.

An entirely similar set-up may be introduced for a material system $\mathscr{S}$ formed by N point particles $\mathbf{P}_1, \ldots, \mathbf{P}_N$, with masses $m_1, \ldots, m_N$. In place of $\mathscr{V}_4$ we have now the $(3N+1)$-dimensional manifold $t : \mathscr{V}_{3N+1} \to \mathbb{R}$ defined as the *fibered product* of the space-time manifolds associated with the points of $\mathscr{S}$.

Once again, the introduction of an arbitrary frame of reference $\mathscr{I}$ determines a representation of $\mathscr{V}_{3N+1}$ as a Cartesian product $\mathbb{R} \times \underbrace{\mathscr{E}_3 \times \ldots \times \mathscr{E}_3}_{\text{N times}}$, thus giving rise to a *relativization process* $\{\mathbf{x}_i : \mathscr{V}_{3N+1} \to \mathscr{E}_3,\ \ i=1, \ldots, N\}$ assigning to each configuration $\mathscr{P} \in \mathscr{V}_{3N+1}$ the positions $\mathbf{x}_i(\mathscr{P})$ of the points of $\mathscr{S}$, at the instant $t(\mathscr{P})$, in the reference space $\mathscr{E}_3$ associated with $\mathscr{I}$.

Exactly as in the case of a single point P, the axiom of absolute space is summarized into the fact that each fiber $t=\text{const.}$ in $\mathscr{V}_{3N+1}$ can be given an intrinsic euclidean geometry, with a "distance function" d now expressed by the relation

$$(d(\mathscr{P}, \mathscr{Q}))^2 := \sum_{i=1}^{N} m_i |\mathbf{x}_i(\mathscr{P}) - \mathbf{x}_i(\mathscr{Q})|^2 \tag{1.1}$$

for all possible choices of the relativization process $\mathbf{x}_i(.)$.

In what follows, we shall indicate by $V(\mathscr{V}_{3N+1})$ the *vertical bundle* over $\mathscr{V}_{3N+1}$, defined in the usual way, as the sub-bundle of $T(\mathscr{V}_{3N+1})$ formed by the totality of vectors $\mathbf{U}$ *tangent* to the fibers $t=\text{const.}$ ([10], [11]).

The scalar product on $V(\mathscr{V}_{3N+1})$ induced by the metric (1.1) will be denoted by $(\ ,\ )$. Every local coordinate system $t, \xi^1, \ldots, \xi^{3N}$ in $\mathscr{V}_{3N+1}$ consistent with the fibration $t : \mathscr{V}_{3N+1} \to \mathbb{R}$ (*i.e.*, including the absolute time t among the coordinate functions), will be called *admissible*.

In admissible coordinates, the verticality of a vector field $\mathbf{U} \in \mathscr{D}^1(\mathscr{V}_{3N+1})$ is expressed by the condition

$$\langle \mathbf{U}, dt \rangle = 0 \iff \mathbf{U} = U^\alpha \frac{\partial}{\partial \xi^\alpha} \tag{1.2}$$

In a similar way, the representation of the scalar product between vertical vectors relies on the introduction of the quantities

$$g_{\alpha\beta} := \left(\frac{\partial}{\partial \xi^\alpha}, \frac{\partial}{\partial \xi^\beta}\right) = \sum_{i=1}^{N} m_i \frac{\partial \mathbf{x}_i}{\partial \xi^\alpha} \cdot \frac{\partial \mathbf{x}_i}{\partial \xi^\beta}, \qquad \alpha, \beta = 1, \ldots, 3\mathrm{N} \qquad (1.3)$$

$\mathbf{x}_i = \mathbf{x}_i(t, \xi^1, \ldots, \xi^{3N})$ denoting the relativization of $\mathscr{V}_{3N+1}$ to an arbitrary frame of reference $\mathscr{I}$.

In particular, as a consequence of the stated definitions, it is easily seen that assigning a vertical vector $\mathbf{U} \in \mathrm{V}(\mathscr{V}_{3N+1})$ is mathematically equivalent to assigning an N-tuple of ordinary vectors $\mathbf{u}_1, \ldots, \mathbf{u}_N$ in the three-space $\mathscr{E}_3$ associated with $\mathscr{I}$, on the basis of the identification

$$\mathbf{u}_i := d\mathbf{x}_i(\mathbf{U}) = \mathrm{U}^\alpha \frac{\partial \mathbf{x}_i}{\partial \xi^\alpha}, \qquad i = 1, \ldots, \mathrm{N} \qquad (1.4a)$$

with inverse

$$\sum_{i=1}^{N} m_i \mathbf{u}_i . \frac{\partial \mathbf{x}_i}{\partial \xi^\alpha} = g_{\alpha\beta} \mathrm{U}^\beta = \left(\mathbf{U}, \frac{\partial}{\partial \xi^\alpha}\right), \qquad \alpha = 1, \ldots, 3\mathrm{N} \qquad (1.4b)$$

$d\mathbf{x}_i(.)$ denoting the *differential* of the map $\mathbf{x}_i(.)$. The proof is straightforward, and is left to the reader.

(*b*) Any evolution of the system $\mathscr{S}$ is represented by a corresponding *section* $\hat{\gamma} : \mathbb{R} \to \mathscr{V}_{3N+1}$ and conversely. This leads to a natural identification of the first jet bundle $j_1(\mathscr{V}_{3N+1})$ with the *velocity space* of $\mathscr{S}$. The jet-extension of the section $\hat{\gamma}$ will be indicated by $j_1(\hat{\gamma}) : \mathbb{R} \to j_1(\mathscr{V}_{3N+1})$.

The argument is completed by the following remarks:

(i) the jet-projection $\pi : j_1(\mathscr{V}_{3N+1}) \to \mathscr{V}_{3N+1}$ makes $j_1(\mathscr{V}_{3N+1})$ into an affine bundle over $\mathscr{V}_{3N+1}$, modelled on the vector bundle $\mathrm{V}(\mathscr{V}_{3N+1})$ [10];

(ii) every frame of reference $\mathscr{I}$ determines a corresponding global section $\sigma : \mathscr{V}_{3N+1} \to j_1(\mathscr{V}_{3N+1})$ assigning to each $\mathscr{P} \in \mathscr{V}_{3N+1}$ the state of instantaneous *rest* of the system $\mathscr{S}$ in the configuration $\mathscr{P}$ in the frame of reference $\mathscr{I}$.

Properties (i) and (ii), together give rise to a *frame-dependent* identification of the velocity space $j_1(\mathscr{V}_{3N+1})$ with the vertical bundle $\mathrm{V}(\mathscr{V}_{3N+1})$, *i.e.*, in view of equation (1.4*a*), to a representation of each element of $j_1(\mathscr{V}_{3N+1})$ in terms of a corresponding N-tuple of spatial vectors $\mathbf{v}_1, \ldots, \mathbf{v}_N$, expressing the *relative velocities* of the points of $\mathscr{S}$ in the given frame of reference.

In addition to this, another important implication of property (i) is the fact that the fibration $\pi : j_1(\mathscr{V}_{3N+1}) \to \mathscr{V}_{3N+1}$ determines its own *vertical bundle* $\mathrm{V}(j_1(\mathscr{V}_{3N+1})) \subset \mathrm{T}(j_1(\mathscr{V}_{3N+1}))$. As it is well known [10], the latter is essentially identical to the vector bundle over $j_1(\mathscr{V}_{3N+1})$ obtained by pulling back the vertical space $\mathrm{V}(\mathscr{V}_{3N+1})$ through the projection π. Its fibers are therefore *isomorphic* to those of $V(\mathscr{V}_{3N+1})$, thus inheriting from these a natural operation of scalar product.

(*c*) Every admissible coordinate system $t, \xi^1, \ldots, \xi^{3N}$ in $\mathscr{V}_{3N+1}$ induces jet-coordinates $t, \xi^1, \ldots, \xi^{3N}, \dot{\xi}^1, \ldots, \dot{\xi}^{3N}$ in $j_1(\mathscr{V}_{3N+1})$ in an obvious way.

The resulting group of jet-transformations has the form

$$\bar{t}=t+c; \qquad \bar{\xi}^\alpha=\bar{\xi}^\alpha(t, \xi^1, \ldots, \xi^{3N}); \qquad \dot{\bar{\xi}}^\alpha=\frac{\partial\bar{\xi}^\alpha}{\partial\xi^\beta}\dot{\xi}^\beta+\frac{\partial\bar{\xi}^\alpha}{\partial t},$$

thus ensuring the invariance of the 1-form dt, as well as of the module $\hat{\mathscr{D}}_1$ generated locally by the 1-forms

$$\{\hat{\omega}^\alpha=d\xi^\alpha-\dot{\xi}^\alpha\,dt,\ \alpha=1, \ldots, 3N\} \tag{1.5}$$

Given any frame of reference $\mathscr{I}$, the corresponding relativization process, extended to the *velocities* of the points of $\mathscr{S}$ in the sense indicated in (*b*), is expressed by the equations

$$\mathbf{x}_i=\mathbf{x}_i(t, \xi^1, \ldots, \xi^{3N}), \qquad \mathbf{v}_i=\frac{\partial\mathbf{x}_i}{\partial\xi^\alpha}\dot{\xi}^\alpha+\frac{\partial\mathbf{x}_i}{\partial t}, \qquad i=1, \ldots, N \tag{1.6}$$

The verticality of a vector $\mathbf{X}\in T_x(j_1(\mathscr{V}_{3N+1}))$ is characterized by the conditions

$$\langle\mathbf{X}, (dt)_x\rangle=0, \qquad \langle\mathbf{X}, (\hat{\omega}^\alpha)_x\rangle=0, \quad \alpha=1, \ldots, 3N$$

mathematically equivalent to the representation

$$\mathbf{X}=X^\alpha\left(\frac{\partial}{\partial\dot{\xi}^\alpha}\right)_x \tag{1.7}$$

while the identification of the fibers of $V(j_1(\mathscr{V}_{3N+1}))$ with the fibers of $V(\mathscr{V}_{3N+1})$ is expressed by the correspondence

$$\left(\frac{\partial}{\partial\dot{\xi}^\alpha}\right)_x \leftrightarrow \left(\frac{\partial}{\partial\xi^\alpha}\right)_{\pi(x)}, \qquad \forall x\in j_1(\mathscr{V}_{3N+1})$$

In particular, the scalar product on $V(j_1(\mathscr{V}_{3N+1}))$ is based on the identifications

$$\left(\left(\frac{\partial}{\partial\dot{\xi}^\alpha}\right)_x, \left(\frac{\partial}{\partial\dot{\xi}^\beta}\right)_x\right)=\left(\left(\frac{\partial}{\partial\xi^\alpha}\right)_{\pi(x)}, \left(\frac{\partial}{\partial\xi^\beta}\right)_{\pi(x)}\right), \qquad \alpha, \beta=1, \ldots, 3N$$

Together with equations (1.3), (1.6), these yield the explicit formulae

$$\left(\frac{\partial}{\partial\dot{\xi}^\alpha}, \frac{\partial}{\partial\dot{\xi}^\beta}\right)=g_{\alpha\beta}=\sum_{i=1}^{N} m_i\frac{\partial\mathbf{x}_i}{\partial\xi^\alpha}\cdot\frac{\partial\mathbf{x}_i}{\partial\xi^\beta}=\sum_{i=1}^{N} m_i\frac{\partial\mathbf{v}_i}{\partial\dot{\xi}^\alpha}\cdot\frac{\partial\mathbf{v}_i}{\partial\dot{\xi}^\beta} \tag{1.8}$$

Finally, we have again the result that assigning a vertical vector $\mathbf{X}$ on $j_1(\mathscr{V}_{3N+1})$ is mathematically equivalent to assigning N vectors $\mathbf{z}_1, \ldots, \mathbf{z}_N$

in the three-space $\mathscr{E}_3$ associated to an arbitrary frame of reference $\mathscr{I}$, the correspondence being now expressed by the relation

$$\mathbf{z}_i = \mathrm{X}^\alpha \frac{\partial \mathbf{v}_i}{\partial \dot{\xi}^\alpha} = \mathrm{X}^\alpha \frac{\partial \mathbf{x}_i}{\partial \xi^\alpha}, \qquad i=1, \ldots, \mathrm{N} \tag{1.9a}$$

with inverse

$$\sum_{i=1}^{\mathrm{N}} m_i \mathbf{z}_i \cdot \frac{\partial \mathbf{x}_i}{\partial \xi^\alpha} = g_{\alpha\beta} \mathrm{X}^\beta = \left(\mathbf{X}, \frac{\partial}{\partial \xi^\alpha} \right), \qquad \alpha = 1, \ldots, 3\mathrm{N} \tag{1.9b}$$

[*see* the analogous equations (1.4*a*), (1.4*b*)]. The proof follows from equations (1.7), (1.8), and is left to the reader.

