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Elementary acceleration and multisummability I (1)

by

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Lorsqu'il suit le bon *rayon* vers la périphérie, le promeneur peut découvrir...
André Hardellet, « Périphérie »

This paper is extracted from the contents of a forthcoming book by the same authors [MR 3]. Paragraphs 1 to 3 joined to chapter 2 of [MR 2] form a more or less self-contained set. We recall basic definitions about *Borel-summability* (*Borel* [Bo 1], [Bo 2]), and its natural generalization *k-summability* (*Leroy* [Le], *Nevanlinna* [Ne], *Ramis* [Ra 1]). We describe the "*elementary acceleration*" introduced by *Ecalle* [E 4] and different summability operators related to it. If one compares to [E 4] our description is slightly modified in order to fit with our "*geometric*" interpretations [MR 2], [MR 3]. In paragraph 4 as an example of application we give a "*natural*", *simple* and *general*, definition of ***Stokes multipliers*** (2), using a result (3) of *Ramis* [Ra 3] (*cf.* also [Ra 2]), and derive a new proof of a theorem of *Ramis* ([Ra 4], [Ra 5]) about the computation of the *differential Galois group* of a linear differential equation. As a byproduct we get (4)

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(1) Part I of this paper contains paragraphs 1 to 4 (a preliminary manuscript version has been distributed during a Luminy Conference, in september 1989); paragraphs 5 and 6 will appear in *Elementary acceleration and multisummability II.* The second author has exposed part II at a R.C.P. 25 meeting dedicated to *R. Thom* (Strasbourg, 1989). *See* also [LR 3].

(2) Compare with the program of [Me]. Relations between our description of *Stokes phenomenon* and the *cohomological approach* [Ma 3], [Ma 4], [Si], [De 3], [J], [BJL], [BV], will be explained in 4.

(3) The main steps of one proof of this result, using Gevrey asymptotic expansions technics, are detailed in paragraph 5. *Cf.* also [LR 1] for another approach.

(4) *Multisummability* (in its *analytical* formulation or its *"wild-Cauchy"* formulation) is ***not necessary*** in order to obtain this description (which can be derived from *Malgrange-Sibuya* results using algebraic tools from [LR 1]) but it allows an interesting presentation, fundamental for *non-linear* extensions.

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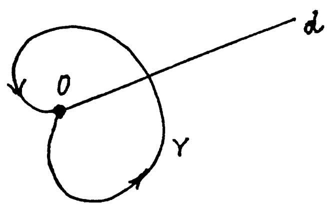
also a description of the ***meromorphic classification*** of ***meromorphic linear differential equations*** on a Riemann surface by the finite dimensional ***linear representations*** of a ***"wild fundamental group"***. (This is a *natural* generalization of the ***Riemann-Hilbert correspondence***.) Paragraph 6 is *very sketchy*; we describe "*infinitesimal neighbourhoods*" of *analytic geometry* (following an idea of *Deligne* [De 4]) and sheaves of "*analytic functions*" (***weakly*** analytic and ***wild*** analytic functions) on these neighbourhoods. Afterwards we are able to give "***geometric interpretations***" ***of elementary acceleration, summability and Stokes phenomena*** ([5]) and to get various generalizations (the sum of a formal power series is now a *wild* analytic function) important for extensions to *non-linear* situations.

1. BOREL SUMMABILITY, BOREL AND LAPLACE TRANSFORMS

We denote by B_d the Borel transform in the direction d.

$$\mathrm{B}_d f(\xi) = \mathbf{f}(\xi) = \frac{1}{2i\pi}\int_\gamma f(x)\,(e^{\xi/x}\,dx/x^2).$$



This formula makes sense with "good" hypothesis on f [MR 2]. We will omit d and write $\mathrm{B}f$ if $\mathrm{B}_d f$ is independent of d (up to analytic continuation).

If $\hat{\varphi}$ is a convergent power *series* ($\hat{\varphi} \in \mathbf{C}\{\xi\}$), we will denote by $\varphi(\xi) = \mathrm{S}\hat{\varphi}(\xi)$ its *sum* on a "small disc" centered at zero.

If f is an analytic function in a "small disc" centered at zero, or, more generally, in a "small sector" bisected by the direction d, we will denote by $\cdot_d f$ its *analytic continuation* (if it exists) along d. In the following, when we write $\cdot_d f$, we will *always* suppose that $\cdot_d f$ is defined on a *sector* bisected by d with infinity radius.

([5]) Partially based upon a ***cohomological version*** of Phragmén-Lindelöf theorem. (Similar and more precise results have been obtained by Malgrange [Ma 5], using Fourier transform; *cf.* also Il'Yashenko's Luminy Conference lectures.) The first cohomological version of Phragmén-Lindelöf theorem is due to Lin [Li].

Operators S and $\cdot_d$ are clearly *injective homomorphisms of differential algebras* (laws being addition and multiplication, and derivation being $d/d\xi$ or $\xi^2\, d/d\xi$) or of "*convolution* [6] *differential algebras*" (laws being addition and convolution, and derivation being multiplication by ξ).

If $\lambda > 0$ and $f(x) = x^\lambda$, we get

$\mathrm{B}_d f(\xi) = \mathrm{B} f(\xi) = \xi^{\lambda-1}/\Gamma(\lambda)$; in particular, for $\lambda = n \in \mathbf{N}^*$, $f(x) = x^n$ $(n \in \mathbf{N})$:

$$\mathrm{B} f(\xi) = \xi^{n-1}/\Gamma(n) = \xi^{n-1}/(n-1)!$$

If we introduce

$\mathbf{B}_d f = \mathrm{B}_d f(\xi)\, d\xi$; then for $f(x) = 1$, we get as a natural generalization:

$$\mathbf{B}_d f = \delta \text{ (Dirac distribution).}$$

We can now define a "*formal Borel transform*" $\hat{\mathrm{B}}$:

For $\hat{f} \in \mathbf{C}[[x]]$, $\hat{f}(x) = \sum_{n \geq 1} a_n x^n$

$$\hat{\mathbf{f}}(\xi) = \hat{\mathrm{B}} \hat{f}(\xi) = \sum_{n \geq 1} a_n \xi^{n-1}/(n-1)!$$

This definition can be extended, replacing $\mathbf{N}$ as a set of indices for the expansion $\hat{f}$ by a more general semi-group (contained in $\mathbf{R}$): $\Lambda^* = \Lambda - \{0\}$,

$$\hat{f}(x) = \sum_{\lambda \in \Lambda^*} a_\lambda x^\lambda, \quad \hat{\mathrm{B}} \hat{f}(\xi) = \sum_{\lambda \in \Delta^*} a_\lambda \xi^{\lambda-1}/\Gamma(\lambda).$$

We will also use later formal expansions indexed by $\lambda \in \alpha + \mathbf{N}$ $(\alpha \in \mathbf{C})$, and the corresponding *generalized asymptotic expansions* (simply named "*asymptotic expansions*" in the following). (Regular parts of such expansions will be called "*polynomials*".)

LEMMA 1. — *We have an isomorphism of differential algebras:*

Differential algebra [7] $\mathbf{C}\{x\}$ *of convergent power series without constant term.*	$\xrightarrow{\mathrm{B}}$	*Convolution differential algebra of entire functions of order* ≤ 1.

Let $\mathbf{f}$ be holomorphic with *exponential growth* of order ≤ 1 in a "small" sector bisected by the direction d (or, more generally, infinitely differentiable on d [8] with an *exponential growth* of order ≤ 1). We can define its

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[6] The convolution law is defined by $\varphi * \psi = \int_0^\xi \varphi(t)\, \psi(\xi - t)\, dt$ in the analytic case and $\hat{\varphi} * \hat{\psi}$ is deduced, in the formal case, from the identities $(\xi^{m-1}/\Gamma(m)) * (\xi^{n-1}/\Gamma(n)) = \xi^{m+n-1}/\Gamma(m+n)$.

[7] The differential is $x^2\, d/dx$.

[8] A function "infinitely differentiable on d" is infinitely differentiable on the right at zero, by convention.

Laplace transform along d:

$$f(x)=\mathrm{L}_d\,\mathbf{f}(x)=\int_d \mathbf{f}(\xi)\,(e^{-\xi/x}\,d\xi)$$

If $f\in\mathbf{C}\{x\}$ (resp. $\mathbf{f}$ entire of order $\leqq 1$):

$$\mathrm{LB}\,f=f \qquad \text{and} \qquad \mathrm{BL}\,\mathbf{f}=\mathbf{f}.$$

With "good hypothesis":

$$\mathrm{L}_d\,\mathrm{B}_d=\mathrm{id} \qquad \textit{and} \qquad \mathrm{B}_d\,\mathrm{L}_d=\mathrm{id}\ [\mathrm{MR}\ 2].$$

Example. — For $\mathbf{f}(\xi)=\xi^{\mu}\,(\mu>-1)$, *we have* $\mathrm{L}\,\mathbf{f}(x)=\Gamma(\mu+1)\,x^{\mu+1}$.

Let $\hat{f}$ be a formal power series, of *Gevrey order* (9) 1 ($\hat{f}\in\mathbf{C}[[x]]_1$). Then

$$\hat{\mathrm{B}}\,\hat{f}=\hat{\mathbf{f}}\in\mathbf{C}\{\xi\}.$$

If $\mathbf{f}=S\,\hat{\mathbf{f}}$ can be analytically extended along some direction d in a fonction $\mathbf{f}={}_{\cdot d}S\,\hat{\mathbf{f}}$ which is analytic with exponential growth of order $\leqq 1$ on a small sector bisected by d, we can define

$$f_d(x)=\mathrm{L}_d\,{}_{\cdot d}S\hat{\mathbf{f}}=\mathrm{L}_d\,{}_{\cdot d}S\,\hat{\mathrm{B}}\,\hat{f}.$$

By definition f_d is the "*Borel sum*" of $\hat{f}$ in the *direction d* ($\hat{f}$ is *Borel-summable* in the *direction d*).

Clearly if $\hat{f}\in\mathbf{C}\{x\}$, $S\,\hat{\mathrm{B}}=\mathrm{B}$ *and* $f_d(x)=S\,\hat{f}(x)$. So $S_d=\mathrm{L}_d\,{}_{\cdot d}S\,\hat{\mathrm{B}}$ *extends* the operator S.

LEMMA 2. — *The operator S_d is an injective morphism of differential algebras:*

Differential algebra (10) *of Borel summable series in the direction d.*	$\xrightarrow{S_d}$	*Differential algebra* (11) *of germs of holomorphic functions on sectors bisected by d.*

So Borel-summability is "*natural*" (*i.e.* "Galois").

Let $R>0$ and d a direction.

Let $D_{R;d}=\{t\in\mathbf{C}/\,|\operatorname{Arg}t-\operatorname{Arg}d|<\frac{\pi}{2}$ *and* $\operatorname{Re}(e^{i\operatorname{Arg}d}/t)>1/R\}$.

We denote by γ_R the boundary of $D_{R;d}$ oriented in the positive sense. We write

$$\mathrm{B}_d\,f(\xi)=\mathbf{f}(\xi)=\frac{1}{2i\pi}\int_{\gamma_R} f(x)\,(e^{\xi/x}\,dx/x^2),\ \text{if } f(x)=o(x^2),$$

and $\mathrm{B}_d\,f(\xi)=1$, if $f(x)=x$. Then we have defined $\mathrm{B}_d\,f$ for $f(x)=o(x)$.

(9) For definitions and notations *see* [MR 1].

(10) The differential is $x^2\,d/dx$.

(11) Idem.

Later we will need the "well known"

LEMMA 3. – *The map*

Convolution differential algebra of functions infinitely differentiable on d with an exponential growth of order ≤ 1 *at infinity.*	$\overset{\mathrm{L}}{\rightarrow}$	*Differential algebra* ([12]) *of functions analytic on open discs* $D_{R;\,d}$ *(*$R>0$ *arbitrary), with an asymptotic expansion* ([13]) (without *constant term) at zero.*

is an ***isomorphism*** *of differential algrebras.*

Let f be an analytic function on the open Borel-disc $D_{R;\,d}$ with an asymptotic expansion (without constant term) at zero. Then, using Fubini's theorem and the formula

$$\mathrm{L}(e^{-\xi})(x)=\frac{x}{x+1}, \text{ we get easily } \mathrm{LB}\,f=f \qquad (\textit{see } [\text{Bo } 2]).$$

Let $\mathbf{f}$ be infinitely differentiable on d with an exponential growth of order ≤ 1 at infinity. If $\mathrm{L}\,\mathbf{f}=0$, then $\mathbf{f}=0$ (using inversion of *Fourier transform*).

Now, from $\mathrm{L}(\mathrm{BL}\,\mathbf{f})=\mathrm{LB}(\mathrm{L}\,\mathbf{f})=\mathrm{L}\,\mathbf{f}$, we deduce $\mathrm{BL}\,\mathbf{f}=\mathbf{f}$. That ends the proof of *lemma* 3.

2. k-SUMMABILITY, k-BOREL AND k-LAPLACE TRANSFORMS

Using B_d, L_d, $\cdot_d$, S and *ramification operators* $\rho_k\,(k>0)$ it is easy to build *new* operators $\mathrm{B}_{k;\,d}$ and $\mathrm{L}_{k;\,d}$ (and the formal operator $\hat{\mathrm{B}}_k$ corresponding to $\mathrm{B}_{k;\,d}$):

We will use the notation ($k>0$): $\rho_k f(x)=f(x^{1/k})$ (x is varying onto the Riemann surface of Logarithm); $\rho_{1/k}=\rho_k^{-1}$.

If d^k corresponds to d by the ramification ρ_k, we will set:

$$\mathrm{B}_{k;\,d}=\rho_k^{-1}\,\mathrm{B}_{d^k}\,\rho_k$$

and

$$\mathrm{L}_{k;\,d}=\rho_k^{-1}\,\mathrm{L}_{d^k}\,\rho_k.$$

We have (in general we will simplify our notations: $\mathbf{f}_k=\mathbf{f}$, $\xi_k=\xi$):

$$\mathrm{B}_{k;\,d}\,f(\xi_k)=\mathbf{f}_k(\xi_k)=\frac{1}{2i\pi}\int_{\gamma_k} f(x)\,(k e^{\xi_k^k/x^k}\,dx/x^{k+1})$$

$$\mathrm{L}_{k;\,d}\,\mathbf{f}_k(x)=f(x)=\int_l \mathbf{f}_k(\xi_k)\,(k e^{-\xi_k^k/x^k}\,\xi_k^{k-1}\,d\xi_k).$$

([12]) Idem.

([13]) Uniform on closed subdiscs $\bar{\mathrm{D}}_{\mathrm{R}';\,d}\,(\mathrm{R}'<\mathrm{R})$.

The operator $L_{k;d}$ can be applied to functions holomorphic with an exponential growth of order $\leqq k$ on a small sector bisected by d and an asymptotic expansion at the origin (indexed by the set $1-k+\mathbf{N}$). These functions form a *k-convolution* differential algebra:

The k-convolution $*_k$ is defined by:

$$\mathbf{f}_k *_k \mathbf{g}_k = \rho_k^{-1}((\rho_k \mathbf{f}_k) * (\rho_k \mathbf{g}_k)).$$

Operations are: $+$, $*_k$, and derivation $\partial_k = \mathrm{B}_k(x^2 d/dx)\mathrm{L}_k$ (∂_k will be explicitely described later; ∂_1 is multiplication by $-\xi$).

LEMMA 4. — *We have an isomorphism of differential algebras:*

$$\begin{array}{ccc} \textit{Differential algebra } (\mathbf{C}\{x\},\ x^2 d/dx) & \xrightarrow{\mathrm{B}_k} & \textit{k-convolution differential algebra:} \\ \textit{of convergent power series} & & \xi^{1-k}\{\textit{entire functions} \\ \textit{vanishing at } 0. & & \textit{of order } \leqq k\}. \end{array}$$

We will use the following notations:

$\mathbf{C}[[x]]_{1/k}$ is the differential algebra of formal power series of Gevrey *order* $1/k$ (Gevrey *level k*) ([14]);

$\mathbf{C}\{x\}_{1/k;d}$ is the differential algebra of formal power series k-summable in the direction d (definition is given just below);

$\mathbf{C}\{x\}_{1/k}$ is the differential algebra of k-summable series (that is of formal power series k-summable in *every direction but* perhaps a *finite* number).

Let $\hat{f} \in \mathbf{C}[[x]]_{1/k}$. Then $\hat{\mathbf{f}}_k = \hat{\mathrm{B}}_k \hat{f} \in \mathbf{C}\{\xi_k\}$. If $\mathbf{f}_k = S\hat{\mathbf{f}}_k$ can be analytically extended along some direction d in a function ${}_{\cdot d}\mathbf{f}_k = {}_{\cdot d}S\hat{\mathbf{f}}_k$ analytic with exponential growth of order $\leqq k$ on a small sector bisected by d, we can set:

$$f_{k;d}(x) = \mathrm{L}_{k;d} \cdot_d S\hat{\mathbf{f}}_k = \mathrm{L}_{k;d} \cdot_d S\hat{\mathrm{B}}_k \hat{f}.$$

By definition $f_{k;d}$ is the "k-sum" of $\hat{f}$ in the direction d ($\hat{f}$ is k-summable in the direction d). It is clear that $S_{k;d} = \mathrm{L}_{k;d} \cdot_d S\hat{\mathrm{B}}_k$ extends the operator S (defined for $\hat{f} \in \mathbf{C}\{x\}$).

LEMMA 5. — *The operator $S_{k;d}$ is an injective morphism of differential algebras:*

$$\begin{array}{ccc} \textit{Differential algebra of} & \xrightarrow{S_{k;d}} & \textit{Differential algebra of germs of} \\ \textit{k-summable series} & & \textit{holomorphic functions} \\ \textit{in the direction d.} & & \textit{on sectors bisected by d.} \end{array}$$

So k-summability is "*natural*" (*i.e.* "Galois").

We have built a *one parameter family* ($k \in \mathbf{R}$, $k>0$) of *summation processes*. We will now compare these processes for different values of the

([14]) Notations of [MR 2]. (Be careful, these notations differ from those of [Ra 1], [Ra 2], [Ra 7].)

parameter $k>0$: if a formal power series is summable by *two processes* then *the two sums are equal*, but this is quite exceptional because k_1-summability and k_2-summability for $k_1 \neq k_2$ requires in some sense *very different* conditions. More precisely:

PROPOSITION 1. – *Let $k, k'>0$ with $k<k'$ and $\hat{f} \in \mathbf{C}[[x]]$ k-summable and k'-summable in the direction d. Then:*

(i) $S_{k;d}\hat{f} = S_{k';d}\hat{f}$;

(ii) *The power series $\hat{f}$ is k'-summable in every direction d' with* $\arg d' \in]\arg d - \pi/2k + \pi/2k', \arg d + \pi/2k - \pi/2k'[$ *and the sums $S_{k';d'}\hat{f}$ glue together by analytic continuation;*

(iii) *The power series $\hat{f}$ is k''-summable in every direction d'' with* $\arg d'' \in]\arg d - \pi/2k + \pi/2k'', \arg d + \pi/2k - \pi/2k''[$, *for $k<k''<k'$.*

Moreover $S_{k'';d''}\hat{f} = S_{k';d''}\hat{f}$.

PROPOSITION 2. – *Let $k, k'>0$ with $k<k'$ and $\hat{f} \in \mathbf{C}[[x]]_{1/k'}$. If $\hat{f}$ is k-summable, then $\hat{f}$ is a convergent power series*

$$(\text{i.e. } \mathbf{C}[[x]]_{1/k'} \cap \mathbf{C}\{x\}_{1/k} = \mathbf{C}\{x\}).$$

This result, announced in [Ra 2], is proved in [Ra 5] (for a particular case and example, *see* [RS 1]).

From such a result it is easy to understand that summation operators $S_{k;d}$ (with d and $k>0$), if very useful, are *not sufficient* if one wants to deal with quite simple situations as "*non* generic" linear algebraic differential equations:

A formal power series solution of a "*generic*" linear algebraic equation is *k-summable* for some $k>0$ [Ra 2], [MR 2], [MR 3]. Let now $\hat{f}_1, \hat{f}_2 \in \mathbf{C}[[x]]$ be *divergent* power series, where $\hat{f}_1$ *is* k_1*-summable* and $\hat{f}_2$ k_2*-summable* $(k_1 \neq k_2)$. Then $\hat{f} = f_1 + f_2$ is *divergent* (proposition 2) and there exists ***no*** $k>0$ such that $\hat{f}$ is *k-summable* (proposition 1 and 2). If we suppose moreover that there exists $D_1, D_2 \in \mathbf{C}[x][d/dx]$ *such that* $D_1\hat{f}_1 = 0$, $D_2\hat{f}_2 = 0$, then there exists $D \in \mathbf{C}[x][d/dx]$ such that $D\hat{f} = 0$ (for an explicit exemple *see* [RS 1]).

Any formal power series solution of ***any*** analytic linear differential equation ***can be summed*** using a "*blend*" of a *finite set* of processes of k-summability (*cf.* 4, 6, infra). The corresponding values for k are *computable* using a Newton polygon [Ra 1], [Ra 7]. We get in this way a *process of summability* (consisting in replacing each *formal* power series in the blend by its *k-sum choosing the "good" k*). This method gives an *injective morphism of differential algebras* but is *purely theoretical* (*i. e. not* explicit). This motivates the introduction of a more general tool, that is *multisummability*. Multisummability [due to *Ecalle*] ([15]) is *effective* and a "blend" of

([15]) It is a particular case of his concept of "*accelerosummability*".

k-summable power series is multisummable. Here we have slightly modified *Ecalle's* presentation in order to be as near as possible of our *geometric description* of multisummability ([16]) (*cf.* 6, infra).

3. ACCELERATION AND MULTISUMMABILITY

We will introduce here only a *very elementary acceleration* (for a more general theory cf. *Ecalle* [E 4]). It is sufficient for our applications (and easy to generalize along the same lines [MR 3]). Following *Ecalle, accelerating operators* are *first* defined using *Laplace, Borel and ramification* operators; afterwards we get an equivalent definition using an *integral formula*. The *important fact* is that this integral formula lead to a ***natural extension of the domain*** of the corresponding operator.

Let $\alpha \geqq 1$. Formally the operator $\boldsymbol{\rho}_\alpha$ of α-*acceleration* is the *conjugate* of the *ramification* operator ρ_α by the *Laplace transform:*

$$\boldsymbol{\rho}_\alpha = \mathrm{L}^{-1}\,\rho_\alpha\,\mathrm{L} = \mathrm{B}\,\rho_\alpha\,\mathrm{L}.$$

The operator ρ_α is an *isomorphism of differential algebras*, therefore the operator $\boldsymbol{\rho}_\alpha$ is *an isomorphism of convolution differential algebras*. More precisely:

$\boldsymbol{\rho}_\alpha = \mathrm{L}_{d^\alpha}^{-1}\,\rho_\alpha\,\mathrm{L}_d$, and:

Convolution differential algebra of analytic functions on sectors bisected by d with an exponential growth of order $\leqq 1$ at infinity and an asymptotic expansion at zero.	$\xrightarrow{\boldsymbol{\rho}_\alpha}$	Convolution differential algebra of analytic functions on sectors with opening $> \pi(\alpha - 1)$, bisected by d^α with an exponential growth of order $\leqq 1$ at infinity and an "asymptotic expansion" at zero ([17]).

is an *isomorphism*.

As ρ_α the operator $\boldsymbol{\rho}_\alpha$ moves the direction d. It is useful to introduce operators of "*normalized acceleration*" not moving d:

$$\mathrm{A}_\alpha = \rho_{1/\alpha}\,\boldsymbol{\rho}_\alpha = \rho_\alpha^{-1}\,\mathrm{L}^{-1}\,\rho_\alpha\,\mathrm{L} = (\mathrm{L}\,\rho_\alpha)^{-1}\,\rho_\alpha\,\mathrm{L} = \mathrm{B}_\alpha\,\mathrm{L}.$$

Then A_α is the *commutator* of $\mathrm{B} = \mathrm{L}^{-1}$ and $\rho_{1/\alpha} = \rho_\alpha^{-1}$.

([16]) As *analytic continuation* along rays starting from the origin *across the "analytic halo"*.

([17]) This asymptotic expansion is in powers of $x^{1/\alpha}$.

The operator $\mathbf{A}_\alpha$ gives an *isomorphism* of "convolution" *differential algebras:*

Convolution differential algebra of analytic functions, on sectors bisected by d with an exponential growth of order $\leqq 1$ at infinity and an asymptotic expansion at zero.	$\xrightarrow{\mathbf{A}_\alpha}$	α-convolution differential algebra of analytic functions, on sectors, with opening $>\pi/\beta=\pi\dfrac{\alpha-1}{\alpha}$, bisected by d with an exponential growth of order $\leqq \alpha$ at infinity and an "asymptotic expansion" at zero.

For the proof of this statement see below the more general case of $\mathbf{A}_{k',k}$.
If necessary we will denote more precisely the operator $\mathbf{A}_\alpha$ by $\mathbf{A}_{\alpha;d}$.
The operator $\mathbf{A}_\alpha$ is clearly related to *level* 1. We need now to introduce similar operators for *arbitrary* levels $k>0$. Let $k'>k$, $\alpha=k'/k$, we will denote:

$$\mathbf{A}_{k',k}=\rho_{1/k}\,\mathbf{A}_\alpha\,\rho_k=(\rho_k)^{-1}\,(\rho_{k'/k})^{-1}\,\mathrm{L}^{-1}\,\rho_{k'/k}\,\mathrm{L}\,\rho_k$$
$$\mathbf{A}_{k',k}=(\rho_{k'})^{-1}\,\mathrm{L}^{-1}\,\rho_{k'/k}\,\mathrm{L}\,\rho_k=(\rho_{k'})^{-1}\,\mathrm{L}^{-1}\,\rho_{k'}\,(\rho_k)^{-1}\,\mathrm{L}\,\rho_k.$$

The operator $\mathbf{A}_{k',k}$ gives an *isomorphism* of "convolution" *differential algebras*

k-convolution differential algebra of analytic functions on sectors bisected by d with an exponential growth of order $\leqq k$ at infinity and an "asymptotic expansion" at zero	$\xrightarrow{\mathbf{A}_{k',k}}$	k'-convolution differential algebra of analytic functions, on sectors with opening $>\pi/\kappa=\pi\dfrac{k'-k}{kk'}$ bisected by d with an exponential growth of order $\leqq k'$ at infinity and an "asymptotic expansion" at zero.

If necessary we will denote more precisely the operator $\mathbf{A}_{k',k}$ by $\mathbf{A}_{k',k;d}$.
We have:

$$\mathbf{A}_{k',k}(f *_k g)=\rho_{k'}^{-1}\,\mathrm{L}^{-1}\,\rho_{k'/k}\,\mathrm{L}\,\rho_k\,\rho_k^{-1}\,((\rho_k f)*(\rho_k g))$$
$$\mathbf{A}_{k',k}(f *_k g)=\rho_{k'}^{-1}\,\mathrm{L}^{-1}\,\rho_{k'/k}\,\mathrm{L}\,((\rho_k f)*(\rho_k g))$$
$$\mathbf{A}_{k',k}(f *_k g)=\rho_{k'}^{-1}\,\mathrm{L}^{-1}\,\rho_{k'/k}\,((\mathrm{L}\,\rho_k f)\,(\mathrm{L}\,\rho_k g))$$
$$\mathbf{A}_{k',k}(f *_k g)=\rho_{k'}^{-1}\,(\mathrm{L}^{-1}\,(\rho_{k'/k}\,\mathrm{L}\,\rho_k f))*(\mathrm{L}^{-1}\,(\rho_{k'/k}\,\mathrm{L}\,\rho_k g))$$
$$\mathbf{A}_{k',k}(f *_k g)=\mathbf{A}_{k',k} f *_{k'} \mathbf{A}_{k',k}\, g.$$

In order to prove that $\mathbf{A}_{k',k}$ is an *isomorphism* it suffices to remark that L_d is an *isomorphism* between the convolution differential algebra of analytic functions on sectors bissected by d with an exponential growth of order $\leqq 1$ at infinity and an asymptotic expansion at zero, and the differential algebra of functions analytic on sectors with opening $>\pi$

bisected by d, and with an asymptotic expansion (without constant term) at zero.

It is natural to set:

$$\mathbf{A}_{\infty, k}=\mathrm{L}_k$$
$$\mathbf{A}_{\infty, 1}=\mathrm{L}.$$

We have

$$\mathbf{A}_{k, 1}=\mathbf{A}_k \qquad \text{and} \qquad \mathbf{A}_{k, k}=\mathrm{id}.$$

Let $k''>k'>k>0$. When the formula makes sense we get:

$$\mathbf{A}_{k'', k'}\, \mathbf{A}_{k', k}=\mathbf{A}_{k'', k}.$$

We will use later the above formula to *extend* the operator $\mathbf{A}_{k'', k}$:

The first step is to extend the domain of the operator $\mathbf{A}_{k', k}$ and the second to replace $\mathbf{A}_{k', k}$ in the formula by $\cdot_d \mathbf{A}_{k', k; d}$: $\mathbf{A}_{k'', k'; d} \cdot_d \mathbf{A}_{k', k; d}=\mathbf{A}_{k'', k', k; d}$ (*definition*).

More generally, let $k_1>k_2>\ldots>k_r>0$. When the formula makes sense, we get:

$$\mathbf{A}_{k_1, k_2}\, \mathbf{A}_{k_2, k_3} \ldots \mathbf{A}_{k_{r-1}, k_r}=\mathbf{A}_{k_1, k_r}.$$

With this formula we will later *extend* the operator $\mathbf{A}_{k_1, k_r}$, using extensions of the operators

$$\mathbf{A}_{k_i, k_{i+1}; d} \quad (i=1, \ldots, r-1)$$

and

$$\mathbf{A}_{k_1, k_2; d} \cdot_d \mathbf{A}_{k_2, k_3; d} \ldots \cdot_d \mathbf{A}_{k_{r-1}, k, d}=\mathbf{A}_{k_1, k_2, \ldots, k_r; d} \quad \text{(definition)}.$$

Let $k'>k$, when the formula make sense we get:

$$\mathrm{L}_{k'} \mathbf{A}_{k', k}=\mathrm{L}_k \quad (\text{or } \mathbf{A}_{\infty, k'} \mathbf{A}_{k', k}=\mathbf{A}_{\infty, k}).$$

So we can *extend* the operator L_k using $\mathrm{L}_{k'} \cdot_d \mathbf{A}_{k', k}$. Then

$$\mathrm{id}=\mathrm{L}_k\, \mathrm{B}_k=\mathrm{L}_{k'}\, \mathbf{A}_{k', k}\, \mathrm{B}_k$$
$$S=\mathrm{L}_{k'}\, \mathbf{A}_{k', k}\, S\, \hat{\mathrm{B}}_k, \text{ and, more generally, for } k_1>k_2>\ldots>k_r:$$
$$S=\mathrm{L}_{k_1}\, \mathbf{A}_{k_1, k_2} \ldots \mathbf{A}_{k_{r-1}, k_r}\, S\, \hat{\mathrm{B}}_{k_r}.$$

Then it is natural to *extend* the domain $\mathbf{C}\{x\}$ of the *summation* operator S using the new summation operator (along the direction d):

$$S_{k_1, k_2, \ldots, k_r; d}=\mathrm{L}_{k_1; d} \cdot_d \mathbf{A}_{k_1, k_2; d} \ldots \cdot_d \mathbf{A}_{k_{r-1}, k_r; d} \cdot_d S\, \hat{\mathrm{B}}_{k_r}$$

(In this formula we have written $\mathbf{A}_{k_i, k_{i+1}; d}$ for an *extension* of $\mathbf{A}_{k_i, k_{i+1}; d}$ that we will define precisely below.)

The domain of definition of the operator $\mathbf{A}_{k', k; d}$ is the set

$\{$analytic functions on sectors bisected by d with an exponential growth of order $\leqq k$ at infinity and an asymptotic expansion at zero$\}$.

We will now see that there exists a *natural extension* of this operator to the *larger* domain

$$\left\{\begin{array}{l}\text{analytic functions on sectors bisected by } d \text{ with an exponential growth}\\ \quad\text{of order } \leqq \kappa = \dfrac{kk'}{k'-k} \text{ at infinity and an "asymptotic expansion" at zero}\end{array}\right\};$$

$$1/k' + 1/\kappa = 1/k; \qquad \kappa = k\frac{k'}{k'-k} > k.$$

It is clearly sufficient to understand how to *extend* the operator $\mathbf{A}_{\alpha;\,d}\,(\alpha > 1)$ defined on the domain

$$\left\{\begin{array}{l}\text{analytic functions on sectors bisected by } d \text{ with an exponential growth}\\ \quad\text{of order } \leqq 1 \text{ at infinity and an "asymptotic expansion" at zero}\end{array}\right\}$$

to the *larger* domain

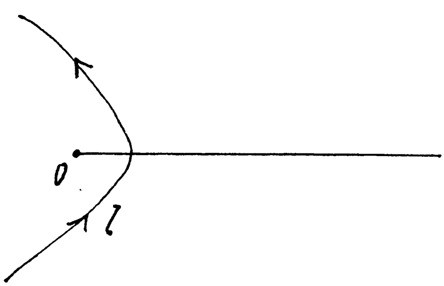
$$\left\{\begin{array}{l}\text{analytic functions on sectors bisected by } d \text{ with an exponential growth}\\ \quad\text{of order } \leqq \beta = \dfrac{\alpha}{\alpha-1} \text{ at infinity and an "asymptotic expansion" at zero}\end{array}\right\},$$

$1/\alpha + 1/\beta = 1$.

