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Gravitational scattering of electromagnetic field by Schwarzschild black-hole

by

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ABSTRACT. — We study the electromagnetic scattering by a spherical black-hole. Maxwell's equations are written in Schwarzschild coordinates: the electromagnetic tensor is replaced with electric and magnetic fields in a three dimensional absolute space. We introduce a set of wave operators, $W_0^{\pm}$, $W_1^{\pm}$, yielding an electromagnetic field given an asymptotic behavior far from the black-hole, $W_0^{\pm}$, and near the Schwarzschild radius, $W_1^{\pm}$, as universal time $t \to \pm\infty$. The long range interactions are eliminated by identifying the radial coordinate in the asymptotic Minkowski space with the Regge-Wheeler parameter. After a separation of variables thanks to the generalised vector spherical harmonics of Gel'fand and Šapiro, the existence of the scattering operator is proved by using a Birman-Kato method, in particular, the asymptotic completeness of $W_1^{\pm}$ implies the Damour-Znajeck condition: near the horizon, the fields of finite redshifted energy are described by ingoing plane waves. The Membrane Paradigm is justified: the scattering operator can be approximated by putting the impedence condition on the stretched horizon. We interpret these results on the Kruskal universe: the existence of W_0^-, W_1^- assures the characteristic Cauchy problem with data on the past horizons is well posed in the Schwarzschild submanifold, and the asymptotic completeness of W_0^+, W_1^+ allows to define the solution on the future horizons.

RÉSUMÉ. — On étudie la diffraction du champ électromagnétique par un trou noir sphérique. Les équations de Maxwell sont exprimées dans le système de coordonnées de Schwarzschild : le tenseur électromagnétique est décomposé sur un espace absolu tridimensionnel, en champs vectoriels électriques et magnétiques dont on étudie l'évolution au cours d'un temps universel t.

On introduit un ensemble d'opérateurs d'onde, $W_0^{\pm}$, $W_1^{\pm}$, associant un champ électromagnétique à un comportement asymptotique donné, quand $t \to \pm\infty$, respectivement loin du trou noir, $W_0^{\pm}$, et près du rayon de Schwarszchild, $W_1^{\pm}$. Les interactions à longue portée sont éliminées en identifiant la coordonnée radiale de l'espace de Minkowski asymptotique avec le paramètre de Regge-Wheeler. Après une séparation des variables à l'aide des harmoniques sphériques vectorielles généralisées de Gel'fand et Šapiro, l'existence de l'opérateur de diffraction est prouvée par une méthode de Birman-Kato, en particulier, la complétude asymptotique de $W_1^{\pm}$ exprime la condition de Damour-Znajeck : près de l'horizon, les champs d'énergie (décalée vers le rouge) finie, sont décrits par des ondes planes rentrantes. On justifie le paradigme de la membrane : l'opérateur de diffraction est approché en imposant la condition d'impédance sur un horizon élargi. On interpréte ces résultats sur l'univers de Kruskal : l'existence de W_0^-, W_1^- assure que le problème de Cauchy caractéristique à données sur les horizons passés est bien posé dans la sous-variété de Schwarzschild, et la complétude de W_0^+, W_1^+ permet de définir la trace de la solution sur les horizons futurs.

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1. INTRODUCTION

This paper develops the mathematical foundations of the scattering theory of electromagnetic field by a spherical black-hole, that is, we study Maxwell's equations in Schwarzschild Universe $(\mathscr{S}, ds^2_{\mathscr{S}})$.

$\mathscr{S}$ is a four-dimensional globaly hyperbolic pseudoriemannian manifold, that admits a split into an universal time $t \in \mathbb{R}$, and an absolute riemannian space (V, ds^2_V):

$$\mathscr{S} = \mathbb{R}_t \times V_x, \tag{1}$$

$$ds^2_{\mathscr{S}} = \alpha^2(x)\, dt^2 - ds^2_V. \tag{2}$$

V can be described by the spherical coordinates

$$V = I \times S^2, \tag{3}$$

where S^2 is the euclidian two-sphere

$$S^2 \simeq [0, \pi]_\theta \times [0, 2\pi[_\varphi, \tag{4}$$

and I is an open real interval.