(*d*) Exactly in the same way as $j_1(\mathscr{V}_{3\mathrm{N}+1})$ is characterized in terms of velocities, the second jet bundle $j_2(\mathscr{V}_{3\mathrm{N}+1})$ admits a natural interpretation in terms of *accelerations*.

Once again, the argument is completed by the following remarks:

(i) the jet projection $\pi : j_2(\mathscr{V}_{3\mathrm{N}+1}) \to j_1(\mathscr{V}_{3\mathrm{N}+1})$ makes $j_2(\mathscr{V}_{3\mathrm{N}+1})$ into an affine bundle over $j_1(\mathscr{V}_{3\mathrm{N}+1})$, modelled on the vector bundle $\mathrm{V}(j_1(\mathscr{V}_{3\mathrm{N}+1}))$. As such, $j_2(\mathscr{V}_{3\mathrm{N}+1})$ is canonically isomorphic to the submanifold of $\mathrm{T}(j_1(\mathscr{V}_{3\mathrm{N}+1}))$ formed by the totality of vectors tangent to jet-extensions $j_1(\hat{\gamma})$ of sections $\hat{\gamma} : \mathbb{R} \to \mathscr{V}_{3\mathrm{N}+1}$, on the basis of the identification

$$j_2(\mathscr{V}_{3\mathrm{N}+1}) \simeq \{ \mathbf{X} \,|\, \mathbf{X} \in \mathrm{T}(j_1(\mathscr{V}_{3\mathrm{N}+1})),\ \langle \mathbf{X}, dt \rangle = 1,\ \langle \mathbf{X}, \omega \rangle = 0,\ \forall\, \omega \in \hat{\mathscr{D}}_1 \} \tag{1.10}$$

$\hat{\mathscr{D}}_1$ denoting the module generated locally by the 1-forms (1.5).

(ii) every frame of reference $\mathscr{I}$ determines a corresponding section $\mathbf{A} : j_1(\mathscr{V}_{3\mathrm{N}+1}) \to j_2(\mathscr{V}_{3\mathrm{N}+1})$, assigning to each $x \in j_1(\mathscr{V}_{3\mathrm{N}+1})$ the unique element $\mathbf{A}(x) \in \pi^{-1}(x)$ expressing the (istantaneous) vanishing of the *relative* accelerations of all points $\mathbf{P}_i \in \mathscr{S}$, $i=1, \ldots, \mathrm{N}$ in the kinetic state described by x in the frame of reference $\mathscr{I}$.

In view of (i), every section $\mathbf{Z} : j_1(\mathscr{V}_{3\mathrm{N}+1}) \to j_2(\mathscr{V}_{3\mathrm{N}+1})$ determines a one-to-one correspondence $\rho_{\mathbf{z}} : j_2(\mathscr{V}_{3\mathrm{N}+1}) \to \mathrm{V}(j_1(\mathscr{V}_{3\mathrm{N}+1}))$, sending each $\mathbf{Y}_x \in j_2(\mathscr{V}_{3\mathrm{N}+1})_{|x}$ into the difference

$$\rho_{\mathbf{z}}(\mathbf{Y}_x) := \mathbf{Y}_x - \mathbf{Z}_{|x} \in \mathrm{V}(j_1(\mathscr{V}_{3\mathrm{N}+1}))_{|x}, \tag{1.11}$$

the identification (1.10) being implicitly understood. We call $\rho_{\mathbf{z}}$ the *verticalizer* of $j_2(\mathscr{V}_{3\mathrm{N}+1})$ *induced* by $\mathbf{Z}$.

Comparison with equation (1.9*a*) shows that every verticalizer accomplishes a *representation* of the elements of $j_2(\mathscr{V}_{3\mathrm{N}+1})$ in terms of N-tuples of ordinary vectors.

In particular, taking assertion (ii) into account, it is easily seen that every frame of reference $\mathscr{I}$ determines its own verticalizer $\rho_{\mathbf{A}}$, thus yielding back – through equation (1.9*a*) – the corresponding representation of the elements of $j_2(\mathscr{V}_{3\mathrm{N}+1})$ as N-tuples $\mathbf{a}_1, \ldots, \mathbf{a}_{\mathrm{N}}$ of *relative accelerations*.

More generally, every section $\mathbf{Z}: j_1(\mathscr{V}_{3N+1}) \to j_2(\mathscr{V}_{3N+1})$ may be viewed in relative terms as a prescription

$$\mathbf{a}_i = \mathbf{a}_i(t, \mathbf{x}_1, \ldots, \mathbf{x}_N, \mathbf{v}_1, \ldots, \mathbf{v}_N), \qquad i=1, \ldots, N \tag{1.12}$$

assigning the accelerations of the points of $\mathscr{S}$ in the frame of reference $\mathscr{I}$ as functions of positions, velocities and time.

On the other hand, due to the identification (1.10), $\mathbf{Z}$ is automatically also a map $j_1(\mathscr{V}_{3N+1}) \to T(j_1(\mathscr{V}_{3N+1}))$, *i.e.* a *vector field* over $j_1(\mathscr{V}_{3N+1})$: as such, $\mathbf{Z}$ will be called more specifically a *free dynamical flow* for the system $\mathscr{S}$.

It is then an easy matter to verify that the determination of the *integral curves* of the field $\mathbf{Z}$ is precisely what is meant by *solving* the system (1.12), viewed as an *inverse kinematical problem.*

The situation is made more transparent by referring $j_2(\mathscr{V}_{3N+1})$ to local *jet-coordinates* t, ξ^α, $\dot{\xi}^\alpha$, $\ddot{\xi}^\alpha$. Every section $\mathbf{Z}: j_1(\mathscr{V}_{3N+1}) \to j_2(\mathscr{V}_{3N+1})$ is then described locally through the equations

$$\ddot{\xi}^\alpha = Z^\alpha(t, \xi^1, \ldots, \xi^{3N}, \dot{\xi}^1, \ldots, \dot{\xi}^{3N}), \qquad \alpha=1, \ldots, 3N \tag{1.13a}$$

mathematically equivalent to the vector representation

$$\mathbf{Z} = \frac{\partial}{\partial t} + \dot{\xi}^\alpha \frac{\partial}{\partial \xi^\alpha} + Z^\alpha \frac{\partial}{\partial \dot{\xi}^\alpha} \tag{1.13b}$$

From this one sees once again that a vector field over $j_1(\mathscr{V}_{3N+1})$ is a dynamical flow if and only if its integral curves are *jet-extensions* of (local) sections of $\mathscr{V}_{3N+1}$, and that, given any dynamical flow $\mathbf{X}$, a necessary and sufficient condition for $\mathbf{Y}$ to be a dynamical flow too is that the difference $\mathbf{X}-\mathbf{Y}$ be a *vertical* vector field on $j_1(\mathscr{V}_{3N+1})$.

Given any frame of reference $\mathscr{I}$, the relativization process for the accelerations is summarized into the identifications

$$\mathbf{a}_i = \frac{\partial \mathbf{v}_i}{\partial t} + \dot{\xi}^\alpha \frac{\partial \mathbf{v}_i}{\partial \xi^\alpha} + \ddot{\xi}^\alpha \frac{\partial \mathbf{v}_i}{\partial \dot{\xi}^\alpha}, \qquad i=1, \ldots, N \tag{1.14}$$

$\mathbf{v}_i = \mathbf{v}_i(t, \xi, \dot{\xi})$ denoting the relative velocities (1.6).

In particular, in view of equations (1.13a), (1.13b), the distribution of relative accelerations (1.12) associated with an arbitrary dynamical flow $\mathbf{Z}$ may be written explicitly as

$$\mathbf{a}_i = \frac{\partial \mathbf{v}_i}{\partial t} + \dot{\xi}^\alpha \frac{\partial \mathbf{v}_i}{\partial \xi^\alpha} + Z^\alpha \frac{\partial \mathbf{v}_i}{\partial \dot{\xi}^\alpha} := \mathbf{Z}(\mathbf{v}_i), \qquad i=1, \ldots, N \tag{1.15}$$

The same result may be stated more geometrically in terms of the vertical vector field $\rho_{\mathbf{A}}(\mathbf{Z}) := \mathbf{Z} - \mathbf{A}$ associated with $\mathbf{Z}$ through the verticalization process induced by the frame of reference $\mathscr{I}$. Indeed, denoting by $T = \frac{1}{2} \sum m_i v_i^2$ the *relative kinetic energy* of $\mathscr{S}$ in the given frame of reference, and recalling equations (1.6), (1.9 *a*), (1.9 *b*), (1.13 *b*), one can easily verify that equation (1.15) is mathematically equivalent to the identification

$$\left(\rho_{\mathbf{A}}(\mathbf{Z}), \frac{\partial}{\partial \xi^\alpha}\right) = \sum_{i=1}^{N} m_i \mathbf{a}_i \frac{\partial \mathbf{x}_i}{\partial \xi^\alpha} = \sum_{i=1}^{N} m_i \mathbf{Z}(\mathbf{v}_i) . \frac{\partial \mathbf{x}_i}{\partial \xi^\alpha} = \mathbf{Z}\left(\frac{\partial \mathrm{T}}{\partial \xi^\alpha}\right) - \frac{\partial \mathrm{T}}{\partial \xi^\alpha}. \tag{1.16}$$

The proof is straightforward, and is left to the reader.

(*e*) The previous remarks are especially relevant in a *dynamical* context, the assignment (1.12) being then determined by the knowledge of the total forces $\mathbf{F}_i$ acting on the points P_i in the frame of reference $\mathscr{I}$, through Newton's *second law*

$$m_i \mathbf{a}_i = \mathbf{F}_i(t, \mathbf{x}_1, \ldots, \mathbf{x}_{\mathrm{N}}, \mathbf{v}_1, \ldots, \mathbf{v}_{\mathrm{N}})$$

In this respect, the *problem of motion* for the system $\mathscr{S}$ has therefore a natural, *frame-independent* counterpart in the study of the integral curves of a suitable *dynamical flow* **Z**, completely characterized in terms of the *interactions*.

In the following we shall systematically pursue this *dynamical* viewpoint. The vertical vector $\rho_{\mathbf{A}}(\mathbf{Z}) = \mathbf{Z} - \mathbf{A}$, henceforth denoted by **F**, will be called the *total force* acting on $\mathscr{S}$ in the frame of reference $\mathscr{I}$.

Comparison with equations (1.9 *a*), (1.9 *b*) yields the identifications

$$\frac{\mathbf{F}_i}{m_i} = \mathrm{F}^\alpha \frac{\partial \mathbf{x}_i}{\partial \xi^\alpha} \tag{1.17 a}$$

with inverse

$$\mathrm{F}_\alpha := \left(\mathbf{F}, \frac{\partial}{\partial \xi^\alpha}\right) = g_{\alpha\beta} \mathrm{F}^\beta = \sum_{i=1}^{N} \mathbf{F}_i . \frac{\partial \mathbf{x}_i}{\partial \xi^\alpha} \tag{1.17 b}$$

The special dynamical flow $\mathbf{Z}_0$ expressing the complete *absence* of interactions on the points of $\mathscr{S}$ will be called the *inertial flow* associated with $\mathscr{S}$.