This is done using an *integral formula* for $\mathbf{A}_{\alpha;\,d}$ discovered by *Ecalle* [E 4]:

We introduce a family of "*special functions*" $C_\alpha\,(\alpha > 1)$, the "*accelerating functions*":

$$C_\alpha(t) = \frac{1}{2i\pi}\int_l e^{u - tu^{1/\alpha}}\,du; \text{ the path } l \text{ being a Hankel contour:}$$



It is easy to see that C_α is an *entire function* and to compute its analytic expansion at the origin:

$$C_1 = \frac{1}{\pi}\sum_{n\geqq 0}\sin\frac{n\pi}{\beta}\,\frac{\Gamma(1+n/\alpha)}{\Gamma(1+n)}\,t^n;$$

with $1/\alpha + 1/\beta = 1$.

Example:

$$\alpha=\beta=2; \quad \text{then } C_2(t)=\frac{1}{2\sqrt{\pi}}te^{-t^2/4}.$$

Functions C_α are *resurgent* at ∞ [E 4], [MA], [C]. If $\alpha \in \mathbf{Q}$ these functions are related to *Mejer G-functions* and *solutions of linear differential equations* (*cf.* below "*Formulae about accelerating functions*").

LEMMA 6 ([E 4], [MR 3] (18))

Let $\beta>1$, *and* $\alpha=\frac{\beta}{\beta-1}$. *Let* $0<\theta<\frac{\pi}{\beta}$.

Let $V_\theta=\left\{t\in\mathbf{C}/|\operatorname{Arg} t|<\frac{\theta}{2}\right\}$. *Then (on* V_θ*):*

$$|C_\alpha(t)|\leqq\frac{K_\alpha}{\sqrt{\cos\beta\theta}}|t^{\beta/2}e^{-(t/c_\alpha)^\beta}|; \qquad \textit{with} \quad K_\alpha>0 \quad \textit{and} \quad c_\alpha=\beta(\alpha-1)^{1/\alpha}.$$

PROPOSITION 3. – *Let* $\alpha>1$. *Let* $A_{\alpha;d}=(L_{d^\alpha}\rho_\alpha)^{-1}\rho_\alpha L_d$ *and* φ *be an analytic function on a sector bisected by d and with an asymptotic expansion at zero (or, more generally an infinitely differentiable function on d with an exponential growth of order* $\leqq 1$ *at infinity). Then*

$$A_{\alpha;d}\varphi(\zeta)=\zeta^{-\alpha}\int_d C_\alpha(\xi/\zeta)\varphi(\xi)\,d\xi.$$

DEFINITION 1. – *Let* $\alpha>1$ *and* φ *an infinitely differentiable function on* (19) *a direction d. If the integral* $\int_d C_\alpha(\xi/\zeta)\varphi(\xi)\,d\xi$ *exists, we will say that* φ *is* α*-accelerable in the direction d.*

The operator $A_{\alpha;d}=(L_{d^\alpha}\rho_\alpha)^{-1}\rho_\alpha L_d$ is defined on the domain
{analytic functions on sectors bisected by d with an exponential growth of order $\leqq 1$ at infinity and an asymptotic expansion at the origin},
but we have $\beta>1$ and the operator $\varphi\to\int_d C_\alpha(\zeta/\xi)\varphi(\xi)\,d\xi$ is defined on the ***larger*** domain
{analytic functions on sectors bisected by d with an exponential growth

(18) More precisely, using saddlepoint method, it is possible to get an *asymptotic expansion* for the function C_α on the sector V_θ (and even in $|\operatorname{Arg} t|<\pi/2$), *cf.* [HL], p. 45, [Bak], p. 84, [MR 3].

(19) The function φ is defined on $d-\{0\}$. We do not suppose it differentiable at the origin.

of order $\leqq \beta$ at infinity and an asymptotic expansion at the origin }.
(More generally a function infinitely differentiable on d with an exponential growth of order $\leqq \beta$ at infinity is *α-accelerable.*)

Then we get from *proposition* 3 the searched *extension* for the operator $\mathbf{A}_{\alpha;\, d}$. (In the following we will still denote this extension by $\mathbf{A}_{\alpha;\, d}$.)

Now using

$\mathbf{A}_{k',\, k;\, d}\, \psi(\zeta) = \zeta^{-k'} \int_d \psi(\xi)\, C_\alpha(\xi^k/\zeta^k)\, k\, \xi^{k-1}\, d\xi$ where ψ is analytic on a sector bisected by d, with an exponential growth of order $\leqq 1$ at infinity, it is possible to extend the operator $\mathbf{A}_{k',\, k;\, d}$ to the *larger* domain

$$\Big\{\text{analytic functions on sectors bisected by } d \text{ with an exponential growth of order } \leqq \kappa = \frac{kk'}{k'-k} \text{ at infinity and an asymptotic expansion at the origin}\Big\}.$$

We can definie now the notion of $(k_1, k_2, \ldots, k_r)$-*summability* in a direction d and the corresponding summability operator $\mathrm{S}_{k_1,\, k_2,\, \ldots,\, k_r;\, d}$. (In the following definition operators $\mathbf{A}_{k_i,\, k_{i+1};\, d}$ must be interpreted in the *extended* sense, that is as *integral operators.*)

DEFINITION 2. – *Let* $k_1 > k_2 > \ldots > k_r > 0$ *and a direction d. A formal power series* $\hat{f} \in \mathbf{C}[[x]]$ *is called* $(k_1, k_2, \ldots, k_r)$-*summable in the direction d if the following conditions are satisfied:*

(0) $\hat{f} \in \mathbf{C}[[x]]_{1/k_r}$.

(1) $S\hat{\mathbf{B}}_{k_r}\hat{f}$ *can be* ***analytically extended*** *along d to a function* $\bullet_d S\hat{\mathbf{B}}_{k_r}\hat{f}$ *analytic on a sector bisected by d with an* ***exponential growth of order*** $\leqq \dfrac{k_{r-1}k_r}{k_{r-1}-k_r}$.

(2) $\mathbf{A}_{k_{r-1},\, k_r;\, d} \bullet_d S\hat{\mathbf{B}}_{k_r}\hat{f}$ *can* ***analytically extended*** *along d to a function* $\bullet_d \mathbf{A}_{k_{r-1},\, k_r;\, d} \bullet_d S\hat{\mathbf{B}}_{k_r}\hat{f}$ *with an* ***exponential growth of order*** $\leqq \dfrac{k_{r-2}k_{r-1}}{k_{r-2}-k_{r-1}}$.

. . . .

(i) $\mathbf{A}_{k_{r-i+1},\, k_{r-i+2};\, d} \ldots \bullet_d \mathbf{A}_{k_{r-1},\, k_r;\, d} \bullet_d S\hat{\mathbf{B}}_{k_r}\hat{f}$ *can be* ***analytically extended*** *along d to a function*
$\bullet_d \mathbf{A}_{k_{r-i+1},\, k_{r-i+2};\, d} \ldots \bullet_d \mathbf{A}_{k_{r-1},\, k_r;\, d} \bullet_d S\hat{\mathbf{B}}_{k_r}\hat{f}$ *with an* ***exponential growth of order*** $\leqq \dfrac{k_{r-i}k_{r-i+1}}{k_{r-i}-k_{r-i+1}}$.

. . . .

(r) $\mathbf{A}_{k_1,\, k_2;\, d} \ldots \bullet_d \mathbf{A}_{k_{r-1},\, k_r;\, d} \bullet_d S\hat{\mathbf{B}}_{k_r}\hat{f}$ *can be* ***analytically extended*** *along d to a function*
$\bullet_d \mathbf{A}_{k_1,\, k_2;\, d} \ldots \bullet_d \mathbf{A}_{k_{r-1},\, k_r;\, d} \bullet_d S\hat{\mathbf{B}}_{k_r}\hat{f}$ *with an* ***exponential growth of order*** $\leqq k_1$.

If a formal power serie $\hat{f} \in \mathbf{C}[[x]]$ is $(k_1, k_2, \ldots, k)$-*summable* in the direction d, then:

$\mathbf{L}_{k_1; d} \cdot_d \mathbf{A}_{k_1, k_2; d} \cdots \cdot_d \mathbf{A}_{k_{r-1}, k_r; d} \cdot_d S \hat{\mathbf{B}}_{k_r} \hat{f}$ is defined and analytic in a sector bisected by d.

We will set

$$S_{k_1, k_2, \ldots, k_r; d} = \mathbf{L}_{k_1; d} \cdot_d \mathbf{A}_{k_1, k_2; d} \cdots \cdot_d \mathbf{A}_{k_{r-1}, k_r; d} \cdot_d S \hat{\mathbf{B}}_{k_r};$$

$S_{k_1, k_2, \ldots, k_r; d} \hat{f}$ is the $(k_1, k_2, \ldots, k_r)$-*sum* of $\hat{f}$ in the direction d.

If $\hat{f} \in \mathbf{C}[[x]]$ is $(k_1, k_2, \ldots, k_r)$-*summable* in the direction d, we will write it

$$f \in \mathbf{C}\{x\}_{1/k_1, 1/k_2, \ldots, 1/k_r; d}.$$

If $\hat{f} \in \mathbf{C}[[x]]$ is $(k_1, k_2, \ldots, k_r)$-*summable* in all directions, but perhaps a *finite* number, we will write it

$\hat{f} \in \mathbf{C}\{x\}_{1/k_1, 1/k_2, \ldots, 1/k_r}$, and say that $\hat{f}$ is $(k_1, k_2, \ldots, k_r)$-*summable.*

LEMMA 7. — *Let $k_1, k_2, \ldots, k_r > 0$ and let d be a given direction. Then*

(i) $\mathbf{C}\{x\}_{1/k_1, 1/k_2, \ldots, 1/k_r; d}$ and $\mathbf{C}\{x\}_{1/k_1, 1/k_2, \ldots, 1/k_r}$ are differential subalgebras of $\mathbf{C}[[x]]$;

(ii) The subalgebra of $\mathbf{C}[[x]]$ generated by the differential algebras $\mathbf{C}\{x\}_{1/k_1; d}$, $\mathbf{C}\{x\}_{1/k_2; d}, \ldots, \mathbf{C}\{x\}_{1/k_r; d}$, is a differential subalgebra of $\mathbf{C}\{x\}_{1/k_1, 1/k_2, \ldots, 1/k_r; d}$. Moreover if

$f = \sum_{i \in \mathrm{I}} f_{i,1} \ldots f_{i,r}$, with I finite and $f_{ij} \in \mathbf{C}[[x]]_{1/k_j; d}$ ($i \in \mathrm{I}$, and $j = 1, \ldots, r$), then

$S_{k_1, k_2, \ldots, k_r; d} f = \sum_{i \in \mathrm{I}} S_{k_1; d} f_{i,1} \ldots S_{k_r; d} f_{i,r}$; in particular the analytic function $\sum_{i \in \mathrm{I}} S_{k_1; d} f_{i,1} \ldots S_{k_r; d} f_{i,r}$ is ***independant*** *of the "decomposition" $\sum_{i \in \mathrm{I}} f_{i,1} \ldots f_{i,r}$ of the formal power series* ([20]).

PROPOSITION 4. — *Let $k' > k > 0$. The operator $\mathbf{A}_{k', k}$ interpreted in the extended sense (that is as an integral operator) gives an* ***injective morphism*** *of "convolution" differential algebras:*

k-convolution differential algebra of analytic functions on sectors bisected by d with an exponential growth of order $\leqq \kappa = \dfrac{kk'}{k'-k}$ at infinity, and with an "asymptotic expansion" at zero.	$\xrightarrow{\mathbf{A}_{k' k}}$	*k'-convolution differential algebra of analytic functions on sectors with opening $> \pi/\kappa = \pi \dfrac{k'-k}{kk'}$ and arbitrary radius bisected by d, and with an "asymptotic expansion" at zero.*

([20]) This was proved in [Ra 5] using a different method, answering a question of [Ra 2].

Let f and g be infinitely differentiable (as functions of a real variable) on d with complex values. If f and g have a growth of order $\leqq k$ (in particular if f and g have a compact support) we have

$$\mathbf{A}_{k',k}(f *_k g) = \rho_{k'}^{-1} \mathrm{L}^{-1} \rho_{k'/k} \mathrm{L}((\rho_k f) * (\rho_k g))$$
$$\mathbf{A}_{k',k}(f *_k g) = \rho_{k'}^{-1} \mathrm{L}^{-1} \rho_{k'/k}((\mathrm{L}\,\rho_k f)(\mathrm{L}\,\rho_k g))$$
$$\mathbf{A}_{k',k}(f *_k g) = \mathbf{A}_{k',k} f *_{k'} \mathbf{A}_{k',k} g.$$

We get the *same* formula when f and g have *only* a groth $\leqq \kappa$ by a *density argument*. Then $\mathbf{A}_{k',k}$ is a morphism of "convolution differential algebras".

The proof of *injectivity* is a little more subtle. We need a little bit of *Ecalle's "deceleration theory"* [E 4]:

We have (definition)

$$\mathbf{A}_\alpha^{-1} = \mathbf{D}_\alpha = (\rho_\alpha \mathrm{L})^{-1} \mathrm{L}\,\rho_\alpha = \mathrm{L}^{-1} \rho_\alpha^{-1} \mathrm{L}\,\rho_\alpha$$

and

$$\mathbf{A}_{k',k}^{-1} = \mathbf{D}_{k',k} = \rho_k^{-1} \mathrm{L}^{-1} \rho_{k/k'} \mathrm{L}\,\rho_{k'} \text{ (formally } \mathbf{D}_{k',k} = \mathbf{A}_{k',k}).$$

There exists *integral formulae* for the operators of *"normalized deceleration"* $\mathbf{D}_\alpha$, $\mathbf{D}_{k',k}$. To get them we need a new family of "special functions" C^α $(\alpha > 1)$, the *"decelerating functions"*:

$$\mathrm{C}^\alpha(t) = \int_{\mathbf{R}^+} e^{-u + t u^{1/\alpha}} u^{-1/\beta}\, du.$$

It is easy to see that C^α is an *entire function* and to compute its *analytic expansion at zero:*

$$\sum_{n \geqq 0} \frac{\Gamma((n+1)/\alpha)}{\Gamma(n+1)} t^n.$$

Example:

$\alpha = \beta = 2$; then $\mathrm{C}^2(t) = 2 e^{t^2/4} \int_{-t/2}^{+\infty} e^{-u^2} du$. This function is related to *"error functions"* [21]:

$$\mathrm{Erfc}(\sigma) = \frac{2}{\sqrt{\pi}} \int_\sigma^{+\infty} e^{-v^2} dv = 1 - \mathrm{Erf}(\sigma).$$

Functions C^α are *resurgent* at ∞ [E 4], [Ma 8], [C]. If $\alpha \in \mathbf{Q}$ these functions are related to *Mejer G-functions* and *solutions of linear differential equations* (*cf.* below *"Formulae about decelerating functions"*).

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[21] The function C^3 is simply related to Airy function Ai and to Bessel function $\mathrm{K}_{1/3}$ (*cf.* [Bak], p. 98).

Ecalle's functions C^α are particular cases ([22]) of *Faxén's* integrals:

$$Fi(\mu,\ \nu;\ t)=\int_{\mathbf{R}^+} e^{-u+tu^\lambda} u^{\mu-1}\,du \quad (\textit{see}\ [\mathrm{O\,1}],\ [\mathrm{Fa}],\ [\mathrm{BHL}])$$

$$Fi(\alpha^{-1},\ \alpha^{-1};\ t)=C^\alpha(t).$$

There is in fact a very interesting family of functions:

$$F_{P;\,\pm}(\alpha;\ \beta;\ y)=\int_{\gamma\pm} e^{P(\nu^\alpha)\pm\nu y}\,\nu^\beta\,d\nu; \qquad \text{with} \quad \alpha\in\mathbf{R},\ \beta\in\mathbf{C},\ P\in\mathbf{C}[w],$$

and $\gamma_\pm$ a convenient path.

There are many occurences of particular cases of these functions in the literature: the main sources are *arithmetic* (in connection with exponential sums: *cf.* the *Hardy-Littlewood's* paper on Waring's problem [HL] ([23]), and more recently works of *N. Katz* [Ka 4], *Deligne*, ...), *physic* (*Airy*, *Kelvin*, *Brillouin* ([24]), ...), *analysis* (study of accelerating and decelerating functions, study of Laplace transform: *cf.* [Ma 5]), and *probabilities* (up to variable and function rescalings, *stable densities* are real parts of accelerating functions, *cf.* [Fe], p. 548). If $\alpha\in\mathbf{Q}$ the function $F_{P;\,\pm}(\alpha;\ \beta;\ y)$ is solution of a differential equation (obtained by a method similar to the derivation of Gauss-Manin connection). These functions ([25]) would certainly deserve a thoroughful study.

LEMMA 8 [E 4], [MR 3] ([26]).

Let $R'>0$ *and* $\beta>1$; *we set* $\alpha=\dfrac{\beta}{\beta-1}$.

Let $D'_{\beta,\,R'}=\{t\in\mathbf{C}/|\operatorname{Arg} t|<\dfrac{\pi}{2\beta}$ *and* $\operatorname{Re}(t^\beta)\geqq 1/R'^\beta\}$. *Then* (*on* $D'_{\beta,\,R'}$):

$$|C^\alpha(t)|\leqq K^\alpha R'^{\beta/2}\,|t^{\beta-1}e^{(t/c_\alpha)^\beta}|; \quad \textit{with}\ K^\alpha>0\ \textit{and}\ c_\alpha=\beta(\alpha-1)^{1/\alpha}.$$

This Lemma is proved using *saddlepoint method.*

DEFINITION 3. – *Let* $\alpha>1$, $\beta=\dfrac{\alpha}{\alpha-1}$, $R>0$, *and a direction d.*

Let ψ *be a function analytic on the open* β-*Borel disc*

$$D_{\beta,\,R;\,d}=\{t\in\mathbf{C}/|\operatorname{Arg}\zeta-\operatorname{Arg} d|<\frac{\pi}{2\beta}$$

([22]) This was mentionned to us by *A. Duval.*

([23]) *Cf.* also *Bakhoom* [Bak].

([24]) *Cf.* also [AS], p. 1002.

([25]) And the similar functions obtained when we replace the *Laplace* transform by the *Mellin* transform in the definition (*cf.* functions Γ_P studied in [Du]).

([26]) More precisely it is possible, using saddlepoint method, to get an *asymptotic expansion* for the function C^α on the domain $D'_{\beta,\,R'}$ (*cf.* [MR 3]).

and

$$\operatorname{Re}(\zeta e^{-i\operatorname{Arg} d})^{-\beta} > 1/R^{\beta}\},$$

and continuous on the closure of $D_{\beta, R; d}$.

If we denote by γ_R *the boundary of* $D_{\beta, R; d}$ *oriented in the positive sense, we will say that* ψ *is* α*-decelerable in the direction d if the integral*

$$\varphi(\xi) = \frac{1}{2 i \pi} \int_{\gamma R} \psi(\zeta) \zeta^{\alpha} C^{\alpha}(\xi/\zeta) \, d\zeta/\zeta^2 \text{ exists} \qquad (\text{for } \xi \in d, \text{ arbitrary}).$$

PROPOSITION 5. — *Let* $\alpha > 1$, $\beta = \dfrac{\alpha}{\alpha - 1}$. *Let* ψ *be an analytic function on a sector, with opening* $> \dfrac{\pi}{\beta}$, *bisected by d, with exponential growth of order* $\leqq \alpha$ *at infinity and an "asymptotic expansion" at zero. Then* ψ *is* α*-decelerable in the direction d and:*

$$\mathbf{D}_{\alpha} \psi(\xi) = \mathrm{L}^{-1} \rho_{\alpha}^{-1} \mathrm{L} \rho_{\alpha} \psi(\xi) = \frac{1}{2 i \pi} \int_{\gamma R} \psi(\zeta) \zeta^{\alpha} C^{\alpha}(\xi/\zeta) \, d\zeta/\zeta^2.$$

If the function ψ is analytic on a sector V with opening $> \dfrac{\pi}{\beta}$, bisected by d, and if ψ is sufficiently flat at zero, that is if there exists $\lambda > 0$ such that

$$\psi = o(\zeta^{1+\beta-\alpha+\lambda}) \quad \text{on } V,$$

then it is α-*decelerable in the direction d* and $\mathbf{D}_{\alpha} \psi$ is analytic on a sector bisected by d, with an exponential growth of order $\leqq \beta$ at infinity.

If a function ψ is analytic on $D_{\beta, R; d}$ and admits an "asymptotic expansion" at zero and if there exists a "polynomial" P such that $\psi = \psi_0 + \mathrm{P}$, where ψ_0 is α-*decelerable* in the direction d, we will *still* say that ψ is α-*decelerable* in the direction d and we will write

$$\mathbf{D}_{\alpha} \psi = \mathbf{D}_{\alpha} \psi_0 + \mathbf{D}_{\alpha} \mathrm{P}$$

(where $\mathbf{D}_{\alpha} \mathrm{P}$ is computed "*formally*": *see* formulae at the end of this paragraph).

The operator $\mathbf{D}_{\alpha; d} = \mathrm{L}^{-1} \rho_{\alpha}^{-1} \mathrm{L} \rho_{\alpha}$ is defined on the domain

$$\left\{ \begin{array}{l} \text{analytic functions on sectors with opening } > \dfrac{\pi}{\beta} \text{ bisected by } d, \\ \qquad \text{with an exponential growth of order} \leqq \alpha \text{ at infinity} \\ \qquad\qquad \text{and an "asymptotic expansion" at the origin} \end{array} \right\}.$$

The operator $\psi \to \dfrac{1}{2i\pi}\displaystyle\int_{\gamma_R} \psi(\zeta)\,\zeta^\alpha C^\alpha(\xi/\zeta)\,d\zeta/\zeta^2$ is defined on the *larger* domain

$$\left\{\text{analytic functions on sectors with opening} > \frac{\pi}{\beta} \text{ with arbitrary radius bisected by } d, \text{ with an asymptotic expansion at the origin}\right\}.$$

So, *proposition* 5 gives an extension for the operator $\mathbf{D}_{\alpha;\,d}$.

LEMMA 9. — *The function* C^α *is* α*-accelerable in the direction* $\mathbf{R}^+$ *and*

$$A_\alpha C^\alpha(\zeta) = \zeta/\zeta^\alpha(1-\zeta).$$

PROPOSITION 6. — *Let* $\alpha > 1$, $\beta = \dfrac{\alpha}{\alpha - 1}$.

(i) *If a function* ψ *is* α*-decelerable in the direction d, then* $D_\alpha \psi$ *is* α*-accelerable in the direction d and:*

$$A_\alpha D_\alpha \psi = \psi.$$

(ii) *If a function* φ *is infinitely differentiable on d, with an exponential growth of order* $\leqq \beta$ *at infinity, then* $A_\alpha \varphi$ *is* α*-decelerable in the direction d and:*

$$D_\alpha A_\alpha \varphi = \varphi.$$

The proof of (i) is easy, using Fubini's theorem and *lemma* 9.

To prove (ii), using *lemma* 3, we first prove it when ψ is infinitely differentiable on d with an exponential growth of order $\leqq 1$ at infinity (in particular when ψ has a compact support). Then, for ψ with *only* an exponential growth of order $\leqq \beta$, we conclude by a *density argument*.

From proposition 5 (ii) we deduce the *injectivity* of $A_{\alpha;\,d}$. The *injectivity* of $\mathbf{A}_{k',\,k;\,d}$ follows. That ends the proof of *proposition* 6.

The following result is ***essential***:

THEOREM 1. — *Let* $k_1 > k_2 > \ldots > k_r > 0$, *and d a given direction. Then the* ***summation*** *operator*

$$\mathbf{C}\{x\}_{1/k_1,\,1/k_2,\,\ldots,\,1/k_r;\,d} \xrightarrow{\;S_{k_1,\,k_2,\,\ldots,\,k_r;\,d}\;} \begin{matrix}\textit{Differential algebra of germs}\\ \textit{of analytic functions}\\ \textit{on sectors bisected by d.}\end{matrix}$$

is an ***injective morphism of differential algebras.***

Operators S and $\cdot_d$ are *isomorphisms* of differential algebras and of k-convolution differential algebras. Operator $\hat{\mathbf{B}}_{k_r}$ is an *isomorphism* of differential algebras between the differential algebra $(x\,\mathbf{C}[[x]],\, x^2\,d/dx)$ and the k_r-convention differential algebra $(\xi^{1-k_r}\mathbf{C}[[x]],\, \partial_k)$. Operator L_{k_1} is an *isomorphism* between the convolution differential algebra of analytic functions on sectors bisected by d with an exponential growth of order $\leqq k_1$ at infinity and an "asymptotic expansion at zero", and the differential algebra of analytic functions on sectors with opening $>\pi/k_1$, bisected by d, and with an "asymptotic expansion" (without constant term) at zero. We can now end the proof of theorem 1, using *proposition* 4 with $k'=k_{i-1}$, $k=k_i\,(i=r, \ldots, 2)$.

In fact it follows from this proof that the image of the operator $S_{k_1, k_2, \ldots, k_r; d}$ is *contained* in the differential algebra of analytic functions on sectors with opening $>\pi/k_1$, bisected by d, and with an asymptotic expansion (without constant term) at zero.

It is possible to extend *proposition* 2:

PROPOSITION 7. – *Let* $k'>k_1>k_2>\ldots>k_r>0$. *Then:*

$$\mathbf{C}[[x]]_{1/k'} \cap \mathbf{C}\{x\}_{1/k_1, 1/k_2, \ldots, 1/k_r} = \mathbf{C}\{x\}.$$

PROPOSITION 8. – *Let* $k'_1, k'_2, \ldots, k'_{r'}>0$ *and* $k''_1, k''_2, \ldots, k''_{r''}>0$. *If*

$$\{k_1, k_2, \ldots, k_r\} = \{k'_1, k'_2, \ldots, k'_{r'}\} \cap \{k''_1, k''_2, \ldots, k''_{r''}\},$$

with $k_1>k_2>\ldots>k_r>0.(r\leqq r', r'')$, then:

$$\mathbf{C}\{x\}_{1/k'_1, 1/k'_2, \ldots, 1/k'_{r'}} \cap \mathbf{C}\{x\}_{1/k''_1, 1/k''_2, \ldots, 1/k''_{r''}} = \mathbf{C}\{x\}_{1/k_1, 1/k_2, \ldots, 1/k_r}.$$

If $\hat{f}\in\mathbf{C}[[x]]$ is $(k'_1, k'_2, \ldots, k'_{r'})$-summable, the *smallest set* $\{k_1, k_2, \ldots, k_r\}$ (with $k_1>k_2>\ldots>k_r>0$) such that $\hat{f}$ is $(k_1, k_2, \ldots, k_r)$-summable, is a subset of $\{k'_1, k'_2, \ldots, k'_{r'}\}$ and depends *only* on $\hat{f}$. The numbers $k_1, k_2, \ldots, k_r$ are ***the singular levels*** of $\hat{f}$:

$$\{k_1, k_2, \ldots, k_r\} = \mathrm{N}\Sigma(\hat{f}) \subset]0, +\infty[\ (\textit{definition}).$$

The situation is *very* different if $\hat{f}\in\mathbf{C}[[x]]$, is $(k'_1, k'_2, \ldots, k'_{r'})$*-summable* in a *direction d*. It is easy to prove then that there exists $\varepsilon<0$, such that $\hat{f}$ is $(k'_1-\varepsilon', k'_2-\varepsilon', \ldots, k'_{r'}-\varepsilon')$-summable in the direction d *for every* $\varepsilon'\in[0, \varepsilon]$.

We identify the real analytic blow-up of the origin in the complex plane ([27]) with the circle S^1. Then we introduce the "***analytic halo***" of the origin in the complex plane:

$$\mathbf{HA}_0 =]0, +\infty] \times S^1 = \{(k, d)/k\in]0, +\infty], d\in S^1\} \text{ (definition)}.$$

The complex plane with an analytic halo at zero is:

$$\mathbf{CH}_0 = \{0\} \cup \mathbf{HA}_0 \cup \mathbf{C}^* = ((\{``0"\} \cup ``]0, +\infty]") \cup]0, +\infty[) \times S^1)/\mathscr{R};$$

where the relation $\mathscr{R}$ corresponds to the identification of $\{``0"\}\times S^1$ with a point $\{``0"\}$.

([27]) If we use *polar coordinates* for the points of $\mathbf{C}^*$:

$\mathbf{C}^* = \{(\rho, \theta)/\rho>0, \theta\in S^1\} =]0, +\infty[\times S^1$, *this set corresponds to* $\{0\}\times S^1$.

On the set $\{``0"\} \cup ``]0, +\infty]") \cup]0, +\infty[)$ we put the ordering relation:

Ordinary ordering relation on $]0, +\infty[$ and "$]0, +\infty]$", $\rho > 0 > k$, if $\rho \in]0, +\infty[$, and $k \in$ "$]0, +\infty]$" ("$+\infty$" is identified with 0). We endow $\{``0"\} \cup \mathbf{HA}_0 \cup \mathbf{C}^*$ with the corresponding topology (quotient of the product topology). We will consider $\{``0"\} \times S^1$ as the "*real blow up*" of 0 in $\mathbf{C}\,\mathbf{H}_0$ (that is the set of directions starting from 0 in $\mathbf{C}\,\mathbf{H}_0$).

The universal covering of $(S^1, 1)$ is $(\mathbf{R}, 0)$. We will interpret $\mathbf{HA}_0 =]0, +\infty] \times (\mathbf{R}, 0)$ as the "*universal covering* of $\mathbf{HA}_0$ pointed on the direction "$\mathbf{R}^+$"$\in \{``0"\} \times S^1$".

Let $U \subset S^1$ be an open arc. Let $k_1 > k_2 > \ldots > k_r > 0$. If $\hat{f} \in \mathbf{C}[[x]]$ is $(k_1, k_2, \ldots, k_r)$-*summable* in every direction $d \in U$, then the sums $f_{k_1, k_2, \ldots, k_r; d}$ glue together in a function f analytic on a "sector" with opening equal to

$$(\text{opening of } U + \pi/k_1).$$

If now $U \subset S^1$ is an open arc bisected by d, let

$$U^+ = \{d^+ \in U / \operatorname{Arg} d^+ > \operatorname{Arg} d\},$$

and

$$U^- = \{d^- \in U / \operatorname{Arg} d^- < \operatorname{Arg} d\}.$$

If $\hat{f} \in \mathbf{C}[[x]]$ is $(k_1, k_2, \ldots, k_r)$-*summable* in every direction $d' \in U - \{d\}$, then we write

$$f^+_{k_1, k_2, \ldots, k_r; d} = S^+_{k_1, k_2, \ldots, k_r; d} \hat{f}$$

and

$$f^-_{k_1, k_2, \ldots, k_r; d} = S^-_{k_1, k_2, \ldots, k_r; d} \hat{f}$$

for the sums of $\hat{f}$ for $d^+ \in U^+$ and $d^- \in U^-$ respectively.

They are in particular defined on a *common "sector"* bisected by d, with opening equal to π/k_1

If $\hat{f} \in \mathbf{C}[[x]]$ is $(k_1, k_2, \ldots, k_r)$-summable, then $S^+_{k_1, k_2, \ldots, k_r; d} \hat{f}$ and $S^-_{k_1, k_2, \ldots, k_r; d} \hat{f}$ are defined for *every* direction $d \in S^1$.

We can define along the same lines operators $\mathrm{L}^{\varepsilon}_{k_1; d}$ and $\mathbf{A}^{\varepsilon}_{k_{j-1}, k_j; d}$, for $\varepsilon \in \{1, -1\}$.

Using *decelerating* operators we get easily the *very important:*

LEMMA 10. — *Let $k_1 > k_2 > \ldots > k_r > 0$ and d a given direction. Then if $\hat{f} \in \mathbf{C}[[x]]$, $(k_1, k_2, \ldots, k_r)$-summable in every direction of $U - \{d\}$, the following conditions are equivalent:*

(i) *$\hat{f}$ is $(k_1, k_2, \ldots, k_r)$-summable in the direction d;*

(ii) *$S^+_{k_1, k_2, \ldots, k_r; d} \hat{f} = S^-_{k_1, k_2, \ldots, k_r; d} \hat{f}$ on a "sector" bisected by d.*

Moreover if these conditions are satisfied, then

$$S^+_{k_1, k_2, \ldots, k_r; d} \hat{f} = S^-_{k_1, k_2, \ldots, k_r; d} \hat{f} = S_{k_1, k_2, \ldots, k_r; d} \hat{f}.$$

If the conditions of *lemma* 10 are *not* satisfied we will say that d is a *singular direction* for the formal power series $\hat{f}$, and we will write $d \in \Sigma(\hat{f})$; the "*singular support*" $\Sigma(\hat{f})$ of $\hat{f}$ is clearly *finite*, and $\Sigma(\hat{f}) = \varnothing$ is equivalent to $\hat{f} \in \mathbf{C}\{x\}$. We will see below that the "*jump*" from $S^{+}_{k_1, k_2, \ldots, k_r; d}\hat{f}$ to $S^{-}_{k_1, k_2, \ldots, k_r; d}\hat{f}$ is a natural generalisation of the classical "***Stokes phenomenon***" for solutions of linear differential equations.