The Schwarzschild coordinates are $t \in \mathbb{R}$, $(\theta, \varphi) \in S^2$, $r \in I$ and

$$I =]r_0, +\infty[_r, \tag{5}$$

where $r_0 > 0$ is the black-hole radius. Then the lapse function α and riemannian metric $ds^2_{\mathscr{S}}$ are respectively defined by

$$\alpha(r, \theta, \varphi) = \alpha(r) = \left(1 - \frac{r_0}{r}\right)^{1/2}, \tag{6}$$

$$ds^2_V = \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2, \tag{7}$$

and the Schwarzschild metric has the form

$$ds^2_{\mathscr{S}} = \left(1 - \frac{r_0}{r}\right) dt^2 - \left(1 - \frac{r_0}{r}\right)^{-1} dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\varphi^2. \tag{8}$$

In fact there is no canonical choice of radial parameter, we could take the proper radial distance R with

$$I =]0, +\infty[_R, \tag{9}$$

$$R = r\left(1 - \frac{r_0}{r}\right)^{1/2} + r_0 \operatorname{Log}\left[\left(\frac{r}{r_0} - 1\right)^{1/2} + \left(\frac{r}{r_0}\right)^{1/2}\right], \tag{10}$$

$$ds^2_V = dR^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2. \tag{11}$$

Another nice choice is the tortoise coordinate of A. Wheeler r_*, for which

$$I =]-\infty, +\infty[_{r_*}, \tag{12}$$

$$r_* = r + r_0 \operatorname{Log}(r - r_0), \tag{13}$$

$$ds^2_{\mathscr{S}} = \left(1 - \frac{r_0}{r}\right)(dt^2 - dr_*^2) - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\varphi^2. \tag{14}$$

A very important property of r_* is that the radial null geodesics are described by the same formula of their flat analogs:

$$\pm t = r_* + \text{Const.} \tag{15}$$

The splitting of the Schwarzschild Universe with Galilean type universal time t and absolute space V, is very fit for the study of the Cauchy problem and the scattering theory, where we consider the black-hole as a perturbation living in a three dimensional space, it is the "3+1 *view point*" of the Caltech Paradigm Society (to *see* e. g. [14]). Following this formalism, the electric and magnetic fields, **E**, **B**, are vector fields on V, defined physically by measurements made by fiducial observers (Fido's of [14]), at rest in V, and Maxwell's equations are:

$$\frac{\partial \mathbf{E}}{\partial t} = \nabla_V \times (\alpha \mathbf{B}), \tag{16.1}$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla_V \times (\alpha \mathbf{E}), \tag{16.2}$$

$$\nabla_V . \mathbf{E} = 0, \tag{16.3}$$

$$\nabla_V . \mathbf{B} = 0, \tag{16.4}$$

where $\nabla_V \times$ and $\nabla_V .$ are respectively the *curl* and *divergence* in (V, ds_V^2).

Now we emphasize some significant properties of the Schwarzschild metric, relevant for the scattering problem, by discussing about the null geodesics $\gamma(t) = (t, r(t), \theta(t), \varphi(t))$.

1. It is tempting to introduce a boundary of V, or "horizon" of black-hole:

$$\Gamma = \mathbb{R}_t \times \{r = r_0\} \times S^2. \tag{17}$$

In Schwarzschild coordinates, no null geodesic γ reaches Γ at finite time t: if $r(t) \to r_0$ along γ, then $|t| \to \infty$. This fact has important consequences:

(i) Γ is not a time-like submanifold and no boundary condition on Γ is necessary to solve (16): we are concerned by a true Cauchy problem and no a mixed problem.

(ii) We know that the solutions of hyperbolic systems with constant coefficients in $\mathbb{R}^{n+1}$ admit an asymptotic profile satisfying an outgoing radiation condition, so we may expect such an asymptotic polarization, ingoing with respect to the black-hole, as $r(t) \to r_0$ along γ.

In fact, the electromagnetic field satisfies the famous "Damour-Znajeck condition" ([4], [5], [6], [24]): for any radial null geodesic γ, we have

$$\alpha \mathbf{E} \times \mathbf{n} - \mathbf{n} \times (\alpha \mathbf{B} \times \mathbf{n}) \to 0, \quad \textit{along } \gamma, \ r \to r_0, \tag{18}$$

where $\mathbf{n}$ is the unit spatial vector $\alpha \frac{\partial}{\partial r}$.