According to the *principle of inertia*, the equality $\mathbf{Z}_0 = \mathbf{A}$ characterizes the class of the *inertial* frames of reference. More generally, the total force $\mathbf{Z}_0 - \mathbf{A}$ determined by $\mathbf{Z}_0$ in an arbitrary *non inertial* frame $\mathscr{I}$ provides a description of the "apparent forces" in $\mathscr{I}$.

All this can be conveniently embodied into the geometrical set-up, by including $\mathbf{Z}_0$ among the *attributes* of the manifold $\mathscr{V}_{3\mathrm{N}+1}$. The resulting geometrical object – still denoted by $\mathscr{V}_{3\mathrm{N}+1}$ – will be called the *dynamical space-time* associated with the system $\mathscr{S}$.

In terms of $\mathbf{Z}_0$, any other dynamical flow **Z** may be decomposed into

$$\mathbf{Z} = \mathbf{Z}_0 + \hat{\mathbf{F}} \tag{1.18}$$

The splitting (1.18) has now an intrinsic, *frame-independent* character. The vertical vector

$$\hat{\mathbf{F}}=\mathbf{Z}-\mathbf{Z}_0=g^{\alpha\beta}\,\hat{\mathbf{F}}_\alpha\frac{\partial}{\partial\dot{\xi}^\beta}$$

provides an invariant description of the *interactions*, and is in fact related to the *real physical forces* $\hat{\mathbf{F}}_i$ acting on the points P_i through the identifications [analogous to equation (1.17 *a*), (1.17 *b*)]

$$\hat{\mathbf{F}}_\alpha=\left(\hat{\mathbf{F}},\frac{\partial}{\partial\dot{\xi}^\alpha}\right)=\sum_{i=1}^{N}\hat{\mathbf{F}}_i\cdot\frac{\partial\mathbf{x}_i}{\partial\xi^\alpha},\qquad\frac{\hat{\mathbf{F}}_i}{m_i}=\hat{\mathbf{F}}^\alpha\frac{\partial\mathbf{x}_i}{\partial\xi^\alpha}\tag{1.19}$$

We let the reader verify that the previous arguments summarize the content of *Relative Mechanics.*

1.2. Constraints

(*a*) By definition, the presence of *constraints* upon the system $\mathcal{S}$ is related to the existence of *restrictions* placed on the set of possible *evolutions* of $\mathcal{S}$, *i.e.* on the set of admissible sections $\hat{\gamma}:\mathbb{R}\to\mathcal{V}_{3N+1}$.

Quite often, the situation may be geometrized by introducing two submanifolds (without boundaries) $i:\mathcal{V}_{n+1}\to\mathcal{V}_{3N+1}$ and $\psi:\mathcal{A}\to j_1(\mathcal{V}_{3N+1})$, both fibered over $\mathbb{R}$ through the absolute time function, and satisfying the requirements:

(i) the totality of time evolutions allowed by the constraints coincides with the totality of sections $\hat{\gamma}:\mathbb{R}\to\mathcal{V}_{3N+1}$ whose first jet-extension $j_1(\hat{\gamma})$ has image contained in $\psi(\mathcal{A})$;

(ii) $\mathcal{A}$ is a fibered manifold over $\mathcal{V}_{n+1}$, diffeomorphic to a sub-bundle of $j_1(\mathcal{V}_{n+1})$.

Using j_1 and π to denote jet-extensions and projections respectively, the stated properties are summarized into the commutative diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "h"] \arrow[d, "\pi"] \arrow[rr, bend left, "\psi"] & j_1(\mathcal{V}_{n+1}) \arrow[r, "j_1(i)"] \arrow[d, "\pi"] & j_1(\mathcal{V}_{3N+1}) \arrow[d, "\pi"] \\
\mathcal{V}_{n+1} \arrow[r, phantom, "="] & \mathcal{V}_{n+1} \arrow[r, "i"] & \mathcal{V}_{3N+1}
\end{tikzcd}\tag{1.20}$$

in which all columns denote fibrations, while h is an embedding, fibered over the identity map.

In what follows, we shall restrict our attention to the class of constraints indicated above. Following the standard usage, we shall denote by $t, q^1, \ldots, q^n$ a local coordinate system on $\mathcal{V}_{n+1}$, and by t,

$q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n$ the associated system on $j_1(\mathscr{V}_{n+1})$. Also, we shall denote by $t, q^1, \ldots, q^n, z^1, \ldots, z^r$ a local coordinate system on $\mathscr{A}$, fibered over $t, q^1, \ldots, q^n$.

Keeping the same notation as in Section 1.1 for the coordinates on $j_1(\mathscr{V}_{3N+1})$, the various maps indicated in the diagram (1.20) are then summarized into the equations

$$\mathscr{V}_{n+1} \to \mathscr{V}_{3N+1} \qquad : \xi^\alpha = \xi^\alpha(t, q^1, \ldots, q^n) \tag{1.21 a}$$

$$j_1(\mathscr{V}_{n+1}) \to j_1(\mathscr{V}_{3N+1}) : \xi^\alpha = \xi^\alpha(t, q^1, \ldots, q^n), \quad \dot{\xi}^\alpha = \frac{\partial \xi^\alpha}{\partial q^i}\dot{q}^i + \frac{\partial \xi^\alpha}{\partial t} \tag{1.21 b}$$

$$\mathscr{A} \to j_1(\mathscr{V}_{n+1}) \qquad : q^i = q^i, \quad \dot{q}^i = \dot{q}^i(t, q^1, \ldots, q^n, z^1, \ldots, z^r) \tag{1.22}$$

with

$$\operatorname{rank} \| \partial(\xi^1, \ldots, \xi^{3N})/\partial(q^1, \ldots, q^n) \| = n$$

and

$$\operatorname{rank} \| \partial(\dot{q}^1, \ldots, \dot{q}^n)/\partial(z^1, \ldots, z^r) \| = r.$$

Equivalently, equation (1.22) may be replaced by the implicit representation

$$g_\sigma(t, q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n) = 0, \qquad \sigma = 1, \ldots, n-r \tag{1.23}$$

with rank $\| \partial(g_1, \ldots, g_{n-r})/(\partial(\dot{q}^1, \ldots, \dot{q}^n) \| = n - r$.

Given an arbitrary frame of reference $\mathscr{I}$, the relativization process (1.6), restricted to the submanifold $j_1(i): j_1(\mathscr{V}_{n+1}) \to j_1(\mathscr{V}_{3N+1})$ will be written synthetically as

$$\mathbf{x}_i = \mathbf{x}_i(t, q^1, \ldots, q^n), \qquad \mathbf{v}_i = \sum_{k=1}^{n} \frac{\partial \mathbf{x}_i}{\partial q^k}\dot{q}^k + \frac{\partial \mathbf{x}_i}{\partial t}, \qquad i = 1, \ldots, N \tag{1.24}$$

The further restriction to the submanifold $h : \mathscr{A} \to j_1(\mathscr{V}_{n+1})$ is then achieved in the obvious way, either by pulling back the $\dot{q}^k$'s through equations (1.22), or by subjecting them to the implicit conditions (1.23).

For simplicity, we shall systematically identify $\mathscr{A}$ with its image $h(\mathscr{A}) \subseteq j_1(\mathscr{V}_{n+1})$. The manifold $\mathscr{V}_{n+1}$ will be called the *configuration space-time* of the system $\mathscr{S}$; the fiber-bundle $\pi : \mathscr{A} \to \mathscr{V}_{n+1}$ will be called the *space of admissible velocities* of $\mathscr{S}$; a section $\gamma : \mathbb{R} \to \mathscr{V}_{n+1}$ will be said to be consistent with the constraints or *admissible* if and only if the corresponding jet-extension $j_1(\gamma)$ has image contained in $\mathscr{A}$; the totality of admissible sections of $\mathscr{V}_{n+1}$ will be denoted by $\mathscr{H}(\mathscr{V}_{n+1})$.

With these definitions, it is easily seen that the totality of time evolutions of $\mathscr{S}$ allowed by the constraints is in one-to-one correspondence with the class $\mathscr{H}(\mathscr{V}_{n+1})$ of admissible sections through the composition $\hat{\gamma} = i \cdot \gamma$.

As already pointed out, the previous framework embodies a large class of constraints both of positional and of kinetic character, provided only that they are *two-sided* and sufficiently smooth.

Concerning the effect of the inclusion $\mathscr{A} \subseteq j_1(\mathscr{V}_{n+1})$, the simplest situation occurs when the equality $\mathscr{A} = j_1(\mathscr{V}_{n+1})$ holds. In this case $\mathscr{S}$ is called a *holonomic system* with *n degrees of freedom*; any section $\gamma : \mathbb{R} \to \mathscr{V}_{n+1}$ is automatically admissible, in agreement with the fact that the restrictions imposed by the constraints are merely *positional.*

The strict inclusion $\mathscr{A} \subset j_1(\mathscr{V}_{n+1})$ indicates, on the contrary, the presence of *kinetic* constraints, *i.e.* of restrictions placed directly on the *velocities* of the points of the system.

Depending on the type of problem in study, these may include *integrable* constraints, as well as truly *non-holonomic* ones, possibly of *non-linear* nature: all cases will be dealt with on a unified basis in the subsequent discussion.

This flexibility is reflected in the fact that, in general, the same physical situation may admit several different geometrizations, each involving its own choice of the fibration $\pi : \mathscr{A} \to \mathscr{V}_{n+1}$, the freedom relying on the possibility of replacing arbitrary positional constraints with equivalent integrable kinetic ones, and conversely.

As a result, in the case $\mathscr{A} \subset j_1(\mathscr{V}_{n+1})$, the dimension of the manifold $\mathscr{V}_{n+1}$ has no longer a definite physical meaning. In particular, the concept of *number of degrees of freedom* of the system is now less immediate than in the holonomic case, and has to be related with the dimension of the slicing (if any!) induced in $\mathscr{V}_{n+1}$ by the equivalence relation

$$(x, t) \sim (y, t') \quad \Leftrightarrow \quad \exists\, \varphi \in \mathscr{H}(\mathscr{V}_{n+1}), \qquad \varphi(t) = x, \quad \varphi(t') = y.$$

(*b*) The geometrical concepts discussed in Section 1.1 in the case of free systems have obvious counterparts in the presence of constraints. Let us examine the situation in detail:

(i) the distance (1.1) determines a Riemannian structure on each fiber $t = \text{const.}$ in $\mathscr{V}_{n+1}$ or, what is the same, a scalar product for vertical vectors in $\mathscr{V}_{n+1}$ (verticality being here understood with respect to the fibration $t : \mathscr{V}_{n+1} \to \mathbb{R}$). Using equations (1.3), (1.21 *a*), (1.24), we get the explicit representation

$$\left(\frac{\partial}{\partial q^i}, \frac{\partial}{\partial q^j}\right) = a_{ij}(t, q^1, \ldots, q^n) \tag{1.25}$$

with

$$a_{ij} := g_{\alpha\beta} \frac{\partial \xi^\alpha}{\partial q^i} \frac{\partial \xi^\beta}{\partial q^j} = \sum_{k=1}^{N} m_k \frac{\partial \mathbf{x}_k}{\partial q^i} \cdot \frac{\partial \mathbf{x}_k}{\partial q^j} \tag{1.26}$$

(ii) both fibrations $\pi : \mathscr{A} \to \mathscr{V}_{n+1}$ and $\pi : j_1(\mathscr{V}_{n+1}) \to \mathscr{V}_{n+1}$ determine their own vertical bundles, denoted respectively by $\mathrm{V}(\mathscr{A}) \subset \mathrm{T}(\mathscr{A})$, and $\mathrm{V}(j_1(\mathscr{V}_{n+1})) \subset \mathrm{T}(j_1(\mathscr{V}_{n+1}))$. The differentials of the maps

$h : \mathscr{A} \rightarrow j_1(\mathscr{V}_{n+1})$ and $j_1(i) : j_1(\mathscr{V}_{n+1}) \rightarrow j_1(\mathscr{V}_{3N+1})$ are easily seen to preserve this type of verticality, thus giving rise to injective immersions

$$V(\mathscr{A}) \rightarrow V(j_1(\mathscr{V}_{n+1})) \rightarrow V(j_1(\mathscr{V}_{3N+1})) \tag{1.27}$$

In view of these, each fiber of $V(\mathscr{A})$ may be regarded as a vector subspace of a corresponding fiber in $V(j_1(\mathscr{V}_{n+1}))$, and the latter, in turn, as a vector subspace of a fiber in $V(j_1(\mathscr{V}_{3N+1}))$: both submanifolds $\mathscr{A}$ and $j_1(\mathscr{V}_{n+1})$ thus inherit from $j_1(\mathscr{V}_{3N+1})$ a natural operation of *scalar product* for vertical vectors.