We will give below (*cf.* 6) a very natural *interpretation of multisummability:*

A formal power series $\hat{f} \in \mathbf{C}[[x]]$ is *multisummable in the direction d* (that is there exist $k_1 > k_2 > \ldots > k_r > 0$ such that $\hat{f}$ is $(k_1, k_2, \ldots, k_r)$*-summable in the direction d*) if and only if it is "*analytic*" ("*wild* analytic") in an "*infinitesimal disc*" ([28]) and can be "***extended analytically***" along d across the "*infinitesimal neighbourhood*" ([29]) in a wild analytic function on a sector bisected by d with a "non infinitesimal" radius $R > 0$.

Then, just like one can give a *direct* (that is *without* using Borel and Laplace transforms) definition of *Borel-summability* and *k-summability* using ***Gevrey estimates*** [Ra 2], [MR 1], [MR 2], [MR 3], it is also possible to give a *direct* (that is *without* any use of *Ecalle's* acceleration operators) definition of *multisummability* using the ***wild Cauchy theory*** recently introduced by the authors [MR 3]. This "*geometric*" definition is *easier to check* in the *usual applications.* Conversely the "*analytic*" definition gives an "*explicit*" way for the computation of the sum (for instance if one has in mind *numerical computations).*

Let $U \subset S^1$ be an open arc bisected by *d.* Let $k_1 > k_2 > k \ldots > k_r > 0$ and let $\hat{f} \in \mathbf{C}[[x]]$ *be* $(k_1, k_2, \ldots, k_r)$*-summable* in every direction $d' \in U - \{d\}$. There is a natural way to generalize the sums $S^{+}_{k_1, k_2, \ldots, k_r; d}\hat{f}$ and $S^{-}_{k_1, k_2, \ldots, k_r; d}\hat{f}$:

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$, *with* $\varepsilon_i \in \{1, -1\}$ $(i = 1, \ldots, r)$. We will say that $(d; \varepsilon)$ defines a "*path*" ([30]) We can now introduce the notion of $(k_1, k_2, \ldots, k_r)$*-summability along the path* $(d; \varepsilon)$:

Definition 3. — *Let* $U \subset S^1$ *be an open arc bisected by d. Let* $k_1 > k_2 > \ldots > k_r > 0$ *and* let $\hat{f} \in \mathbf{C}[[x]]$, *be* $(k_1, k_2, \ldots, k_r)$*-summable in every direction* $d' \in U - \{d\}$. Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$, *with* $\varepsilon_i \in \{1, -1\}$ $(i = 1, \ldots, r)$. *We will say that* $\hat{f}$ *is* $(k_1, k_2, \ldots, k_r)$***-summable along the path*** $(d; \varepsilon)$ *if*

$$S^{\varepsilon}_{k_1, k_2, \ldots, k_r; d}\hat{f} = \mathbf{L}^{\varepsilon}_{k_1, k_2; d \cdot d^{\varepsilon_1}} \mathbf{A}^{\varepsilon_2}_{k_1, k_2; d} \ldots \mathbf{A}^{\varepsilon_r}_{k_r - 1, k_r; d \cdot d^{\varepsilon_r}} S\,\hat{\mathbf{B}}_{k_r}\hat{f}$$

([28]) The corresponding punctured disc has a radius $\geqq$"k">"0" in the analytic halo at zero.

([29]) This infinitesimal neighborhood is the union of zero and the analytic halo at zero.

([30]) Later we will see that such a $(d; \varepsilon)$ corresponds to a *wild homotopy class of paths* in the analytic halo of the origin, avoiding *"singularities"* of f in this halo.

exists. Then $S^{\varepsilon}_{k_1, k_2, \ldots, k_r; d}\hat{f}$ *is the* ***sum*** *of* $\hat{f}$ ***along the path*** $(d; \varepsilon)$.

THEOREM 2. – *Let* $k_1 > k_2 > \ldots > k_r > 0$, $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$, *with* $\varepsilon_i \in \{1, -1\}$ $(i = 1, \ldots, r)$. *Let d be a given direction. Then the* ***summation*** *operator*

$$\begin{array}{ccc} (k_1, k_2, \ldots, k_r)\text{-summable} & S^{\varepsilon}_{k_1, k_2, \ldots, k_r; d} & \text{Differential algebra of germs of} \\ \text{along the path } (d; e) & \longrightarrow & \text{analytic functions on sectors} \\ \text{power series } f \in \mathbf{C}[[x]] & & \text{bisected by } d. \end{array}$$

is an ***injective morphism of differential algebras***

Comparison between $S^{\varepsilon}_{k_1, k_2, \ldots, k_r; d}\hat{f}$ and $S^{\varepsilon'}_{k_1, k_2, \ldots, k_r; d}\hat{f}$ for different ε, ε' will give birth to a "*generalized Stokes phenomenon*".

We will finish this paragraph with a small list of ***useful formulae***:

Let k, k', λ, $\mu > 0$. Then:

$$\rho_k(x^\lambda) = u^{\lambda/k}, \qquad \boldsymbol{\rho}_\alpha(\xi^\mu) = \frac{\Gamma(1+\mu)}{\Gamma((1+\mu)/\alpha)}\zeta^{(1+\mu-\alpha)/\alpha}$$

$$\mathrm{B}_k(x^\lambda) = \xi^{\lambda-k}/\Gamma(\lambda/k), \qquad \mathrm{L}_k(\xi^\mu) = \Gamma(1+\mu/k)\, x^{\mu+k}$$

$$\mathrm{A}_\alpha(\xi^\mu) = \frac{\Gamma(1+\mu)}{\Gamma((1+\mu)/\alpha)}\zeta^{1+\mu-\alpha}$$

$$\mathrm{A}_{k', k}(\xi^\mu) = \frac{\Gamma((k+\mu)/k)}{\Gamma((k+\mu)/k')}\zeta^{\mu+k-k'}$$

$$\mathrm{D}_\alpha(\zeta^\nu) = \frac{\Gamma(1+\nu/\alpha)}{\Gamma(\nu+\alpha)}\xi^{\nu-1+\alpha}$$

$$\mathrm{D}_{k', k}(\zeta^\nu) = \frac{\Gamma((k'+\nu)/k')}{\Gamma((k'+\nu)/k)}\xi^{\nu+k'-k}$$

$$\begin{aligned}\partial_k(\xi^\lambda) &= (\lambda+k)\Gamma(1+\lambda/k)\,\xi^{\lambda+1}/\Gamma([(\lambda+1)/k]+1) \\ &= ((\lambda+k)\Gamma(1+\lambda/k)\,\xi/\Gamma([(\lambda+1)/k]+1))\,\xi^\lambda\end{aligned}$$

$$\xi^\lambda \ast \xi^\mu = \frac{\Gamma(1+\lambda)\Gamma(1+\mu)}{\Gamma(1+\lambda+\mu)}\xi^{\lambda+\mu}.$$

$$\xi^\lambda \ast_k \xi^\mu = \frac{\Gamma(1+\lambda/k)\Gamma(1+\mu/k)}{\Gamma(2+(\lambda+\mu)/k)}\xi^{\lambda+\mu+k}.$$

Formulae about accelerating and decelerating functions.

The following results were obtained recently (january 1990) by *A. Duval*:

$$\mathrm{C}_3(t) = i\sqrt{3}\, G^{2,\,0}_{0,\,2}\left((t/3)^3 \,\middle|\, {}^{1}_{1/3,\ 2/3}\right);$$

$$\mathrm{C}^2(t) = \frac{1}{\sqrt{\pi}}\, G^{2,\,1}_{1,\,2}\left((t/2)^2 \,\middle|\, {}^{1/2}_{0,\ 1/2}\right) = \psi(1/2,\ 1/2;\ t^2/4);$$

G is a *Mejer* G-*function* [Lu].

$$\mathrm{C}_\alpha(t)=\int_{+\infty}^{(0^-)}\frac{\Gamma(-s)}{\Gamma(-s/\alpha)}t^s\,ds \quad \text{(Hankel type contour around } \mathbf{R}^+\text{)},$$

$$\mathrm{C}^\alpha(t)=\frac{1}{2i\pi}\int_{+\infty}^{(0^-)}\Gamma(-s)\,\Gamma\left(\frac{s+1}{\alpha}\right)(-t)^s\,ds.$$

If $\alpha=p/q$, with p and q positive integers, $q>p>0$, $(p, q)=1$:

$$\mathrm{C}_{q/p}(t)=\frac{1}{2i\pi}\sqrt{pq\,(2\pi)^{p-q}}\int_{+\infty}^{(0^-)}\frac{\prod_{j=1,\,\ldots,\,q-1}\Gamma(-s+j/q)}{\prod_{j=1,\,\ldots,\,p-1}\Gamma(s+j/p)}(p^p\,(t/q)^q)^s\,ds$$

$$\mathrm{C}_{q/p}(t)=\sqrt{pq\,(2\pi)^{p-q}}\,G_{p-1,\,q-1}^{q-1,\,0}\left(p^p\,(t/q)^q\,\middle|\,{}_{1/q,\,2/q,\,\ldots,\,(q-1)/q}^{1/p,\,2/p,\,\ldots,\,(p-1)/p}\right);$$

$$\mathrm{C}^{q/p}(t)=-\frac{i\,p^{p/q}\sqrt{q}}{\sqrt{p\,(2\pi)^{q+p}}}\int_{+\infty}^{(0^-)}\prod_{j=0,\,\ldots,\,q-1}\Gamma(-s+j/q)\prod_{j=0,\,\ldots,\,p-1}\Gamma(s+1/q+j/p)\,(p^p\,(-t/q)^q)^s\,ds$$

$$\mathrm{C}^{q/p}(t)=\frac{2\pi\,p^{p/q}\sqrt{q}}{\sqrt{p\,(2\pi)^{q+p}}}\,G_{p,\,q}^{q,\,p}\left(p^p\,(-t/q)^q\,\middle|\,{}_{0,\,1/q,\,\ldots,\,1-1/q}^{1/p-1/q,\,2/p-1/q,\,\ldots,\,1-1/q}\right).$$

Accelerating functions $\mathrm{C}_{q/p}$ are *solutions* of the *differential operators* (respectively of order $q-1$ and q):

$$q\prod_{j=1,\,\ldots,\,q-1}(\delta-j)-(-1)^{q-p}\,p\,t^q\prod_{j=1,\,\ldots,\,p-1}\left(\frac{p}{q}\delta+j\right)\quad(\delta=t\,d/dt),$$

and

$$\mathrm{D}^q-(-1)^{q-p}\prod_{j=1,\,\ldots,\,p}\left(\frac{p}{q}t\,\mathrm{D}+j\right)\quad(\mathrm{D}=d/dt).$$

We get in particular, for $q=n$, $p=1$:

$$\mathrm{D}^n+(-1)^n\left(\frac{1}{n}t\,\mathrm{D}+1\right).$$

Decelerating functions $\mathrm{C}^{q/p}$ are *solutions* of *differential operators*

$$\mathrm{D}^q-\prod_{j=0,\,\ldots,\,p-1}\left(\frac{p}{q}t\,\mathrm{D}+\frac{p}{q}+j\right).$$

We get in particular, for $q=n$, $p=1$:

$$\mathrm{D}^n-\frac{1}{n}(t\,\mathrm{D}+1).$$

4. STOKES MULTIPLIERS

Let $\Delta = d/dx - A$, with $A \in \mathrm{End}(n; \mathbf{C}\{x\}[x^{-1}])$, be a germ of *meromorphic differential operator* at the origin of the complex plane $\mathbf{C}$.

It is well known [Ma 2] that Δ admits a ***formal** fundamental solution* (31) :

$$\hat{F}(x) = \hat{H}(u)\, u^{\nu L} e^{Q(1/u)},$$

with $u^\nu = x$ (for some $\nu \in \mathbf{N}^*$), $L \in \mathrm{End}(n; \mathbf{C})$, $\hat{H} \in GL(n; \mathbf{C}[[u]][u^{-1}])$, and Q a diagonal matrix with entries in $u^{-1}\mathbf{C}[u^{-1}]$, invariant, up to permutations of the diagonal entries, by the transformation corresponding to $u \to e^{2i\pi/\nu} u\, (x \to e^{2i\pi} x)$ and satisfying $[e^{2i\pi\nu L}, Q] = 0$. (If $\nu = 1$ $[L, Q] = 0$, and L can be supposed in *Jordan form*.)

If $Q = \mathrm{Diag}\{q_1, q_2, \ldots, q_n\}$, then the set $\{q_1, q_2, \ldots, q_n\}$ is a subset of $u^{-1}\mathbf{C}[u^{-1}]$ which is *independent* of the choice of the fundamental solution $\hat{F}$ (ν is chosen *minimal*).

We will set $\{q_1, q_2, \ldots, q_n\} = \mathbf{q}(Q) = \mathbf{q}(\Delta)$; the ***set*** $\mathbf{q}(\Delta)$ is clearly a *formal invariant* of Δ (invariant by the transformation $\mathbf{q}(\Delta)(u) \to \mathbf{q}(\Delta)(e^{2i\pi/\nu} u)$).

PROPOSITION 9. — *Let $k_1 > k_2 > \ldots > k_r > 0$, and $\nu \in \mathbf{N}^*$. Let d be a fixed direction. Let $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbf{C}$, and $q_1, q_2, \ldots, q_n \in x^{1/\nu}\mathbf{C}[x^{1/\nu}]$. Then the summation operator*

$$\mathbf{C}\{x\}_{1/k_1,\, 1/k_2, \ldots,\, 1/k_r;\, d} \xrightarrow{S_{k_1, k_2, \ldots, k_r; d}} \text{Differential algebra of germs of analytic functions on sectors bisected by } d.$$

can be ***uniquely extended*** *to a summation operator (still denoted by $S_{k_1, k_2, \ldots, k_r; d}$)*

$$\mathbf{C}\{x\}_{1/k_1,\, 1/k_2, \ldots,\, 1/k_r;\, d}\langle x^{\alpha_i}, e^{q_j}, \mathrm{Log}\, x\rangle \ (i = 1, \ldots, m;\ j = 1, \ldots, n) \to \text{Differential algebra of germs of analytic functions on sectors bisected by } d.$$

such that [a "branch" of Log *x being fixed* (32) *:*

$$S_{k_1, k_2, \ldots, k_r; d}(x^{\alpha_i}) = e^{\alpha_i \mathrm{Log}\, x}, \qquad S_{k_1, k_2, \ldots, k_r; d}(e^{q_j}) = e^{q_j},$$

and

$$S_{k_1, k_2, \ldots, k_r; d}(\mathrm{Log}\, x) = \mathrm{Log}\, x.$$

This operator is ***an injective morphism of differential algebras.***

It is easy to extend the definition of the operator $S_d = S_{k_1, k_2, \ldots, k_r; d}$ to the elements of $\mathbf{C}\{x\}_{1/k_1,\, 1/k_2, \ldots,\, 1/k_r;\, d}\langle x^{\alpha_i}, \mathrm{Log}\, x\rangle$ $(i = 1, \ldots, m)$.

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(31) *Cf.* infra for a more precise description of $\hat{F}$ when $\nu \geq 2$ ("*ramified* case").

(32) Log x is "formal" in the "left expression" and is an actual function in the "right expression".

Then, using *asymptotic expansions* (the inverse of S_d, restricted to $\mathrm{Im}\, S_d$, is the *asymptotic expansion* operator in the classical sense), we get

$$\mathbf{C}\{x\}\langle e^{q_j}\rangle \cap \mathbf{C}\{x\}_{1/k_1,\, 1/k_2,\, \ldots,\, 1/k_r;\, d}\langle x^{\alpha_i}, \mathrm{Log}\, x\rangle = \mathbf{C}\{x\}$$
$$(i=1, \ldots, m;\, j=1, \ldots, n).$$

The result follows.

THEOREM 3. — *Let* $\Delta = d/dx - \mathrm{A}$, *with* $\mathrm{A} \in \mathrm{End}(n; \mathbf{C}\{x\}[x^{-1}])$, *be a germ of meromorphic differential operator at the origin of the complex plane* **C**.

We denote by $k_1 > k_2 > \ldots > k_r$ *the positive (non zero) slopes of the Newton polygon of the* (rank n^2) *differential operator*

$$\mathrm{End}\,\Delta = d/dx - [\mathrm{A}, .].$$

Let $\hat{\mathrm{F}}$ *be a* ***formal*** *fundamental solution of* Δ. *Then there exists a "natural decomposition"* (33).

$\hat{\mathrm{H}} = \hat{\mathrm{H}}_1 \hat{\mathrm{H}}_2 \ldots \hat{\mathrm{H}}_r$, *where* $\hat{\mathrm{H}}_i \in \mathrm{GL}(n; \mathbf{C}[[u]][u^{-1}])$ *is* k_i-***summable*** *as a "function" of* x (i.e. νk_i*-summable as a "function" of* u), *for* $i=1, \ldots, r$, *and such that*

(i) $\hat{\mathrm{F}}^i(u^\nu) = \hat{\mathrm{H}}_i(u)\hat{\mathrm{H}}_{i+1}(u)\ldots\hat{\mathrm{H}}_r(u)\, u^{\nu \mathrm{L}} e^{\mathrm{Q}(1/u)}$ *is a* ***formal fundamental solution*** *of a* ***meromorphic*** *differential operator* $\Delta_\nu^i = d/dx - \mathrm{A}_\nu^i$, *with*

$$\mathrm{A}_\nu^i \in \mathrm{End}(n; \mathbf{C}\{u\}[u^{-1}]), \qquad for \quad i=1, \ldots, r \quad (34);$$

(ii) *If* $\Sigma(\hat{\mathrm{F}}) = \Sigma(\hat{\mathrm{H}}) = \bigcup_{i=1,\ldots,r} \Sigma(\hat{\mathrm{H}}_i)$, $\mathrm{H}_{i;\, d} = S_{k_i;\, d}\, \hat{\mathrm{H}}_i$ *(for* $i=1, \ldots, r$*)*

and

$$\mathrm{H}_d = \mathrm{H}_{1;\, d}\mathrm{H}_{2;\, d} \ldots \mathrm{H}_{r;\, d},$$

then, for $d \notin \Sigma(\hat{\mathrm{H}})$, *and every determination of* $\mathrm{Log}\, x$ ($u = e^{(\mathrm{Log}\, x)/\nu}$ *and* $u^{\mathrm{L}} = e^{\mathrm{L}\, \mathrm{Log}\, u}$):

$\mathrm{F}_d(x) = \mathrm{H}_d(u)\, u^{\nu \mathrm{L}} e^{\mathrm{Q}(1/u)}$ *is an* ***actual*** *analytic fundamental solution of the operator* Δ *on a sector bisected by* d.

From this result (using *proposition* 9) it is easy to deduce the

THEOREM 4. — *Let* $\Delta = d/dx - \mathrm{A}$, *with* $\mathrm{A} \in \mathrm{End}\,(n; \mathbf{C}\{x\}[x^{-1}])$, *be a germ of meromorphic differential operator at the origin of the complex plane* **C**. *Let* $\hat{\mathrm{F}}$ *be a* ***formal fundamental solution*** *of* Δ. *If we denote by* $\mathbf{C}\{x\}[x^{-1}]\langle \hat{\mathrm{F}}\rangle$ *the differential field generated, on* $\mathbf{C}\{x\}[x^{-1}]$, *by the*

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(33) Unique up to "natural" analytic transformations (*see* [Ra 4]); in particular, the matrices H_i are well defined up to analytic (in u) conjugation.

(34) Moreover the matrices A_ν^i and $\hat{\mathrm{H}}_{i+1}$ have a common "blockstructure" and Δ_ν^i can be reduced by a transform "$\mathrm{Y} = \mathrm{Exp}(\mathrm{Q}_i) Z$" to a differential operator whose *Katz's invariant* [De 1] is k_{i+1}; Q_i being a diagonal matrix whose entries are monomials in u (fixed for each block) of degree νk_i [J], [Ra 6].

entries of $\hat{\mathrm{F}}$, *then, for* $d \notin \Sigma(\hat{\mathrm{F}})$, *the map*

$$\mathbf{C}\{x\}[x^{-1}]\langle \hat{\mathrm{F}} \rangle \rightarrow \begin{array}{c}\textit{Differential field generated,}\\ \textit{on } \mathbf{C}\{x\}[x^{-1}],\\ \textit{by the analytic solutions}\\ \textit{of the operator } \Delta \textit{ in a germ}\\ \textit{of sector bisected by } d,\end{array}$$

defined by "identity" on $\mathbf{C}\{x\}[x^{-1}]$ *and* $\hat{\mathrm{F}} \rightarrow \mathrm{F}_d$, *is an* ***isomorphism of differential fields.***

We will first admit *theorem* 3, and will go back in 5 to some indications about its proof, after some applications. It is very easy to deduce *theorem* 4 from *theorem* 3, using ***multisummability*** (other ways to do that are explained in [Ra 5], [Ra 6], and [De 4] (35) :

From *theorem* 3 and *lemma* 7 we get

THEOREM 5. – *Let* $\Delta = d/dx - \mathrm{A}$, *with* $\mathrm{A} \in \mathrm{End}\,(n; \mathbf{C}\{x\}[x^{-1}])$, *be a germ of meromorphic differential operator at the origin of the complex plane* **C**. *Let* $\hat{\mathrm{F}}$ *be a* ***formal fundamental solution*** *of* Δ. *We denote by* $k_1 > k_2 > \ldots > k_r$ *the positive (non zero) slopes of the Newton polygon of the operator*

$$\mathrm{End}\,\Delta = d/dx - [\mathrm{A}, .].$$

Then $\hat{\mathrm{F}}$ *is* $(k_1, k_2, \ldots, k_r)$*-summable in every direction, but perhaps a finite number belonging to* $\Sigma(\hat{\mathrm{F}}) \subset \mathrm{S}^1$.

Clearly (using *lemma* 7) the sums (in a common non singular direction) given by *theorems* 2 and 4 are the same.

If $d \notin \Sigma(\hat{\mathrm{F}})$, the operator $S_{k_1, k_2, \ldots, k_r; d}$ is injective and *Galois-differential.* So *theorem* 4 follows from *theorem* 5. Moreover we have got an ***"explicit"*** method of ***summation of formal solutions of linear differential equations*** (36). It is interesting to remark that $k_1, k_2, \ldots, k_r$ are ***rational numbers,*** so $k_i/k_{i-1} = \alpha_i \in \mathbf{Q}$ and $\mathbf{C}_{\alpha_i}$ $(i = 1, \ldots, r)$ is *a solution of a linear differential equation;* moreover all the functions written under $\int$ when we apply the successive computations of the resummation algorithm are *solutions of linear differential equations.* A consequence is that, for numerical computations, we can apply efficient algorithms in order to compute the successive *analytic continuations* $\cdot_d$ (Runge-Kutta algorithm, Chudnovskys algorithm [Chu], . . .).

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(35) The methods differs by the respective proportions of analysis and algebra used.

(36) There exists an algorithm for the explicit computation of the levels $k_1, k_2, \ldots, k_r$ [Ma 2]. An effective computation is possible on a computer using the systems "Reduce", "Desir" and "D5" [Tou]. For the ("generic") one-level case there are efficient numerical algorithms of summation [Th]; for the multilevelled case, algorithms are studied by *Thomann.*

Let now $d \in \Sigma(\hat{\mathrm{F}})$ be a *singular* direction:
Then (a "*branch*" of Logarithm being choosen)

$$S^{+}_{k_1, k_2, \ldots, k_r; d} \hat{\mathrm{F}} \quad \text{and} \quad S^{-}_{k_1, k_2, \ldots, k_r; d} \hat{\mathrm{F}}$$

are (*different*) ***actual*** *fundamental solutions* of Δ, *analytic on a* ***common*** *sector bisected by d, with opening* π/k_1, on the Riemann surface of Logarithm. So we get $\mathrm{F}^{+}_{d} = \mathrm{F}^{-}_{d}\, St_d$, with $St_d \in \mathrm{GL}(n; \mathbf{C})$. By definition St_d is ***the Stokes matrix*** associated to the *formal fundamental solution* $\hat{\mathrm{F}}$ of Δ, to the *direction d*, and to the *choice of branch of Logarithm*.

The operator $(S^{+}_{k_1, k_2, \ldots, k_r; d})^{-1}(S^{-}_{k_1, k_2, \ldots, k_r; d}) = \mathrm{St}_d$ is clearly a K-automorphism of the differential extension $\mathbf{C}\{x\}[x^{-1}]\langle \hat{\mathrm{F}} \rangle$ (which is a *Picard-Vessiot extension* of $\mathbf{C}\{x\}[x^{-1}]$ associated to Δ [Kap], [Kol]), that is an element of the *Galois differential group*, clearly *independent* of the choice of $\hat{\mathrm{F}}$). Later we will *systematically* write the operation of elements of St_d, and, more generally, of differential automorphisms, ***on the right*** (and ask the reader to be careful with the *ordering* of compositions...). We will also denote by $\mathrm{St}_{\mathbf{d}}$ the induced automorphism (this automorphism depends on $d \in \mathrm{S}^1$ and on the choice of branch of Logarithm ([37]), that is on $\mathbf{d} \in (\mathbf{R}, 0)$ *universal covering* of $(\mathrm{S}^1, 0)$ *"above" d*) of the $\mathbf{C}$-*vector space of formal solutions* of Δ (the matrix of this automorphism in the basis formed by the columns of $\hat{\mathrm{F}}$ is St_d). So the ***Stokes matrix*** $St_{\mathbf{d}}$ ***is an element of the representation of "the" differential Galois group*** $\mathrm{Gal}_{\mathrm{K}}(\Delta) = \mathrm{Aut}_{\mathrm{K}}\mathrm{K}\langle \hat{\mathrm{F}} \rangle$ $(\mathrm{K} = \mathbf{C}\{x\}[x^{-1}])$ in $\mathrm{GL}(n; \mathbf{C})$ given by *the formal fundamental solution* $\hat{\mathrm{F}}$.

Here one must be very careful: *Stokes matrices defined by our method* (very near of *Stokes original method* [Sto] (*cf.* references and comments in [MR 2], chapter 3)) are "in" the Galois differential group, but this is in general *completely false for "classical" Stokes matrices*. Classical definition, starting from asymptotic expansions in Poincaré's sense ([38]), is "unnatural" and corresponds to a misunderstanding of the original Stokes ideas (Stokes was working by numerical computations with in mind something like an idea of "*exact asymptotic expansions*").

Remark. – Stokes operators $\mathrm{St}_{\mathbf{d}}$ and Stokes matrices $St_{\mathbf{d}}$ are ***unipotent*** (*see* infra), so we can define their *logarithms* st_d and st_d respectively (the idea of a systematical use of these logarithms seems essentially due to

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([37]) Up to conjugation by the "formal monodromy" (*cf.* infra).

([38]) Asymptotic expansions in Poincaré's sense must be replaced by "*transasymptotic expansions*" (*Ecalle's* terminology): the transasymptotic expansion map is the *inverse* of the summation map). Transasymptotic expansions can only make *"exponentially small jump"* on *singular lines* ("anti Stokes lines"), but Poincaré asymptotic expansions can only make "*jumps*" on "*Stokes lines*" (consequence of transasymptotic expansion "jumps", in "*quadrature of phasis*").

Ecalle in a more general context):

$$\mathrm{St}_{\mathbf{d}} = \mathrm{Exp}\, \mathrm{st}_{\mathbf{d}} \quad \text{and} \quad St_{\mathbf{d}} = \mathrm{Exp}\, st_{\mathbf{d}}.$$

Then

$$\mathrm{F}_{\mathbf{d}} = \mathrm{F}_{\mathbf{d}}^{+} \,\mathrm{Exp}\left(-\frac{1}{2} st_{\mathbf{d}}\right) = \mathrm{F}_{\mathbf{d}}^{-} \,\mathrm{Exp}\left(\frac{1}{2} st_{\mathbf{d}}\right),$$

and we can choose $\mathrm{F}_{\mathbf{d}}$ as *sum* of $\hat{\mathrm{F}}$ in the *singular direction* **d** (this idea is already in *Dingle*'s book [Din]; this has been recently extended to *extremely general* situations by *Ecalle: "sommation médiane"*). If the differential operator Δ is ***real,*** if $\hat{\mathrm{G}}$ is a ***real formal fundamental solution,*** and if $\mathbf{d} = \mathbf{R}^{+}$, then we can choose the *fundamental determination* of the Logarithm, and *"median sum"* $\mathrm{G}_{\mathbf{d}}$ is ***real*** (this can be applied to *Airy* equation at infinity, *cf.* [*MR* 2], chapter 3). Moreover $\mathrm{st}_{\mathbf{d}}$ is a ***Galois derivation*** (i. e. *commuting* with the *derivation* of the differential field) of the differential field $\mathrm{K}\langle \hat{\mathrm{G}} \rangle$, and $\mathrm{Exp}\left(\frac{1}{2}\mathrm{st}_{\mathbf{d}}\right) \in \mathrm{Aut}_{\mathrm{K}}\mathrm{K}\langle \hat{\mathrm{G}} \rangle$, then, when the reality conditions given above are satisfied, the map $\mathbf{R}\{x\}[x^{-1}]\langle \hat{\mathrm{G}} \rangle \to$ *germs of real meromorphic functions at* $0 \in]0, +\infty[$, defined by

$$\hat{\mathrm{G}} \to \mathrm{G}_{\mathbf{d}} \text{ on } \hat{\mathrm{G}},$$

and equal to S on $\mathbf{R}\{x\}[x^{-1}]$ is ***an injective morphism of differential fields.***

The following generalization of a *Schlesinger's* theorem ([39]) [Sch] was first proved in [Ra 4], [Ra 5], using a different method ([40]).

THEOREM 6. – *Let* $\mathrm{K} = \mathbf{C}\{x\}[x^{-1}]$. *Let* $\Delta = d/dx - \mathrm{A}$, *with* $\mathrm{A} \in \mathrm{End}(n;\, \mathrm{K})$, *be a germ of meromorphic differential operator at the origin of the complex plane* $\mathbf{C}$. *Let* $\hat{\mathrm{F}}$ *be a* ***formal fundamental solution*** *of* Δ. *Let* H *be the subgroup of* $\mathrm{GL}(n;\, \mathbf{C})$ *generated by the* ***formal monodromy*** *matrix* $\hat{\mathrm{M}}$, ***the exponential torus*** T, *and the* ***Stokes matrices*** *of* Δ *associated to the given formal fundamental solution* $\hat{\mathrm{F}}$. *Then the representation of the* ***Galois differential group*** $\mathrm{Gal}_{\mathrm{K}}(\Delta)$ *of* Δ *in* $\mathrm{GL}(n;\, \mathbf{C})$, *given by* $\hat{\mathrm{F}}$, *is the* ***Zariski closure*** *of* H *in* $\mathrm{GL}(n;\, \mathbf{C})$.

Using *"Galois correspondence"* [Kap], it suffices to prove that the *invariant field* of H (that is the subfield of $\mathrm{K}\langle \hat{\mathrm{F}} \rangle$ consisting of the *invariant* elements by H) is K.

First we must define the *"formal monodromy"* and the *"exponential torus"* of Δ.

([39]) Schlesinger's theorem is for the case of Fuchsian equations.

([40]) A second proof has been given by *Deligne* using "Tannakian" ideas [De 4], and, during Luminy conference (september 1989), I have learned from *Y. Il'Yashenko* that he has also recently got another proof...

Replacing u by $ue^{2i\pi}$ in $\hat{F}(u)$, we get a (in general *new*) *fundamental solution* of the differential operator Δ:

$\hat{F}(ue^{2i\pi})=\hat{F}(u)\hat{M}$, with $\hat{M}\in GL(n; \mathbf{C})$. By definition $\hat{M}$ is the *formal monodromy matrix* associated to Δ and to the fondamental solution $\hat{F}$. The corresponding element $\hat{M}$ of $\mathrm{Aut}_K K\langle\hat{F}\rangle$ is clearly *independant* of the choice of $\hat{F}$ and is a *formal invariant* of Δ; it is the *formal monodromy* of Δ. (We will later systematically write the operation of $\hat{M}$ on the ***right***.)

We will now define the "*exponential torus*".

Let $\hat{\mathbb{K}}=\hat{K}_\nu\langle u^L, e^Q\rangle$ the differential field generated by $\hat{K}_\nu=\mathbf{C}[[u]][u^{-1}]$ *and the entries of the matrices* u^L and e^Q.

Let $\hat{\mathbb{L}}_\nu=\hat{K}_\nu\langle e^Q\rangle=\hat{K}_\nu\langle e^{q_1}, e^{q_2}, \ldots, e^{q_n}\rangle\subset\hat{\mathbb{K}}$.