(18) is formally analogous at the impedence condition for a membrane Γ' with a surface resistivity $377\,\Omega$ in the euclidian space:

$$\mathbf{E} \times \mathbf{n} - \mathbf{n} \times (\mathbf{B} \times \mathbf{n}) = 0 \quad \textit{on } \Gamma', \tag{19}$$

$\mathbf{n}$ being the outward normal. But we emphasize that, unlike (19), Damour-Znajeck property (18) is a *consequence* of Maxwell's equations (16). In fact, horizon Γ defined by (17) is more a matter of mental or verbal picture, imposed by our euclidian intuition and by the choice of the Schwarzschild coordinates which yields a fictitious singularity at $r = r_0$. The Kruskal coordinates allow to define a meaningful concept of horizon, as null submanifold of Kruskal Universe. We shall establish a natural connection with the scattering theory: this horizon is the domain in which live the asymptotic profiles.

2. There exists closed null geodesics: all the great circles of photons-sphere $\left\{ r = \frac{3}{2} r_0 \right\} \times S^2$. There exists so null geodesics asymptotic to the

photons-sphere. Then some singularities of the fields are trapped inside these null characteristics and cannot escape as $|t| \to \infty$. Such a situation is well-known in the classical scattering theory by obstacle in the euclidian space and yields some difficulties in the study of: the decay of local energy, the spectral properties of the hamiltonian... We prove that, despite the Schwarzschild metric is trapping, the only time periodic electric or magnetic fields are stationnary and their set is one dimensional like the second cohomological space associated with a connected obstacle in the euclidian space.

3. The Schwarzschild metric is asymptotically flat and the spatial projection of null geodesics tends to a straigh-line as $r \to \infty$, then we expect the field is asymptotic to a free electromagnetic field in the Minkowski-space far from the black-hole. A subtle problem is to choose coordinates in the Minkowski space. It is natural to identify Schwarzschild coordinates (θ, φ) with the usual angular coordinates of euclidian space, and Schwarzschild universal time t with the Minkowski cosmic time. The choice is much less clear as regards the radial parameter, between e. g. r, R, or r_*. It follows from (10) (13) (15) that the radial null geodesics are asymptotically straight as $r \to \infty$ in (t, r_*) coordinates but *no* in (t, R) or (t, r) coordinates. So, long range interactions between the gravitationnal and electromagnetic fields appear if we identify the radial coordinate in euclidian space with r or R and we let as an open problem the existence of Dollard-modified wave operators in these both cases. In opposite we shall prove the existence of classical wave-operator related to the flat infinity by comparing the electromagnetism dynamics in the Schwarzschild Universe and the Minkowski space-time with the metric

$$dt^2 - dr_*^2 - r_*^2\, d\theta^2 - r_*^2 \sin^2\theta\, d\varphi^2, \qquad 0 \leqq r_*. \tag{20}$$

Therefore there are two ways to study the scattering by a Black-Hole:

(i) The *conformal approach* with the Kruskal coordinates;

(ii) The *quantum scattering approach* with the Schwarzschild coordinates.

In the conformal approach we take advantage of the conformal invariance of the Maxwell equations to pose the Cauchy problem in the Kruskal manifold which is a globally hyperbolic curved spacetime. Since Maxwell's equations are an hyperbolic system and posses a well posed initial value formulation, given smooth initial data on a Cauchy surface in Schwarzschild space time, the resulting solution exists and is smooth on the horizons; hence we obtain easily the existence of the Sommerfeld condition at the flat infinity and the Damour-Znajeck condition at the black-hole horizon. Nevertheless this method presents some disadvantages: it is necessary to make some technical assumptions of smoothness of the data to define the limit of the field at each horizons; since the timelike and spacelike infinity are singular in the Penrose conformal spacetime the Characteristic Cauchy Problem does not follow from standard previous results; moreover we obtain directly the asymptotic behaviour of the field only along a fixed null geodesic, but no information on uniform decay; at last this method cannot be applied for massive field.