In local coordinates, introducing the differential forms

$$\omega^i := dq^i - \dot{q}^i\, dt, \qquad i = 1, \ldots, n \tag{1.28}$$

as well as their pull-backs

$$h^*(\omega^i) = dq^i - \dot{q}^i(t, q^1, \ldots, q^n, z^1, \ldots, z^r)\, dt, \qquad i = 1, \ldots, n \tag{1.29}$$

we have the representations

$$V(j_1(\mathscr{V}_{n+1})) \\ = \{ X \mid X \in T(j_1(\mathscr{V}_{n+1})), \langle X, dt \rangle = 0, \langle X, \omega^i \rangle = 0, i = 1, \ldots, n \} \tag{1.30}$$

$$V(\mathscr{A}) \\ = \{ X \mid X \in T(\mathscr{A}), \langle X, dt \rangle = 0, \langle X, h^*(\omega^i) \rangle = 0, i = 1, \ldots, n \} \tag{1.31}$$

or explicitly

$$X \in V(j_1(\mathscr{V}_{n+1})) \quad \Leftrightarrow \quad X = X^i \frac{\partial}{\partial \dot{q}^i} \tag{1.32}$$

$$X \in V(\mathscr{A}) \quad \Leftrightarrow \quad X = X^A \frac{\partial}{\partial z^A} \tag{1.33}$$

The effect of the injections (1.27) is summarized into the identifications

$$\frac{\partial}{\partial z^A} = \frac{\partial \dot{q}^i}{\partial z^A} \frac{\partial}{\partial \dot{q}^i}, \qquad \frac{\partial}{\partial \dot{q}^i} = \frac{\partial \dot{\xi}^\alpha}{\partial \dot{q}^i} \frac{\partial}{\partial \dot{\xi}^\alpha} = \frac{\partial \xi^\alpha}{\partial q^i} \frac{\partial}{\partial \dot{\xi}^\alpha} \tag{1.34}$$

Together with equations (1.8), (1.22), (1.24), (1.26), these yield the representations

$$\left(\frac{\partial}{\partial \dot{q}^i}, \frac{\partial}{\partial \dot{q}^j} \right) = \frac{\partial \xi^\alpha}{\partial q^i} \frac{\partial \xi^\beta}{\partial q^j} g_{\alpha\beta} = a_{ij}(t, q^1, \ldots, q^n) = \sum_{k=1}^{N} m_k \frac{\partial \mathbf{v}_k}{\partial \dot{q}^i}, \frac{\partial \mathbf{v}_k}{\partial \dot{q}^j} \tag{1.35}$$

$$\left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B} \right) = a_{ij} \frac{\partial \dot{q}^i}{\partial z^A} \frac{\partial \dot{q}^j}{\partial z^B} = \sum_{k=1}^{N} m_k \frac{\partial \mathbf{v}_k}{\partial z^A}, \frac{\partial \mathbf{v}_k}{\partial z^B} := a_{AB} \tag{1.36}$$

If, rather than the description (1.22) for the submanifold $\mathscr{A} \subset j_1(\mathscr{V}_{n+1})$, one adopts the implicit one (1.23), equation (1.31) provides an identification of the vertical bundle $V(\mathscr{A})$ with the subspace of $V(j_1(\mathscr{V}_{n+1}))$ formed by the totality of vectors $\mathbf{X}=\mathbf{X}^i(\partial/\partial\dot{q}^i)_x$ $(x\in\mathscr{A})$ satisfying the conditions

$$\mathbf{X}(g_\sigma)=\mathbf{X}^i\left(\frac{\partial g_\sigma}{\partial \dot{q}^i}\right)_x=0, \qquad \sigma=1, \ldots, n-r. \tag{1.37}$$

The scalar product in $\mathrm{V}(\mathscr{A})$ is then expressed directly in terms of equation (1.35).

(c) For dynamical purposes, it is important to characterize the *admissible accelerations* of the system $\mathscr{S}$ in the presence of constraints.

Matters are quite straightforward in the *holonomic* case, the relevant space being then the second jet-bundle $j_2(\mathscr{V}_{n+1})$. Indeed, in local jet-coordinates, starting with equation (1.24), and proceeding as in Section 1.1, it is easily seen that every frame of reference $\mathscr{I}$ determines a corresponding representation of the points of $j_2(\mathscr{V}_{n+1})$ in terms of N-tuples of *relative accelerations*, through a standard *relativization process*, now given explicitly by

$$\mathbf{a}_i=\frac{\partial \mathbf{v}_i}{\partial t}+\dot{q}^k\frac{\partial \mathbf{v}_i}{\partial q^k}+\ddot{q}^k\frac{\partial \mathbf{v}_i}{\partial \dot{q}^k}, \qquad i=1, \ldots, \mathrm{N}. \tag{1.38}$$

Once again, $j_2(\mathscr{V}_{n+1})$ is an affine bundle over $j_1(\mathscr{V}_{n+1})$, modelled on the vertical bundle $\mathrm{V}(j_1(\mathscr{V}_{n+1}))$, and canonically isomorphic to a submanifold of the tangent space $\mathrm{T}(j_1(\mathscr{V}_{n+1}))$, through the usual identification

$$j_2(\mathscr{V}_{n+1}) =\{\mathbf{Y}\,|\,\mathbf{Y}\in\mathrm{T}(j_1(\mathscr{V}_{n+1})),\ \langle\mathbf{Y}, dt\rangle=1,\ \langle\mathbf{Y}, \omega^i\rangle=0,\ i=1, \ldots, n\} \tag{1.39}$$

with ω^i defined by equation (1.28).

Every section $\tilde{\mathbf{Z}}: j_1(\mathscr{V}_{n+1})\to j_2(\mathscr{V}_{n+1})$, viewed as a vector field over $j_1(\mathscr{V}_{n+1})$, will be called a *holonomic flow* for the system $\mathscr{S}$. In local coordinates, this corresponds to the representation

$$\tilde{\mathbf{Z}}=\frac{\partial}{\partial t}+\dot{q}^i\frac{\partial}{\partial q^i}+\mathbf{Z}^i\frac{\partial}{\partial \dot{q}^i} \tag{1.40}$$

with $\mathbf{Z}^i=\mathbf{Z}^i(t, q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$.

Exactly as in Section 1.1, every holonomic flow $\tilde{\mathbf{Z}}$ is easily seen to determine a corresponding *verticalization* $\rho_{\tilde{\mathbf{Z}}}: j_2(\mathscr{V}_{n+1})\to\mathrm{V}(j_1(\mathscr{V}_{n+1}))$, sending each $\mathbf{Y}_x\in j_2(\mathscr{V}_{n+1})_{|x}$ into the difference

$$\rho_{\tilde{\mathbf{Z}}}(\mathbf{Y}_x):=\mathbf{Y}_x-\tilde{\mathbf{Z}}_{|x}\in\mathrm{V}(j_1(\mathscr{V}_{n+1}))_{|x}, \tag{1.41}$$

the identification (1.39) being implicitly understood.

At the same time, taking equations (1.24), (1.38) into account, the flow $\tilde{\mathbf{Z}}$ may also be viewed in relative terms as an assignment

$$\mathbf{a}_i=\frac{\partial \mathbf{v}_i}{\partial t}+\dot{q}^h\frac{\partial \mathbf{v}_i}{\partial q^h}+\mathbf{Z}^h\frac{\partial \mathbf{v}_i}{\partial \dot{q}^h}:=\tilde{\mathbf{Z}}(\mathbf{v}_i), \qquad i=1, \ldots, \mathrm{N} \tag{1.42}$$

expressing the admissible accelerations of the points of $\mathscr{S}$ with respect to $\mathscr{I}$ in terms of (admissible) positions, velocities and time, admissibility being considered here solely with respect to the restrictions placed by the *holonomic* constraints.

Once again, as a consequence of equation (1.42), one can easily recover the identities

$$\sum m_i \mathbf{a}_i . \frac{\partial \mathbf{x}_i}{\partial q^k} = \tilde{\mathbf{Z}}\left(\frac{\partial \mathrm{T}}{\partial \dot{q}^k}\right) - \frac{\partial \mathrm{T}}{\partial q^k} = a_{hk}\,\mathrm{Z}^h + \left(\frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i}\right)\left(\frac{\partial \mathrm{T}}{\partial \dot{q}^k}\right) - \frac{\partial \mathrm{T}}{\partial q^k} \qquad (1.43)$$

T denoting now the relative kinetic energy of $\mathscr{S}$, *restricted* to the submanifold $j_1(\mathscr{V}_{n+1})$.

Conversely, for any given distribution $\mathbf{a}_i = \mathbf{a}_i(t, q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)$ of relative accelerations, the determination of a corresponding holonomic flow $\tilde{\mathbf{Z}}$ relies on the solution of the (over-determined) system (1.42) for the unknowns Z^h. In this respect, the compatibility of the equations is the mathematical counterpart of the requirement of admissibility of the $\mathbf{a}_i$'s. Moreover, for admissible $\mathbf{a}_i$'s, the system (1.42) is algebraically equivalent to the subsystem (1.43), thus showing that the entire information on the original distribution of accelerations is summarized into the knowledge of the n quantities $\sum m_i \mathbf{a}_i . \dfrac{\partial \mathbf{x}_i}{\partial q^k}$. We shall return on this point in Section 3.

(*d*) When the kinetic constraints are explicitly accounted for, the space of admissible accelerations for the system $\mathscr{S}$ does no longer coincide with the entire jet-bundle $j_2(\mathscr{V}_{n+1})$, but is restricted to the subset $\tau(\mathscr{A}) \subset j_2(\mathscr{V}_{n+1})$ formed by the totality of vectors $\mathbf{Y}$ *tangent* to jet extensions $j_1(\gamma)$ of admissible sections $\gamma \in \mathscr{H}(\mathscr{V}_{n+1})$.

Regarding both $\mathrm{T}(\mathscr{A})$ and $j_2(\mathscr{V}_{n+1})$ as subsets of $\mathrm{T}(j_1(\mathscr{V}_{n+1}))$, this corresponds to the identification

$$\tau(\mathscr{A}) = \mathrm{T}(\mathscr{A}) \cap j_2(\mathscr{V}_{n+1})$$

In local coordinates, recalling equations (1.29), (1.39), we have the explicit representation

$$\tau(\mathscr{A}) = \{\mathbf{Y} \mid \mathbf{Y} \in \mathrm{T}(\mathscr{A}), \langle \mathbf{Y}, dt \rangle = 1, \langle \mathbf{Y}, h^*(\omega^i) \rangle = 0, i = 1, \ldots, n\} \qquad (1.44)$$

From this, taking equations (1.31) into account, it is easily seen that $\tau(\mathscr{A})$ has the nature of an *affine bundle* over $\mathscr{A}$, modelled on the vertical bundle $\mathrm{V}(\mathscr{A})$. In the following we shall refer $\tau(\mathscr{A})$ to local coordinates t, q^k, z^A, $\dot{z}^\mathrm{A}$, the functions $\dot{z}^\mathrm{A}$ denoting the *formal time derivatives* of the z^A's, $(\mathrm{A} = 1, \ldots, r)$, defined in the usual way by the requirement

$$\dot{z}^\mathrm{A} \cdot j_2(\gamma) = \frac{d}{dt}(z^\mathrm{A} \cdot j_1(\gamma))$$

for all admissible sections $\gamma \in \mathscr{H}(\mathscr{V}_{n+1})$.