If μ is the dimension of the (free) abelian **Z**-*module* $\mathbf{E}(\Delta)\subset u^{-1}\mathbf{C}[u^{-1}]$ generated by $q_1, q_2, \ldots, q_n$, the *Galois differential group* $\mathrm{Aut}_{\hat{K}_\nu}\hat{\mathbb{L}}_\nu=\mathrm{Aut}_{K_\nu}\mathbb{L}_\nu$ *is a torus* $\mathscr{T}(Q)=\mathscr{T}_\nu(Q)=\mathscr{T}(\mathbf{q}(\Delta))$ *isomorphic to* $(\mathbf{C}^*)^\mu$ (clearly $\mu\leqq n$). (We have set $K_\nu=\mathbf{C}\{u\}[u^{-1}]$ and $\mathbb{L}_\nu=K_\nu\langle e^Q\rangle$.)

We have $\hat{\mathbb{L}}_\nu\cap\hat{K}_\nu\langle u^L\rangle=\hat{K}_\nu$. Then $\mathscr{T}(Q)$ can be identified with a subgroup of $\mathrm{Aut}_{\hat{K}_\nu}\hat{\mathbb{K}}$ leaving $\hat{K}_\nu\langle u^L\rangle$ *fixed* (still denoted by $\mathscr{T}(Q)$).

We have $K\langle\hat{F}\rangle\subset\hat{\mathbb{K}}$, and $K\langle\hat{F}\rangle$ are invariant by $\mathscr{T}(Q)$; so $\mathscr{T}(Q)$ can be identified with a subgroup of $\mathrm{Aut}_K K\langle\hat{F}\rangle=\mathrm{Gal}_K(\Delta)$. This group is clearly independent of the choice of $\hat{F}$. By *definition* we call this group "*the exponential torus*" *of* Δ. It will be denoted by $T(\Delta)$ (it depends only on $\mathbf{q}(\Delta)$ and is a *formal invariant* of Δ). Its representation in $GL(n; \mathbf{C})$ given by the fundamental solution $\hat{F}$ will be denoted by $T=T(\Delta)=T(Q(\Delta))$ (and still named "*exponential torus*").

Let $K'_\nu=\mathbf{C}\{u\}_{1/\nu k_1, 1/\nu k_2, \ldots, 1/\nu k_r}$. We have

$$K\langle\hat{F}\rangle\subset K'_\nu\langle u^L, e^Q\rangle=\mathbb{K}'.$$

Let now $\xi\in K\langle\hat{F}\rangle$ be an element ***invariant*** by H (more precisely by the subgroup of $\mathrm{Aut}_K K\langle\hat{F}\rangle$ corresponding to H). If $x=u^\nu$, then ξ is invariant by $\hat{M}^\nu$, that is by the *formal monodromy* "in u", so $\xi\in K'_\nu\langle e^Q\rangle$. But ξ is also invariant by the *exponential torus* and $\xi\in K'_\nu$. From the *invariance* of ξ by the *Stokes matrices* we deduce that the $(k_1, k_2, \ldots, k_r)$-*summable* power series ξ admits ***no singular direction*** (*Lemma* 10), so ξ is ***convergent*** and $\xi\in K_\nu$. The action of the monodromy matrix $\hat{M}$ on $\xi\in K_\nu$ is the same as the action of the (*ordinary*) Galois group $\mathrm{Aut}_K K_\nu$ (isomorphic to $\mathbf{Z}/\nu\mathbf{Z}$), so ξ is invariant by $\mathrm{Aut}_K K_\nu$ and $\xi\in K$ (by the *ordinary* Galois correspondence). That ends the proof of *Theorem* 5.

Examples. – From fundamental systems of solutions at infinity ($z=x^{-1}$; $x=0$) for *Airy* and *Kummer differential equations it is possible* to compute *formal monodromies, exponential tori* and *Stokes multipliers*. From these results it is possible to compute the Galois differential groups of our differential equations ([41]). *See* [MR 3]).

For a deeper study of germs of analytic linear differential equations we need now a little "*toolbox*" ([42]) (built with elementary linear algebra).

Let $\mathbf{E}_\nu = x^{-1/\nu}\mathbf{C}\{x^{-1/\nu}\}$ $(\nu \in \mathbf{N}^*)$ *and* $\mathbf{E} = \bigcup_{\nu \in \mathbf{N}^*} \mathbf{E}_\nu$. *If*

$$\mathbf{q} = \{q_1, q_2, \ldots, q_n\} \subset \mathbf{E},$$

we denote by

$$\mathbf{E}(\mathbf{q}) = \mathbf{Z}_{q_1} + \mathbf{Z}_{q_2} + \ldots + \mathbf{Z}_{q_n} \subset \mathbf{E}$$

the sublattice of $\mathbf{E}$ *generated by* $q_1, q_2, \ldots, q_n$. The *smallest* integer ν such that $\mathbf{E}(\mathbf{q}) \subset x^{-1/\nu}\mathbf{C}\{x^{-1/\nu}\}$ is, by *definition*, the *ramification* of $\mathbf{q}$, *or* $\mathbf{E}(\mathbf{q})$. We have:

$$\mathbf{E} = \bigcup_{\mathbf{q}} \mathbf{E}(\mathbf{q}) = \varinjlim_{\mathbf{q}} \mathbf{E}(\mathbf{q}).$$

We define an action of the (classical) Galois group $\mathrm{Aut}_K K_\nu \approx \mathbf{Z}/\nu\mathbf{Z}$ on a sublattice $\mathbf{E}'$ of $\mathbf{E}_\nu$, by $q(x^{-1/\nu}) \to q(e^{-2i\pi/\nu}x^{-1/\nu})$ (corresponding to $x \to e^{-2i\pi}x$). If $\mathbf{E}'$ is *invariant* by this action we will say that $\mathbf{E}'$ is *Galois invariant*. The lattice $\mathbf{E}(\mathbf{q})$ is Galois invariant if and only if the *set* $\mathbf{q}$ is invariant by the corresponding action (Galois invariant).

If $q \in \mathbf{E}(\mathbf{q})$, its "*degree*" $\delta(q)$ is the rational number $m/\nu \in \frac{1}{\nu}\mathbf{Q}$, where m is the degree of q as a polynomial in $x^{1/\nu}$. There is a natural filtration of $\mathbf{E}$ by the degree, that is by the sublattices

$$\mathbf{E}^m = \{q \in \mathbf{E} / \delta(q) \leqq m\}.$$

We identify the *universal covering* of $(S^1, 1)$ to $(\mathbf{R}, 0)$. By definition the "*front*" $\mathrm{Fr}(q)$ of $q \in \mathbf{E}$ $(q \neq 0)$ is the subset of $(\mathbf{R}, 0)$ whose elements are the "*lines of maximal decrease*" of e^q (we will also call "front" the natural projection of this set on the ν-covering of $(S^1, 1)$, identified with another copy of $(S^1, 1)$). The front of q depends clearly *only* on the monomial of maximal degree $\delta(q)$ of q. If $\mathbf{d}$ is a direction belonging to the *front* of q

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([41]) "Classical computation" of the Galois differential group of Airy equation is in [Kap]; the computation of the Galois differential group of Kummer equations is, as far as we know, new (it is possible to do the computations "classical", using improvements of Kovacic's algorithm [Kov], [DLR], [MR 3]).

([42]) A first version ot these tools was first introduced by *Balser*, *Jurkat*, *Lutz* [BJL 1], [J]. In our presentation we have also used ideas of *Deligne*, *Malgrange* [De 3], [Ma 3], [Ma 4], *Babbitt*, *Varadarajan* [BV], and the systematic treatment of *M. Loday-Richaud* [LR 1].

(or of its projection on S^1), or if $q=0$, we will say that q is "*carried*" by $\mathbf{d}$.

If $x=u^\nu$, we write $K_\nu=\mathbf{C}\{u\}[u^{-1}]$, *and* $\hat{K}_\nu=\mathbf{C}[[u]][u^{-1}]$.

Let $\mathbb{L}_\nu=K_\nu\langle e^{q_1}, e^{q_2}, \ldots, e^{q_n}\rangle$, and $\hat{\mathbb{L}}_\nu=\hat{K}_\nu\langle e^{q_1}, e^{q_2}, \ldots, e^{q_n}\rangle$. As above we write $\mathrm{Aut}_{\hat{K}_\nu}\hat{\mathbb{L}}_\nu=\mathrm{Aut}_{K_\nu}\mathbb{L}_\nu=\mathcal{T}(\mathbf{q})$.

To each $q\in\mathbf{E}(\mathbf{q})$ we can associate a *character* of the exponentiel torus $\mathcal{T}(\mathbf{q})$, that is a (continuous) homomorphism of groups (still denoted by q):

$$q:\ \mathcal{T}(\mathbf{q})\to\mathbf{C}^*$$
$$q:\ \theta\to q(\theta),$$

with

$$(e^q)\,\theta=q(\theta)\,e^q \qquad (e^q\in\mathbb{L}_\nu \text{ and } \theta \text{ acts on } \mathbb{L}_\nu).$$

Let $(p_1, p_2, \ldots, p_\nu)$ be a **Z**-*basis* of the lattice $\mathbf{E}(\mathbf{q})$

We get an *isomorphism*

$$(p_1, p_2, \ldots, p_\nu):\ \mathcal{T}(\mathbf{q})\to(\mathbf{C}^*)^\nu$$
$$(p_1, p_2, \ldots, p_\nu):\ \theta\to(p_1(\theta), p_2(\theta), \ldots, p_\nu(\theta)).$$

In the following the *exponential lattice* $\mathbf{E}(\mathbf{q})$ will be identified with the *lattice of* ***characters*** *on the exponential torus* $\mathcal{T}(\mathbf{q})$.

Let $\mathbf{d}\in(\mathbf{R}, 0)$ [the universal covering of $(S^1, 1)$], we set

$\mathbf{E}_\mathbf{d}(\mathbf{q})=\{q\in\mathbf{E}(\mathbf{q})/q$ is carried by $\mathbf{d}\}$; $\mathbf{E}_\mathbf{d}(\mathbf{q})$ is a *semi-lattice* of $\mathbf{E}(\mathbf{q})$, and depends clearly only on the projection d of $\mathbf{d}$ on the ν-*covering* of S^1:

$$\mathbf{E}_\mathbf{d}(\mathbf{q})=\mathbf{E}_d(\mathbf{q}).$$

To the set $\mathbf{q}=\{q_1, q_2, \ldots, q_n\}\subset\mathbf{E}$, after the choice of an ***ordering***, we associate the diagonal matrix e^Q, with $Q=\mathrm{Diag}\{q_1, q_2, \ldots, q_n\}$.

We will use *ordering relations* associated to a direction $\mathbf{d}\in(\mathbf{R}, 0)$:

$q\gg_\mathbf{d}q'$, if and only if $q'-q\in\mathbf{E}_d(\mathbf{q})$ (*i.e.* $q'-q$ is carried by $\mathbf{d}$):

$q>_\mathbf{d}q'$, if and only if $e^{q'-q}$ is *infinitely flat on d*;

$q\geqq_\mathbf{d}q'$, if and only if $e^{q'-q}$ is bounded on d.

Clearly, if $q\gg_\mathbf{d}q'$, then $q>_\mathbf{d}q'$; and, if $q>_\mathbf{d}q'$, then $q\geqq_\mathbf{d}q'$.

We will also use an *equivalence relation* on the space $\mathbf{E}$ associated to a rational number $k>0$, $k\in\mathbf{Q}$:

$q=_kq'$ *if and only if* $\delta(q-q')<k$ [if $\delta(q-q')\geqq k$, we will write $q\neq_kq'$].

To a rational number $k>0$ we associate the *partition* of the set $\mathbf{q}=\{q_1, q_2, \ldots, q_n\}$, defined by the relation $=_k$. This partition is named the "*k-partition*". The only "significative" values for k are in the set $\{k_1, k_2, \ldots, k_r\}=\mathrm{N}\Sigma(\mathbf{q})$ of values taken by $\delta(q_i-q_j)(q_i\neq q_j)$. We will ***always*** suppose in the following that we have chosen an ordering on $q_1, q_2, \ldots, q_n$ such that, for every $k>0$, $k\in\mathbf{Q}$, the elements of each subset of the k-partition are *consecutive*. Then, there exists a *unique block-decomposition* (by definition the *k-block-decomposition*) of the matrix Q which is invariant by transposition and induces the *k-partition* on the diagonal. For $k=k_1, k_2, \ldots, k_r$ we get, by definition, the "iterated block-decomposition" (*cf.* [BJL 1], [J]). If a matrix A admits the same k-block-decomposition than Q, we will say that A admits a (Q, k)-*block-structure*. Moreover, a direction **d** being fixed, it is possible to choose an *indexation* (called by definition a **d**-*indexation*) of the elements q_i of **q** such that:

$$q_1 \leqq_{\mathbf{d}} q_2 \leqq_{\mathbf{d}} \ldots \leqq_{\mathbf{d}} q_n.$$

The corresponding ordering on **q** satisfies the above conditions; the corresponding iterated block-decomposition is named a *d-iterated block-decomposition*.

The set **q** and the direction **d** being fixed, and an *ordering* (perhaps depending on **d**) being chosen on **q**, the diagonal matrix Q is defined. To this matrix and a ***fixed direction*** $\mathbf{d} \in (\mathbf{R}, 0)$, we will associate families of subgroups of GL (n; **C**), indexed by $k_m \in \{k_1, k_2, \ldots, k_r\} = \mathrm{N}\Sigma(\mathbf{q})$ (*isotropy* groups, and *Stokes* groups).

All these groups are ***unipotent***. More precisely, if P is a matrix belonging to one of these group, all the diagonal terms of P are equal to 1, and I − P is ***nilpotent*** (if the order on **q** corresponds to a **d**-indexation, then P is ***upper-triangular***).

Let $\Lambda(\mathrm{Q}; \mathbf{d}) = \{\mathrm{C} = (c_{ij}) / \textit{if } i=j,\ c_{ij}=1, \textit{ and, if } i \neq j, \textit{ and } c_{ij} \neq 0, \textit{ then } q_i <_{\mathbf{d}} q_j\}$; $\Lambda(\mathrm{Q}; \mathbf{d})$ is a subgroup of GL (n; **C**), named the *isotropy subgroup* in the direction **d**. Let $\mathrm{Sto}(\mathrm{Q}; \mathbf{d}) = \{\mathrm{C} = (c_{ij}) / \textit{if } i=j,\ c_{ij}=1, \textit{ and, if } i \neq j, \textit{ and } c_{ij} \neq 0, \textit{ then } q_i \ll_{\mathbf{d}} q_j\}$; Sto (Q; **d**) is a subgroup of Λ (Q; **d**), named the *Stokes subgroup* in the direction **d**. Let now $k_m \in \{k_1, k_2, \ldots, k_r\} = \mathrm{N}\Sigma(\mathbf{q})$. We set:

$$\Lambda^{\geq k_m}(\mathrm{Q}; \mathbf{d}) = \{\mathrm{C} = (c_{ij}) / \textit{if } i=j,\ c_{ij}=1, \text{ and, if } i \neq j \text{ and } c_{ij} \neq 0, \text{ then } q_i <_{\mathbf{d}} q_j \text{ and } q_i \neq_{k_m} q_j\};$$

$$\Lambda^{k_m}(\mathrm{Q}; d) = \{\mathrm{C} = (c_{ij}) / \textit{if } i=j,\ c_{ij}=1, \text{ and, if } i \neq j \text{ and } c_{ij} \neq 0, \text{ then } q_i <_{\mathbf{d}} q_j,\ q_i \neq_{k_m} q_j \text{ and } q_i =_{k_{m-1}} q_j\};$$

$$\Lambda^{<k_m}(\mathrm{Q}; \mathbf{d}) = \{\mathrm{C} = (c_{ij}) / \textit{if } i=j,\ c_{ij}=1, \textit{ and, if } i \neq j \textit{ and } c_{ij} \neq 0, \textit{ then } q_i <_{\mathbf{d}} q_j \text{ and } q_i =_{k_m} q_j\};$$

and

$$\mathrm{Sto}^{\geq k_m}(\mathrm{Q}; \mathbf{d}) = \{\mathrm{C} = (c_{ij}) / \textit{if } i=j,\ c_{ij}=1, \textit{ and, if } i \neq j, \textit{ and } c_{ij} \neq 0, \textit{ then } q_i \ll_{d} q_j \text{ and } q_i \neq_{k_m} q_j\};$$

$$\mathrm{Sto}^{k_m}(\mathrm{Q}; \mathbf{d}) = \{\mathrm{C} = (c_{ij}) / \textit{if } i=j,\ c_{ij}=1, \textit{ and, if } i \neq j, \textit{ and } c_{ij} \neq 0, \textit{ then } q_i \ll_{\mathbf{d}} q_j,\ q_i \neq_{k_m} q_j \text{ and } q_i =_{k_{m-1}} q_j\};$$

$$\mathrm{Sto}^{<k_m}(\mathrm{Q}; \mathbf{d}) = \{\mathrm{C} = (c_{ij}) / \textit{if } i=j,\ c_{ij}=1, \textit{ and, if } i \neq j, \textit{ and } c_{ij} \neq 0, \textit{ then } q_i \ll_{\mathbf{d}} q_j \text{ and } q_i =_{k_m} q_j\}.$$

PROPOSITION 10. — *Let* Q *be a diagonal matrix with entries in* **E**, *and* $\mathbf{d} \in (\mathbf{R}, 0)$ *be a fixed direction. Then, for every* $k > 0$, $k \in \mathbf{Q}$, *the four sequences*

$$\{\mathrm{id}\} \to \Lambda^{\geqq k_m}(\mathrm{Q};\, \mathbf{d}) \to \Lambda(\mathrm{Q};\, \mathbf{d}) \to \Lambda^{<k_m}(\mathrm{Q};\, \mathbf{d}) \to \{\mathrm{id}\},$$
$$\{\mathrm{id}\} \to \Lambda^{k_m}(\mathrm{Q};\, \mathbf{d}) \to \Lambda^{\leqq k_m}(\mathrm{Q};\, \mathbf{d}) \to \Lambda^{<k_m}(\mathrm{Q};\, \mathbf{d}) \to \{\mathrm{id}\},$$
$$\{\mathrm{id}\} \to \mathrm{Sto}^{\geqq k_m}(\mathrm{Q};\, \mathbf{d}) \to \mathrm{Sto}(\mathrm{Q};\, \mathbf{d}) \to \mathrm{Sto}^{<k_m}(\mathrm{Q};\, \mathbf{d}) \to \{\mathrm{id}\},$$
$$\{\mathrm{id}\} \to \mathrm{Sto}^{k_m}(\mathrm{Q};\, \mathbf{d}) \to \mathrm{Sto}^{\leqq k_m}(\mathrm{Q};\, \mathbf{d}) \to \mathrm{Sto}^{<k_m}(\mathrm{Q};\, \mathbf{d}) \to \{\mathrm{id}\},$$

are ***split exact sequences*** *of (algebraic) groups.*

Maps are evident inclusions and evident "projections" (by "suppression" of some entries). The sequences are split by the inclusion maps $\Lambda^{<k_m}(\mathrm{Q};\, \mathbf{d}) \to \Lambda(\mathrm{Q};\, \mathbf{d}), \ldots$

Proposition 9 consists of *"block variations"* on the

LEMMA 11. — *Let* D_n *be the subgroup of* $\mathrm{GL}(n;\, \mathbf{C})$ *of diagonal invertible matrices. Let* T_n *be the subgroup of* $\mathrm{GL}(n;\, \mathbf{C})$ *of upper triangular invertible matrices. Let* B_n *be the subgroup of* $\mathrm{GL}(n;\, \mathbf{C})$ *of upper triangular unipotent matrices. Then we have a split exact sequence of groups:*

$$\{\mathrm{id}\} \to B_n \to T_n \to D_n \to \{\mathrm{id}\}.$$

The map $T_n \to D_n$ is the evident "projection" (we replace by zero the off diagonal entries), and the map $B_n \to T_n$ is the natural injection; the natural inclusion $D_n \to T_n$ gives the splitting.

Then T_n is the semi-direct product of B_n and D_n. We will write

$$T_n = D_n \ltimes B_n;$$

$\Lambda(\mathrm{Q};\, \mathbf{d})$ is the semi-direct product of $\Lambda^{\geqq k_m}(\mathrm{Q};\, \mathbf{d})$ and $\Lambda^{<k_m}(\mathrm{Q};\, \mathbf{d})$, we will write

$$\Lambda(\mathrm{Q};\, \mathbf{d}) = \Lambda^{<k_m}(\mathrm{Q};\, \mathbf{d}) \ltimes \Lambda^{\geqq k_m}(\mathrm{Q};\, \mathbf{d}), \ldots$$

LEMMA 12. — *If*

$$\{k_1, k_2, \ldots, k_r\} = \{\delta(q_i - q_j)/i, j = 1, \ldots, n \text{ and } q_i - q_j \neq 0\}$$

$(k_1 > k_2 > \ldots > k_r > 0)$, *we have:*

$$\Lambda(\mathrm{Q};\, \mathbf{d}) = \Lambda^{k_r}(\mathrm{Q};\, \mathbf{d}) \ltimes \Lambda^{k_{r-1}}(\mathrm{Q};\, \mathbf{d}), \ltimes \ldots \ltimes \Lambda^{k_1}(\mathrm{Q};\, \mathbf{d}).$$

If $C \in \Lambda(\mathrm{Q};\, \mathbf{d})$, *there exists a* ***unique decomposition:***

$$C = C_r C_{r-1} \ldots C_1, \qquad \text{with} \quad C_i \in \Lambda^{k_i}(\mathrm{Q};\, \mathbf{d}).$$

We can now go back to *linear differential equations*. We need a more precise version of *theorem* 3. (Beware of the slight change of notation for H.)

Let $\Delta = d/dx - \mathrm{A}$, with $\mathrm{A} \in \mathrm{End}(n;\, \mathbf{C}\{x\}[x^{-1}])$, be a germ of *meromorphic differential operator* at the origin of the complex plane **C**.

The operator Δ admits a ***formal** fundamental solution:*

$$\hat{F}(x)=\hat{H}(x)\,x^{L}\,U\,e^{Q(1/u)},$$

with: $u^{\nu}=x$ (for some $\nu\in\mathbf{N}^*$), $L\in\operatorname{End}(n;\mathbf{C})$, *in Jordan form*, $\hat{H}\in GL(n;\mathbf{C}[[x]][x^{-1}])$, Q a diagonal matrix with entries in $u^{-1}\mathbf{C}[u^{-1}]$, Galois invariant, unique up to permutations of the diagonal entries, *and* $U\in\operatorname{End}(n;\mathbf{C})$ a "*universal*" matrix (depending only on Q) [BJL 1], [J] (ν is chosen *minimal*).

Let $\hat{M}=U^{-1}e^{2i\pi L}U$. We have:

$$\hat{F}(e^{2i\pi}x)=\hat{H}(x)\,x^{L}\,U\,\hat{M}\,e^{Q(\exp(-2i\pi/\nu)/u)}=\hat{F}(x)\,\hat{M},$$

and

$$e^{Q(\exp(-2i\pi/\nu)/u)}=\hat{M}^{-1}e^{Q(1/u)}\hat{M},\qquad [\hat{M}^{\nu},Q]=0.$$

THEOREM 7. — *Let $\Delta=d/dx-A$, with $A\in\operatorname{End}(n;\mathbf{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane* **C**.

We denote by $k_1>k_2>\ldots>k_r$ the positive (non zero) slopes of the Newton polygon of the (rank n^2) *differential operator*

$$\operatorname{End}\Delta=d/dx-[A,.].$$

*Let $\hat{F}$ be a **formal** fundamental solution of Δ as above. Then there exists a "natural decomposition" (unique up to "meromorphic transforms"* [Ra 4]) $\hat{H}=\hat{H}_1\hat{H}_2\ldots\hat{H}_r$, *where $\hat{H}_i\in GL(n;\mathbf{C}[[x]][x^{-1}])$, is k_i-**summable** for $i=1,\ldots,r$, and such that*

(i) $\hat{F}^i(x)=\hat{H}_i(x)\hat{H}_{i+1}(x)\ldots\hat{H}_r(x)\,x^{L}\,U\,e^{Q(1/u)}$ *is a **formal fundamental solution** of a **meromorphic** differential operator $\Delta^i=d/dx-A^i$, with*

$$A^i\in\operatorname{End}(n;\mathbf{C}\{x\}[x^{-1}]),\qquad \textit{for}\quad i=1,\ldots,r;$$

(ii) *If* $\Sigma(\hat{F})=\Sigma(\hat{H})=\bigcup_{i=1,\ldots,r}\Sigma(\hat{H}_i)$, $H_{i;d}=S_{k_i;d}\hat{H}_i$ *(for* $i=1,\ldots,r$*)*

$$H_d=H_{1;d}H_{2;d}\ldots H_{r;d},$$

then, for $d\notin\Sigma(H)$, and every determination of $\operatorname{Log}x\,(u=e^{(\operatorname{Log}x)/\nu}$ *and* $x^{L}=e^{L\operatorname{Log}x})$:

$F_{\mathbf{d}}(x)=H_d(x)\,x^{L}\,U\,e^{Q(1/u)}$ *is an **actual** analytic fundamental solution of the operator Δ in a sector bisected by d* [$\mathbf{d}\in(\mathbf{R},0)$ *"above"* d corresponds to the given branch of Logarithm].

Moreover $\hat{H}^i$ admits a (Q,k_{i-1})-block-structure $(i=2,\ldots,r)$ and A^i admits a (Q,k_i)-block-structure $(i=1,\ldots,r)$.

We define $F^i_{\mathbf{d}}(x)=H_{i;d}H_{i+1;d}\ldots H_{r;d}\,x^{L}\,U\,e^{Q(1/u)}$; $F^i_{\mathbf{d}}(x)$ is an *actual analytic fundamental solution* of the operator Δ^i in a sector bisected by $d\,(i=1,\ldots,r)$, and admits a (Q,k_{i-1})-*block-structure* $(i=2,\ldots,r)$.

We have:

$$F^i_{\mathbf{d}}=H_{i;d}F^{i+1}_{\mathbf{d}}\ (i=1,\ldots,r-1),\ \text{and we set}\ (i=1,\ldots,r):$$

$$H^+_{i;\,d}\, F^{i+1}_{\mathbf{d}^+} = H^-_{i;\,d}\, F^{i+1}_{\mathbf{d}^+}\, S_{i;\,\mathbf{d}}.$$

We have $S_{i;\,\mathbf{d}} \in \mathrm{GL}(n;\, \mathbf{C})\,(i=1, \ldots, r)$ and $St_d = S_{r;\,d}\, S_{r-1;\,d} \ldots S_{1;\,d}$.

LEMMA 13. – *Let* $\mathbf{q} = \{q_1, q_2, \ldots, q_n\} \subset \mathbf{E}$, *and, after an ordering, let* Q *be the diagonal matrix* $Q = \mathrm{Diag}\{q_1, q_2, \ldots, q_n\}$. *Let* $C \in \mathrm{End}(n;\, \mathbf{C})$, *and* $\mathbf{d}$ *a fixed direction* $[\mathbf{d} \in (\mathbf{R}, 0)]$:

(i) *The following conditions are equivalent:*

(*a*) $e^{Q} C e^{-Q} = I + \Phi$, *with* Φ ***infinitely flat*** *on* $\mathbf{d}$.

(*b*) $C \in \Lambda(Q;\, \mathbf{d})$.

(ii) *The following conditions are equivalent:*

(*a*) $e^{Q} C e^{-Q} = I + \Phi$, *with* Φ ***exponentially flat of order*** $\geqq k$ *on* $\mathbf{d}$.

(*b*) $C \in \Lambda^{\geqq k}(Q;\, \mathbf{d})$.

(iii) *The following conditions are equivalent:*

(*a*) $e^{Q} C e^{-Q} = I + \Phi$, *with* Φ ***exponentially flat of order exactly*** k *on* $\mathbf{d}$.

(*b*) $C \in \Lambda^{k}(Q;\, \mathbf{d})$.

(iv) *The following conditions are equivalent:*

(*a*) $e^{Q} C e^{-Q} = I + \Phi$, *with* Φ ***exponentially flat of order*** $\geqq k$ *on an open sector* ***with opening*** π/k, ***bisected*** *by* $\mathbf{d}$.

(*b*) $e^{Q} C e^{-Q} = I + \Phi$, *with* Φ ***exponentially flat of order exactly*** k *on an open sector* ***with opening*** π/k, ***bisected*** *by* $\mathbf{d}$.

(*c*) $C \in \mathrm{Sto}^{k}(Q;\, \mathbf{d})$.

THEOREM 8. – *Let* $\Delta = d/dx - A$, *with* $A \in \mathrm{End}(n;\, \mathbf{C}\{x\}[x^{-1}])$, *be a germ of meromorphic differential operator at the origin of the complex plane* $\mathbf{C}$.

We denote by $k_1 > k_2 > \ldots > k_r$ *the positive (non zero) slopes of the Newton polygon of the differential operator*

$$\mathrm{End}\,\Delta = d/dx - [A,\ .],$$

Let $\hat{F}(x) = \hat{H}(x)\, x^{L}\, U\, e^{Q(1/u)}$, *be a* ***formal*** *fundamental solution of* Δ *as above, and*

$$\hat{H} = \hat{H}_1\, \hat{H}_2 \ldots \hat{H}_r, \text{ a decomposition like in theorem 7.}$$

Let $S_{i;\,\mathbf{d}} \in \mathrm{GL}(n;\, \mathbf{C})\,(i=1, \ldots, r)$ *defined as above. Then:*

(i) $S_{i;\,\mathbf{d}} \in \mathrm{Sto}^{k_i}(Q;\, \mathbf{d})\,(i=1, \ldots, r)$.

(ii) $\mathrm{St}_{\mathbf{d}} \in \mathrm{Sto}(Q;\, \mathbf{d})$ *and* $\mathrm{St}_{\mathbf{d}} = S_{r;\,\mathbf{d}}\, S_{r-1;\,\mathbf{d}} \ldots S_{1;\,\mathbf{d}}$ *is the unique decomposition of* $\mathrm{St}_{\mathbf{d}}$ *corresponding to*

$$\Lambda(Q;\, \mathbf{d}) = \Lambda^{k_r}(Q;\, \mathbf{d}) \ltimes \Lambda^{k_{r-1}}(Q;\, d) \ltimes \ldots \ltimes \Lambda^{k_1}(Q;\, \mathbf{d}).$$

Assertion (i) is a consequence of *lemma* 13 (iv):

We have $(H_{i;\,d^-})^{-1}\, H_{i;\,d^+} = I + \Psi$, *with* Ψ *exponentially flat of order* $\geqq k_i$ *on an open "sector" with opening* π/k_i *bisected by* d (H_i is k_i*-summable*). We set

$$G_i = H_{i+1;\,d} \ldots H_{r;\,d}\, x^{L}\, U;$$

it is clear that G_i and G_i^{-1} are analytic on an open "sector" with opening π/k_{i+1} $(\pi/k_{i+1} > \pi/k_i)$ bisected by d, and admit a *moderate* growth at the origin on this sector. Then $e^Q S_{i;d} e^{-Q} = G_i (I+\Psi) G_i^{-1} = I + \Phi$, where Φ *is exponentially flat of order* $\geqq k_i$ on an open "sector" with *opening* π/k_i, *bisected* by d. Assertion (ii) follows from (i) and *lemma* 12.

Stokes matrices $S_{i;\mathbf{d}}$ are *a priori* defined in a *transcendental* way. *Theorem* 8 says that we can get them by an *algebraic algorithm* from the *knowledge of* $\mathrm{St}_\mathbf{d}$ ***and*** Q. We will give later an "*infinitesimal version*" of this computation.

LEMMA 14. — *Let* $k'_1 > k'_2 > \ldots > k'_{r'} > k' > 0$. *Let* $d = \mathbf{R}^+$. *Then:*

$$e^{-1/x^{k'}} = L_{k'_1;d} A_{k'_1, k'_2; d} \ldots A_{k'_{r-1} k'_r; d} B_{k'_{r'}} (e^{-1/x^{k'}}).$$

From this *lemma* and *theorem* 8, we get

THEOREM 9. — *Let* $\Delta = d/dx - A$, *with* $A \in \mathrm{End}(n; \mathbf{C}\{x\}[x^{-1}])$, *be a germ of meromorphic differential operator at the origin of the complex plane* $\mathbf{C}$.

We denote by $k_1 > k_2 > \ldots > k_r$ *the positive (non zero) slopes of the Newton polygon of the differential operator*

$$\mathrm{End}\,\Delta = d/dx - [A, .].$$

Let $\hat{F}(x) = \hat{H}(x) x^L J e^{Q(1/u)}$ *be a* ***formal*** *fundamental solution of* Δ *as above, and*

$$\hat{H} = \hat{H}_1 \hat{H}_2 \ldots \hat{H}_r, \textit{ a decomposition like in theorem 7.}$$

Let $S_{i;\mathbf{d}} \in \mathrm{GL}(n; \mathbf{C})$ $(i = 1, \ldots, r)$ *defined as above.*

Let $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r)$, *and* $\varepsilon' = (\varepsilon'_2, \ldots, \varepsilon'_r)$, *with* ε_i, $\varepsilon'_i \in \{1, -1\}$ $(i = 1, \ldots, r)$.