The quantum scattering approach has been used by Dimock [7] and Dimock and Kay ([8], [9]) to study the scattering for the scalar wave equation on the Schwarzschild metric. In this paper we adopt this approach to study Maxwell's equations, and in section 7 we establish the connection between the Characteristic Cauchy problem in the Penrose conformal spacetime and the existence of wave operators. This method presents plenty of advantages: we can study the fields with finite (redshifted) energy without supplementary regularity and we obtain sharp results on decay and asymptotic behaviour. According to this viewpoint, we study the behaviour of the field on Cauchy surface $\{t = \text{Const.}\}$ as Schwarzschild time t tends to infinity, by introducing waves operators and S-matrix and by using the spectral machinery well known in quantum mechanics. The difficulties become from the tensor nature of fields: we must choose carefully the Hilbert spaces to avoid stationnary fields; on the other hand, there is a very interesting phenomenon: unlike the scalar case, a *long range* interaction at infinity appears, and we know that the classical wave operators do not exist for the *scalar* equations with such a perturbation; but we are concerned by a *system* and the long range terms affect essentially the *radial* components which decay as t^{-2} along the null radial geodesics; this fundamental consequence of the spherical invariance of Maxwell' equations cancels the long range effects and allows to construct the wave operators. At last, thanks to the theorems of separation of variables of Carter and Teukolsky, we may expect to generalise this method to the linear fields, even massive Dirac fields, on the Kerr-Neuman background (the De Sitter-Schwarzschild space time is investigated in [1]).

This paper is organized as follows:

Section 2 presents the $3+1$ formulation of Maxwell's equations, we prove the self-adjointness of the hamiltonian in the finite redshifted energy space, so the global Cauchy problem is solved by Stone's theorem; we establish the point spectrum is empty on the subspace of free divergence fields without stationnary part.

In Section 3 we introduce the wave operator related to the flat infinity, which yields solutions with given asymptotic behaviour as $r \to +\infty$.

In Section 4 the wave operator describing the behavior of field near the black-hole is constructed and we study the Rindler approximation. We deduct the existence of infalling fields, similar to disappearing solutions in dissipative scattering.

The existence of the scattering operator is proved in Section 5. In particular the asymptotic completeness of the wave operator related to the horizon is equivalent to Damour-Znajeck property (18). The key of the proof is a variables separation thanks to the generalised vector spherical harmonics of Gel'fand and Šapiro [11], then we apply the results of Birman to a one dimensional wave equation with a short range potential and the two Hilbert spaces scattering theory of Kato [22].

The Membrane Paradigm is investigated in Section 6: we consider Maxwell's equations outside the stretched horizon $\{r=r_0+\varepsilon\}\times S^2$ on which we impose the impedence condition. We prove the existence of the scattering operator S_ε associated to this dissipative hyperbolic problem. As $\varepsilon\to 0$, S_ε tends to the operator describing the scattering at infinity by the true Black-Hole, for an incoming wave.

In Section 7, the whole scattering theory of electromagnetic field by a Schwarzschild black-hole is interpreted in terms of characteristic Cauchy problem in the Kruskal Universe.

To end this introduction, we give some bibliographic information. As regards the Schwarzschild metric and the astrophysical aspects of black-holes theory, to *see* e. g. the classical books of S. Chandrasekar [3], S. W. Hawking and G. F. R. Ellis [13], C. W. Misner, K. S. Thorne and J. A. Wheeler [18]. The recent book of I. Novikov and V. Frolov [19] treats the Physics of Black-Hole. The independent works of Th. Damour [4] [5] [6] and R. L. Znajeck [24] lay the foundations of the black-holes electrodynamic theory. Here, we use the "3+1 formalism" of D. Mac Donald and Kip S. Thorne ([16], [17]), to see so ([14], [15]. The scattering by black-holes is studied with astrophysical view point in [10]. The Bondi type expansions of fields in powers of r^{-1} at infinity are investigated by J. Porril, J. M. Steward [21], B. G. Schmidt and J. M. Stewart [23]. The scalar wave equation in Schwarzschild space-time is studied by J. Dimock and B. S. Kay ([7], [8], [9]) who construct the wave operators. As regard mathematical background in Scattering Theory, to *see* e. g. V. Petkov [20] and M. Reed, B. Simon [22].

Notations

Throughout this paper we use the conventions of [18]: we denote vectors and tensors by bold-face letters, e. g. **E**, **B**, **F**, $\mathbf{e}_\mu$, their components multiplet in some local basis $\mathbf{e}_\mu$ is noted E^μ, B^μ, . . .:

$$\mathbf{e}_\mu=\frac{\partial}{\partial x^\mu},\qquad \mathbf{E}=E^\mu\,\mathbf{e}_\mu,\qquad E=(E^\mu),$$

and we use the Einstein summation convention: any index that is repeated in a produit is automatically summed on. The caret or "hat" is used to indicate the components of a vector or tensor in a local Lorentz frame for which

$$ds^2_{\mathscr{S}} = \eta_{\hat{\mu}, \hat{\nu}}\, dx^{\hat{\mu}}\, dx^{\hat{\nu}}, \qquad (\eta_{\hat{\mu}, \hat{\nu}}) = \mathrm{diag}(1, -1, -1, -1). \tag{21}$$