Exactly as it happens for $j_2(\mathscr{V}_{n+1})$, the choice of an arbitrary frame of reference $\mathscr{I}$ gives rise to a representation of the points of $\tau(\mathscr{A})$ as N-tuples of relative accelerations, the correspondence being now expressed by the equations

$$\mathbf{a}_i=\frac{\partial \mathbf{v}_i}{\partial t}+\dot{q}^k\frac{\partial \mathbf{v}_i}{\partial q^k}+\dot{z}^{\mathrm{A}}\frac{\partial \mathbf{v}_i}{\partial z^{\mathrm{A}}}, \qquad i=1, \ldots, \mathrm{N} \tag{1.45}$$

with $\dot{q}^k$ given by equation (1.22).

A vector field $\mathbf{Z}$ on $\mathscr{A}$, with values in $\tau(\mathscr{A})$, will be called an *effective flow* for the system $\mathscr{S}$. In local coordinates, taking equation (1.44) into account, we have the representation

$$\mathbf{Z}=\frac{\partial}{\partial t}+\dot{q}^i(t, q^1, \ldots, q^n, z^1, \ldots, z^r)\frac{\partial}{\partial q^i}+\mathrm{Z}^{\mathrm{A}}\frac{\partial}{\partial z^{\mathrm{A}}} \tag{1.46}$$

with $\mathrm{Z}^{\mathrm{A}}=\mathrm{Z}^{\mathrm{A}}(t, q^1, \ldots, q^n, z^1, \ldots, z^r)$.

Once again, every effective flow $\mathbf{Z}:\mathscr{A}\to\tau(\mathscr{A})$ may be viewed in relative terms as a prescription

$$\mathbf{a}_i=\frac{\partial \mathbf{v}_i}{\partial t}+\dot{q}^k\frac{\partial \mathbf{v}_i}{\partial q^k}+\mathrm{Z}^{\mathrm{A}}\frac{\partial \mathbf{v}_i}{\partial z^{\mathrm{A}}}, \qquad i=1, \ldots, \mathrm{N} \tag{1.47}$$

assigning the *admissible* accelerations of the points of $\mathscr{S}$ in the frame of reference $\mathscr{I}$ in terms of (admissible) positions, velocities and time, admissibility being now understood with respect to the entire set of constraints.

Conversely, given any distribution $\mathbf{a}_i=\mathbf{a}_i(t, q^k, z^{\mathrm{A}})$ of relative accelerations, the determination of a corresponding effective flow $\mathbf{Z}$ relies on the solutions of the (over-determined) system (1.47) for the unknowns Z^{A}. Once again, the compatibility of the equations is the mathematical counterpart of the requirement of admissibility of the $\mathbf{a}_i$'s. When this is the case, taking equation (1.36) into account, the system (1.47) is algebraically equivalent to the subsystem

$$\begin{aligned}\sum_i m_i\,\mathbf{a}_i\cdot\frac{\partial \mathbf{v}_i}{\partial z^{\mathrm{A}}}&=\sum_i m_i\left(\frac{\partial \mathbf{v}_i}{\partial t}+\frac{\partial \mathbf{v}_i}{\partial q^k}\dot{q}^k+\frac{\partial \mathbf{v}_i}{\partial z^{\mathrm{B}}}\mathrm{Z}^{\mathrm{B}}\right)\cdot\frac{\partial \mathbf{v}_i}{\partial z^{\mathrm{A}}}\\ &=a_{\mathrm{AB}}\,\mathrm{Z}^{\mathrm{B}}+\sum_i m_i\left(\frac{\partial \mathbf{v}_i}{\partial t}+\frac{\partial \mathbf{v}_i}{\partial q^k}\dot{q}^k\right)\cdot\frac{\partial \mathbf{v}_i}{\partial z^{\mathrm{A}}}, \qquad \mathrm{A}=1, \ldots, r,\end{aligned} \tag{1.48}$$

thus showing that the knowledge of the r quantities $\sum m_i\,\mathbf{a}_i\cdot\frac{\partial \mathbf{v}_i}{\partial z^{\mathrm{A}}}$ as functions of t, q^k, z^{A} summarizes the whole information contained in the original distribution of accelerations.

(*e*) The differentials of the maps $h:\mathscr{A}\to j_1(\mathscr{V}_{n+1})$ and $j_1(i):(\mathscr{V}_{n+1})\to j_1(\mathscr{V}_{3\mathrm{N}+1})$ give rise to injective immersions

$$\tau(\mathscr{A})\to j_2(\mathscr{V}_{n+1})\to j_2(\mathscr{V}_{3\mathrm{N}+1}) \tag{1.49}$$

By abuse of language, we shall regard the latter as effective *inclusions*, thus identifying the spaces $\tau(\mathscr{A})$ and $j_2(\mathscr{V}_{n+1})$ with their respective images.

Keeping the same notation as in Section 1.1, this means e.g. that, if $\hat{\mathbf{Z}}: j_1(\mathscr{V}_{3N+1}) \to j_2(\mathscr{V}_{3N+1})$ is any free dynamical flow for the system $\mathscr{S}$, the difference $\mathbf{Y}_x - \hat{\mathbf{Z}}_{|x}$ makes sense at each $x \in \mathscr{A}$ and for each $\mathbf{Y}_x \in \tau_x(\mathscr{A})$ [as well as at each $x \in j_1(\mathscr{V}_{n+1})$ and for each $\mathbf{Y}_x \in j_2(\mathscr{V}_{n+1})_{|x}$], and is automatically a *vertical* vector on $j_1(\mathscr{V}_{3N+1})$. In a similar way, if $\tilde{\mathbf{Z}}$ is any *holonomic flow* on $j_1(\mathscr{V}_{n+1})$, the difference $\mathbf{Y}_x - \tilde{\mathbf{Z}}_{|x}$, $x \in \mathscr{A}$, $\mathbf{Y}_x \in \tau_x(\mathscr{A})$ is a vertical vector on $j_1(\mathscr{V}_{n+1})$, etc.

All this is easily rephrased in terms of the "verticalizers" (1.11), (1.41). A straightforward argument shows that, at each $x \in \mathscr{A}$, the image space

$$\rho_{\hat{\mathbf{Z}}}(\tau_x(\mathscr{A})) = \{\mathbf{Y}_x - \hat{\mathbf{Z}}_{|x} \mid \mathbf{Y}_x \in \tau_x(\mathscr{A})\}$$

is an *affine space*, modelled on the vertical space $\mathrm{V}_x(\mathscr{A})$, *i.e.* a *plane* in $\mathrm{V}_x(j_1(\mathscr{V}_{3N+1}))$, not necessarily containing the origin, and *parallel* to the subspace $\mathrm{V}_x(\mathscr{A}) \subset \mathrm{V}_x(j_1(\mathscr{V}_{3N+1}))$.

From this, one can easily infer the existence of an element of *minimal norm* in $\rho_{\hat{\mathbf{Z}}}(\tau_x(\mathscr{A}))$, namely, the *unique* element $\mathbf{N}_x \in \rho_{\hat{\mathbf{Z}}}(\tau_x(\mathscr{A}))$ *orthogonal* to $\mathrm{V}_x(\mathscr{A})$. The same reasoning also implies that $\mathbf{N}_x$ depends differentiably on x, since both $\rho_{\hat{\mathbf{Z}}}(\tau_x(\mathscr{A}))$ and $\mathrm{V}_x(\mathscr{A})$ do.

The unique section $\mathbf{Z}: \mathscr{A} \to \tau(\mathscr{A})$ defined by the requirement

$$|\mathbf{Z}_{|x} - \hat{\mathbf{Z}}_{|x}| = \min_{\mathbf{Y}_x \in \tau_x(\mathscr{A})} |\mathbf{Y}_x - \hat{\mathbf{Z}}_{|x}|, \qquad \forall x \in \mathscr{A} \tag{1.50}$$

will be denoted by $\mathscr{P}_{\mathscr{A}}(\hat{\mathbf{Z}})$, and will be called the *orthogonal projection* of the flow $\hat{\mathbf{Z}}$ on the submanifold $\mathscr{A}$.

Quite similar results apply to the image spaces

$$\rho_{\hat{\mathbf{Z}}}(j_2(\mathscr{V}_{n+1})_{|x}) \subset \mathrm{V}_x(j_1(\mathscr{V}_{3N+1})) \qquad (x \in j_1(\mathscr{V}_{n+1})),$$

and

$$\rho_{\tilde{\mathbf{Z}}}(\tau_x(\mathscr{A})) \subset \mathrm{V}_x(j_1(\mathscr{V}_{n+1})) \qquad (x \in \mathscr{A}),$$

$\tilde{\mathbf{Z}}$ denoting any holonomic flow on $j_1(\mathscr{V}_{n+1})$. In addition to $\mathscr{P}_{\mathscr{A}}(\hat{\mathbf{Z}})$, we may therefore introduce two further orthogonal projections acting on flows, namely

$$\hat{\mathbf{Z}} \to \mathscr{P}_{j_1(\mathscr{V}_{n+1})}(\hat{\mathbf{Z}}) \qquad \text{and} \qquad \tilde{\mathbf{Z}} \to \mathscr{P}_{\mathscr{A}}(\tilde{\mathbf{Z}}),$$

with an obvious meaning of the symbols.

We let the reader verify the composition rule

$$\mathscr{P}_{\mathscr{A}}(\hat{\mathbf{Z}}) = \mathscr{P}_{\mathscr{A}} \cdot \mathscr{P}_{j_1(\mathscr{V}_{n+1})}(\hat{\mathbf{Z}}) \tag{1.51}$$

All this will play a central role in the construction of a *dynamical scheme* in the presence of constraints.

2. REACTIVE FORCES

2.1. Mechanical determinism

In this section we shall examine the implementation of the *principle of determinism* within the framework described in Section 1.

To this end, we shall consider once again a constrained system $\mathscr{S}$, formed by N point particles $\mathbf{P}_1, \ldots, \mathbf{P}_N$, moving under the action of given active forces.

The geometrical set-up will therefore include:

(i) the dynamical space-time $\mathscr{V}_{3N+1}$ associated with $\mathscr{S}$, with all the attributes indicated in Section 1.1 (vertical metric, inertial flow, etc.);

(ii) the fiber bundle $\pi : \mathscr{A} \to \mathscr{V}_{n+1}$, summarizing the geometrical and kinematical restrictions imposed by the constraints, and related to the dynamical space-time $\mathscr{V}_{3N+1}$ by the fibered map

$$\begin{array}{ccccc} \mathscr{A} & \xrightarrow{h} & j_1(\mathscr{V}_{n+1}) & \xrightarrow{j_1(i)} & j_1(\mathscr{V}_{3N+1}) \\ \downarrow{\scriptstyle\pi} & & {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathscr{V}_{n+1} & = & \mathscr{V}_{n+1} & \xrightarrow{i} & \mathscr{V}_{3N+1} \end{array} \tag{2.1}$$

(iii) the (unconstrained) dynamical flow

$$\hat{\mathbf{Z}} := \frac{\partial}{\partial t} + \dot{\xi}^\alpha \frac{\partial}{\partial \xi^\alpha} + \hat{Z}^\alpha \frac{\partial}{\partial \dot{\xi}^\alpha} = \mathbf{Z}_0 + \hat{\mathbf{F}} \tag{2.2}$$

expressed as the sum of the inertial flow $\mathbf{Z}_0$ and of the active force $\hat{\mathbf{F}}$ in accordance with equation (1.18).