Then, for every direction $\mathbf{d} \in (\mathbf{R}, 0)$:

(i) $\hat{H}$ *is* $(k_1, k_2, \ldots, k_r)$***-summable along the paths*** $(d; \varepsilon)$ *and* $(d; \varepsilon')$.

(ii) *If* $S^{\varepsilon}_{k_1, k_2, \ldots, k_r; \mathbf{d}} \hat{F} = S^{\varepsilon'}_{k_1, k_2, \ldots, k_r; \mathbf{d}} \hat{F}\, \mathrm{St}^{\varepsilon, \varepsilon'}_\mathbf{d}$, $\mathrm{St}^{\varepsilon, \varepsilon'}_\mathbf{d} \in \mathrm{GL}(n; \mathbf{C})$, *and if* $\varepsilon = (-, -, \ldots, -) = -$: *then*

$$\mathrm{St}^{\varepsilon, \varepsilon'}_\mathbf{d} = \varepsilon'(S_{r;\mathbf{d}})\, \varepsilon'(S_{r-1;\mathbf{d}}) \ldots \varepsilon'(S_{1;\mathbf{d}}),$$

with

$$\varepsilon'(S_{i;\mathbf{d}}) = S_{i;\mathbf{d}} \quad \textit{if } \varepsilon'_i = +, \qquad \textit{and} \qquad \varepsilon'(S_{i;\mathbf{d}}) = I \quad \textit{if } \varepsilon'_i = -.$$

(iii) *If* $\varepsilon = (-, -, \ldots, -)$ *and* $\varepsilon' = (-, -, \ldots, +, \ldots, -)$, *with a + only at the index i, then* $\mathrm{St}^{\varepsilon, \varepsilon'}_\mathbf{d} = S_{i;\mathbf{d}}$, *and* $S_{i;\mathbf{d}}$ ***is in the representation in*** $\mathrm{GL}(n; \mathbf{C})$ ***of the differential Galois group*** $\mathrm{Gal}_K(\Delta)$ *given by the fundamental formal solution* $\hat{F}$ $(i = 1, \ldots, r)$.

We will write $S_{i;\mathbf{d}} = \mathrm{St}_{d; k_i}$.

Our aim now is to use preceding results and concepts to give a ***"purely combinatorial" description*** of the category of ***germs of meromorphic connections*** at the origin of the complex plane, as simple as possible. In "down to earth terms" a germ of meromorphic connection is a germ of differential system *up to meromorphic equivalence* [De 1], [Ma 4], [MR 2]; so the searched combinatorial description is equivalent to a *meromorphic classification of germs of differential systems.*

Such a result is well known for the *regular singular* case. It is given by the ***Riemann-Hilbert correspondance*** [De 1], [Ka 2], [MR 2]:

Germs of ***Fuchsian*** *connections at the origin of* **C**.	→	*Finite dimensional linear representations of the local fundamental group* (43).
Germ of meromorphic fuchsian differential operator Δ, *up to meromorphic equivalence.*	→	*Monodromy* M (Δ) "*around* 0".

This map is ***bijective***, moreover it is an *equivalence of Tannakian categories* [Saa], [De Mi], [De 2]. The result is *false* if we *suppress* the *fuchsian* hypothesis.

The now "classical" meromorphic classification of germs of meromorphic differential operators is given in terms of *cohomology of sheaves of groups* (isotropy groups of a "normal form") on S^1 [Si], [Ma 3], [Ma 4], [De 3], [MR 1] (44). We have in mind a "*better*" description (particularly adapted to the computation of differential Galois groups), *extending the Riemann-Hilbert correspondence to the irregular case*, that is a description of connections in terms of ***representations of groups***:

Germs of connections at the origin of **C**.	→	*Finite dimensional linear representations of the local "****wild*** *fundamental group".*
Germ of meromorphic differential operator Δ, *up to meromorphic equivalence.*	→	*????*

We will call "*Gevrey front*" of $q \in \mathbf{E}\,(q \neq 0)$ the set

$$\mathrm{Gfr}\, q = \{(\mathbf{d}, k)/\mathbf{d} \in \mathrm{Fr}\, q, k = \delta(q)\} \subset \widetilde{\mathbf{HA}}_0,$$

universal covering of the analytic halo $\mathbf{HA}_0$.

We write

$$\mathrm{Fr}(\mathbf{q}) = \bigcup_{ij} \mathrm{Fr}\, q_{ij} \quad (q_{ij} = q_i - q_j \neq 0),$$

$$\mathrm{Gfr}(\mathbf{q}) = \bigcup_{ij} \mathrm{Gfr}\, q_{ij},$$

and denote by $\Sigma(\mathbf{q})$ the *projection* on S^1 of $\mathrm{Fr}(\mathbf{q})$.

(43) Generated by a *loop* turning "one time" around the origin and isomorphic to **Z**.

(44) We will recall this description in part 5.

We define an action of the group $(\hat{\gamma}_0)$ generated ([45]) by $\hat{\gamma}_0$ on the (non abelian) free group generated by the $\gamma'_{\mathbf{d}}s$ ($\mathbf{d} \in \mathrm{Fr}(\mathbf{q})$) by

$\hat{\gamma}_0$: $\gamma_{\mathbf{d}} \to \gamma_{\exp(-2i\pi)\,\mathbf{d}}$ ($\exp(-2i\pi)$. is a translation of -2π in $(\mathbf{R}, 0)$).

We denote by $\Pi(\mathbf{q})$ the corresponding *semi-direct* product

$$\Pi = (\hat{\gamma}_0) \ltimes \big(\underset{d \in \mathrm{Fr}(\mathbf{q})}{*} (\gamma_{\mathbf{d}}) \big).$$

In $\Pi(\mathbf{q})$ we have $\hat{\gamma}_0 \gamma_{\mathbf{d}} \hat{\gamma}_0^{-1} = \gamma_{\exp(-2i\pi)\,\mathbf{d}}$.

We define an action of the free group $(\hat{\gamma}_0)$ generated by $\hat{\gamma}_0$ on the (non abelian) free group generated by the $\gamma'_a s$ ($a \in \mathrm{Gfr}(\mathbf{q})$) by

$$\hat{\gamma}_0 : \gamma_a \to \gamma_{\exp(-2i\pi)\,a} \quad (a = (\mathbf{d}, k), \exp(-2i\pi)\,a = (\exp(-2i\pi)\,\mathbf{d}, k)).$$

We denote by $\mathrm{G}\Pi(\mathbf{q})$ the corresponding *semi-direct* product

$$\mathrm{G}\Pi(\mathbf{q}) = (\hat{\gamma}_0) \ltimes \big(\underset{a \in \mathrm{Gfr}(\mathbf{q})}{*} (\gamma_a) \big).$$

In $\mathrm{G}\Pi(\mathbf{q})$ we have $\hat{\gamma}_0 \gamma_a \hat{\gamma}_0^{-1} = \gamma_{\exp(-2i\pi)\,a}$.

The groups $\Pi(\mathbf{q})$, and $\mathrm{G}\Pi(\mathbf{q})$ are "first approximations" of the "wild local fundamental group" ([46]). We can identify $\Pi(\mathbf{q})$ to a subgroup of $\mathrm{G}\Pi(\mathbf{q})$ by

$$\gamma_{\mathbf{d}} = \gamma_{a_r} \gamma_{a_{r-1}} \ldots \gamma_{a_1} \quad (a_\iota = (\mathbf{d}, k_\iota);\ \iota = 1, \ldots, r).$$

We will obtain below a classification in terms of *linear representations* of these groups ([47]). Unfortunately there are ***conditions*** ("*Stokes conditions*") on these representations in order that *they come from a connection.* That is *unsatisfying*: we want a "*wild fundamental group*" whose ***all*** finite dimensional linear representations come from a connection, just like for the Riemann-Hilbert correspondance. We will be led to the "good" group $\pi_{1,s}(\mathbf{C}^*, 0)$ by a "***Fourier analysis***" of the (Galois differential) "*unfolding*" of the ***Stokes phenomena*** under the ***adjoint action of the exponential torus***. Moreover we will see that this approach gives ([48]) a *very natural* interpretation of *Ecalle's* ***resurgence*** [E 4].

Let $\Delta = d/dx - \mathrm{A}$, with $\mathrm{A} \in \mathrm{End}(n; \mathbf{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane $\mathbf{C}$.

([45]) Here $\hat{\gamma}_0$ and the $\gamma'_{\mathbf{d}}s$, $\gamma'_a s$ are "labels"; later $\hat{\gamma}_0$ and γ_a will be interpreted as *loops* turning around respectively 0 and *a*.

([46]) The terminology "wild π_1" (in french "π_1-sauvage") was suggested to the second author by *B. Malgrange* for the group G II [Ma 7].

([47]) If we consider "*isoformal*" families, that is if we fix the "*formal form*". If we leave it free, we need to "add" a representation of the "formal fundamental group".

([48]) With paragraph 6 tools this approach will lead us to an essentially *"geometric"* description of the resurgence where *Laplace transform* and *convolution* no longer play the central characters... The second author was led to this description particularly by *Malgrange's* account of a part of *Ecalle's* work [Ma 8].

Let $\hat{F}(x)=\hat{H}(x)x^{L}U e^{Q(1/u)}$, be a ***formal*** *fundamental solution* of Δ as above. We set

$F_0(x)=x^{L}U e^{Q(1/u)}$ (Hermite formal fundamental solution [J]).

For $\hat{P}\in GL(n; \mathbf{C}[[x]][x^{-1}])$, we set

$$A^{\hat{P}}=\hat{P}A\hat{P}^{-1}+\frac{d\hat{P}}{dx}\hat{P}^{-1}$$

and $\Delta^{\hat{P}}=d/dx-A^{\hat{P}}$, and we say that *the differential operators* Δ and $\Delta^{\hat{P}}$ are ***formally*** *equivalent*. If $P\in GL(n; \mathbf{C}\{x\}[x^{-1}])$, we will say that *the differential operators* Δ and Δ^{P} are ***analytically*** *equivalent*. We have $(\Delta^{\hat{P}_2})^{\hat{P}_1}=\Delta^{\hat{P}_1\hat{P}_2}$.

It is easy to check [BJL 1] that F_0 is a fundamental solution of a *rational differential operator* $\Delta_0=d/dx-A_0$, *with* $A_0\in End(n; \mathbf{C}(x)[x^{-1}]$, *which is formally equivalent to* $\Delta\,(\Delta=\Delta_0^{\hat{H}})$.

We will write

$$\mathscr{I}_0(\hat{F})=\{C\in GL(n; \mathbf{C})/C\hat{F}=\hat{F}C\},$$

and $\mathscr{I}(F)=\{C\in GL(n; \mathbf{C})/$ *there exists* $\hat{G}\in GL(n; \mathbf{C}[[x]][x^{-1}])$ *such that* $\hat{G}\hat{F}=\hat{F}C\}$; $\mathscr{I}_0(\hat{F})$ and $\mathscr{I}(\hat{F})$ are *algebraic subgroups* of $GL(n; \mathbf{C})$ [BV], and $\mathscr{I}_0(\hat{F})\subset\mathscr{I}(\hat{F})$.

We will write:

$\mathscr{I}(\Delta)=\{\hat{G}\in GL(n; \mathbf{C}[[x]][x^{-1}])/$ *there exists* $C\in GL(n; \mathbf{C})$ *such that* $\hat{G}\hat{F}=\hat{F}C\}$; $\mathscr{I}(\Delta)$ is a subgroup of $GL(n; \mathbf{C}[[x]][x^{-1}])$. It is easy to check that $\mathscr{I}(\Delta_0)$ is a subgroup of $GL(n; \mathbf{C}(x)[x^{-1}])$ *containing* $\mathscr{I}_0(F_0)$. It is clear that $\Delta^{\hat{G}}=\Delta$ is equivalent to $\hat{G}\in\mathscr{I}(\Delta)$ ($\mathscr{I}(\Delta)$ is *independant* of the choice of $\hat{F}$).

We leave now Δ_0 *fixed* and we want to ***classify, up to meromorphic equivalence***, all the ***meromorphic differential operators*** Δ ***formally equivalent to*** Δ_0. Moreover we are also interested in the classification of the "***marked pairs***" $(\Delta, \hat{H})$ such that $\Delta^{\hat{H}}=\Delta_0$.

To a differential operator Δ *formally equivalent to* Δ_0 (*a fundamental solution* F_0 of Δ_0 being *fixed*) we can associate representations $\rho_{irr}(\Delta)$ of the groups $\Pi(\mathbf{q})$ and $G\Pi(\mathbf{q})$ in $GL(n; \mathbf{C})$ defined by:

$$\rho_{irr}(\Delta)(\hat{\gamma}_0)=\hat{M},\qquad \rho_{irr}(\Delta)(\gamma_d)=St_d(\Delta),\qquad \rho_{irr}(\Delta)(\gamma_a)=St_{d,k}(\Delta)$$

$(a=(d, k))$. (We use the formulae:

$$\hat{M}St_{\mathbf{d}}(\Delta)\hat{M}^{-1}=St_{\exp(-2i\pi)d}(\Delta),\quad \textit{and}\quad \hat{M}St_a(\Delta)\hat{M}^{-1}=St_{\exp(-2i\pi)a}(\Delta).)$$

These representations are clearly submitted to the ***constraints***:

$$\rho_{irr}(\Delta)(\gamma_{\mathbf{d}})\in Sto(Q; \mathbf{d}),\qquad \text{and}\qquad \rho_{irr}(\Delta)(\gamma_{\mathbf{a}})\in Sto^{k}(Q; \mathbf{d})=(a=(\mathbf{d}, k)).$$

We will name these conditions "*Stokes conditions*". These representations are defined up the action (by conjugacy) of $\mathscr{I}(F_0)$: if $\hat{F}=\hat{H}F_0$ is a formal fundamental solution of Δ, C an element of $\mathscr{I}(F_0)$, and $\hat{G}\in GL(n; \mathbf{C}\{x\}[x^{-1}])$ the corresponding element of $\mathscr{I}(\Delta_0)$, then

$\hat{F} C = \hat{H} F_0 C = \hat{H} \hat{G} F_0$ is also a formal fundamental solution of Δ. These representations do *not* change if we replace Δ by *a meromorphically* equivalent operator ($\hat{H}$ is then changed in $P\hat{H}$, with $P \in GL(n; \mathbf{C}\{x\}[x^{-1}])$, and $\rho_{irr}(\Delta)$ depends *only* on the *connection* ∇ associated to Δ and of the *choice* of F_0; we can set $\rho_{irr}(\nabla) = \rho_{irr}(\Delta)$.

THEOREM 10. — *Let Δ_0 be a fixed differential operator with a fixed fundamental solution $F_0 = x^L U e^{Q(1/u)}$. We denote by ∇_0 the meromorphic connection defined by Δ_0. We set $\mathbf{q} = \mathbf{q}(Q)$, and denote by n the rank of Δ_0.*

(i) *The natural map*

Meromorphic** connections ∇* ***formally *equivalent to ∇_0.*	$\overset{\rho_{irr}}{\rightarrow}$	*Representations* of the group $G\Pi(\mathbf{q})$ *in* $GL(n; \mathbf{C})$ *satisfying* ***Stokes conditions**, up to the action of $\mathscr{I}(F_0)$.*

$$\nabla \rightarrow \rho_{irr}(\nabla)$$

*is a **bijection**.*

(ii) *The natural map*

Meromorphic** connections ∇* ***formally *equivalent to ∇_0.*	$\overset{\rho_{irr}}{\rightarrow}$	*Representations of the group* $\Pi(\mathbf{q})$ *in* $GL(n; \mathbf{C})$ *satisfying* ***Stokes conditions**, up to the action of $\mathscr{I}(F_0)$.*

*is a **bijection**.*

This result is ***non trivial***. We deduce its proof from the (non trivial...) *classification of isoformal meromorphic connections* in the form given by *Malgrange* and *Sibuya*, [Ma 3], [Si] ([49]). We need before to recall some definitions and results (we will return to this topic in more details in 5). In the following we will systematically consider a function f (with values in a **C**-vector space) holomorphic on an *open sector* V as an "object" on the *open arc U* corresponding to *V* in S^1 (the real analytic blow-up of the origin in **C**) as in [Ma 3]. We define in this way on S^1 the sheaf $\mathscr{A}$ of holomorphic functions (with values in **C**) on sectors, admitting an asymptotic expansion at the origin (with Taylor expansion in $\mathbf{C}[[x]][x^{-1}]$). We denote by Λ_I the subsheaf of $\mathrm{End}(n; \mathscr{A})$ of germs of analytic matrices which are ***asymptotic to identity***; Λ_I is a sheaf of (non abelian) *groups*. If $\mathscr{F}$ is a sheaf on S^1 we will denote by $\mathscr{F}_d$ its fiber at $d \in S^1$.

([49]) The first general classification (after the work of *Birkhoff* for the "generic case") is in [BJL 2].

THEOREM 11 (Malgrange, Sibuya [Ma 3], [Si]). — *There exists a natural isomorphism*

$$\mathrm{GL}(n;\, \mathbf{C}\{x\}[x^{-1}])\backslash \mathrm{GL}(n;\, \mathbf{C}[[x]][x^{-1}]) \xrightarrow{\mu} \mathrm{H}^1(S^1;\, \Lambda_{\mathrm{I}}).$$

We recall the definition of the *Malgrange-Sibuya map* μ:

Let $\mathbf{U}=\{\mathrm{U}_i\}_{i\in \mathbf{I}}$ be a finite open covering of S^1 by open arcs. We suppose that $\mathrm{U}_i \cap \mathrm{U}_j \cap \mathrm{U}_k = \varnothing$, if $i, j, k \in \mathbf{I}$ are distinct ([50]).

Let $\hat{\mathrm{A}} \in \mathrm{GL}(n;\, \mathbf{C}[[x]][x^{-1}])$. By *Borel-Ritt theorem* [Wa] we can "*represent*" $\hat{\mathrm{A}}$ by a collection $\{\mathrm{A}_i\}_{i\in \mathbf{I}}$ (A_i being a *holomorphic* matrix on an open sector V_i corresponding to U_i admitting $\hat{\mathrm{A}}$ as *asymptotic expansion* at the origin).

We consider $\{\mathrm{A}_i\}_{i\in \mathbf{I}}$ as a *0-cochain* [with values in $\mathrm{GL}(n;\, \mathscr{A})$] and we take its *coboundary*

$\delta = \{\mathrm{A}_j^{-1}\mathrm{A}_i\}_{ij\in \mathbf{I}} \in \mathrm{Z}^1(\mathbf{U};\, \mathrm{GL}(n;\, \mathscr{A}))$. *We have* $\delta \in \mathrm{Z}^1(\mathbf{U};\, \Lambda_{\mathrm{I}})$ (A_i and A_j have the ***same*** *asymptotic expansion* $\hat{\mathrm{A}}$). We write $\mathrm{A}_j^{-1}\mathrm{A}_i = \mathrm{A}_{ij}$.

By *definition* $\mu(\hat{\mathrm{A}})$ is the image of δ in $\mathrm{H}^1(S^1;\, \Lambda_{\mathrm{I}})$. If $\mathrm{P} \in \mathrm{GL}(n;\, \mathbf{C}\{x\})$, and $\hat{\mathrm{B}} = \mathrm{P}\hat{\mathrm{A}}$, we can choose $\mathrm{A}_i = \mathrm{PA}_i$; then $\mu(\hat{\mathrm{B}}) = \mu(\hat{\mathrm{A}})$. In the following we will set

$\mathbf{I} = [1, \ldots, p]$ (where "$p+1=1$"), the bijection between $\mathbf{I}$ and $[1, \ldots, p]$ being chosen such that $\mathrm{U}_{\iota, \iota+1} = \mathrm{U}_\iota \cap \mathrm{U}_{\iota+1} \neq \varnothing$ ($\iota = 1, \ldots, p$) and such that the bisecting lines of the arcs $\mathrm{U}_{\iota, \iota+1}$ *turn* ***clockwise*** when ι increases.

If $\Sigma = \{d_1, d_2 \ldots, d_p\} \subset S^1$, we will say that the covering $\mathbf{U}$ is "*adapted*" to Σ if

$$\mathrm{U}_{\iota, \iota+1} = \mathrm{U}_\iota \cap \mathrm{U}_{\iota+1} \cap \Sigma = \{d_\iota\} \quad (\iota = 1, \ldots, p).$$

Let $k_1 > k_2 > \ldots > k_r > 0$. Let $\mathrm{A} \in \mathrm{GL}(n;\, \mathbf{C}\{x\}_{1/k_1,\, 1/k_2,\, \ldots,\, 1/k_r}[x^{-1}])$.

If ([51]) $\Sigma = \Sigma(\hat{\mathrm{A}}) = \{d_1, d_2 \ldots, d_p\}$, we can built a covering $\mathbf{U} = \{\mathrm{U}_\iota\}_{\iota\in \mathbf{I}}$ *adapted to* Σ, where $\mathrm{U}_\iota \cap \mathrm{U}_{\iota+1}$ is *bisected* by d, with *opening* $\leqq \pi/k_1$ ($\iota = 1, \ldots, p$); such a covering is said to be k_1*-adapted to* Σ. We can choose

$\mathrm{A}_\iota = S_{k_1, k_2, \ldots, k_r;\, d}\hat{\mathrm{A}}$ (*where* $d \in \mathrm{U}_\iota$ *is arbitrary* ([52]) *between* d_ι *and* $d_{\iota+1}$; $\iota = 1, \ldots, p$). Then the 1-cocycle

$\mathbf{St}(\mathbf{U};\, \hat{\mathrm{A}}) = \{\mathrm{A}_{\iota+1}^{-1}\mathrm{A}_\iota\}_{\iota\in \mathbf{I}}$ is well defined; the image of $\mathbf{St}(\mathbf{U};\, \hat{\mathrm{A}})$ in $\mathrm{H}^1(S^1;\, \Lambda_{\mathrm{I}})$ is clearly $\mu(\hat{\mathrm{A}})$. We will denote by $\mathbf{St}(\hat{\mathrm{A}})$ the 1-cocyle $\mathbf{St}(\mathbf{U};\, \hat{\mathrm{A}})$ up to the choice of $\mathbf{U}$ (satisfying our hypothesis), and *identify* it to $\{(\mathrm{A}_{\iota, \iota+1})_{d_\iota}\}_{\iota\in \mathbf{I}}$.

([50]) We will make this hypothesis for *all* the coverings in the following.

([51]) More generally we can also take $\Sigma(\mathrm{A}) \subset \Sigma$ *finite*.

([52]) The values of A_i obtained for the different d glue together by analytic continuation in an analytic matrix still denoted by A_i.

If **U** is an open covering of S^1, and $\mathscr{F}$ a sheaf of groups on S^1, we denote by

$$i_{\mathbf{U}}:\quad Z^1(\mathbf{U};\mathscr{F}) \to H^1(S^1;\mathscr{F}) \text{ the } \textit{natural map}.$$

Let $k>0$. We denote by $\Lambda^{\geqq k}$ the subsheaf of Λ_I of germs $I+\Phi$ *where* Φ *is* ***exponentially flat of order*** $\geqq k$.

DEFINITION 5. – Let $k>0$. *Let* $\Sigma=\{d_1, d_2 \ldots, d_p\}\subset S^1$, *and an open covering* **U** *"adapted" to* Σ. *A* 1-*cochain* $\delta\in C^1(\mathbf{U};\Lambda_I)$ *is said to be "k-summable", if* $\delta=\{A_{\iota,\iota+1}\}_{\iota\in I}$ *with* $A_{\iota,\iota+1}\in\Gamma(U_{\iota,\iota+1};\Lambda^{\geqq k})$, *and if each* $A_{\iota,\iota+1}$ *can be (uniquely of course) "analytically" extended to an element of* $\Gamma(V_{\iota,\iota+1};\Lambda^{\geqq k})$ *where* $V_{\iota,\iota+1}$ *is an open arc of* $(\mathbf{R}, 0)$ *with* ***opening*** π/k *"containing"* $U_{\iota,\iota+1}$ $(\iota=1, \ldots, p)$.

We will denote by $H^{1;\geqq k}(S^1;\Lambda^{\geqq k})\subset H^1(S^1;\Lambda_I)$ the subset consisting of the images of *k-summable* 1-*cocycles*.

THEOREM 12 (Martinet-Ramis [MR 1], I-6. – *Let* $k>0$.

(i) *The Malgrange-Sibuya isomorphism*

$$GL(n;\mathbf{C}\{x\}[x^{-1}])\diagdown GL(n;\mathbf{C}[[x]][x^{-1}]) \overset{\mu}{\to} H^1(S^1;\Lambda_I).$$

induces an isomorphism

$$GL(n;\mathbf{C}\{x\}[x^{-1}])\diagdown GL(n;\mathbf{C}\{x\}_{1/k}[x^{-1}]) \overset{\mu}{\to} H^{1;\geqq k}(S^1;\Lambda^{\geqq k}).$$

(ii) *If* $\delta\in Z^1(\mathbf{U};\Lambda^{\geqq k})$ *is a k-summable* 1-*cocycle, then*

$$\mathbf{St}(\mathbf{U};\mu^{-1} i_{\mathbf{U}}(\delta))=\delta.$$

Let now Δ be a differential operator. We denote by $\Lambda(\Delta)$ the sheaf (on S^1) of solutions of $\operatorname{End}\Delta$ and by $\Lambda_I(\Delta)$ the subsheaf of solutions of $\operatorname{End}\Delta$ which are ***asymptotic to identity***; $\Lambda_I(\Delta_0)$ is a subsheaf of Λ_I.

Let now Δ_0 be a differential operator with a fundamental solution $F_0=x^L U e^{Q(1/\mu)}$. We denote by ∇_0 the meromorphic connection defined by Δ_0, and write $\mathbf{q}=\mathbf{q}(Q)$; $N\Sigma(\mathbf{q})=\{k_1, k_2, \ldots, k_r\}$ is the set of values taken by $\delta(q_i-q_j)$ $(q_i\neq q_j)$, and n the rank of Δ_0. We write as above $\operatorname{End}\Delta_0=d/dx-[A_0, .]$.

Let $\mathbf{d}\in(\mathbf{R}, 0)$ be a direction and $d\in S^1$ its projection. To the choice of $\mathbf{d}\in(\mathbf{R}, 0)$ corresponds a "*branch*" of Logarithm and a "*sum*" $F_{0,\mathbf{d}}$ of $F_0=x^L U e^{Q(1/u)}$, which is analytic on an open sector bisected by d.

The map

$$\begin{aligned}\lambda_{\mathbf{d}}:&\quad GL(n;\mathbf{C})\to\Lambda(\Delta_0)_d\\ \lambda_{\mathbf{d}}:&\quad C\to F_{0,\mathbf{d}}C(F_{0,\mathbf{d}})^{-1}\end{aligned}$$

is an ***isomorphism*** *of groups*.

Let

$$\Lambda(\Delta_0;\mathbf{d};F_0)=\lambda_{\mathbf{d}}(\Lambda(Q;d))$$

$$\Lambda^{k}(\Delta_0; \mathbf{d}; F_0)=\lambda_{\mathbf{d}}(\Lambda^{k}(Q; d))$$
$$\Lambda^{\geqq k}(\Delta_0; \mathbf{d}; F_0)=\lambda_{\mathbf{d}}(\Lambda^{\geqq k}(Q; d))$$
$$\Lambda^{<k}(\Delta_0; \mathbf{d}; F_0)=\lambda_{\mathbf{d}}(\Lambda^{<k}(Q; d)).$$

It is easy to see that $\Lambda(\Delta_0; \mathbf{d}; F_0)$, $\Lambda^{k}(\Delta_0; \mathbf{d}; F_0)$, $\Lambda^{\geqq k}(\Delta_0; \mathbf{d}; F_0)$, and $\Lambda^{<k}(\Delta_0; \mathbf{d}; F_0)$ *do **not** depend on the choice of* F_0 and $\mathbf{d}$; moreover $\Lambda(\Delta_0; \mathbf{d}; F_0)=\Lambda_I(\Delta_0)_d$. We can set:

$$\Lambda^{k}(\Delta_0; \mathbf{d}; F_0)=\Lambda^{k}(\Delta_0)_d, \Lambda^{\geqq k}(\Delta_0; \mathbf{d}; F_0)=\Lambda^{\geqq k}(\Delta_0)_d, \quad \Lambda^{<k}(\Delta_0; \mathbf{d}; F_0)=\Lambda^{<k}(\Delta_0)_d.$$

All these *groups* ([53]) are *subgroups* of $\Lambda(\Delta_0)_d$ and when the direction d varies we get *subsheaves* $\Lambda^{k}(\Delta_0)$, $\Lambda^{\geqq k}(\Delta_0)$, and $\Lambda^{<k}(\Delta_0)$ of $\Lambda_I(\Delta_0)$. (When d moves the groups remain "in general" the "*same*". They can "*jump*" only for a finite set of values of d, the "*Stokes lines*".)

Let

$$Sto(\Delta_0; \mathbf{d}; F_0)=\lambda_{\mathbf{d}}(\mathrm{Sto}(Q; d))$$
$$Sto^{k}(\Delta_0; \mathbf{d}; F_0)=\lambda_{\mathbf{d}}(\mathrm{Sto}^{k}(Q; d))$$
$$Sto^{\geqq k}(\Delta_0; \mathbf{d}; F_0)=\lambda_{\mathbf{d}}(\mathrm{Sto}^{\geqq k}(Q; d))$$
$$Sto^{<k}(\Delta_0; \mathbf{d}; F_0)=\lambda_{\mathbf{d}}(\mathrm{Sto}^{<k}(Q; d)).$$

It is easy to see that $Sto(\Delta_0; \mathbf{d}; F_0)$, $Sto^{k}(\Delta_0; \mathbf{d}; F_0)$, $Sto^{\geqq k}(\Delta_0; \mathbf{d}; F_0)$, and $Sto^{<k}(\Delta_0; \mathbf{d}; F_0)$ *do **not** depend on the choice of* F_0 and $\mathbf{d}$. We can set:

$$Sto(\Delta_0; \mathbf{d}; F_0)=Sto(\Delta_0)_d, Sto^{k}(\Delta_0; \mathbf{d}; F_0)=Sto^{k}(\Delta_0)_d, Sto^{\geqq k}(\Delta_0; \mathbf{d}; F_0)=Sto^{\geqq k}(\Delta_0)_d, \quad Sto^{<k}(\Delta_0; \mathbf{d}; F_0)=Sto^{<k}(\Delta_0)_d.$$

If $d\notin\Sigma(\Delta_0)$, then $Sto(\Delta_0)_d$ *reduces to identity.*

From *proposition* 10 and *lemma* 12, we get

PROPOSITION 11. – *Let $d\in S^1$ and $k>0$. Let Δ_0 be a given differential operator with a fixed fundamental solution $F_0=x^{L}Ue^{Q(1/u)}$. We set $\mathbf{q}=\mathbf{q}(Q)$, and $N\Sigma(\mathbf{q})=\{k_1, k_2 \dots, k_r\}$ $(k_1>k_2>\dots>k_r)$. Then:*

(i) *The four sequences*

$$\{\mathrm{id}\}\to\Lambda^{\geqq k}(\Delta_0)_d\to\Lambda(\Delta_0)_d\to\Lambda^{<k}(\Delta_0)_d\to\{\mathrm{id}\},$$
$$\{\mathrm{id}\}\to\Lambda^{k}(\Delta_0)_d\to\Lambda^{\geqq k}(\Delta_0)_d\to\Lambda^{<k}(\Delta_0)_d\to\{\mathrm{id}\},$$
$$\{\mathrm{id}\}\to Sto^{\geqq k}(\Delta_0)_d\to Sto(\Delta_0)_d\to Sto^{<k}(\Delta_0)_d\to\{\mathrm{id}\},$$
$$\{\mathrm{id}\}\to Sto^{k}(\Delta_0)_d\to Sto^{\geqq k}(\Delta_0)_d\to Sto^{<k}(\Delta_0)_d\to\{\mathrm{id}\},$$

*are **split exact sequences** of groups.*

(ii) $$\Lambda(\Delta_0)_d=\Lambda^{k_r}(\Delta_0)_d\ltimes\Lambda^{k_{r-1}}(\Delta_0)_d\ltimes\dots\ltimes\Lambda^{k_1}(\Delta_0)_d.$$

([53]) It is possible to give a "direct" definition of these groups using *Deligne I-filtered structures* (or *Stokes structures*) [Ma 4], [De 3], [De 4].