2. MAXWELL'S EQUATIONS IN THE SCHWARZSCHILD UNIVERSE

The electromagnetic tensor field **F** on $(\mathscr{S}, ds^2_{\mathscr{S}})$ satisfies Maxwell's equations

$$\mathbf{dF} = 0, \tag{22.1}$$

$$\mathbf{d} * \mathbf{F} = 0, \tag{22.2}$$

where $*$ notes the Hodge operator for the metric $ds^2_{\mathscr{S}}$.

Given a vector field **u** on $\mathscr{S}$, we split **F** into electric field **E** and magnetic field **B** by putting

$$E_\mu = F_{\mu, \nu}\, u^\nu, \tag{23}$$

$$B_\mu = -(*\mathbf{F})_{\mu, \nu}\, u^\nu. \tag{24}$$

Obviously, the definition of **E** and **B** is independent of choice of local coordinates, moreover, they completely determine tensor **F** at any point at which **u** is no null for the Schwarzschild metric. This is seen by using a local Lorentz frame from the formula

$$u^{\hat{\mu}}\, u_{\hat{\mu}}\, F_{\hat{\alpha}, \hat{\beta}} = E_{\hat{\alpha}}\, u_{\hat{\beta}} - u_{\hat{\alpha}}\, E_{\hat{\beta}} + B^{\hat{\mu}} u^{\hat{\nu}}\, \varepsilon_{\hat{\mu}, \hat{\nu}, \hat{\alpha}, \hat{\beta}}, \tag{25}$$

where ε is the four-rank completely antisymetric tensor.

Of physical view point, **E** and **B** are the electric and magnetic fields measured by an observer with four velocity **u**. Since we are concerned by the scattering theory, we consider the Black-Hole as a perturbation living in absolute space V and we choose an observer at rest by respect to the Black-Hole (Fiducial observer of [14]), and so:

$$\mathbf{u} = \alpha^{-1} \frac{\partial}{\partial t}. \tag{26}$$

Then, **E** and **B** are tangential to V and we consider them as vector fields on V. In terms of **E**, **B**, Maxwell's equations (22) take familiar form (16). Now, it is convenient to use a local Cartesian coordinate system in (V, ds^1_V), of fiducial observer's proper reference frame, by choosing

$$\mathbf{e}_{\hat{r}} = \alpha \frac{\partial}{\partial r}, \qquad \mathbf{e}_{\hat{\theta}} = \frac{1}{r} \frac{\partial}{\partial \theta}, \qquad \mathbf{e}_{\hat{\varphi}} = \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi}. \tag{27}$$

We put

$${}^t U = {}^t(E^{\hat{r}}, E^{\hat{\theta}}, E^{\hat{\varphi}}, B^{\hat{r}}, B^{\hat{\theta}}, B^{\hat{\varphi}}) = {}^t(E, B) \tag{28}$$

where

$$\mathbf{X}=\mathrm{X}^{\hat{r}}\mathbf{e}_{\hat{r}}+\mathrm{X}^{\hat{\theta}}\mathbf{e}_{\hat{\theta}}+\mathrm{X}^{\hat{\varphi}}\mathbf{e}_{\hat{\varphi}}, \qquad \mathrm{X}=\mathrm{E},\ \mathrm{B}. \tag{29}$$

After tedious but elementary calculations Maxwell's equations (16) become

$$\frac{\partial}{\partial t}\mathrm{U}=-i\mathrm{H}\mathrm{U}, \tag{30}$$

$$\nabla_{\mathrm{V}}.\mathrm{E}=\nabla_{\mathrm{V}}.\mathrm{B}=0, \tag{31}$$

where

$$\mathrm{H}=i\begin{pmatrix} 0 & \nabla_V\times\alpha \\ -\nabla_V\times\alpha & 0 \end{pmatrix}, \tag{32}$$