Within this framework, the idea of *mechanical determinism* may be identified with the requirement that, for each choice of the initial data consistent with the constraints (*i.e.* for each $x \in \mathscr{A}$), the subsequent evolution of the system—expressed as a corresponding admissible section $\gamma \in \mathscr{H}(\mathscr{V}_{n+1})$—be uniquely determined.

According to the discussion in Section 1.2 this is automatically accomplished through the introduction of a suitable *effective flow* $\mathbf{Z}$ on $\mathscr{A}$. More precisely, we may say *the implementation of the principle of determinism in the presence of constraints is mathematically equivalent to the introduction of a rule* χ *assigning to every free dynamical flow* $\hat{\mathbf{Z}}$ *a corresponding effective flow* $\mathbf{Z} = \chi(\hat{\mathbf{Z}})$, *expressing in a frame independent way the equations of evolution of the system.*

Such a rule will be called a *constitutive characterization of the constraints.* As it is clear from the previous discussion, the latter is not included in the geometrical set-up described above, but represents an *additional* piece of information, accounting for the influence of the constraints on the dynamics of the system, and therefore depending explicitly on the *physical properties* of the devices involved.

This viewpoint is formalized by considering the difference

$$\varphi := \chi(\hat{\mathbf{Z}}) - \hat{\mathbf{Z}}, \tag{2.3}$$

between the effective dynamical flow and the original, unconstrained one, evaluated at each $x \in \mathscr{A}$, and therefore viewed as a field on $\mathscr{A}$ with values in $\mathrm{T}(j_1(\mathscr{V}_{3N+1}))$.

For each choice of the constitutive characterization χ, the field (2.3) is automatically vertical. Therefore, in any frame of reference $\mathscr{I}$, it determines an N-tuple of vectors $\Phi_1, \ldots, \Phi_N$ applied in the points $\mathbf{P}_1, \ldots, \mathbf{P}_N$, on the basis of the identifications

$$\varphi_\alpha = \left(\varphi, \frac{\partial}{\partial \xi^\alpha}\right) = \sum_{i=1}^{N} \Phi_i \cdot \frac{\partial \mathbf{x}_i}{\partial \xi^\alpha}, \qquad \frac{\Phi_i}{m_i} = \varphi^\alpha \frac{\partial \mathbf{x}_i}{\partial \xi^\alpha} \tag{2.4}$$

the notation $\varphi = \varphi^\alpha \frac{\partial}{\partial \dot{\xi}^\alpha} = g^{\alpha\beta}\varphi_\beta \frac{\partial}{\partial \dot{\xi}^\alpha}$ being implicitly understood [*see* the analogous equation (1.19)].

The interpretation of the vectors Φ_i as *reactive forces* follows at once from equations (2.2), (2.3). In view of these, in fact, the effective flow $\mathbf{Z} = \chi(\hat{\mathbf{Z}})$ may be expressed as

$$\mathbf{Z} = \hat{\mathbf{Z}} + \varphi = \mathbf{Z}_0 + \hat{\mathbf{F}} + \varphi \tag{2.5}$$

thus showing that, in the presence of constraints, the physical forces acting on the points of $\mathscr{S}$ are described by the sum $\hat{\mathbf{F}} + \varphi$, rather than by $\hat{\mathbf{F}}$ alone.

By abuse of language, the field φ itself will be called the *reactive force*.

2.2. Ideal constraints

Within the scheme indicated in Section 2.1, we shall now describe a special constitutive characterization, essentially equivalent to Gauss' criterion of *minimal constraint* ([2], [7], [8], [9]).

As we shall see, in addition to being geometrically simple, the latter is powerful enough to cover a wide variety of applications, and to provide a natural extension of Lagrangian Dynamics to arbitrary non-holonomic systems, independently of any assumption of linearity of the constraints.

To start with we observe that, in view of equation (1.50), no matter how one chooses the constitutive characterization χ, the reactive force $\varphi = \chi(\hat{\mathbf{Z}}) - \hat{\mathbf{Z}}$ associated with an arbitrary unconstrained dynamical flow $\hat{\mathbf{Z}}$ is necessarily subject to the inequality

$$|\varphi_x| = |\chi(\hat{\mathbf{Z}})_{|x} - \hat{\mathbf{Z}}_x| \geqq \min_{\mathbf{Y}_x \in \tau_x(\mathscr{A})} |\mathbf{Y}_x - \hat{\mathbf{Z}}_{|x}|, \qquad \forall x \in \mathscr{A} \tag{2.6}$$

since, by definition, $\chi(\hat{\mathbf{Z}})_{|x}$ itself belongs to $\tau_x(\mathscr{A})$.

DEFINITION 2.1. – A set of constraints is said to be *ideal* if and only if the equality sign holds identically in equation (2.6), *i.e.* if the corresponding constitutive characterization χ satisfies

$$|\chi(\hat{\mathbf{Z}})_{|x}-\hat{\mathbf{Z}}_x| = \min_{\mathbf{Y}_x \in \tau_x(\mathscr{A})} |\mathbf{Y}_x-\hat{\mathbf{Z}}_{|x}| \tag{2.7}$$

for all $x \in \mathscr{A}$.

Note 2.1. – Recalling the connection between the vector φ and the reactive forces $\Phi_1, \ldots, \Phi_N$ acting on the points of the system, as well as the definition (1.9) of the scalar product for vertical vectors, it may be seen that the requirement (2.7) is indeed equivalent to the original formulation of Gauss' Principle of minimal constraint:

"For a natural system subject to ideal constraints, the actual motion under the action of given external forces is selected among the totality of the kinematically admissible evolutions, as the one for which, at any instant t, the quantity

$$|\varphi|^2 = \sum_{\alpha,\,\beta} g^{\alpha\beta}\varphi_\alpha\varphi_\beta = \sum_i \frac{1}{m_i}|\Phi_i|^2 = \sum_i \frac{1}{m_i}|m_i\mathbf{a}_i-\mathbf{F}_i|^2$$

attains a minimum" [2].

With the terminology introduced at the end of Section 1.2, the condition (2.7) may be expressed more synthetically as

$$\chi(\hat{\mathbf{Z}}) = \mathscr{P}_{\mathscr{A}}(\hat{\mathbf{Z}}) \tag{2.8}$$

$\mathscr{P}_{\mathscr{A}}$ being the orthogonal projection defined through equation (1.50).

Also, in view of equation (1.51), the correspondence $\hat{\mathbf{Z}} \to \chi(\hat{\mathbf{Z}})$ may be factorized into two subsequent steps, the first one

$$\hat{\mathbf{Z}} \to \tilde{\mathbf{Z}} := \mathscr{P}_{j_1(\mathscr{V}_{n+1})}(\hat{\mathbf{Z}}) \tag{2.9 a}$$

yielding an intermediate *holonomic* flow $\tilde{\mathbf{Z}}$ on $j_1(\mathscr{V}_{n+1})$ in terms of the original free flow $\hat{\mathbf{Z}}$ and of the holonomic constraints placed on the system, and the second one

$$\tilde{\mathbf{Z}} \to \mathbf{Z} = \mathscr{P}_{\mathscr{A}}(\tilde{\mathbf{Z}}) \tag{2.9 b}$$

expressing the effective flow $\mathbf{Z} = \chi(\hat{\mathbf{Z}})$ in terms of the holonomic one, and the additional kinetic constraints.

In this connection, it is also worth noticing that, by the very definition of the projections $\mathscr{P}_{j_1(\mathscr{V}_{n+1})}$ and $\mathscr{P}_{\mathscr{A}}$, both steps (2.9 *a*), (2.9 *b*) are separately characterized by a corresponding "minimality criterion" of Gauss' type. We shall return on this point in Section 3.

In terms of reactive forces, the characterization (2.8) is mathematically equivalent to the orthogonality condition $\varphi_x \perp \mathrm{V}_x(\mathscr{A})$, $\forall x \in \mathscr{A}$, *i.e.*

$$(\varphi_x, \mathbf{X}_x) = 0, \qquad \forall \mathbf{X}_x \in \mathrm{V}_x(\mathscr{A}), \quad \forall x \in \mathscr{A}. \tag{2.10}$$

Now, as pointed out in Section 1.2 [equations (1.33), (1.34), (1.37)], the most general vector $\mathbf{X}_x \in V_x(\mathscr{A})$ may be written locally as

$$\mathbf{X}_x = X^k \left(\frac{\partial}{\partial \dot{q}^k}\right)_{|x} = X^k \left(\frac{\partial \xi^\alpha}{\partial \dot{q}^k} \frac{\partial}{\partial \dot{\xi}^\alpha}\right)_{|x}$$

with the components X^k subject to the restrictions

$$X^k \frac{\partial g_\sigma}{\partial \dot{q}^k} = 0, \qquad \sigma = 1, \ldots, n-r, \tag{2.11 a}$$

mathematically equivalent to the representation

$$X^k = X^A \left(\frac{\partial \dot{q}^k}{\partial z^A}\right)_{|x}, \qquad k = 1, \ldots, n \tag{2.11 b}$$

for arbitrary X^A, $A = 1, \ldots, r$.

The content of equation (2.10) is therefore summarized into the relation

$$0 = \left(\varphi, X^k \frac{\partial \xi^\alpha}{\partial \dot{q}^k} \frac{\partial}{\partial \dot{\xi}^\alpha}\right) = X^k \varphi_\alpha \frac{\partial \xi^\alpha}{\partial \dot{q}^k} \tag{2.12}$$

for all choices of X^k consistent with equations (2.11 *a*), (2.11 *b*).

In any frame of reference $\mathscr{I}$, recalling equations (1.24), (2.4), equation (2.12) may be given the more expressive form

$$\sum_{k=1}^{n} \left(\sum_{i=1}^{N} \Phi_i . \frac{\partial \mathbf{x}_i}{\partial q^k}\right) X^k = 0. \tag{2.13}$$

Concerning the representation (2.13) of the principle of minimal constraint, some comments are in order. First of all, from equation (2.13) one can easily recover the traditional principle of *virtual work*, by performing the formal substitution $X^k \to \delta q^k$, and defining the *virtual displacement* $(\delta \mathbf{x}_1, \ldots, \delta \mathbf{x}_N)$ of the system $\mathscr{S}$ on the basis of the equations

$$\delta \mathbf{x}_i = \sum_{k=1}^{n} \frac{\partial \mathbf{x}_i}{\partial q^k} \delta q^k, \qquad i = 1, \ldots, N \tag{2.14}$$

with the δq^k's subject to the restrictions coming from equations (2.11 *a*), (2.11 *b*), namely

$$\sum_{k=1}^{n} \frac{\partial g_\sigma}{\partial \dot{q}^k} \delta q^k = 0, \qquad \sigma = 1, \ldots, n-r. \tag{2.15}$$

With this definition – which yields back the standard one in the holonomic and linear non-holonomic cases, and agrees with Chetaev' one in the more general case ([3] to [6]) – equation (2.13) takes the familiar form

$$\sum_{i=1}^{N} \Phi_i . \delta \mathbf{x}_i = 0 \tag{2.16}$$

for any virtual displacement of the system ("principle of virtual work").

[Following a slightly different procedure, one might equally well introduce the *virtual velocities*

$$\mathbf{u}_i = \sum_{k=1}^{n} \frac{\partial \mathbf{x}_i}{\partial q^k} \mathrm{X}^k = \sum_{k=1}^{n} \frac{\partial \mathbf{v}_i}{\partial \dot{q}^k} \mathrm{X}^k, \quad i=1, \ldots, \mathrm{N} \tag{2.17}$$

and then regard equation (2.13) as requiring the vanishing of the *virtual power* $\sum_i \Phi_i . \mathbf{u}_i$ of the reactive forces, for any choice of the X^k's consistent with equations (2.11 *a*), (2.11 *b*)].