THEOREM 13 (Malgrange, Sibuya, Babbitt-Varadarajan [Ma 3], [Si], [BV] *Let* Δ_1 *be a meromorphic differential operator. We denote by* ∇_1 *the meromorphic connection defined by* Δ_1. *Let* Δ_0 *be a differential operator with a fixed fundamental solution* $F_0 = x^L U e^{Q(1/u)}$. *We denote by* ∇_0 *the meromorphic connection defined by* Δ_0. *Then:*

(i) *There is a natural isomorphism* $\nu = \nu_{\nabla_1}$:

$$\left\{\begin{array}{c}\textit{Marked pairs } (\nabla, \hat{\xi}), \textit{ where} \\ \nabla \textit{ is a } \textbf{meromorphic} \textit{ connections} \\ \textit{which is } \textbf{formally} \textit{ equivalent} \\ \textit{to } \nabla_1 \textit{ and } \hat{\xi} \textit{ is an } \textbf{isomorphism} \\ \textit{between } \nabla \textit{ and } \nabla_1.\end{array}\right\} \overset{\nu}{\to} H^1(S^1; \Lambda(\Delta))$$

(ii) *If* $\nabla_1 = \nabla_0$ *the natural isomorphism* ν *induces an isomorphism:*

$$\left\{\begin{array}{c}\textbf{Meromorphic} \textit{ connections } \nabla \\ \textit{which are} \\ \textbf{formally} \textit{ equivalent to } \nabla_0.\end{array}\right\} \overset{\nu}{\to} \mathscr{I}(\Delta_0)\backslash H^1(S^1; \Lambda(\Delta_0))$$

[*The group* $\mathscr{T}(\Delta_0)$ *is acting by* ***conjugacy*** *on the sheaf* $\Lambda(\Delta_0)$.]

DEFINITION 6. — *Let* Δ_0 *be a given differential operator with a fixed fundamental solution* $F_0 = x^L U e^{Q(1/u)}$. *We set*

$$\mathbf{q} = \mathbf{q}(Q), \qquad N\Sigma(\mathbf{q}) = \{k_1, k_2, \ldots, k_r\},$$

and denote by $\Sigma(\mathbf{q}) = \{d_1, d_2 \ldots, d_p\}$ *the projection of* $\mathrm{Fr}(\mathbf{q})$ *on* S^1. *Let* $\mathbf{U} = \{U_\iota\}_{\iota \in I}$ *be an open covering* k_1*-adapted to* $\Sigma(\mathbf{q})$. *Then, a* 1*-cochain*

$$\delta \in C^1(\mathbf{U}; \Lambda(\Delta_0)) = Z^1(\mathbf{U}; \Lambda(\Delta_0))$$

*is called a "****Stokes cochain****" if*

$$\delta = \{A_{\iota, \iota+1}\}_{\iota \in I} (I = \{1, \ldots, p\})$$

with

$$(A_{\iota, \iota+1})_{d_\iota} \in Sto(\Delta_0)_{d_\iota} \quad (\iota = 1, \ldots, p).$$

Let $\mathbf{d} \in \mathrm{Fr}(\mathbf{q})$, *let* d be its projection on S^1, and let ρ be a representation of $\Pi(\mathbf{q})$ in $GL(n; \mathbf{C})$. It is easy to check that $\lambda_{\mathbf{d}}(\rho(\gamma_{\mathbf{d}})) \in \Lambda(\Delta_0)_d$ depends *only* on $d \in S^1$.

LEMMA 15. — *Let* Δ_0 *be a fixed differential operator with a fundamental solution* $F_0 = x^L U e^{Q(1/u)}$.

We set $\mathbf{q} = \mathbf{q}(Q)$, $N\Sigma(\mathbf{q}) = \{k_1, k_2, \ldots, k_r\}$ $(k_1 > k_2 > \ldots > k_r)$, *and denote by* $\Sigma(\mathbf{q})$ *the projection of* $\mathrm{Fr}(\mathbf{q})$ *on* S^1. *Let* $\mathbf{U} = \{U_\iota\}_{\iota \in I}$, *be an open covering which is* k_1*-adapted to* $\Sigma(\mathbf{q})$.

The natural map

$$\textit{Representations of } \Pi(\mathbf{q}) \textit{ in } GL(n; \mathbf{C}) \overset{z_{\mathbf{U}}}{\to} \{\textbf{Stokes} \textit{ cocycles of } Z^1(\mathbf{U}; \Lambda(\Delta_0))\}$$

$$\rho \overset{z_{\mathbf{U}}}{\to} \text{“}\{\lambda_{\mathbf{d}}(\rho(\gamma_{\mathbf{d}}))\}\text{”} (d \in \Sigma(\mathbf{q}))$$

*is a **bijection**.*

THEOREM 14. – *Let Δ_0 be a given differential operator with a fixed fundamental solution $F_0 = x^L U e^{Q(1/u)}$. We set*

$$\mathbf{q} = \mathbf{q}(Q), \qquad N\Sigma(\mathbf{q}) = \{k_1, k_2, \ldots, k_r\} \quad (k_1 > k_2 > \ldots > k_r),$$

and denote by $\Sigma(\mathbf{q}) = \{d_1, d_2 \ldots, d_p\}$ the projection of $\mathrm{Fr}(\mathbf{q})$ *on S^1. Let $\mathbf{U} = \{U_\iota\}_{\iota \in I}$, be an open covering k_1-adapted to $\Sigma(\mathbf{q})$. Then:*

(i) *Let* (54)*:*

$$\hat{H} = \hat{H}_1 \hat{H}_2 \ldots \hat{H}_r,$$

*where $\hat{H}_i \in GL(n; \mathbf{C})[[x]][x^{-1}])$ is k_i-**summable** for $i = 1, \ldots, r$. We suppose that $F = H x^L U e^{Q(1/u)}$ is a **formal** fundamental solution of a **meromorphic** differential operator Δ. Then the 1-cocycle $\mathbf{St}(\mathbf{U}; \hat{H})$ is a **Stokes cocycle**.*

(ii) *Let $\delta \in C^1(\mathbf{U}; \Lambda(\Delta_0)) = Z^1(\mathbf{U}; \Lambda(\Delta_0))$ be a **Stokes cocycle**. Then, $\Lambda(\Delta_0) \subset \Delta_I$, $\delta \in Z^1(\mathbf{U}; \Lambda_I)$, and if $\hat{H} = \mu^{-1} i_{\mathbf{U}}(\delta)$:*

(*a*) $\hat{H} = \hat{H}_1 \hat{H}_2 \ldots \hat{H}_r$, *where $\hat{H}_i \in GL(n; \mathbf{C})[[x]][x^{-1}])$ is k_i-**summable** for $i = 1, \ldots, r$;*

(*b*) $\hat{F} = \hat{H} x^L U e^{Q(1/u)}$ *is a formal fundamental solution of a **meromorphic** differential operator Δ which is **formally** equivalent to Δ_0.*

Moreover: $\delta = \mathbf{St}(\mathbf{U}; \hat{H}) = \mathbf{St}(\mathbf{U}; \mu^{-1} i_{\mathbf{U}}(\delta))$, and if ∇ is the meromorphic connection associated to Δ, then $\nu(\nabla) = i_{\mathbf{U}}(\delta)$.

(iii) *Let $\alpha \in H^1(S^1; \Lambda(\Delta_0))$. Then there exist **one** and **only one Stokes cocycle** $\delta \in Z^1(\mathbf{U}; \Lambda(\Delta_0))$ such that $\alpha = i_{\mathbf{U}}(\delta)$ (that is, representing α)* (55).

We will first prove *assertion* (i).

Using the construction of *theorem* 10, we can associate to $\hat{F} = \hat{H}F_0$ a *representation* $\rho(H)$ of $\Pi(\mathbf{q})$ in $GL(n; \mathbf{C})$, satisfying *Stokes conditions*. We have $\mathbf{St}(\mathbf{U}; \hat{H}) = z_{\mathbf{U}}(\rho(\hat{H}))$, *and* $\mathbf{St}(\mathbf{U}; \hat{H})$ is a *Stokes cocycle*.

We will ***admit*** *assertion* (ii), *for a moment*.

Assertion (iii) follows easily from *assertions* (ii) and (iii):

Let $\alpha \in H^1(S^1; \Lambda(\Delta_0))$. From *theorem* 13, we get a meromorphic connection $\nabla = \nu^{-1}(\alpha)$, wich is formally equivalent to ∇_0. We choose a differential operator Δ representing ∇; then there exists a fundamental solution $\hat{F} = \hat{H}F_0$ of Δ, with $\hat{H} \in GL(n; \mathbf{C})[[x]][x^{-1}])$. From *theorem* 7 we get a decomposition

$$\hat{H} = \hat{H}_1 \hat{H}_2 \ldots \hat{H}_r,$$

(54) It is important to notice that this definition is stated in such a way that it is ***not*** *necessary to know theorem* 5 *or theorem* 7 to apply it (*see* footnote below). Of course one can also apply it in the situation of *theorem* 5 *or theorem* 7...

(55) Assertion (iii) is due to *M. Loday-Richaud* [LR 1]. Her proof is *completely different:* she gives an *explicit algebraic **algorithm*** in order to *compute explicitely* δ from α. She uses *Malgrange-Sibuya* theory but ***not*** Gevrey asymptotics and multisummability; so it is possible, using her result and noting that assertions (i) and (ii) are proved here ***without*** *any use of theorem* 5 *or theorem* 7, to get a ***new proof of theorem*** 7 [LR 1]. *Cf.* also [BV].

where $\hat{H}_i \in GL(n; \mathbf{C})[[x]][x^{-1}])$ *is* k_i-***summable*** *for* $i=1, \ldots, r$.

We have $\rho(\hat{H}) = \rho_{irr}(\nabla)$. Let $z_{\mathbf{U}}(\rho(\hat{H})) = \delta \in Z^1(\mathbf{U}; \Lambda(\Delta_0))$. We have $i_{\mathbf{U}}(\delta) = \alpha$, and δ is a *Stokes cocycle representing* α.

It remains to prove *unicity*. Let $\delta \in Z^1(\mathbf{U}; \Lambda(\Delta_0))$, with $i_{\mathbf{U}}(\delta) = \alpha$. From *assertion* (ii) we get $\delta = \mathbf{St}(\mathbf{U}; \mu^{-1} i_{\mathbf{U}}(\delta)) = \mathbf{St}(\mathbf{U}; \mu^{-1}(\alpha))$, but $\mathbf{St}(\mathbf{U}; \mu^{-1}(\alpha))$ depends *only* on α; *unicity* of δ follows.

Before we prove *assertion* (ii) we will give some consequences of *theorem* 14.

PROPOSITION 12. — *Let* Δ_0 *be a given differential operator with a fixed fundamental solution* $F_0 = x^L U e^{Q(1/u)}$. *We set*

$$\mathbf{q} = \mathbf{q}(Q), \qquad N\Sigma(\mathbf{q}) = \{k_1, k_2, \ldots, k_r\} \quad (k_1 > k_2 > \ldots > k_r),$$

and denote by $\Sigma(\mathbf{q}) = \{d_1, d_2, \ldots, d_p\}$ *the projection of* $\mathrm{Fr}(\mathbf{q})$ *on* S^1. *Let* $\mathbf{U} = \{U_\iota\}_{\iota \in I}$ *be an open covering which is* k_1*-adapted to* $\Sigma(\mathbf{q})$. *Then the natural map*

$$\begin{array}{c} \textit{Representations of the group } \Pi(\mathbf{q}) \\ \textit{in } GL(n; \mathbf{C}), \textit{ satisfying the} \\ \textbf{\textit{Stokes conditions}} \end{array} \rightarrow H^1(S^1; \Lambda(\Delta_1))$$

$$\rho \rightarrow z_{\mathbf{U}}(\delta)$$

is a ***bijection*** *commuting with the action of* $(\mathscr{I}(F_0); \mathscr{I}(\Delta_0))$.

Theorem 10 follows from *theorem* 13 and *proposition* 12.

It remains now to prove *assertion* (ii) of *theorem* 14.

Let Δ_0 be a *given* differential operator with a fixed fundamental solution $F_0 = x^L U e^{Q(1/u)}$. We set

$$\mathbf{q} = \mathbf{q}(Q), \qquad N\Sigma(\mathbf{q}) = \{k_1, k_2, \ldots, k_r\} \quad (k_1 > k_2 > \ldots > k_r),$$

and denote by $\Sigma(\mathbf{q}) = \{d_1, d_2 \ldots, d_p\}$ the projection of $\mathrm{Fr}(\mathbf{q})$ on S^1. Let $\mathbf{U} = \{U_\iota\}_{\iota \in I}$ $(\mathbf{I} = \{1, \ldots, p\})$, be an open covering which is k_1-*adapted* to $\Sigma(\mathbf{q})$.

Let $\delta \in (\mathbf{U}; \Lambda(\Delta_0)) = Z^1(\mathbf{U}; \Lambda(\Delta_0))$ *be a* ***Stokes cocycle***. *Then,* $\Lambda(\Delta_0) \subset \Lambda_I$, $\delta \in Z^1(\mathbf{U}; \Lambda_I)$. Let $\hat{H} = \mu^{-1} i_{\mathbf{U}}(\delta)$. We will prove that δ is a ***Stokes cocycle*** by *a* ***descending*** *induction* on $i = r, r-1, \ldots, 1$.

Our *induction hyothesis* is:

(Hyp i) *Let* $\delta^i = \{A_{\iota, \iota+1}\}_{\iota \in I} \in C^1(\mathbf{U}; \Lambda(\Delta_0)) = Z^1(\mathbf{U}; \Lambda(\Delta_0))$ *be a Stokes cocycle* ***satisfying***:

$(A_{\iota, \iota+1})_{d_\iota} \in Sto^{k_i}(\Delta_0)_{d_\iota}$ $(\iota = 1, \ldots, p;$ $Sto^{\leqq k_i} = Sto^{< k_{i-1}}$, *if* $i > 1$, *and* $Sto^{\leqq k_1} = Sto)$.

Then, if $\hat{H}^i = \mu^{-1} i_{\mathbf{U}}(\delta^i)$:

(a_i) $\hat{H}^i = \hat{H}_i \hat{H}_{i+1} \ldots \hat{H}_r$, *where* $\hat{H}_j \in GL(n; \mathbf{C}[[x]][x^{-1}])$ *is* k_j-***summable*** *for* $j = i, \ldots, r$.

(b_i) $\hat{F}^i = \hat{H}^i x^L U e^{Q(1/u)}$ *is a formal fundamental solution of a* ***meromorphic*** *differential operator* Δ^i *which is* ***formally*** *equivalent to* Δ_0.

Moreover:

$\delta^i = \mathbf{St}(\mathbf{U}; \hat{H}^i) = \mathbf{St}(\mathbf{U}; \mu^{-1} i_{\mathbf{U}}(\delta^i))$, *and, if* ∇^i *is the meromorphic connection associated to* Δ^i, *then* $\nu(\nabla^i) = i_{\mathbf{U}}(\delta^i)$

Assertion (ii) *is* (Hyp 1).

We will first prove (Hyp r).

Let $\delta^r = \{A_{\iota, \iota+1}\}_{\iota \in \mathbf{I}} \in C^1(\mathbf{U}; \Lambda(\Delta_0)) = Z^1(\mathbf{U}; \Lambda(\Delta_0))$ *be a Stokes cocycle* ***with:***

$$(A_{\iota, \iota+1})_{d_\iota} \in Sto^{k_r}(\Delta_0)_{d_\iota}.$$

We have (for $\mathbf{d}_\iota \in (\mathbf{R}, 0)$ *"above"* d_1)

$\lambda_{\mathbf{d}_\iota}^{-1}(A_{\iota, \iota+1})_{d_\iota} = C_{\mathbf{d}_\iota; r}$, or $(A_{\iota, \iota+1})_{d_\iota} = F_{0, \mathbf{d}_\iota} C_{\mathbf{d}_\iota; r}(F_{0, \mathbf{d}_\iota})^{-1}$; if $V_{\iota, \iota+1}$ is the open arc of $(\mathbf{R}, 0)$ bisected by d_ι, with opening π/k_r, *then* $C_{\mathbf{d}_\iota; r} \in \mathrm{Sto}^{k_r}(Q; d)$, *and* $F_{0, \mathbf{d}_\iota} C_{\mathbf{d}_\iota; r}(F_{0, \mathbf{d}_\iota})^{-1}$ is the germ of a function belonging to $\Gamma(V_{\iota, \iota+1}; \Lambda^{\geq k_r})$. So the 1-cocycle δ^i is k_r*-summable*. It follows from *theorem* 12 that $\hat{H}^r = \hat{H}_r$ is k_r*-summable* and (a_r) is proved; (b_r) follows from *theorem* 13.

We suppose now that (Hyp j) is true for $r \geqq j \geqq i > 1$, and will prove (Hyp $i-1$).

Let $\delta^{i-1} = \{A_{\iota, \iota+1}^{i-1}\}_{\iota \in \mathbf{I}} \in C^1(\mathbf{U}; \Lambda(\Delta_0)) = Z^1(\mathbf{U}; \Lambda(\Delta_0))$ *be a Stokes cocycle* ***with:***

$$(A_{\iota, \iota+1}^{i-1})_{d_\iota} \in Sto^{\leqq k_{i-1}}(\Delta_0)_{d_\iota}.$$

Let $\{C_{\mathbf{d}_\iota}^{i-1}\} = z_{\mathbf{U}}^{-1}(\delta^{i-1})$. We have $C_{\mathbf{d}_\iota}^{i-1} \in \mathrm{Sto}^{\leqq k_{i-1}}(Q; d)$, and from the *decomposition* (*Lemma* 12):

$$\Lambda^{\leqq k_{i-1}}(Q; d) = \Lambda^{k_r}(Q; d) \ltimes \Lambda^{k_{r-1}}(Q; d) \ltimes \ldots \ltimes \Lambda^{k_{i-1}}(Q; d),$$

we get, for $C_{\mathbf{d}_\iota}^{i-1} \in \Lambda^{\leqq k_{i+1}}(Q; d)$, *a decomposition:*

$$C_{\mathbf{d}_\iota}^{i-1} = C_{\mathbf{d}_\iota; r} C_{\mathbf{d}_\iota; r-1} \ldots C_{\mathbf{d}_\iota; i-1}, \qquad \text{with} \quad C_{\mathbf{d}_\iota; j} \in \Lambda^{k_j}(Q; d)$$

$(j = r, \ldots, i-1)$.

We have

$$C_{\mathbf{d}_\iota}^{i-1} = C_{\mathbf{d}_\iota}^{i} C_{\mathbf{d}_\iota; i-1},$$

with $C_{\mathbf{d}_\iota}^{i} \in \Lambda^{\leqq k_i}(Q; d)$, *and* $C_{\mathbf{d}_\iota; i-1} \in \Lambda^{k_{i-1}}(Q; d)$.

Whe have $(A_{\iota, \iota+1}^{i})_{d_\iota} = \lambda_{\mathbf{d}_\iota}(C_{\mathbf{d}_\iota}^{i})$ (which is *independant* of the choice of $\mathbf{d}_\iota \in (\mathbf{R}, 0)$ "above" d_ι), and $\delta^i = \{A_{\iota, \iota+1}^{i}\}_{\iota \in \mathbf{I}} \in Z^1(\mathbf{U}; \Lambda(\Delta_0))$. If $\hat{H}^i = \mu^{-1} i_{\mathbf{U}}(\delta^i)$; then $\delta^i = \mathbf{St}(\mathbf{U}; \hat{H}^i)$.

It we set:

$$S_{k_i, k_{i+1}, \ldots, k_r; \mathbf{d}_\iota}^{+} \hat{H}^i = H_{\mathbf{d}_\iota}^{i+},$$

and

$$S_{k_i, k_{i+1}, \ldots, k_r; \mathbf{d}_\iota}^{-} \hat{H}^i = H_{\mathbf{d}_\iota}^{i-},$$

we get:

$$(\mathrm{H}_{\mathbf{d}_\iota}^{i-})^{-1}\,\mathrm{H}_{\mathbf{d}_\iota}^{i+}=(\mathrm{A}_{\iota,\,\iota+1}^{i})_{d_\iota}=\lambda_{\mathbf{d}_\iota}(C_{\mathbf{d}_\iota}^{i}),$$

or

$$\mathrm{H}_{\mathbf{d}_\iota}^{i+}\,\mathrm{F}_{0,\,\mathbf{d}_\iota}=\mathrm{H}_{\mathbf{d}_\iota}^{i-}\,\mathrm{F}_{0,\,\mathbf{d}_\iota}\,C_{\mathbf{d}_\iota}^{i}.$$

We set

$$(\mathrm{B}_{\iota,\,\iota+1})_{d_\iota}=\mathrm{H}_{\mathbf{d}_\iota}^{i+}\,\lambda_{\mathbf{d}_\iota}(C_{\mathbf{d}_\iota;\,i-1})\,(\mathrm{H}_{\mathbf{d}_\iota}^{i+})^{-1}.$$

Let $\mathrm{V}_{\iota,\,\iota+1}^{i}$ and $\mathrm{V}_{\iota,\,\iota+1}^{i-1}$ be open arcs of $(\mathbf{R}, 0)$ bisected by d_ι with respective openings π/k_i and π/k_{i-1} ($\mathrm{V}_{\iota,\,\iota+1}^{i-1}$ is contained in $\mathrm{V}_{\iota,\,\iota+1}^{i}$). Then the germ $\lambda_{\mathbf{d}_\iota}(C_{\mathbf{d}_\iota;\,i-1})$ is the germ at d_ι of a function $\mathrm{B}'_{\iota,\,\iota+1}$ belonging to $\Gamma(\mathrm{V}_{\iota,\,\iota+1}^{i-1};\,\Lambda^{\geq k_{i-1}})$ (this follows from $\mathrm{C}_{\mathbf{d}_\iota;\,i-1}\in\Lambda^{k_{i-1}}(\mathrm{Q};\,d)$). The germ $\mathrm{H}_{\mathbf{d}_\iota}^{i+}$ is the germ at d_ι of a function H^{i+} belonging to $\Gamma(\mathrm{V}_{\iota,\,\iota+1}^{i};\,\Lambda)$ *asymptotic to* $\hat{\mathrm{H}}^{i}$ on $\mathrm{V}_{\iota,\,\iota+1}^{i}$ (and, *a fortiori*, on $\mathrm{V}_{\iota,\,\iota+1}^{i-1}$). We conclude that the germ $(\mathrm{B}_{\iota,\,\iota+1})_{d_\iota}$ is the germ at d_ι of a function $\mathrm{B}_{\iota,\,\iota+1}$ belonging to $\Gamma(\mathrm{V}_{\iota,\,\iota+1}^{i-1};\,\Lambda^{\geq k_{i-1}})$.

We have built a k_{i-1}-*summable cochain* $\beta=\{\mathrm{B}_{\iota,\,\iota+1}\}_{\iota\in\mathrm{I}}$. We check easily that

$$\beta\in\mathrm{Z}^1(\mathbf{U};\,\Lambda(\Delta^i)).$$

Then it follows from *theorem* 12 that $\hat{\mathrm{H}}_{i-1}=\mu^{-1}\,i_{\mathbf{U}}(\beta)$ is k_{i-1}-*summable*, and from *theorem* 13 (i) that $(\Delta^i)^{\hat{\mathrm{H}}_{i-1}}=\Delta^{i-1}$ (***definition*** of Δ^{i-1}) is a ***meromorphic*** differential operator. We define

$$\hat{\mathrm{H}}^{i-1}=\hat{\mathrm{H}}_{i-1}\,\hat{\mathrm{H}}^{i}=\hat{\mathrm{H}}_{i-1}\,\hat{\mathrm{H}}_{i}\ldots\hat{\mathrm{H}}_{r}.$$

Then

$$\Delta^{i-1}=(\Delta^i)^{\hat{\mathrm{H}}_{i-1}}=(\Delta_0^{\hat{\mathrm{H}}^i})^{\hat{\mathrm{H}}_{i-1}}=\Delta_0^{\hat{\mathrm{H}}_{i-1}\,\hat{\mathrm{H}}^i}=\Delta_0^{\hat{\mathrm{H}}^{i-1}},$$

and

$$\hat{\mathrm{F}}^{i-1}=\hat{\mathrm{H}}^{i-1}\,x^{\mathrm{L}}\,\mathrm{U}\,e^{\mathrm{Q}(1/u)}$$

is a ***formal*** *fundamental solution of the* ***meromorphic*** *differential operator* Δ^{i-1}, ***formally*** *equivalent to* Δ_0.

Let

$$\mathrm{H}_{\mathbf{d}_\iota;\,i-1}^{+}=S_{k_{i-1},\,\ldots,\,k_r;\,\mathbf{d}_\iota}^{+}\,\hat{\mathrm{H}}_{i-1}\quad\text{and}\quad\mathrm{H}_{\mathbf{d}_\iota;\,i-1}^{-}=S_{k_{i-1},\,\ldots,\,k_r;\,\mathbf{d}_\iota}^{-}\,\hat{\mathrm{H}}_{i-1}.$$

We get:

$$\mathrm{H}_{\mathbf{d}_\iota;\,i-1}^{+}\,\mathrm{H}_{\mathbf{d}_\iota}^{i+}\,\mathrm{F}_{0,\,\mathbf{d}_\iota}=\mathrm{H}_{\mathbf{d}_\iota;\,i-1}^{-}\,\mathrm{H}_{\mathbf{d}_\iota}^{i-}\,\mathrm{F}_{0,\,\mathbf{d}_\iota}\,C_{\mathbf{d}_\iota}^{i}\,C_{\mathbf{d}_\iota;\,i-1}$$
$$\mathrm{H}_{\mathbf{d}_\iota;\,i-1}^{+}\,\mathrm{H}_{\mathbf{d}_\iota}^{i+}\,\mathrm{F}_{0,\,\mathbf{d}_\iota}=\mathrm{H}_{\mathbf{d}_\iota;\,i-1}^{-}\,\mathrm{H}_{\mathbf{d}_\iota}^{i-}\,\mathrm{F}_{0,\,\mathbf{d}_\iota}\,C_{\mathbf{d}_\iota}^{i-1};$$
$$\mathrm{H}_{\mathbf{d}_\iota}^{i-1+}\,\mathrm{F}_{0,\,\mathbf{d}_\iota}=\mathrm{H}_{\mathbf{d}_\iota}^{i-1-}\,\mathrm{F}_{0,\,\mathbf{d}_\iota}\,C_{\mathbf{d}_\iota}^{i-1}.$$

Then $\delta^{i-1}=\mathbf{St}(\mathbf{U};\,\hat{\mathrm{H}}^{i-1})=\mathbf{St}(\mathbf{U};\,\mu^{-1}\,i_{\mathbf{U}}(\delta^{i-1}))$. We have got (Hyp $i-1$) and *assertion* (ii) of *theorem* 14 is proved by induction. That ***concludes*** the proof of *theorem* 14.

Examples. — To illustrate the preceding constructions, it is possible to compute the "wild groups" and their representations for *Airy equation* and *Kummer equations*. This is a simple reformulation of computations of [MR 2], *chapter* 3.

Remark. — For $\mathbf{d} \in (\mathbf{R}, 0)$ *the "label"* $\gamma_{\mathbf{d}} \in \Pi(\mathbf{q})$ will later (*see* 6, infra) correspond to a *loop* pointed at "$\mathbf{R}^+$" = ("0", $\mathbf{R}^+$) $\in$ { "0" } $\times (\mathbf{R}, 0)$ ("$\mathbf{R}^+$" is a point belonging to the universal covering of the real blow-up of the origin {"0"} $\times S^1$ in the analytic halo).

We start from "$\mathbf{R}^+$" and go (along {"0"} $\times (\mathbf{R}, 0)$) to

$$(\text{"0"}, \mathbf{d}) \in \{\text{"0"}\} \times (\mathbf{R}, 0).$$

Then we turn clockwise around "$]0, +\infty]$" $\times \{\mathbf{d}\}$ onto the universal covering of $\mathbf{C}^*$ with an analytic halo at zero and go back to ("0", $\mathbf{d}$). Afterwards we return to "$\mathbf{R}^+$" (along "{ 0 }" $\times (\mathbf{R}, 0)$). Then groups $\Pi(\mathbf{q})$ and $\mathrm{G}\Pi(\mathbf{q})$ are interpreted as "wild fundamental groups *pointed* at

$$\text{"}\mathbf{R}^+\text{"} \in \{\text{"0"}\} \times S^1\text{"}.$$

The *Stokes operator* $St_{\mathbf{d}}(\Delta_0)$ corresponds to a *"wild monodromy"* along the loop $\gamma_{\mathbf{d}}$ for the vector space of "germs of solutions of the differential operator Δ at "$\mathbf{R}^+$"", *modulo an isomorphism* between this linear space and the linear space of *formal* solutions of Δ_0 (in order to get this isomorphism we use the *"analyticity"* of $\hat{\mathrm{H}}$ near 0 *in the analytic halo* and choose the principal determination for the Logarithm: $\Delta = \Delta_0^{\hat{\mathrm{H}}}$). The *"wild connections"* induced by ∇_0 and ∇ in a *"small"* sector of the *universal covering of the analytic halo* bisected by $\mathbf{R}^+$ are *isomorphic* (H is a *wild analytic* function in such a sector), then the representation $\rho_{\mathrm{irr}}(\nabla)$ *in* $\mathrm{GL}(n.\mathbf{C})$, up to the action of $\mathscr{T}(\mathrm{F}_0)$, can be interpreted as a *representation* of $\Pi(\mathbf{q}(\nabla))$ in the linear group $\mathrm{GL}(\mathrm{V})$ of the vector space $\mathrm{V} = \mathrm{Sol}_{\text{"}\mathbf{R}^+\text{"}}(\nabla)$ of *germs of horizontal sections of* ∇ "at "$\mathbf{R}^+$" $\in$ {"0"} $\times S^1$" (V can be identified with a subspace of $(\mathrm{K}\langle x^{\mathrm{L}}\rangle \otimes_{\mathrm{K}} \hat{\mathrm{K}}\langle e^{\mathrm{Q}}\rangle)^n$, where $\mathrm{K}\langle x^{\mathrm{L}}\rangle$ is identifed with a space of germs of meromorphic functions on sectors bisected by $\mathbf{R}^+$, and a *class* modulo $\mathscr{T}(\mathrm{F}_0)$ corresponds to a *uniquely determined* representation in $\mathrm{GL}(\mathrm{V})$). Finally we get a "*wild monodromy*" (which does not depend on the choices of Δ_0 and F_0). This "wild monodromy" expresses the "*difference*" between ∇ and ∇_0. In fact we want to understand ∇ *independantly* of ∇_0. In order to do that we will first *translate* ∇_0 in terms of *linear representation.*

Let

$$\mathsf{E} = \bigcup_{\mathbf{q}} \mathsf{E}(\mathbf{q}) = \varinjlim_{\mathbf{q}} \mathsf{E}(\mathbf{q}).$$

Let $\mathscr{T}(\mathbf{q})$ *be the exponential torus associated to*

$$\mathbf{q} = \{q_1, q_2, \ldots, q_n\} \subset \mathsf{E} \qquad (\mathscr{T}(\mathbf{q}) = \mathrm{Aut}_{\mathrm{K}_{\mathrm{v}}} \mathbb{L}_{\mathrm{v}}).$$

To *natural injections*

$$\mathbf{E}(\mathbf{q}) \to \mathbf{E}$$

correspond *natural projections*

$$\mathcal{T}(\mathbf{q}) \to \mathcal{T}.$$

We write $\mathcal{T} = \varprojlim_{\mathbf{q}} \mathcal{T}(\mathbf{q})$. By definition $\mathcal{T}$ is the ***exponential torus***; it is a ***commutative*** *group*. The algebraic tori $\mathcal{T}(\mathbf{q})$ are endowed with the *Zariski topology*, and $\mathcal{T}$ is endowed with the corresponding *inverse limit topology*.

LEMMA 16. – (i) *Let* $\kappa: \mathcal{T} \to \mathbf{C}^*$ *be a continuous homomorphism of groups. Then there exists a uniquely determined* $q \in \mathbf{E}$, *such that* κ *is equal to the composition of the natural projection* $\mathcal{T} \to \mathcal{T}(\mathbf{q})$ $(\mathbf{q} = \{q\})$ *and of the character* $q: \mathcal{T}(\mathbf{q}) \to \mathbf{C}^*$. (*We will identify* κ *and* q.)

(ii) *Let* V *be a finitely dimensional* $\mathbf{C}$-*vector space* ($n = \dim_{\mathbf{C}} \mathrm{V}$), *and* $\theta: \mathcal{T} \to \mathrm{GL}(\mathrm{V})$ *be a continuous homomorphism of groups. Let* $\mathrm{G} = \theta(\mathcal{T})$.

Then there exists a basis of V *such that the subgroup* G *of* GL(V), *identified by the choice of this basis to* $\mathrm{GL}(n; \mathbf{C})$, *is* ***diagonal.*** *If* $\varphi_1, \varphi_2, \ldots, \varphi_n: \mathrm{G} \to \mathbf{C}^*$ *are the corresponding homomorphisms of groups* [*if* $g \in \mathrm{G}$, $\varphi_1(g)$ *is the first entry of* g *on the diagonal*...], *and if* q_i *is associated to* $\kappa_i = \varphi_i \theta$, *like in* (i) *it is possible to associate to* κ *the set* $\mathbf{q} = \{q_1, q_2, \ldots, q_n\} \subset \mathbf{E}$, which is ***independent*** *of the choice of the basis of* V, *and* θ *is the composition of the natural projection* $\mathcal{T} \to \mathcal{T}(\mathbf{q})$ *and of*

$$(q_1, q_2, \ldots, q_n): \mathcal{T}(\mathbf{q}) \to \mathrm{GL}(n; \mathbf{C}) = \mathrm{GL}(\mathrm{V}).$$

For $\tau \in \mathcal{T}$, $\theta(\tau) = \mathrm{Diag}(q_1(\tau), q_2(\tau), \ldots, q_n(\tau))$.