$$\nabla_{\mathrm{V}}\times\alpha=\begin{pmatrix} 0 & -\dfrac{\alpha}{r\sin\theta}\partial_{\varphi} & \dfrac{\alpha}{r\sin\theta}\partial_{\theta}\sin\theta \\ \dfrac{\alpha}{r\sin\theta}\partial_{\varphi} & 0 & -\dfrac{\alpha}{r}\partial_r r\alpha \\ -\dfrac{\alpha}{r}\partial_{\theta} & \dfrac{\alpha}{r}\partial_r r\alpha & 0 \end{pmatrix}$$

and

$$\nabla_{\mathrm{V}}.\mathrm{X}=\frac{\alpha}{r^2}\,\frac{\partial}{\partial_r}(r^2\mathrm{X}^{\hat{r}})+\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\,\mathrm{X}^{\hat{\theta}})+\frac{\partial}{\partial\varphi}\mathrm{X}^{\hat{\varphi}}\right], \tag{33}$$
$$\mathrm{X}=\mathrm{E},\ \mathrm{B}.$$

If there is no Black-Hole, $\alpha=1$ and we find the free dynamic in the Minkowski space time with spherical coordinates.

Now, we choose our Hilbert space of finite energy fields. The locally measured energy density is given by

$$(\mathrm{E}^{\hat{\mu}}\mathrm{E}_{\hat{\mu}}+\mathrm{B}^{\hat{\mu}}\mathrm{B}_{\hat{\mu}})d\mathrm{V}=|\mathrm{U}|^2\,d\mathrm{V} \tag{34}$$

where

$$d\mathrm{V}=\alpha^{-1}r^2\sin\theta\,dr\,d\theta\,d\varphi=\alpha^{-1}r^2\,dr\,d\omega \tag{35}$$

is the volume element in V and $|.|$ notes the euclidian norm in $\mathbb{C}^6$. But the corresponding energy integral is not conserved. So, we consider the energy at infinity, or redshifted energy, that is conserved; its density is

$$\alpha(\mathrm{E}^{\hat{\mu}}\mathrm{E}_{\hat{\mu}}+\mathrm{B}^{\hat{\mu}}\mathrm{B}_{\hat{\mu}})\,d\mathrm{V}=|\mathrm{U}|^2\,r^2\,dr\,d\omega. \tag{36}$$

Therefore we introduce the Hilbert space of finite redshifted energy:

$$\mathscr{H}=[\mathrm{L}^2(]r_0,\ +\infty[_r\times \mathrm{S}^2_{\omega},\ r^2\,dr\,d\omega)]^6 \tag{37}$$

and the subspace of free divergence:

$$\mathscr{H}^{(0)}=\{\mathrm{U}\in\mathscr{H},\ \nabla_{\mathrm{V}}.\mathrm{E}=\nabla_{\mathrm{V}}.\mathrm{B}=0\}. \tag{38}$$

Like the case of an obstacle in Euclidian space where the second cohomological space is non trivial, there exists so stationnary fields with data in $\mathbb{H}^2$:

$$\mathbb{H}^2=\{\mathrm{U}\in\tilde{\mathscr{H}}^{(0)},\ \mathrm{HU}=0\}. \tag{39}$$

We shall see that

$$\mathbb{H}^2=\{\mathrm{U}={}^t(ar^{-2},\ 0,\ 0,\ br^{-2},\ 0,\ 0),\ a,\ b\in\mathbb{C}\} \tag{40}$$

and the scattering theory involves naturally the orthogonal of $\mathbb{H}^2$ in $\tilde{\mathscr{H}}^{(0)}$:

$$\tilde{\mathscr{H}}^{(0)}=\mathscr{H}\oplus\mathbb{H}^2,\qquad \mathscr{H}=\left\{\mathrm{U}\in\tilde{\mathscr{H}}^{(0)},\ \int \mathrm{E}^{\hat{r}}\,dr\,d\omega=\int \mathrm{B}^{\hat{r}}\,dr\,d\omega=0\right\}. \tag{41}$$

We consider H as a differential operator defined in distributions sense on $\tilde{\mathscr{H}}$, we note again H its restriction at, $\tilde{\mathscr{H}}^{(0)}$ or $\mathscr{H}$. The natural domain of H on $\mathscr{E}=\tilde{\mathscr{H}}$, $\tilde{\mathscr{H}}^{(0)}$ or $\mathscr{H}$ is

$$\mathrm{D}(\mathrm{H}_{|\mathscr{E}})=\{\mathrm{U}\in\mathscr{E},\ \mathrm{HU}\in\mathscr{E}\}. \tag{42}$$

THEOREM 2.1. – *For* $\mathscr{E}=\tilde{\mathscr{H}}$, $\tilde{\mathscr{H}}^{(0)}$ *or* $\mathscr{H}$, H *is a selfadjoint operator on* $\mathscr{E}$, *with dense domain* $\mathrm{D}(\mathrm{H}_{|\mathscr{E}})$.