The previous discussion points out the complete equivalence between Gauss' principle of minimal constraint and the principle of virtual work, provided one embodies Chetaev' condition (2.15) into the definition of the virtual displacements.

In this respect, one might therefore choose to *reverse* the logical order, by assuming the principle of virtual work as the starting point for an axiomatic characterization of the class of ideal constraints, and then *proving* Gauss' minimality criterion as an ordinary *theorem.*

As a matter of fact, this is precisely the plan of presentation almost universally followed in the literature (*see* e.g. [2]). Such an approach, however, suffers from the major drawback that—with the exception of the holonomic and linear non-holonomic cases—the concept of virtual displacement as described by equations (2.14), (2.15) has no intrinsic geometrical meaning, and has to be accepted on an *a priori* basis, the ultimate motivation being essentially that, in this way, equation (2.16) reproduces exactly the content of equation (2.13).

In this respect, therefore, the line of approach based on Definition 2.1 is definitely more natural.

An alternative "differential" formulation of the principle of minimal constraint, more directly related to the geometrical set-up developed so far, is obtained by introducing a new class of "virtual" objects, namely the *virtual variations* of the *velocities* of the system at any point $x \in \mathscr{A}$.

These—not to be confused with the virtual velocities (2.17)—are defined according to the equations

$$\delta \mathbf{v}_i := \sum_{k=1}^{n} \frac{\partial \mathbf{x}_i}{\partial q^k} \delta \dot{q}^k = \sum_{k=1}^{n} \frac{\partial \mathbf{v}_i}{\partial \dot{q}^k} \delta \dot{q}^k, \qquad i=1, \ldots, \mathrm{N} \tag{2.18}$$

with the quantities $\delta \dot{q}^k$ subject to the conditions

$$\sum_{k=1}^{n} \frac{\partial g_\sigma}{\partial \dot{q}^k} \delta \dot{q}^k = 0, \qquad \sigma = 1, \ldots, n-r \tag{2.19 a}$$

mathematically equivalent to

$$\delta \dot{q}^k = \frac{\partial \dot{q}^k}{\partial z^A} \delta z^A, \qquad k=1, \ldots, n \tag{2.19 b}$$

for arbitrary δz^A [*see* the analogous equations (2.11 *a*), (2.11 *b*)].

With this definition, equation (2.13) may now be written in the equivalent form

$$\sum_{i=1}^{N} \Phi_i . \delta \mathbf{v}_i = 0 \tag{2.20}$$

for *any* virtual variation $(\delta \mathbf{v}_1, \ldots, \delta \mathbf{v}_N)$.

Of course, giving up the principle of virtual work (2.16) in favour of the (almost identical) formulation (2.20) may look a rather poor improvement, suggested mainly by personal taste.

This is indeed the case as long as one restricts his attention to the class of holonomic and linear non-holonomic systems.

However, as pointed out in the previous discussion, as soon as one tries to improve the applicability of the scheme, by extending it to a wider class of systems, the concept of virtual displacement loses most of its effectiveness, due to the lack of a direct geometrical interpretation for the Chetaev condition (2.15) in the presence of non-linear kinetic constraints.

The quantities (2.18), on the contrary, are always perfectly meaningful, as *differences* – up to higher order terms – between pairs of velocity distributions $(\mathbf{v}_1, \ldots, \mathbf{v}_N)$ and $(\mathbf{v}_1 + \delta \mathbf{v}_1, \ldots, \mathbf{v}_N + \delta \mathbf{v}_N)$, both consistent with the constraints, and issuing from the *same* spatial configuration, at the *same* instant.

In this respect, the concept of virtual variation of the velocities is easily recognized as the natural "infinitesimal" counterpart of the concept of *vertical vector* on $\mathscr{A}$, thus confirming that equation (2.20) is in fact *identical* to the original equation (2.10).

To sum up we conclude that, unlike the principle of virtual work, the analogous principle based on equation (2.20) may systematically be used in place equation (2.7) as an equivalent, fully satisfactory *definition* of the class of ideal constraint, without any loss in generality or difficulty of interpretation.

3. THE EQUATIONS OF MOTION

3.1. Intrinsic formulation

In this Section, we complete our analysis, by discussing the *equations of motion* for a material system $\mathscr{S}$ subject to ideal constraints.

For simplicity, we shall systematically stick to the representation (2.20) of the constitutive characterization, regarding the frame of reference $\mathscr{I}$ as given, and denoting by $\mathbf{F}_i$ and Φ_i respectively the active forces (including the inertial effects, when required), and the reactive ones.

In other words, we shall regard the accelerations of the points of $\mathscr{S}$ relative to $\mathscr{I}$ as determined by the equations

$$\mathbf{F}_i + \Phi_i = m_i \mathbf{a}_i, \qquad i=1, \ldots, \mathrm{N} \tag{3.1}$$

with the Φ_i's satisfying the conditions

$$\sum_i \Phi_i \cdot \delta \mathbf{v}_i = 0 \tag{3.2}$$

for any *virtual variation* of the velocities.

By equations (3.1), (3.2) we obtain the *symbolic equation*

$$\sum_i (\mathbf{F}_i - m_i \mathbf{a}_i) \cdot \delta \mathbf{v}_i = 0 \tag{3.3}$$

Making use of the parametric representation (1.22) for the submanifold $\mathscr{A}$, the latter may be written as:

$$\sum_i (\mathbf{F}_i - m_i \mathbf{a}_i) \frac{\partial \mathbf{v}_i}{\partial z^{\mathrm{A}}} \delta z^{\mathrm{A}} = 0$$

for arbitrary δz^{A}, *i.e.*

$$\sum_i (\mathbf{F}_i - m_i \mathbf{a}_i) \frac{\partial \mathbf{v}_i}{\partial z^{\mathrm{A}}} = 0, \qquad \mathrm{A}=1, \ldots, r \tag{3.4}$$

[The same result could have been obtained more directly by a straightforward comparison of the equations of motion (3.1) with the orthogonalitity condition (2.10)].

Equation (3.4) determine a unique *effective flow* $\mathbf{Z} : \mathscr{A} \to \tau(\mathscr{A})$ for the system $\mathscr{S}$. This conclusion, already implicit in equation (2.8) is most easily verified by inserting the content of equation (3.4) in the result (1.48). Denoting by a^{AB} the inverse of the matrix a_{AB}, we obtain the explicit representation

$$\mathbf{Z} = \frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i} + \mathrm{Z}^{\mathrm{A}} \frac{\partial}{\partial z^{\mathrm{A}}} \tag{3.5a}$$

with

$$\mathrm{Z}^{\mathrm{A}} = a^{\mathrm{AB}} \sum_i \left[\mathbf{F}_i - m_i \left(\frac{\partial \mathbf{v}_i}{\partial t} + \frac{\partial \mathbf{v}_i}{\partial q^k} \dot{q}^k \right) \right] \cdot \frac{\partial \mathbf{v}_i}{\partial z^{\mathrm{B}}} \tag{3.5b}$$

In a more traditional language, the previous arguments show that, if we put together equations (3.4) and the equations

$$\frac{dq^i}{dt} = \dot{q}^i(t, q^1, \ldots, q^n, z^1, \ldots, z^r)$$

coming from the representation (1.22) of the constraints, the resulting system of $n+r$ first order differential equations for the unknowns $q^i = q^i(t)$, $z^A = z^A(t)$ gives rise to a *well posed* Cauchy problem, in agreement with the requirement of *mechanical determinism.*

3.2. Connection with the Lagrangian formalism

An important feature of the dynamical set-up discussed above is that, in the study of the problem of motion, the effect of the holonomic constraints may be singled out, and summarized into a constitutive prescription $\hat{\mathbf{Z}} \to \tilde{\mathbf{Z}}$, assigning to each free dynamical flow $\hat{\mathbf{Z}}$ a corresponding *holonomic flow* $\tilde{\mathbf{Z}}$ on $j_1(\mathscr{V}_{n+1})$.

The subsequent evaluation of the effective flow $\mathbf{Z} : \mathscr{A} \to \tau(\mathscr{A})$ then relies entirely on the knowledge of $\tilde{\mathbf{Z}}$, and involves only the part of the diagram (1.20) referring to the kinetic constraints, namely

$$\begin{array}{ccc} \mathscr{A} & \xrightarrow{h} & j_1(\mathscr{V}_{n+1}) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathscr{V}_{n+1} & = & \mathscr{V}_{n+1} \end{array} \tag{3.6}$$

This aspect, already pointed out in Section 2.2 [equations (2.9 *a*), (2.9 *b*)], is easily recovered, starting once again with the symbolic equation (3.3), but considering separately the effects of the restrictions on the $\delta\mathbf{v}_i$'s coming from the positional constraints and from the kinetic ones.

At the first stage (positional constraints only), the definition (2.18) requires

$$\delta\mathbf{v}_i := \sum_{k=1}^{n} \frac{\partial \mathbf{x}_i}{\partial q^k} \delta\dot{q}^k, \qquad i = 1, \ldots, \mathrm{N} \tag{3.7}$$

for arbitrary $\delta\dot{q}^k$. Together with with equation (3.3), this yields the conditions

$$\sum_i (\mathbf{F}_i - m_i \mathbf{a}_i) . \frac{\partial \mathbf{x}_i}{\partial q^k} = 0, \qquad k = 1, \ldots, n \tag{3.8}$$

written more synthetically as

$$\tau_k = \mathrm{F}_k, \qquad k = 1, \ldots, n \tag{3.9}$$

with the identifications

$$
\begin{aligned}
\mathrm{F}_k(t, q, \dot{q}) &:= \sum_i \mathbf{F}_i \cdot \frac{\partial \mathbf{x}_i}{\partial q^k} \\
\tau_k(t, q, \dot{q}, \ddot{q}) &:= \sum_i m_i \mathbf{a}_i \cdot \frac{\partial \mathbf{x}_i}{\partial q^k}
\end{aligned}
\tag{3.10}
$$

The fact that equations (3.8) do indeed determine a *unique* holonomic flow on $j_1(\mathscr{V}_{n+1})$ is well known, and is, in any case, a straightforward consequence of equations (1.43).

Denoting by a^{hk} the inverse of the matrix a_{hk}, we have the explicit representation

$$
\tilde{\mathbf{Z}} = \frac{\partial}{\partial t} + \dot{q}^h \frac{\partial}{\partial q^h} + \mathrm{Z}^h \frac{\partial}{\partial \dot{q}^h} \tag{3.11 a}
$$

with

$$
\mathrm{Z}^h = a^{hk}\left[\mathrm{F}_k - \left(\frac{\partial}{\partial t} + \dot{q}^i \frac{\partial}{\partial q^i}\right)\frac{\partial \mathrm{T}}{\partial \dot{q}^k} + \frac{\partial \mathrm{T}}{\partial q^k}\right] \tag{3.11 b}
$$

The same conclusion may be stated in more traditional terms, by observing that, in view of the identities

$$
\tau_k = \frac{d}{dt}\frac{\partial \mathrm{T}}{\partial \dot{q}^k} - \frac{\partial \mathrm{T}}{\partial q^k}, \qquad k = 1, \ldots, n \tag{3.12}
$$

valid all over $j_2(\mathscr{V}_{n+1})$ $\left[\text{with } \dfrac{d}{dt} \text{ denoting the formal time derivative over } j_1(\mathscr{V}_{n+1})\right]$, the determination of the integral curves of the flow (3.11 a) is mathematically equivalent to the solution of the Lagrange equations

$$
\frac{d}{dt}\frac{\partial \mathrm{T}}{\partial \dot{q}^k} - \frac{\partial \mathrm{T}}{\partial q^k} = \mathrm{F}_k(t, q, \dot{q}), \qquad k = 1, \ldots, n \tag{3.13}
$$

Of course, all this is nothing but a re-statement of the fact that, as far as the holonomic constraints are concerned, Gauss' criterion of ideality is completely equivalent to the principle of virtual work.