In the situation of *lemma* 16 (ii), we will write $\mathbf{q} = \mathbf{q}_\theta$. From a given $\mathbf{q} = \{q_1, q_2, \ldots, q_n\} \subset \mathbf{E}$ we get a diagonal representation $\theta: \mathcal{T} \to \mathrm{GL}(n; \mathbf{C})$, *uniquely determined up to conjugacy*, such that $\mathbf{q} = \mathbf{q}_\theta$.

Let ∇ be a ***formal*** *connection*. There exists a *uniquely determined* (*up to conjugacy*) representation ([56]) $\theta: \mathcal{T} \to \mathrm{GL}(n; \mathbf{C})$, such that $\mathbf{q}(\nabla) = \mathbf{q}_\theta$.

Let $\mathrm{F}_0(x) = x^{\mathrm{L}} \mathrm{U} e^{\mathrm{Q}(1/u)}$, with $u^\nu = x$, be a *formal fundamental solution* of a differential operator whose associate *formal connection* is ∇ [$\mathbf{q}(\nabla)$ is then the set of the diagonal entries of $\mathrm{Q} = \mathrm{Diag}(q_1, q_2, \ldots, q_n)$]. Using F_0 we will identify the space V of *horizontal sections* of ∇ with $\mathbf{C}^n$.

Let $(\hat{\gamma}_0)$ be *the* free group generated by $\hat{\gamma}_0$. We define *an action of the group* $(\hat{\gamma}_0)$ *on the lattice* $\mathbf{E}$ by

$q\hat{\gamma}_0(u) = q(e^{-2i\pi/\nu} u)$, and *an action of the group* $(\hat{\gamma}_0)$ *on the exponential torus* $\mathcal{T}$ by $\hat{\gamma}_0 \tau(q) = \tau(q\hat{\gamma}_0)$, for arbitrary $\tau \in \mathcal{T}$ *and* $q \in \mathbf{E}$.

([56]) In the following *all the representations are supposed to be continuous.*

By *definition* the *wild formal fundamental group* $\pi_{1,\,sf}$ ((**C***, 0); "$\mathbf{R}^+$") of (**C***, 0) pointed at "$\mathbf{R}^+$" is the *semi-direct product*

$$(\hat{\gamma}_0) \ltimes \mathcal{T} \text{ built from the action of } (\hat{\gamma}_0) \text{ on } \mathcal{T}.$$

Let $\hat{M} = U^{-1} e^{2i\pi L} U$ be the formal monodromy matrix associated to F_0. *We set*

$$\hat{\rho}(\hat{\nabla})(\hat{\gamma}_0) = \hat{M}, \quad \text{and, for } \tau \in \mathcal{T},$$
$$\hat{\rho}(\hat{\nabla})(\tau) = \operatorname{Diag}(q_1(\tau), q_2(\tau), \ldots, q_n(\tau)).$$

We have

$$\hat{M}^{-1} Q(1/u) \hat{M} = Q(e^{-2i\pi/\nu}/u)$$
$$\hat{M}^{-1} \operatorname{Diag}(q_1, q_2, \ldots, q_n) \hat{M} = (q_1 \hat{\gamma}_0, q_2 \hat{\gamma}_0, \ldots, q_n \hat{\gamma}_0)$$
$$\hat{M}^{-1} \operatorname{Diag}(q_1(\tau), q_2(\tau), \ldots, q_n(\tau)) \hat{M} = (q_1 \hat{\gamma}_0(\tau), q_2 \hat{\gamma}_0(\tau), \ldots, q_n \hat{\gamma}_0(\tau))$$
$$\hat{\rho}(\hat{\nabla})(\hat{\gamma}_0)^{-1} \hat{\rho}(\hat{\nabla})(\tau) \hat{\rho}(\hat{\nabla})(\hat{\gamma}_0) = \hat{M}^{-1} \hat{\rho}(\hat{\nabla})(\tau) \hat{M} = \hat{\rho}(\hat{\nabla})(\hat{\gamma}_0 \tau).$$

So we have defined a ***linear representation***

$$\hat{\rho}(\hat{\nabla}) : \pi_{1,\,sf}((\mathbf{C}^*, 0); \text{"}\mathbf{R}^+\text{"}) = (\hat{\gamma}_0) \ltimes \mathcal{T} \to \mathrm{GL}(n; \mathbf{C}),$$

*associated to the **formal connection*** $\hat{\nabla}$. (Interpreted as a representation in $\mathrm{GL}(\hat{V})$, where $\hat{V}$ is the vector space $\mathrm{Sol}_{\text{"}\mathbf{R}^+\text{"}}(\hat{\nabla})$ of horizontal sections of $\hat{\nabla}$, this representation is independant of the choice of F_0.)

We will see now that, given a finite dimensional vector space $\hat{V}$ and a linear representation

$$\rho_1: \quad \pi_{1,\,sf}((\mathbf{C}^*, 0); \text{"}\mathbf{R}^+\text{"}) \to \mathrm{GL}(\hat{V}),$$

*there exists a **unique formal connection*** $\hat{\nabla}$, such that $\rho_1 = \hat{\rho}(\hat{\nabla})$ ($\hat{V}$ being identified with the vector space $\mathrm{Sol}_{\text{"}\mathbf{R}^+\text{"}}(\hat{\nabla})$ of horizontal sections of $\hat{\nabla}$).

We set $\rho_1(\hat{\gamma}_0) = \hat{M}$ and $\rho_1(\mathcal{T}) = T_1$. We set $\mathbf{q} = \mathbf{q}_\theta$; θ being the restriction of ρ_1 to $\mathcal{T}$, $\mathbf{q}$ is Galois invariant (it is invariant by the action of $\hat{M}$). We can choose a basis of V in such a way that T_1 is a diagonal group: $T_1 = \{ Q(\tau) = \operatorname{Diag}(q_1(\tau), q_2(\tau), \ldots, q_n(\tau))/\tau \in \mathcal{T} \}$ ($\mathbf{q} = \{ q_1, q_2, \ldots, q_n \}$, and $Q = \operatorname{Diag}(q_1, q_2, \ldots, q_n)$). Using a method of [BJL], [J], we can suppose moreover that we have chosen our basis such that $U \hat{M} U^{-1}$ is in Jordan form. Then let L be such that $e^{2i\pi L} = U \hat{M} U^{-1}$ (L is defined up to multiplication on the right by a diagonal matrix $\operatorname{Diag}(m_1, m_2, \ldots, m_n)$, $m_i \in \mathbf{Z}$). Then $F_0 = x^L U e^Q$ is a fundamental solution of a rational differential operator Δ_0 and the corresponding connection ∇_0 is *independant of the choice* of the *basis* and of the *integers* m_i. We have clearly $\rho(\nabla_0) = \rho_1$ ($\mathbf{C}^n$ being identified by F_0 with the space of horizontal sections of ∇_0).

So we get

THEOREM 15. – *The natural map*

$$\textbf{\textit{Formal meromorphic}}\ \textit{connections} \xrightarrow{\hat{\rho}} \begin{array}{c}\textit{finite dimensional}\\ \textit{linear representations}\\ \textit{of the group}\ \pi_{1,\,sf}((\mathbf{C}^*, 0); \text{"}\mathbf{R}^+\text{"}).\end{array}$$

$$\hat{\nabla} \to \hat{\rho}(\hat{\nabla})$$

is an isomorphism.

This isomorphism is compatible with *sums*, *duality*, *tensor products*,... It is an *isomorphism of Tannakian categories*.

If now ∇ is a *germ of* ***meromorphic*** *connection*, we get from ∇ ***two*** *linear representations* ($V = \mathrm{Sol}_{\text{“}\mathbf{R}^+\text{”}}(\nabla)$) :

$$\hat{\rho}(\nabla):\quad \pi_{1,\,sf}((\mathbf{C}^*, 0);\ \text{“}\mathbf{R}^+\text{”}) \to \mathrm{GL}(V),$$

and

$$\rho_{irr}(\nabla):\quad \mathrm{G}\Pi(\mathbf{q}) \to \mathrm{GL}(V).$$

The respective restrictions of these representations $\hat{\rho}(\nabla)$ and $\rho_{irr}(\nabla)$ to the respective subgroups $(\hat{\gamma}_0)$ of $\pi_{1,\,sf}((\mathbf{C}^*, 0);\ \text{“}\mathbf{R}^+\text{”})$ and $\mathrm{G}\Pi(\mathbf{q})$ are clearly equal.

Conversely, two linear representations

$$\rho_1:\quad \pi_{1,\,sf}((\mathbf{C}^*, 0);\ \text{“}\mathbf{R}^+\text{”}) \to \mathrm{GL}(V),$$

and

$$\rho_2:\quad \mathrm{G}\Pi(\mathbf{q}) \to \mathrm{GL}(V),$$

admitting *equal restrictions* to the subgroups

$$(\hat{\gamma}_0) \subset \pi_{1,\,sf}((\mathbf{C}^*, 0);\ \text{“}\mathbf{R}^+\text{”}) \qquad \text{and} \qquad (\hat{\gamma}_0) \subset \mathrm{G}\Pi(\mathbf{q}),$$

being given, it is general *impossible* to find a germ of *meromorphic connection* ∇ such that $\hat{\rho}(\nabla) = \rho_1$ and $\rho_{irr}(\nabla) = \rho_2 : \rho_1$ and ρ_2 must satisfy a *"Stokes condition"* [checked on $\mathrm{GL}(V)$ in place of $\mathrm{GL}(n;\mathbf{C})$; cf. *theorem* 10]; $V = \mathrm{Sol}_{\text{“}\mathbf{R}^+\text{”}}(\nabla)$.

PROPOSITION 13. — *Let* Δ_0 *be a given differential operator with a fixed fundamental solution* $F_0 = x^L U\, e^{Q(1/u)}$. *Let* ∇_0 *be the connection defined by* Δ_0.

Then the natural map

Germs of ***meromorphic*** *connections* ∇ ***formally*** *equivalent to* ∇_0.	$\to$	*Pairs of representations of the groups* $\pi_{1,\,sf}((\mathbf{C}^*, 0);\ \text{“}\mathbf{R}^+\text{”})$ *and* $\mathrm{G}\Pi(\mathbf{q})$ *in* $\mathrm{GL}(V)$ *coincident on the two subgroups corresponding to* $(\hat{\gamma}_0)$, *such that the first representation corresponds to* $\rho(\nabla_0)$, *and satisfying* ***"Stokes conditions"***.

$$\nabla \to (\rho(\nabla),\ \rho_{irr}(\nabla)),$$

is a ***bijection*** ($V = \mathrm{Sol}_{\text{“}\mathbf{R}^+\text{”}}(\nabla)$).

The next step is now to build a new group $\pi_{1,\,s}((\mathbf{C}^*, 0);\ \text{“}\mathbf{R}^+\text{”})$, the *wild fundamental group* of $(\mathbf{C}^*, 0)$, pointed at "$\mathbf{R}^+$", satisfying the following properties:

(i) The wild fundamental group is a semi-direct product

$$\pi_{1,\,s}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”}) = \pi_{1,\,sf}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”}) \ltimes \mathcal{R}$$

$$\pi_{1,\,s}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”}) = ((\hat{\gamma}_0) \ltimes \mathcal{T}) \ltimes \mathcal{R},$$

where $\mathcal{R}$ (the *resurgent group*) is the *"exponential"* of a *free Lie algebra Lie* $\mathcal{R}$ (the *resurgent Lie algebra*), with infinitely many generators.

(ii) To each germ ∇ of rank *n meromorphic connection* we can associate a linear representation ($V = \mathrm{Sol}_{\text{“}\mathbf{R}^+\text{”}}(\nabla)$):

$$\rho_s(\nabla): \quad \pi_{1,\,s}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”}) \to \mathrm{GL}(V),$$

such that the restriction of $\rho_s(\nabla)$ to $\pi_{1,\,sf}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”})$ is $\hat{\rho}(\nabla)$, and such that, $\hat{\rho}(\nabla)$ *being known*, the knowledge of the restriction of $\rho_s(\nabla)$ to the resurgent group $\mathcal{R}$ is *equivalent* to ghe knowledge of the representation

$$\rho_{\mathrm{irr}}(\nabla): \quad \mathrm{G}\Pi(\mathbf{q}) \to \mathrm{GL}(V)\,(\mathbf{q} = \mathbf{q}(\nabla)).$$

(iii) If a finite dimensional representation ([57]) of the wild fundamental group

$$\rho_0: \quad \pi_{1,\,s}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”}) \to \mathrm{GL}(V), \text{ is given}$$

we denote by ρ_1 the restriction of ρ_0 to $\pi_{1,\,sf}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”})$, and

$$\rho_2: \mathrm{G}\Pi(\mathbf{q}) \to \mathrm{GL}(V)$$

the representation corresponding to the restriction of ρ_0 to the *resurgent group* $\mathcal{R}$ (and to the knowledge of ρ_1...), with $\mathbf{q} = \mathbf{q}_{\rho_0}$. Then the pair (ρ_1, ρ_2) satisfies *"Stokes conditions"*, so there exists (*Proposition* 13) a uniquely determined germ of meromorphic connection ∇ such that

$$(\rho(\nabla), \rho_{\mathrm{irr}}(\nabla)) = (\rho_1, \rho_2),$$

and $(\rho(\nabla), \rho_{\mathrm{irr}}(\nabla))$ can be recovered from the representation

$$\rho_s(\nabla): \quad \pi_{1,\,s}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”}) \to \mathrm{GL}(V)$$

got from ∇ using the construction of (ii) ($V = \mathrm{Sol}_{\text{“}\mathbf{R}^+\text{”}}(\nabla)$).

Let $\mathbf{q} = \{q_1, q_2, \ldots, q_n\} \subset \mathsf{E}$, and, after ordering, let Q denote the diagonal matrix $\mathrm{Diag}\{q_1, q_2, \ldots, q_n\}$. Let $\mathcal{T}(\mathbf{q})$ be the exponential torus associated to $\mathbf{q}$, and let $T(\mathrm{Q})$ be its representation in $\mathrm{GL}(n; \mathbf{C})$ given by Q.

An element $\tau \in \mathcal{T}(\mathrm{Q})$ is *represented* by the matrix

$$\mathrm{Q}(\tau) = \mathrm{Diag}(q_1(\tau), q_2(\tau), \ldots, q_n(\tau)) \in T(\mathrm{Q}) \subset \mathrm{GL}(n; \mathbf{C}).$$

LEMMA 17. – *Let* $\mathbf{q} = \{q_1, q_2, \ldots, q_n) \subset \mathsf{E}$, *and, after ordering, let* Q *be the diagonal matrix* $\mathrm{Q} = \mathrm{Diag}(q_1, q_2, \ldots, q_n\}$. *Let* $\mathrm{C} \in \mathrm{End}(n; \mathbf{C})$,

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([57]) The restriction to $\mathcal{T}$ of such a representation will be *allway* supposed *continuous* in the following.

$C=(c_{ij})\,(q_{ij}=q_i-q_j)$. *Then:*

(i) $\tau C \tau^{-1}=\mathrm{Q}(\tau)\, C\, \mathrm{Q}(\tau)^{-1}=(c_{ij}\, q_{ij}(\tau))$.

(ii) *Let* $q\in \mathbf{E}$, $q\neq 0$ *and*

$$C_q=(a_{ij}),\qquad \text{with}\quad a_{ij}=0 \text{ if } q_i-q_j\neq q,\quad \text{and}\quad a_{ij}=c_{ij} \text{ if } q_{ij}=q.$$

Then:

$$\tau C_q \tau^{-1}=\mathrm{Q}(\tau)\, C_q\, \mathrm{Q}(\tau)^{-1}=q(\tau)\, C_q.$$

(iii) *Let* $\mathrm{Dia}(C)$ *be the diagonal matrix with the same diagonal entries as* C. *Then*

$$\tau C \tau^{-1}=\mathrm{Dia}(C)+\sum_{i,\, j} q_{i,\, j}(\tau)\, C_{q_{i,\, j}}\qquad (\text{with } C_q=0 \text{ if } q=0),$$

and such a decomposition is ***uniquely*** *determined,* i. e. *if*

$$\tau C \tau^{-1}=\mathrm{Dia}(C)+\sum_{i,\, j} q_{i,\, j}(\tau)\, A_{q_{i,\, j}},\quad \text{then } A_{q_{i,\, j}}=C_{q_{i,\, j}}.$$

(iv) *Let* $\mathbf{d}\in(\mathbf{R}, 0)$. *If* $C\in \mathrm{Sto}(\mathrm{Q};\, \mathbf{d})$, *then:*

$$\tau C \tau^{-1}=\mathrm{I}+\sum_q q(\tau)\, C_q, \text{ the sum being extended to } q=q_{i,\, j},$$

with $q_i \ll_{\mathbf{d}} q_j$,

$$\tau C \tau^{-1}=\mathrm{I}+\sum_{q\in \mathbf{E}_{\mathbf{d}}(q)} q(\tau)\, C_q.$$

(v) *Let* $\mathbf{d}\in(\mathbf{R}, 0)$. *If* $C\in \mathrm{Lie\ Sto}(\mathrm{Q};\, \mathbf{d})$, *the Lie algebra of* $\mathrm{Sto}(\mathrm{Q};\, \mathbf{d})$, *then:*

$$\tau C \tau^{-1}=\sum q(\tau)\, C_q,$$

the sum being extended to $q=q_{i,\, j}$, *with* $q_i \ll_{\mathbf{d}} q_j$,

$$\tau C \tau^{-1}=\sum_{q\in \mathbf{E}_{\mathbf{d}}(q)} q(\tau)\, C_q.$$

The only non trivial point is *unicity* in (iii).

Let $(p_1, p_2, \ldots, p_\nu)$ be a **Z**-*basis* of the lattice $\mathbf{E}(\mathbf{q})$.

We have an *isomorphism*

$$(p_1, p_2, \ldots, p_\nu):\quad \mathcal{T}(\mathbf{q})\to(\mathbf{C}^*)^\nu$$
$$(p_1, p_2, \ldots, p_\nu):\quad \tau\to(p_1(\tau), p_2(\tau), \ldots, p_\nu(\tau)).$$

We set $p_k(\tau)=\tau_k\,(k=1, \ldots, \nu)$. Then each $q_{i,\, j}(\tau)$ is a *monomial* in the variables $\tau_k\in\mathbf{C}^*$ and the *distinct* $q_{i,\, j}(\tau)$ are *independant on* **C**.

The decomposition (iii) appears as a "***Fourier decomposition***" of the "*unfolding*" $\tau C \tau^{-1}$ of the matrix C by the *adjoint action of the exponential torus* $\mathcal{T}(\mathbf{q})$.

Let $\Delta=d/dx-\mathrm{A}$, where $\mathrm{A}\in \mathrm{End}(n;\, \mathbf{C}\{x\}[x^{-1}])$, be a germ of meromorphic differential operator at the origin of the complex plane **C**.

Let $\hat{F}(x)=\hat{H}(x)x^{L}U e^{Q(1/u)}$ be a formal fundamental solution of Δ as above. We set

$F_0(x)=x^{L}U e^{Q}$, $\mathbf{q}=\mathbf{q}(Q)$, and we denote by n the rank of Δ.

Let $\mathbf{d}\in \mathrm{Fr}(\mathbf{q})$ and let $\mathrm{St}_{\mathbf{d}}(\Delta)$ be the corresponding Stokes matrix. For ***every*** $\tau\in\mathcal{T}$ the matrix $\tau\,\mathrm{St}_{\mathbf{d}}(\Delta)\,\tau^{-1}$ belongs to the image of the representation of $\mathrm{Gal}_K(\Delta)$ in $\mathrm{GL}(n;\mathbf{C})$ associated to $\hat{F}$, the matrix $\mathrm{St}_{\mathbf{d}}(\Delta)$ is *unipotent* and $\tau\,(\mathrm{Log}\,\mathrm{St}_{\mathbf{d}}(\Delta))\,\tau^{-1}$ belongs to the representation of *Lie* $\mathrm{Gal}_K(\Delta)$ in $\mathrm{End}(n;\mathbf{C})$ associated to $\hat{F}$, that is, it yields a *Galois derivation* of the field $K\langle\hat{F}\rangle$. Then it follows from *Lemma* 17 ($\mathrm{St}_{\mathbf{d}}(\Delta)\in\mathrm{Sto}(Q;\mathbf{d})$) that we have a uniquely determined decomposition

$$\tau\,(\mathrm{Log}\,\mathrm{St}_{\mathbf{d}}(\Delta))\,\tau^{-1}=\sum q(\tau)\,\mathrm{Log}\,\mathrm{St}_{\mathbf{d}}(\Delta)_q,$$

the sum being extended to $q=q_{i,j}$, *with* $q_i \ll_{\mathbf{d}} q_j$, or

$$\tau\,(\mathrm{Log}\,\mathrm{St}_{\mathbf{d}}(\Delta))\,\tau^{-1}=\sum_{q\in \mathbf{E}_{\mathbf{d}}(\mathbf{q})} q(\tau)\,\mathrm{Log}\,\mathrm{St}_{\mathbf{d}}(\Delta)_q,$$

where each $\mathrm{Log}\,\mathrm{St}_{\mathbf{d}}(\Delta)_q$ belongs to the representation of *Lie* $\mathrm{Gal}_K(\Delta)$ in $\mathrm{End}(n;\mathbf{C})$. associated to $\hat{F}$, that is yields a ***Galois derivation*** of the field $K\langle\hat{F}\rangle$. *We have performed a **"Fourier analysis of the infinitesimal Stokes phenomena"**.*

THEOREM 16. – *Let* $\Delta=d/dx-A$, *where* $A\in\mathrm{End}(n;\mathbf{C}\{x\}[x^{-1}])$, *be a germ of meromorphic differential operator at the origin of the complex plane* $\mathbf{C}$. *We set* $\mathbf{q}=\mathbf{q}(\Delta)$, *and* denote by *n the rank of* Δ. *Then* $\tau\,(\mathrm{Log}\,\mathrm{St}_{\mathbf{d}}(\Delta))\,\tau^{-1}$ *belongs to Lie* $\mathrm{Gal}_K(\Delta)$ *for each* $\mathbf{d}\in\mathrm{Fr}(\mathbf{q})$, *and we have a uniquely determined decomposition*

$$\tau\,(\mathrm{Log}\,\mathrm{St}_{\mathbf{d}}(\Delta))\,\tau^{-1}=\sum q(\tau)\,\mathrm{Log}\,\mathrm{St}_{\mathbf{d}}(\Delta)_q,$$

the sum being extended to $q=q_{i,j}$, *with* $q_i \ll_{\mathbf{d}} q_j$, *or*

$$\tau\,(\mathrm{Log}\,\mathrm{St}_{\mathbf{d}}(\Delta))\,\tau^{-1}=\sum_{q\in \mathbf{E}_{\mathbf{d}}(\mathbf{q})} q(\tau)\,\mathrm{Log}\,\mathrm{St}_{\mathbf{d}}(\Delta)_q,$$

with each $\mathrm{Log}\,\mathrm{St}_{\mathbf{d}}(\Delta)_q$ *belonging to Lie* $\mathrm{Gal}_K(\Delta)$.

Moreover

$$\tau\,(\mathrm{Log}\,\mathrm{St}_{\mathbf{d}}(\Delta)_q)\,\tau^{-1}=q(\tau)\,\mathrm{Log}\,\mathrm{St}_{\mathbf{d}}(\Delta)_q,$$

and

$$\hat{M}\,\mathrm{St}_{\mathbf{d}}(\Delta)_q\,\hat{M}^{-1}=\mathrm{St}_{\exp(-2i\pi)\mathbf{d}}(\Delta)_q,\ \textit{for every}\ q\in\mathbf{E}.$$

It is now natural to introduce the ***free complex Lie algebra*** *Lie* $\mathcal{R}$ generated by all "*letters*" $\dot{\Delta}_{q,\mathbf{d}}$ where $(q,\mathbf{d})$ *is such that* $q\in\mathbf{E}$ and $\mathbf{d}\in\mathrm{Fr}\,q$ (*i.e.* such that e^{q} is "*maximally decaying*" on $\mathbf{d}$). We will name it the *resurgent Lie algebra* ([58]).

([58]) Because it contains *all Ecalle's resurgent algebras.*

In the situation of *theorem* 16 we get a *linear representation*

$$\text{Lie}\,\rho_{\text{res}}(\Delta):\quad \text{Lie}\,\mathcal{R} \to \text{End}(n;\mathbf{C})$$
$$\text{Lie}\,\rho_{\text{res}}(\Delta):\quad \dot{\Delta}_{q,\,\mathbf{d}} \to \text{Log}\,\text{St}_{\mathbf{d}}(\Delta)_q \qquad \text{if} \quad \mathbf{d} \in \text{Fr}(\mathbf{q}),$$

and

$$\text{Lie}\,\rho_{\text{res}}(\Delta):\quad \dot{\Delta}_{q,\,\mathbf{d}} \to 0, \qquad \text{if} \quad \mathbf{d} \notin \text{Fr}(\mathbf{q}).$$

We define an action of the *wild formal fundamental group*

$$\pi_{1,\,sf}(\mathbf{C}^*,0);\text{“}\mathbf{R}^+\text{”}) = (\hat{\gamma}_0) \ltimes \mathcal{T}$$

on the *resurgent Lie algebra Lie* $\mathcal{R}$ by

$$\hat{\gamma}_0\,\dot{\Delta}_{q,\,\mathbf{d}}\,\hat{\gamma}_0^{-1} = \dot{\Delta}_{q,\,\exp(-2i\pi)\,\mathbf{d}},$$

and

$$\tau\,\dot{\Delta}_{q,\,\mathbf{d}}\,\tau^{-1} = q(\tau)\,\dot{\Delta}_{q,\,\mathbf{d}}.$$

If we denote by $\hat{\rho}(\Delta)$ the representation

$$\hat{\rho}(\Delta):\ \pi_{1,\,sf}((\mathbf{C}^*,\mathbf{0});\text{“}\mathbf{R}^+\text{”}) \to \text{GL}(n;\mathbf{C})$$

associated to the *formal connection* defined by the differential operator Δ, the above action is "*compatible*" *with the pair of representations* ($\hat{\rho}(\Delta)$, Lie $\rho_{\text{res}}(\Delta)$) (*theorem* 16).

PROPOSITION 14. – *The natural map*

Pairs of representations $(\rho_1, \text{L}\,\rho)$ *of the group* $\pi_{1,\,sf}((\mathbf{C}^*,0);\text{“}\mathbf{R}^+\text{”})$ *in* $\text{GL}(n;\mathbf{C})$ *and of the Lie algebra Lie* $\mathcal{R}$ *in* $\text{End}(n;\mathbf{C})$ *which are "compatible" with the action of* $\pi_{1,\,sf}((\mathbf{C},0);\text{“}\mathbf{R}^+\text{”})$ *on Lie* $\mathcal{R}$.	$\to$	*Pairs of representations of the groups* $\pi_{1,\,sf}((\mathbf{C}^*,0);\text{“}\mathbf{R}^+\text{”})$ *and* $\text{G}\Pi(\mathbf{q}_{\rho_1})$ $\text{GL}(n;\mathbf{C})$ *which coincide on the two subgroups corresponding to* $(\hat{\gamma}_0)$ *and satisfy* ***Stokes conditions.***

$$(\rho_1, \text{L}\,\rho) \to (\rho_1, \rho_2)$$

where

$$\text{Log}\,\rho_2(\gamma_{\mathbf{d}}) = \sum q(\tau)\,\text{L}\,\rho(\dot{\Delta}_{q,\,\mathbf{d}}) \quad \textit{for every} \quad \mathbf{d} \in \text{Fr}(\mathbf{q}_{\rho_1}),$$

is a ***bijection.***

From *Projections* 13 *and* 14, we get a *first version* of the ***"wild Riemann Hilbert correspondence"***:

THEOREM 17. — *The natural map* (*where* V *is a finite dimensional space:* $V = \mathrm{Sol}_{\text{“}\mathbf{R}^+\text{”}}(\nabla)$)

Germs of ***meromorphic*** *connections* ∇ *at the origin.*	$\rightarrow$	*Pairs of* ***representations*** *of the* ***group*** $\pi_{1,\,sf}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”})$ *in* GL(V) *and of the* ***Lie algebra*** *Lie* $\mathscr{R}$ *in* End(V) *which are* ***"compatible"*** *with the action of* $\pi_{1,\,sf}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”})$ *on Lie* $\mathscr{R}$.

$$\nabla \rightarrow (\hat{\rho}(\nabla), \text{Lie } \rho_{\text{res}}(\Delta))$$

is a ***bijection***.

In order to get the wanted result, that is the *classification of germs of meromorphic connections in terms of representations of a* ***group***, it only remains to replace the *resurgent* ***Lie algebra*** *Lie* $\mathscr{R}$ by a ***group***, the *resurgent group* $\mathscr{R}$ (the *"exponential"* of *Lie* $\mathscr{R}$), and the *action* of the *wild formal fundamental group* $\pi_{1,\,sf}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”})$ on the *Lie algebra Lie* $\mathscr{R}$ by an *action* of the same group on the *group* $\mathscr{R}$. Then we will get a pair of representations $(\hat{\rho}(\nabla), \rho_{\text{res}}(\Delta))$ in $\mathrm{GL}(n; \mathbf{C}) = \mathrm{GL}(V)$ of the groups $\pi_{1,\,sf}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”})$ and $\mathscr{R}$ respectively, *compatible* with the *action* of the first group on the second, that is a *representation of the* ***semidirect*** *product* (defined by the *same action*)

$$\pi_{1,\,sf}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”}) \ltimes \mathscr{R} \quad \text{in } \mathrm{GL}(n; \mathbf{C}).$$

Let X be a *set*. We denote [S] (LA 4.10) by L_X the *free complex Lie algebra* on X, by $\hat{L}_X$ its *completion*, by Ass_X the *complex associative algebra* on X, by $\hat{\mathrm{Ass}}_X$ its *completion*, by $\hat{\mathscr{M}}_X$ the ideal generated in $\hat{\mathrm{Ass}}_X$ by X, by Δ: $\hat{\mathrm{Ass}}_X \rightarrow \hat{\mathrm{Ass}}_X \otimes \hat{\mathrm{Ass}}_X$ the *diagonal map*, and by $\hat{G}_X$ the set of $\beta \in I + \hat{\mathscr{M}}_X$ with $\Delta\beta = \beta \otimes \beta$.

There is a natural *isomorphism*

$$\exp\colon \quad \hat{\mathscr{M}}_X \rightarrow I + \hat{\mathscr{M}}_X.$$

We cand *identify* $\hat{L}_X$ with the set of *primitive elements* of $\hat{\mathrm{Ass}}_X$. Then by restriction of the exponential we get an *isomorphism*

$$\exp\colon \quad \hat{L}_X \rightarrow \hat{G}_X.$$

By the *Campbell-Hausdorff formula* we get a *group structure* on $\hat{G}_X$.

If X is the set of "labels" $\dot{\Delta}_{q,\,\mathbf{d}}$, where $(q, \mathbf{d})$ *is* such that $q \in \mathsf{E}$ and $\mathbf{d} \in \mathrm{Fr}\, q$, we write

$$\text{Lie } \mathscr{R} = L_X, \qquad \mathscr{U}\mathscr{R} = \mathrm{Ass}_X,$$
$$\mathscr{U}\hat{\mathscr{R}} = \hat{\mathrm{Ass}}_X, \qquad \mathscr{M}\hat{\mathscr{R}} = \hat{\mathscr{M}}_X, \; \hat{\mathscr{R}} = \hat{G}_X.$$

We get *isomorphisms*

$$\exp\colon\ \mathcal{M}\hat{\mathcal{R}} \to \mathrm{I} + \mathcal{M}\hat{\mathcal{R}}$$
$$\exp\colon\ \mathrm{Lie}\,\hat{\mathcal{R}} \to \hat{\mathcal{R}}.$$

We denote by $\mathcal{R}$ the subgroup of $\hat{\mathcal{R}}$ generated by the image of *Lie* $\mathcal{R}$ by exp; by *definition* $\mathcal{R}$ is the ***resurgent group***.

LEMMA 16. — *We consider the action of the wild formal fundamental group* $\pi_{1,\,sf}((\mathbf{C}^*, 0)$ "$\mathcal{R}^+$") *on the free Lie algebra Lie* $\mathcal{R}$ *defined by*

$$\hat{\gamma}_0\,\dot{\Delta}_{q,\,\mathbf{d}}\,\hat{\gamma}_0^{-1} = \dot{\Delta}_{q,\,\exp(-2i\pi)\,\mathbf{d}}$$
$$\tau\,\dot{\Delta}_{q,\,\mathbf{d}}\,\tau^{-1} = q(\tau)\,\dot{\Delta}_{q,\,\mathbf{d}}.$$

This action can be naturally extended to $\mathcal{U}\hat{\mathcal{R}}$ *and we get (by restriction) an action on* $\hat{\mathcal{R}}$, *leaving* $\mathcal{R}$ *invariant, such that*

$$\hat{\gamma}_0 \exp(\dot{\Delta}_{q,\,\mathbf{d}})\,\hat{\gamma}_0^{-1} = \exp(\dot{\Delta}_{q,\,\exp(-2i\pi)\,\mathbf{d}})$$
$$\tau \exp(\dot{\Delta}_{q,\,\mathbf{d}}\,\tau^{-1} = \exp(q(\tau)\,\dot{\Delta}_{q,\,\mathbf{d}}).$$

The ***wild fundamental group*** of the germ of $\mathbf{C}^*$ at the origin, *pointed at* "$\mathbf{R}^+$", is by *definition* the ***semi-direct*** *product*

$$\pi_{1,\,s}((\mathbf{C}^*, 0); \text{"}\mathbf{R}^+\text{"}) = \pi_{1,\,sf}((\mathbf{C}^*, 0); \text{"}\mathbf{R}^+\text{"}) \ltimes \mathcal{R}$$
$$\pi_{1,\,s}((\mathbf{C}^*, 0); \text{"}\mathbf{R}^+\text{"}) = ((\hat{\gamma}_0) \ltimes \mathcal{T}) \ltimes \mathcal{R}$$

defined by the ***action*** *of* $\pi_{1,\,sf}((\mathbf{C}^*, 0))$; "$\mathbf{R}^+$") *on* $\mathcal{R}$ introduced in *lemma* 16.

Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be **Z**-*independent* elements of Lie $\mathcal{R}$. Then the subgroup of $\hat{\mathcal{R}}$ generated by $\exp \alpha_1$, $\exp \alpha_2, \ldots$, and $\exp \alpha_m$ is *isomorphic to the free group* generated by the m "*letters*" $\exp \alpha_1$, $\exp \alpha_2, \ldots, \exp \alpha_m$. We get:

LEMMA 17. — *If* $(\rho_1, \mathrm{L}\,\rho_2)$ *is a pair of representations of the group* $\pi_{1,\,sf}((\mathbf{C}^*, 0)$; "$\mathbf{R}^+$") *in* $\mathrm{GL}(n; \mathbf{C})$ *and of the Lie algebra Lie* $\mathcal{R}$ *in* $\mathrm{End}(n; \mathbf{C})$ *"compatible" with the action of* $\pi_{1,\,sf}((\mathbf{C}, 0)$; "$\mathbf{R}^+$") *on Lie* $\mathcal{R}$, *then there exists a unique representation*

$$\rho_2\colon\ \mathcal{R} \to \mathrm{GL}(n; \mathbf{C})$$

such that

$$\rho_2(\exp \alpha) = \exp \mathrm{L}\,\rho_2(\alpha) \textit{ for every } \alpha \in \textit{Lie } \mathcal{R}.$$

This representation is compatible with the action of $\pi_{1,\,sf}((\mathbf{C}, 0)$; "$\mathbf{R}^+$") *on* $\mathcal{R}$ *defined in lemma* 16.

We get the "***wild Riemann-Hilbert correspondence***":

THEOREM 18. — *The natural map*

		Finite dimensional
Germs of ***meromorphic***	ρ_s	***linear representations***
connections ∇	$\rightarrow$	*of the*
at the origin.		***wild fundamental group***
		$\pi_{1,\,s}((\mathbf{C}^*, 0); \text{"}\mathbf{R}^{+}\text{"})$.

$$\nabla \rightarrow \rho_s(\nabla)$$

is a ***bijection***.

The *wild Riemann-Hilbert correspondence* in an *equivalence of Tannakian categories*.

Remarks. — 1. There are extensions of the *wild Riemann-Hilbert correspondence* to ***non-linear situations*** in relation with problems of ***analytic classification*** (germs of *non linear analytic differential equations*, germs of *analytic diffeomorphisms*, germs of *analytic vector fields...*) [MR 1], [E 3]. In these generalizations one gets statements which are similar to *theorem* 17. In the case of *differential equations* $\mathbf{C}^n$ is replaced by an analytic *manifold*, End $(n; \mathbf{C})$ by an analytic *vector field* and GL $(n; \mathbf{C})$ by the analytic *pseudogroup* of automorphisms of the manifold. *Theorem* 18 takes a quite technical form...

2. In such situations *Ecalle* introduces "*hidden variables*" ("variables cachées"). We can easily describe [and extend ([60])] his point of view using our technics:

Let ∇ be a germ of meromorphic connection and let $\rho_s(\nabla)$ be the corresponding representation by the wild Riemann-Hilbert correspondence. Let $X(\nabla)$ be the set of "labels" defined by

$$X(\nabla) = \{ \rho_s(\nabla)(\dot{\Delta}_{q,\,\mathbf{d}}) / q \in \mathbf{E} \text{ and } \mathbf{d} \in \operatorname{Fr} q \}.$$

Then there are at most a finite number of values of $(q, \mathbf{d})$ such that the matrix $\rho_s(\nabla)(\Delta_{q,\,\mathbf{d}})$ is non zero. If this matrix is *zero*, we *suppress* the corresponding letter. It remains a finite subset $X'(\nabla)$. We write $\mathrm{Ass}_{X'(\nabla)} = \mathcal{U}\mathcal{R}(\nabla)$.

If f is a horizontal section of ∇, then we set

$$X(\nabla; f) = \{ \rho_s(\nabla)(\dot{\Delta}_{q,\,\mathbf{d}})(f) / q \in \mathbf{E} \text{ and } \mathbf{d} \in \operatorname{Fr} q \},$$

and denote by $X'(\nabla; f)$ the set of "labels" corresponding to $X'(\nabla)$. We write $\mathrm{Ass}_{X'(\nabla;\, f)} = \mathcal{U}\mathcal{R}(\nabla; f)$.

The idea is to *interpret* $\mathcal{U}\hat{\mathcal{R}}(\nabla; f)$ as a "*formal function*" on $\mathcal{U}\hat{\mathcal{R}}$ "*extending*" f. This "function" depends on *new* (***non*** *commutative*) *variables*, the

([59]) We recall that we suppose all the representations *continuous* on $\mathcal{T}$.

([60]) Ecalle uses only particular "one-levelled" lattices.

"*coordinates*" of the elements of $\mathcal{U}\hat{\mathcal{R}}$. These "*hidden variables*" belongs to the *dual* of $\mathcal{U}\hat{\mathcal{R}}$. We will be more precise in *part* 6 below, and interpret $\mathcal{U}\hat{\mathcal{R}}(\nabla; f)$ as giving birth to a "*formal function*" on a ***principal bundle*** *with structure group* $\hat{\mathcal{R}}$, corresponding to an ***actual*** *function extending* f defined on a ***principal bundle*** *with structure group* $\hat{\mathcal{R}}$. Moreover there are *natural actions* of $\pi_{1,\,sf}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”})$ on all these objects.

3. The "*Lie-algebra*" *Lie* $\pi_{1,s}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”})$ of the *wild fundamental group* is the ***semi-direct*** *product of Lie-algebras* (*Lie* $\pi_{1,s}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”}) = Lie\ \mathcal{T}$)

$$Lie\ \mathcal{T} \ltimes Lie\ \mathcal{R},$$

associated to the *action* of the *commutative algebra* (*"Cartan algebra"*) *Lie* $\mathcal{T}$ on the *resurgent algebra Lie* $\mathcal{R}$ defined by

$$[\mathrm{H}, \dot{\Delta}_{q,\mathbf{d}}] = q(\mathrm{H})\,\dot{\Delta}_{q,\mathbf{d}},$$

for $\mathrm{H} \in Lie\ \mathcal{T}$, where

$$q: \quad Lie\ \mathcal{T} \to \mathbf{C}$$

is the *infinitesimal map* associated to

$$q: \quad \mathcal{T} \to \mathbf{C}^*.$$

From the wild monodromy representation ρ_s we get a representation

$$Lie\,\rho_s: \quad Lie\,\pi_{1,s}((\mathbf{C}, 0); \text{“}\mathbf{R}^+\text{”}) \to End(n; \mathbf{C}).$$

The restriction of this representation to *Lie* $\mathcal{R}$ is the map *Lie* ρ_{res} of *theorem* 17. It corresponds to *Ecalle's "**bridge equation**"* ("*equation du pont*").

We will explain now how to *change* the "*base point*" "$\mathbf{R}$" of the wild fundamental group $\pi_{1,s}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”})$.

We will replace "$\mathbf{R}^+$" by

$$\text{“}d\text{”} \in \{\text{“}0\text{”}\} \times S^1\ ((\text{“}0\text{”}, d) = \text{“}d\text{”}) \qquad \text{or} \qquad d \in \{\text{“}+\infty\text{”}\} \times S^1$$

(that we can identify with S^1, the real analytic blow up of the origin in $\mathbf{C}$).

We fix "d" $\in \{\text{“}0\text{”}\} \times S^1$. Let "$c$" be an homotopy class of continuous paths on $\{\text{“}0\text{”}\} \times S^1$ with origin "d" and extremity "$\mathbf{R}$" (corresponding to an homotopy class of paths c on S^1). We set

$$\pi_{1,s}((\mathbf{C}^*, 0); \text{“}d\text{”}) = \{\text{“}c\text{”}\, b\, \text{“}c\text{”}^{-1} / b \in \pi_{1,s}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”})\},$$

and put on this set the evident structure of group; $\pi_{1,s}((\mathbf{C}^*, 0); \text{“}d\text{”})$ is *independant* of the choice of c in a sense that we leave to the reader to explicit.

Let now $d \in \{\text{“}+\infty\text{”}\} \times S^1$. We set

$$\pi_{1,s}((\mathbf{C}^*, 0); d) = \{(\gamma_d^-)^{-1}\, b\, \gamma_d^- / b \in \pi_{1,s}((\mathbf{C}^*, 0); \text{“}d\text{”}))\},$$

where the symbol γ_d^- corresponds to the *multisummation operator* S_d^- in "the" direction d^- (S_d^- is interpreted as an analytic continuation along γ_d^-). We put on $\pi_{1,s}((\mathbf{C}^*, 0); d)$ the evident structure of group.

We can also set

$$\pi_{1,s}((\mathbf{C}^*, 0); d) = \{(\gamma_d^+)^{-1} b \gamma_d^+ / b \in \pi_{1,s}((\mathbf{C}^*, 0); \text{"}d\text{"}))\}:$$

there is a natural isomorphism between the two groups on the right side of our equalities.

We can now replace $\pi_{1,s}((\mathbf{C}^*, 0); \text{"}\mathbf{R}^+\text{"})$ by $\pi_{1,s}((\mathbf{C}^*, 0); \text{"}d\text{"})$ or $\pi_{1,s}((\mathbf{C}^*, 0); d)$ in *theorem* 18 (by definition $\rho_s(\nabla)(\text{"}c\text{"})$ is the analytic isomorphism of solution spaces given by the analytic continuation of a fundamental solution F_0 of "the" *formal normal form* corresponding to ∇ along c, $\rho_s(\nabla)(\gamma_d^-)$ is the isomorphism of solution spaces given by S_d^-). Elements of $\pi_{1,s}((\mathbf{C}^*, 0); d)$ are represented by linear permutations of *actual solutions* in a germ of sector bisected by d.

It is possible now to give a ***global** version* of our *wild fundamental group*.

Let $\mathbf{X}$ be a connected Riemann surface. Let $S = \{a_1, a_2, \ldots, a_m\}$ be a finite subset of $\mathbf{X}$, let x_0 be a base point in $\mathbf{X} - S$, *and, for each* $i = 1, \ldots, m$, let d_i be a fixed direction "starting from a_i". We choose homotopy classes of paths c_i ("in" $\mathbf{X} - S$), with *origin* x_0 and *extremity* a_i, "arriving at a_i *along the direction* d_i" ($i = 1, \ldots, m$). We built, like above, groups

$$G_i = \{c_i b c_i^{-1} / b \in \pi_{1,s}((\mathbf{C}^*, 0); d_i)\}, \qquad i = 1, \ldots, m$$

(these groups are *independent* of the choice of c_i in a sense that we leave to the reader to explicit).

By *definition* (61) the *wild fundamental group* "of" $\mathbf{X} - S$, *pointed at* x_0, is

$$\pi_{1,s}(\mathbf{X} - S, S; x_0) = G_1 * \ldots * G_m \qquad (\textit{free product of groups}),$$

and the *wild fundamental group of* $\mathbf{X}$ *is*

$$\pi_{1,s}(\mathbf{X} - \ldots; .) = \varprojlim_{S} \pi_{1,s}(\mathbf{X} - S; S; .).$$

(There are some trouble with marked point in the limit: we get rid of it as in the classical case...)

It is easy to prove the following results [we define $\rho_s(\nabla)(c_i)$ as the analytic isomorphism of solutions spaces given by the analytic continuation

(61) Be careful: the groups *depends on* $\mathbf{X}$ ***and*** S, not only on $\mathbf{X} - S$.

along c_i]:

We have a ***wild global Riemann-Hilbert correspondence***:

THEOREM 19. — *Let* **X** *be a connected Riemann surface.*
The natural map

$$\begin{array}{ccc} \textbf{\textit{Meromorphic}}\ \textit{connections} & \overset{\rho_s}{\to} & \textit{finite dimensional}\ \textbf{\textit{linear representations}}\ (^{62})\ \textit{of the} \\ \textit{on}\ \mathbf{X}. & & \textbf{\textit{wild fundamental group}}\ \pi_{1,s}(\mathbf{X};\ .). \\ & \nabla \to \rho_s(\nabla) & \end{array}$$

is a ***bijection.***

The *wild global Riemann-Hilbert correspondence* is an *equivalence of Tannakian categories.*

We will call the map $\rho_s(\nabla)$ the ***wild monodromy representation*** of the *connection* ∇.

Let $\rho_m(\nabla)$ be the (classical) ***monodromy representation*** of the connection ∇ (local or global case). It is possible to get (^{63}) the *actual monodromy representation* $\rho_m(\nabla)$ from the *wild monodromy representation* $\rho_s(\nabla)$. If **X** is a connected Riemann surface, we will write

$$\pi_1(\mathbf{X}-\ldots;\ .)=\varprojlim_{S}\pi_1(\mathbf{X}-S;\ .)\qquad (S\ \textit{finite}\ \text{subset of}\ \mathbf{X}).$$

PROPOSITION 15. — (i) *Let* $d\in S^1$ *be a fixed direction. There exists a "natural" functor* $\mathscr{D}$ *from the tensor category of finite dimensional linear representations of* $\pi_{1,s}((\mathbf{C}^*,0);\ d)$ *to the tensor category of finite dimensional linear representations of* $\pi_1((\mathbf{C}^*,0);\ d)$ *such that*

$$\mathscr{D}(\rho_s(\nabla))=\rho_m(\nabla)$$

for every germ of meromorphic connection ∇ *at the origin.*

This functor is defined by

$$\mathscr{D}(\rho)=\rho_1(\gamma_{d_1})\ldots\rho_1(\gamma_{d_p})\,\rho_1,$$

where (ρ_1,ρ_2) *is the pair of representations in* $\mathrm{GL}(n;\mathrm{C})$ *respectively of* $\pi_{1,sf}((\mathbf{C}^*,0);\ d)$ *and* $\mathrm{G}\Pi(\mathbf{q}_{\rho_1})$ *(pointed at* d*) associated to* $\rho(\mathbf{q}=\mathbf{q}_{\rho_1}$, *and* $d_1,\ldots,d_p$ *are the directions of* $\mathrm{Fr}(\mathbf{q})$ *contained in the interval* $[0,2\pi[\subset(\mathbf{R},0)$, ***ordered*** *by the ordering relation induced by* **R**).

(ii) *Let* **X** *be a connected Riemann surface. There exists a "natural" functor* $\mathscr{D}$ *from the tensor category of finite dimensional linear representations of* $\pi_{1,s}(\mathbf{X}-\ldots;\ .)$ *to the tensor category of finite dimensional linear*

(^{62}) We recall that we suppose all the representations *continuous* on $\mathscr{T}$.

(^{63}) In some sense π_1 is contained in a "completion" of $\pi_{1,s}$ and ρ_s can be extended to this completion "by continuity". Then ρ_m is the restriction to π_1 of this extension.

representations of $\pi_1(\mathbf{X} - \ldots; .)$, *such that*

$$\mathcal{D}(\rho_s(\nabla)) = \rho_m(\nabla),$$

for every meromorphic connection ∇.

We can reformulate *theorem* 6 in a more "*geometric form*" (and extend it to the *global* case), *replacing* the *actual monodromy representation* by the *wild monodromy representation* in *Schlesinger's theorem.*

THEOREM 20. – *Let* $K = \mathbf{C}\{x\}[x^{-1}]$. *Let* ∇ *be a germ of meromorphic connection at the origin. We fix a* **C**-*basis of the space of horizontal sections on a germ of sector bisected by a given direction d and identify the Galois differential group* $\mathrm{Gal}_K(\nabla)$ *with its corresponding representation in* $\mathrm{GL}(n; \mathbf{C})$.

Then $\mathrm{Gal}_K(\nabla)$ *is the Zariski closure of the image in* $\mathrm{GL}(n; \mathbf{C})$ *of the wild monodromy representation*

$$\rho_s(\nabla): \quad \pi_{1,s}((\mathbf{C}^*, 0); d) \to \mathrm{GL}(n; \mathbf{C}).$$

THEOREM 21. – *Let* **X** *be a connected Riemann surface. Let* K_X *be the differential field of meromorphic functions on* **X**. *Let* ∇ *be a meromorphic connection on* **X**, *and* $x_0 \in \mathbf{X}$ *a regular point for* ∇. *We fix a* **C**-*basis for the space of horizontal sections of* ∇ *on a germ of small "disc" centered at* x_0 *and we identify the Galois differential group* $\mathrm{Gal}_{K_X}(\nabla)$ *with its corresponding representation in* $\mathrm{GL}(n; \mathbf{C})$.

Then $\mathrm{Gal}_{K_X}(\nabla)$ *is the Zariski closure of the image in* $\mathrm{GL}(n; \mathbf{C})$ *of the wild monodromy representation*

$$\rho_s(\nabla): \quad \pi_{1,s}(\mathbf{X}; .) \to \mathrm{GL}(n; \mathbf{C}).$$

Examples and applications. – It is possible to ***compute explicitely wild monodromy representations*** for ***generalized confluent hypergeometric differential equations*** ([64]) (using results of [DM]). These computations use *elementary functions* and Γ-*function.* It is possible to compute Galois differential groups of generalized confluent hypergeometric differential equations from these representations. This program is partially achieved [DM], [M1], [M2], [M3]. *C. Mitschi* has studied in particular *order* ***seven*** case and got, after *N. Katz* [K3], *generalized confluent hypergeometric differential equations of order seven admitting the* ***exceptional group*** G_2 ***as Galois differential group*** [M2], [M3].

([64]) And for differential equations got from confluent hypergeometric equations by "elementary operations", as, for instance, "Kummer pull-backs" [Kat 3], [M3]. (Differential equations satisfied by accelerating and decelerating functions, and more generally by Faxen's integrals, correspond, when the parameters are rational numbers, to such pull-backs.)

From *theorem* 18 (or *theorem* 17) it is also possible to get an interesting result for the "*inverse problem*" in differential Galois theory [Ra 8]:

THEOREM 22. — *Let* **L** *be a complex semi-simple Lie algebra. Let* L ρ *be a finite dimensional representation of* **L**. *Then:*

(i) *There exists a rational differential equation* **D** *on* $P^1(C)$, *with singularities contained in* $\{0, +\infty\}$, 0 *being regular and* $+\infty$ *irregular, such that* $\mathrm{Gal}_{C(z)}(\mathbf{D})$ *is Zariski connected and such that*

Lie $\mathrm{Gal}_{C(z)}(\mathbf{D}) \approx L\rho(\mathbf{L})$ (*isomorphism of complex Lie-algebras*).

(ii) *There exists a germ of meromorphic differential equation* D *at the origin such that* $\mathrm{Gal}_K(D)$ *is Zariski connected and such that*

$$Lie\,\mathrm{Gal}_K(D) \approx L\rho(\mathbf{L}).$$

We will end this paragraph by a comparison between *N. Katz's* viewpoint and ours.

Let $\mathbf{X}^{an}$ be a compact connected Riemann surface. Let S be a fixed *finite* subset of $\mathbf{X}^{an}$. We denote by D.E. ($\mathbf{X}^{an}$; S)) the tensor category of *meromorphic connections* on $\mathbf{X}^{an}$ *with singularities contained in* S.

To each point z_0 *of* $\mathbf{X}^{an} - S$ we can associate a *fibre functor* $\omega(z_0)$ of the tensor category D.E. ($\mathbf{X}^{an}$; S)):

$\omega(z_0)(\nabla) = \{$ *horizontal sections of* ∇ on a germ of neighbourhood of $\nabla\}$.

We will denote by $\pi_1^{\text{diff}}(\mathbf{X}^{an} - S;\ S;\ z_0)$ the group $\mathrm{Aut}^{an}(\omega(z_0))$ (automorphisms of the fibre functor $\omega(z_0)$).

There is a natural map

$$\pi_{1,s}(\mathbf{X}^{an} - S;\ S;\ z_0) \to \pi_1^{\text{diff}}(\mathbf{X}^{an} - S;\ S;\ z_0):$$

each element of $\pi_{1,s}(\mathbf{X}^{an} - S;\ S;\ z_0)$ defines clearly an automorphism of the fibre functor $\omega(z_0)$.

Let **Y** be a smooth connected **C**-*scheme* such that the corresponding *analytic variety* is a connected *Riemann surface* $\mathbf{X}^{an} - S = \mathbf{Y}^{an}$. We denote by D.E. (**Y**/**C**) the tensor category of *algebraic connections* on **Y**. There is a natural functor

$$\text{D.E.}(\mathbf{Y}/\mathbf{C}) \to \text{D.E.}(\mathbf{X}^{an};\ S))$$
$$\nabla \to \nabla^{an}.$$

If $\mathbf{X}^{an}$ is *compact* it yields *an equivalence of tensor categories* between D.E. (**Y**/**C**) and D.E. ($\mathbf{X}^{an}$; S)).

We denote by $\pi_1^{\text{diff}}(\mathbf{Y}/\mathbf{C};\ z_0)$ the group $\mathrm{Aut}(\omega(z_0))$ [automorphisms of the fibre functor $\omega(z_0)$].

There is a natural morphism $\pi_1^{\text{diff}}(\mathbf{X}^{an} - S;\ S;\ z_0) \to \pi_1^{\text{diff}}(\mathbf{Y}/\mathbf{C};\ z_0)$. If $\mathbf{X}^{an}$ is *compact* it is an *isomorphism*. We get:

PROPOSITION 15. — Let **Y** be a smooth connected **C**-*scheme* such that the corresponding *analytic variety is a connected Riemann surface* $\mathbf{X}^{an} - S = \mathbf{Y}^{an}$, *where* $\mathbf{X}^{an}$ *is a Riemann surface and* S *a finite subset of* $\mathbf{X}^{an}$. *Then*

$\pi_1^{\text{diff}}(\mathbf{Y}/\mathbf{C}; z_0)$ *is an affine pro-algebraic* **C**-*group-scheme and there exists a natural homomorphism of groups*

$$\pi_{1,s}(\mathbf{X}^{an}-\mathrm{S};\ \mathrm{S};\ z_0) \to \pi_1^{\text{diff}}(\mathbf{Y}/\mathbf{C};\ z_0).$$

Even if $\mathbf{X}^{an}$ is *compact* this map is *not onto*. We ignore if it is *injective*. Anyway if $\mathbf{X}^{an}$ is *compact* π_1^{diff} appears as an "*algebraic hull*" of $\pi_{1,s}$, just like π_1^{diff} appears as an *algebraic hull* of π_1 in the *fuchsian* case.

If G is a linear algebraic group we will denote by G^0 the (Zariski) *connected component of the identity*.

If ∇ is a germ meromorphic connection at the origin we will denote by $\rho_{mf}(\nabla)$ the restriction to the subgroup $(\hat{\gamma}_0)$ of the representation $\hat{\rho}(\nabla)$, and by $G_m(\nabla)$ [resp. $G_{mf}(\nabla)$] the *Zariski closure* of the image of $\rho_m(\nabla)$ [resp. $\rho_{mf}(\nabla)$]. If ∇ is a meromorphic connection on a Riemann surface we will denote by $G_m(\nabla)$ the *Zariski closure* of the image of $\rho_m(\nabla)$.

Theorem 19 sounds quite abstract, however (using only *algebraic* methods) we can deduce from it quite interesting results. For instance we get easily a variant of a result of *O. Gabber:*

PROPOSITION 16. — (i) *Let* ∇ *be a germ of meromorphic connection at the origin. Then the map*

$$\pi_1((\mathbf{C}^*, 0); \mathbf{R}^+) \to \mathrm{Gal}_{\mathrm{K}}(\nabla)/\mathrm{Gal}_{\mathrm{K}}(\nabla)^0,$$

induced by the ***monodromy*** *representation* $\rho_m(\nabla)$, *is a* ***surjection.***

(ii) *Let* ∇ *be a germ of meromorphic connection at the origin. Then the map*

$$(\hat{\gamma}_0) \to \mathrm{Gal}_{\mathrm{K}}(\nabla)/\mathrm{Gal}_{\mathrm{K}}(\nabla)^0,$$

induced by the ***formal monodromy*** *representation* $\rho_{mf}(\nabla)$, *is a* ***surjection.***

(iii) *Let* ∇ *be a meromorphic connection on a Riemann surface* **X**. *Let* S *be a discrete subset of* **X** *containing all the singularities of* ∇. *Then the map*

$$\pi_1(\mathbf{X}^{an}-\mathrm{S};.) \to \mathrm{Gal}_{\mathrm{K}_{\mathbf{X}}}(\nabla)/\mathrm{Gal}_{\mathrm{K}_{\mathbf{X}}}(\nabla)^0,$$

induced by the ***monodromy*** *representation* $\rho_m(\nabla)$, *is a* ***surjection.***

Proofs mimic *Gabber's* proof [Kat 1] (1.2.5., p. 18). Proof of assertion (iii) is similar to proof of assertion (i), so we will prove only (i) and (ii). We denote by G the ***finite*** *group* $\mathrm{Gal}_{\mathrm{K}}(\nabla)/\mathrm{Gal}_{\mathrm{K}}(\nabla)^0$. Let ρ' be a *faithful* finite dimensional linear representation of G. Then $\rho'\rho_s(\nabla)$ is a finite dimensional linear representation of the *wild fundamental group* $\pi_{1,s}((\mathbf{C}^*, 0);$ "$\mathbf{R}^+$"), and, using *theorem* 19, we can ***interpret*** it as a meromorphic connection ∇' on **X**, with singularities in $\mathrm{S}(\rho_s(\nabla')=\rho'\rho_s(\nabla))$. Moreover ∇' belongs to the tensor category "*generated*" by ∇.

We have a commutative diagram of homomorphisms of groups

$$\begin{array}{ccc} \pi_{1,s}((\mathbf{C}^*, 0); \text{“}\mathbf{R}^+\text{”}) & \overset{\rho_s(\nabla)}{\rightarrow} & \mathrm{Gal}_K(\nabla) \\ & \underset{\rho_s(\nabla')}{\searrow} & \downarrow \\ & & \mathrm{Gal}_K(\nabla') = G \\ & & = \mathrm{Gal}_K(\nabla)/\mathrm{Gal}_K(\nabla)^0. \end{array}$$

Using *proposition* 14 it yields a new commutative diagram of homomorphisms of groups

$$\begin{array}{ccc} \pi_1((\mathbf{C}^*, 0); \mathbf{R}^+) & \overset{\rho_m(\nabla)}{\rightarrow} & \mathrm{Gal}_K(\nabla) \\ & \underset{\rho_m(\nabla')}{\searrow} & \downarrow \\ & & \mathrm{Gal}_K(\nabla') = G \\ & & = \mathrm{Gal}_K(\nabla)/\mathrm{Gal}_K(\nabla)^0. \end{array}$$

The Galois differential group $\mathrm{Gal}_K(\nabla')$ being ***finite*** the connection ∇' *is* ***fuchsian*** ([65]), then the map $\rho_m(\nabla')$ is ***surjective***. Assertion (i) follows.

We have also a commutative diagram of homomorphisms of groups

$$\begin{array}{ccc} (\hat{\gamma}_0) & \overset{\rho_{mf}(\nabla)}{\rightarrow} & \mathrm{Gal}_K(\nabla) \\ & \underset{\rho_{mf}(\nabla')}{\searrow} & \downarrow \\ & & \mathrm{Gal}_K(\nabla') = G \\ & & = \mathrm{Gal}_K(\nabla)/\mathrm{Gal}_K(\nabla)^0. \end{array}$$

The Galois differential group $\mathrm{Gal}_K(\nabla')$ being ***finite*** the connection ∇' is ***fuchsian***. Then the map $\rho_m(\nabla')$ is *surjective*, *actual monodromy* and *formal monodromy* can be identified, and the map $\rho_{lf}(\nabla')$ is *also* ***surjective***. Assertion (ii) follows.

PROPOSITION 17. – (i) *Let* ∇ *be a germ of meromorphic connection at the origin. Then*

(*a*) *If* $G_m(\nabla)$ *is Zariski connected, then* $\mathrm{Gal}_K(\nabla)$ *is also Zariski connected.*

(*b*) *If* $G_{mf}(\nabla)$ *is Zariski connected, then* $\mathrm{Gal}_K(\nabla)$ *is also Zariski connected.*

(ii) *Let* ∇ *be a meromorphic connection on a Riemann surface* **X**. *Then, if* $G_m(\nabla)$ *is Zariski connected, then* $\mathrm{Gal}_{K_X}(\nabla)$ *is also Zariski connected.*

Be careful, conversely $\mathrm{Gal}_{K_X}(\nabla)$ can be connected and $G_m(\nabla)$ or $G_{mf}(\nabla)$ *not* connected. It is interesting to notice that we can decide if $G_{mf}(\nabla)$ *is Zariski connected* using ***purely algebraic*** methods. This is ***not*** true in general

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([65]) If a connection is ***not*** fuchsian its Galois differential group *contains a* ***non trivial*** *exponential torus* and ***cannot be finite***.

for the connectedness of $G_m(\nabla)$, however there are exceptional (and interesting...) cases (see *examples* below).

Proposition 17 follows immediately from *proposition* 16. There is also a "more elementary" proof:

The *exponential tori* are *connected*, and, if S is a *Stokes multiplier* "in" $\mathrm{Gal}_K(\nabla)$, then S is *unipotent* and the one-parameter group $\{\exp(tS)/t \in \mathbf{C}\}$ *is* ***connected*** *and entirely contained in* $\mathrm{Gal}_K(\nabla)$. Then *exponential tori* are subgroups of $\mathrm{Gal}_K(\nabla)^0$ and *Stokes multipliers* belongs to $\mathrm{Gal}_K(\nabla)^0$. *Proposition* 17 follows.

Example. – Following ideas of *N. Katz* [Kat 1] *proposition* 17 yields elegant methods of computation of some Galois differential groups. Let ∇ be a meromorphic connection on the Riemann sphere with singularities contained in $S=\{0, +\infty\}$, 0 *being regular or regular singular and* $+\infty$ *irregular*. We fix a base point $z_0 \in \mathbf{C}^*$. Monodromies around zero and infinity are inverse each other and *algebraically computable* (using *Frobenius algorithm*). We get in particular interesting situations when $G_m(\nabla)$ *is Zariski connected* (especially when the monodromy around zero is *trivial*) and when the Newton polygon of ∇ at infinity admits only a slope $k>0$, where the rational number *k is not an integer*. Then the monodromy acts *non trivially* by conjugacy on the exponential torus and we get (even if it is not so evidence at first glance...) a lot on information on the connected group $\mathrm{Gal}_{\mathbf{C}(z)}(\nabla)$ (particularly in the *irreducible* case).

As an example of application of these ideas we will give a very easy computation of the Galois differential group for Airy equation $Dy=y''-zy=0$.

The (actual) monodromy of D is *trivial*, then $G_m(D)=\{id\}$ is *Zariski connected* and $\mathrm{Gal}_{\mathbf{C}(z)}(D)$ is also *Zariski connected.* Using a formal fundamental system of solutions for D at infinity we identify $\mathrm{Gal}_{\mathbf{C}(z)}(D)$ with a subgroup of $GL(2; \mathbf{C})$. The Wronskian of a fundamental system of solutions being constant we get more precisely a subgroup of $SL(2; \mathbf{C})$.

If our fundamental system of solutions is "well chosen" [MR 2], then the *exponential torus* is

$$T=\left\{\begin{pmatrix}\alpha & 0\\ 0 & \alpha^{-1}\end{pmatrix}\Big/ \alpha \in \mathbf{C}^*\right\},$$

and the *formal monodromy* matrix is

$$M=\begin{pmatrix}0 & i\\ i & 0\end{pmatrix}.$$

The formal monodromy matrix M acts *non* trivially on the exponential torus T (it permutes the characters). We have

$$TM=\left\{\begin{pmatrix}0 & \beta\\ -\beta^{-1} & 0\end{pmatrix}\Big/ \beta \in \mathbf{C}^*\right\} \subset \mathrm{Gal}_{\mathbf{C}(z)}(D),$$

and

$$T \cup TM \subset \mathrm{Gal}_{C(z)}(D).$$

But the ***only*** *connected subgroup* of SL(2; C) containing T ∪ TM *is* SL(2; C) itself. We get

$$\mathrm{Gal}_{C(z)}(D) = SL(2; C).$$

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