Therefore we can solve easily the Cauchy problem for Maxwell's equations thanks to Stone's theorem: the solution U (*t*) of (30) with initial data U (0) in $\tilde{\mathscr{H}}$ is given by

$$\mathrm{U}(t)=e^{-it\mathrm{H}}\,\mathrm{U}(0)\in\mathrm{C}^0(\mathbb{R}_t,\ \tilde{\mathscr{H}}) \tag{43}$$

where $e^{-it\mathrm{H}}$ is the unitary group on $\tilde{\mathscr{H}}$ generated by H, and if U (0) is in $\mathscr{H}$, the solution of (30) and (31) in $\mathrm{C}^0(\mathbb{R}_t,\ \mathscr{H})$ is given by (43) again.

The hyperbolicity of system (30) implies a classical result of finite velocity dependence, that is nice to express by using Wheeler coordinate r_* (13)

THEOREM 2.2. – *Let* U *be in* $\tilde{\mathscr{H}}$ *such that*

$$\mathrm{supp}\,\mathrm{U}\subset\{r_*^1\leqq r_*\leqq r_*^2\}\times\mathrm{S}^2,$$

then we have

$$\mathrm{supp}\,e^{it\mathrm{H}}\,\mathrm{U}\subset\{r_*^1-|t|\leqq r_*\leqq r_*^2+|t|\}\times\mathrm{S}^2.$$

Despite the fact the Schwarzschild metric is trapping, there does not exist time periodic non stationary solutions:

THEOREM 2.3. – *The point spectrum of* H *on* $\tilde{\mathscr{H}}^{(0)}$ *is* $\{0\}$ *and* $\mathbb{H}^2$ *is given by* (40); *the point spectrum of H on* $\mathscr{H}$ *is empty.*

An useful consequence is that we may approximate any field of free divergence and without stationary part by a field derivated from regular potential:

COROLLARY 2.4. – *For* $k \geqq 1$, *the set:*

$$\{ U = H^k f, f \in (C_0^\infty(V))^6 \}$$

is dense in $\mathscr{H}$.

Proof of Theorem 2.1. – An easy calculation with integrations by parts shows that H is symmetric on $(C_0^\infty(V))^6$. Then to prove H is selfadjoint on $\tilde{\mathscr{H}}$, it is sufficient to verify that $(C_0^\infty(V))^6$ is dense in $D(H_{|\tilde{\mathscr{H}}})$ for the graph norm. Let χ_* be an element of $C_0^\infty(\mathbb{R})$ such that

$$0 \leqq \chi_* \leqq 1, \qquad -1 \leqq r_* \leqq 1 \quad \Rightarrow \quad \chi_*(r_*) = 1. \tag{44}$$

For $j \in \mathbb{N}$ we define $\chi_j \in C_0^\infty(V)$ by putting

$$\chi_j(r, \omega) = \chi_*(j^{-1} r_*), \qquad \forall (r, \omega) \in V. \tag{45}$$

Given f in $D(H_{|\tilde{\mathscr{H}}})$ the sequence

$$f_j = \chi_j f \tag{46}$$

tends obvious to f in $\tilde{\mathscr{H}}$ as $j \to +\infty$.
Now we have

$$H f_j = \chi_j H f + M_j f \tag{47}$$

where M_j is a 6×6 matrix satisfying

$$\| M_j \|_{L^\infty(V)} \to 0, \qquad j \to +\infty. \tag{48}$$

We conclude that $H f_j$ tends to $H f$ in $\tilde{\mathscr{H}}$ as $j \to +\infty$ and then $D(H_{|\tilde{\mathscr{H}}}) \cap \mathscr{E}'(V)$ is dense in $D(H_{|\tilde{\mathscr{H}}})$ where $\mathscr{E}'(V)$ is the space of compact supported distributions.