Pursuing the analogy with Lagrangian Dynamics, we shall now embody the holonomic part of the constraints into the formalism, starting at the outset with the configuration manifold $\mathscr{V}_{n+1}$ as the natural environment for the study of the system $\mathscr{S}$, and summarizing into the holonomic flow $\tilde{\mathbf{Z}}$ the global dynamical effect resulting from the composition of the *active* interactions, and of the reactive forces due to the holonomic constraints themselves.

Among other advantages, a useful aspect of this line of approach is that the geometrical set-up is now completely independent of the value of N, thus allowing a straightfoward extension of the formalism to mechanical systems with a "finite number of degrees of freedom", independently of any restriction on the finiteness of the number of meterial points.

In the presence of kinetic constraints, taking the definitions (3.10), as well as the representation (2.18) of the $\delta \mathbf{v}_i$'s into account, the content of the symbolic equation (3.3) may be rephrased as

$$(\mathrm{F}_k - \tau_k)\,\delta \dot{q}^k = 0 \tag{3.14}$$

in which the $\delta \dot{q}^k$'s are now subject to the admissibility requirements expressed by equations (2.19 *a*) or (2.19 *b*), depending on the type of representation adopted.

In particular, in the alternative (2.19 *a*) – corresponding to the parametric representation (1.22) for the submanifold $\mathscr{A} \subseteq j_1(\mathscr{V}_{n+1})$ – the condition (3.14) yields back equations (3.4), now written more simply as

$$(\mathrm{F}_k - \tau_k)\frac{\partial \dot{q}^k}{\partial z^{\mathrm{A}}} = 0, \qquad \mathrm{A} = 1, \ldots, r$$

Making use of the identities (3.12) we conclude that, in the case in study, the *equations of motion* for the system $\mathscr{S}$ – expressed in the form of $n+r$ first order differential equations for the unknowns $q^i = q^i(t)$, $z^{\mathrm{A}} = z^{\mathrm{A}}(t)$ – may be given the explicit representation

$$\begin{aligned} &\left(\frac{d}{dt}\frac{\partial \mathrm{T}}{\partial \dot{q}^k} - \frac{\partial \mathrm{T}}{\partial q^k} - \mathrm{F}_k\right)\frac{\partial \dot{q}^k}{\partial z^{\mathrm{A}}} = 0 \\ &\frac{dq^i}{dt} = \dot{q}^i(t, q^1, \ldots, q^n, z^1, \ldots, z^r) \end{aligned} \tag{3.15}$$

consisting of r linear combination of the Lagrange equations (3.13), completed with the the representation (1.22) of the restrictions placed by the kinetic constraints.

The geometrical meaning of the previous equations, as well as the well-posedness of the associated Cauchy problem, have already been discussed in Section 3.1.

On the other hand, if we stick to the *implicit representation* (1.23) for the submanifold $\mathscr{A}$, the symbolic equation (3.14), and the restriction (2.19 *b*) on the $\delta \dot{q}^k$'s can be put together by means of the method of Lagrange multipliers, thus giving rise to n independent relations

$$\mathrm{F}_k - \tau_k + \sum_\sigma \lambda^\sigma \frac{\partial g_\sigma}{\partial \dot{q}^k} = 0, \qquad k = 1, \ldots, n \tag{3.16}$$

with rank $\| \partial(g_1, \ldots, g_{n-r})/\partial(\dot{q}^1, \ldots, \dot{q}^n) \| = n - r$ (*see section* 1.2).

Exactly as it happened for equation (3.9), equations (3.16) too determine a unique dynamical flow on $j_1(\mathscr{V}_{n+1})$, now expressed as the sum

$$\mathbf{Z}=\tilde{\mathbf{Z}}+a^{hk}\left(\sum_\sigma \lambda^\sigma \frac{\partial g_\sigma}{\partial \dot{q}^k}\right)\frac{\partial}{\partial \dot{q}^h}, \tag{3.17}$$

$\tilde{\mathbf{Z}}$ denoting the holonomic flow (3.11 *a*), (3.11 *b*).

[As a matter of fact, the notation is slightly abusive, the effective flow $\mathbf{Z}$ associtated with $\mathscr{S}$ being in fact identical to the *restriction* of the field (3.17) to the submanifold $\mathscr{A}$.]

Collecting all previous results, and making use of the identities (3.12), we obtain a representation of the equations of motion, in the form of a system of $2n-r$ equations

$$\begin{gathered}\frac{d}{dt}\frac{\partial \mathrm{T}}{\partial \dot{q}^k}-\frac{\partial \mathrm{T}}{\partial q^k}=\mathrm{F}_k+\sum_\sigma \lambda^\sigma \frac{\partial g_\sigma}{\partial \dot{q}^k}\\ g_\sigma(t, q^1, \ldots, q^n, \dot{q}^1, \ldots, \dot{q}^n)=0\end{gathered} \tag{3.18}$$

for the unknowns $q^i=q^i(t)$, $\lambda^\sigma=\lambda^\sigma(t)$.

The scheme resembles very closely the standard Lagrangian one, the n quantities $\sum \lambda^\sigma \partial g_\sigma/\partial \dot{q}^k$ playing the role of the Lagrangian components of the *reactive forces* due to the kinetic constraints.

Concerning the nature of the system (3.18), some comments are in order.

In the first place, the determination of the multipliers λ^σ relies on the requirement that the dynamical flow (3.17) be effectively *tangent* to the submanifold $\mathscr{A}$. In terms of the implicit representation (1.23), this gives rise to the conditions

$$\tilde{\mathbf{Z}}(g_\rho)+\sum_\sigma a^{hk}\lambda^\sigma \frac{\partial g_\sigma}{\partial \dot{q}^k}\frac{\partial g_\rho}{\partial \dot{q}^h}=0, \qquad \rho=1, \ldots, n-r \tag{3.19}$$

everything being evaluated on the hypersurface $g_\sigma=0$

[The same conclusion may be obtained in a more traditional way, as the requirement of algebraic consistency of the system originating from equations (3.18), with each equation $g_\sigma=0$ replaced by the corresponding formal time derivative $\dfrac{dg_\sigma}{dt}=0.$]

In any case, due to the positive-definiteness of the matrix a^{hk}, and to the condition on the rank of the Jacobian $\left(\dfrac{\partial g_\sigma}{\partial \dot{q}^k}\right)$, equations (3.19) can be solved uniquely for the λ^σ's as functions of t, q, $\dot{q}$. Introducing the notation

$$\mathrm{A}_{\sigma\rho}:=a^{hk}\frac{\partial g_\sigma}{\partial \dot{q}^k}\frac{\partial g_\rho}{\partial \dot{q}^h}; \qquad \mathrm{A}^{\sigma\rho}=(\mathrm{A}^{-1})_{\sigma\rho},$$

we have the result

$$\lambda^\sigma = -A^{\sigma\rho}\,\tilde{\mathbf{Z}}(g_\rho).$$

Together with equation (3.17), this yields a description of the effective flow $\mathbf{Z}$ in the presence of kinetic constraints in the explicit form

$$\mathbf{Z} = \tilde{\mathbf{Z}} - a^{ik}\frac{\partial g_\sigma}{\partial \dot{q}^k}A^{\sigma\rho}\,\tilde{\mathbf{Z}}(g_\rho)\frac{\partial}{\partial \dot{q}^i}, \tag{3.20}$$

involving only the holonomic flow $\tilde{\mathbf{Z}}$ and the representation (1.23) of the constraints themselves.

As a check of inner consistency, let us consider in particular the case of a *holonomic* system $\mathscr{S}$, and let us assume that the associated holonomic flow $\tilde{\mathbf{Z}}$ (now identical with the *effective* flow) admits a first integral of the form $f(t, q, \dot{q}) = \text{const.}$ $(\leftrightarrow \tilde{\mathbf{Z}}\ (f) = 0)$.

At the same time, let $\mathscr{S}'$ be a second system having *the same* holonomic flow $\tilde{\mathbf{Z}}$ on $j_1(\mathscr{V}_{n+1})$ and subject to the restriction $f(t, q, \dot{q}) = \text{const.}$, imposed now as an *a priori constraint*, of non-holonomic type.

Then, in view of equation (3.20), the effective flow $\mathbf{Z}$ associated with $\mathscr{S}'$ satisfies

$$\mathbf{Z} = \tilde{\mathbf{Z}} - \left(a^{rs}\frac{\partial f}{\partial \dot{q}^r}\frac{\partial f}{\partial \dot{q}^s}\right)^{-1} a^{ik}\frac{\partial f}{\partial \dot{q}^k}\tilde{\mathbf{Z}}(f)\frac{\partial}{\partial \dot{q}^i} = \tilde{\mathbf{Z}},$$

in agreement with the idea that imposing a first integral as an *a priori* constraint should have no effect on the dynamical behaviour of the system.

Notice that, although entirely obvious, the previous conclusion can be effectively verified only due to the ability of the formalism to deal with arbitrary kinetic constraints, independently of any assumption of linearity.

The same ability stays also at the basis of another qualifying property of the present scheme, namely the fact that, unlike the more traditional one, it handles all types of constraints – including the non-holonomic ones – in a truly *geometrical* way, focussing attention on the submanifold $\mathscr{A} \subseteq j_1(\mathscr{V}_{n+1})$, independently of the specific choice of the *functions* involved in its *representation*.

As a concluding remark, we shall finally examine a special class of non-holonomic systems, namely those systems for which the associated holonomic flow $\tilde{\mathbf{Z}}$ is derivable from a Lagrangian function $\mathscr{L} = T + U$ in the usual way.

Let

$$H(t, q, \dot{q}) = \frac{\partial \mathscr{L}}{\partial \dot{q}^k}\dot{q}^k - \mathscr{L}$$

denote the corresponding Hamiltonian.

Then, taking the kinetic constraints into accounts, and denoting by $\mathbf{Z}$ the resulting effective flow (3.17), the evolution law for H is given by

$$\mathbf{Z}(\mathrm{H}) = \tilde{\mathbf{Z}}(\mathrm{H}) - a^{ik}\frac{\partial g_\sigma}{\partial \dot{q}^k}\mathrm{A}^{\sigma\rho}\tilde{\mathbf{Z}}(g_\rho)\frac{\partial \mathrm{H}}{\partial \dot{q}^i} = \tilde{\mathbf{Z}}(\mathrm{H}) - \mathrm{A}^{\rho\sigma}\tilde{\mathbf{Z}}(g_\rho)\,\dot{q}^k\frac{\partial g_\sigma}{\partial \dot{q}^k} \quad (3.21)$$

use having been made of the identity

$$\frac{\partial \mathrm{H}}{\partial \dot{q}^i} = a_{ik}\,\dot{q}^k.$$

In particular, if all functions g_ρ happen to be *homogeneous* of the same degree p with respect to the $\dot{q}_k$'s – namely $\dot{q}^k\frac{\partial g_\rho}{\partial \dot{q}^k} = p g_\rho$, $p \in \mathbb{R}_+$, $\rho = 1, \ldots, n-r$ – equation (3.21) reduces to

$$\mathbf{Z}(\mathrm{H}) = \tilde{\mathbf{Z}}(\mathrm{H}) - \mathrm{A}^{\rho\sigma}\tilde{\mathbf{Z}}(g_\rho)\,p g_\sigma$$

whence $\mathbf{Z}(\mathrm{H}) = \tilde{\mathbf{Z}}(\mathrm{H})$ on $\mathscr{A}$.

This shows that, for the given class of systems, the evolution of the Hamiltonian H is not affected by the imposition of *homogeneous* kinetic constraints of arbitrary degree.

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