Now, we consider $f \in D(H_{|\tilde{\mathscr{H}}}) \cap \mathscr{E}'(V)$; there exists $\gamma \in \mathbb{R}$ such that

$$\operatorname{supp} f \subset]\gamma, +\infty[_{r_*} \times S^2_\omega. \tag{49}$$

We introduce the radial parameter s_*

$$s_* = r_* - \gamma + 1. \tag{50}$$

The map L defined by

$$U \in \tilde{\mathscr{H}}, \qquad LU = \alpha r s_*^{-1} U \tag{51}$$

is an isometry from $\tilde{\mathscr{H}}$ onto $\tilde{\mathscr{H}}'$

$$\tilde{\mathscr{H}}' = [L^2(\mathbb{R}_{s_*} \times S^2_\omega, s_*^2 \, ds_* \, d\omega)]^6. \tag{52}$$

We verify easily that

$$LHL^{-1} = ARA \tag{53}$$

where

$$A = \operatorname{diag}(\alpha s_* r^{-1}, 1, 1, \alpha s_* r^{-1}, 1, 1) \tag{54}$$

$$\mathrm{R}=i\begin{pmatrix} 0 & \nabla_* \times \\ -\nabla_* \times & 0 \end{pmatrix} \tag{55}$$

$$\nabla_* \times = \begin{pmatrix} 0 & -\frac{1}{s_* \sin\theta}\partial_\varphi & \frac{1}{s_* \sin\theta}\partial_\theta \sin\theta \\ \frac{1}{s_* \sin\theta}\partial_\varphi & 0 & -\frac{1}{s_*}\partial_{s_*} s_* \\ -\frac{1}{s_*}\partial_\theta & \frac{1}{s_*}\partial_{s_*} s_* & 0 \end{pmatrix} \tag{56}$$

We have

$$\mathrm{L}f \in \tilde{\mathscr{H}}', \ \mathrm{ARAL}\, f \in \tilde{\mathscr{H}}', \ \mathrm{supp}\, \mathrm{L}f \subset]1, +\infty[_{s_*} \times \mathrm{S}^2_\omega. \tag{57}$$

The restriction of $\nabla_* \times$ at $\mathbb{R}^+_{s_*} \times \mathrm{S}^2_\omega$ is the standard *curl* in euclidian space $\mathbb{R}^3$ in spherical coordinates; it is well known that if $u(x)$ is in $\mathrm{L}^2(\mathbb{R}^3_x)$ and

$$\mathrm{curl}\, u \in \mathrm{L}^2(\mathbb{R}^3_x), \qquad \mathrm{supp}\, u \subset \{x \in \mathbb{R}^3, |x| \geqq 1\}$$

then there exists $u_n \in \mathrm{C}^\infty_0(\mathbb{R}^3_x)$ such that

$$\mathrm{supp}\, u_n \subset \{x \in \mathbb{R}^3, |x| \geqq 1/2\}$$
$$u_n \to u \quad \text{in } \mathrm{L}^2(\mathbb{R}^3), \qquad \mathrm{curl}\, u_n \to \mathrm{curl}\, u \quad \text{in } \mathrm{L}^2(\mathbb{R}^3), \qquad n \to \infty.$$

So it follows from (57) that we can approximate f in graph norm of H with a sequence in $(\mathrm{C}^\infty_0(\mathrm{V}))^6$. Hence this space is dense in $\mathrm{D}(\mathrm{H}_{|\tilde{\mathscr{H}}})$.

Q.E.D.

A direct calculation shows that

$$\nabla_\mathrm{V} \cdot \nabla_V \times = 0. \tag{58}$$

Therefore, if $e^{-it\mathrm{H}}$ is the unitary group on $\tilde{\mathscr{H}}$ associated with H by Stone's theorem, its restriction at $\mathscr{E} = \tilde{\mathscr{H}}^{(0)}$ or $\mathscr{H}$ is a unitary group on $\mathscr{E}$; its generator is $\mathrm{H}_{|\mathscr{E}}$ with domain $\mathrm{D}(\mathrm{H}_{|\mathscr{E}})$; thus $\mathrm{H}_{|\mathscr{E}}$ is selfadjoint on $\mathscr{E}$.

Q.E.D.

Proof of Theorem 2.2. – This result of propagation with finite velocity follows immediately from the estimate of local energy:

PROPOSITION 2.5. – *Let* U *be a solution of* (30) *with initial data in* $\tilde{\mathscr{H}}$. *Then, for any time* $\mathrm{T}>0$ *and* r^1_*, r^2_* *in* $\mathbb{R}$, *we have*

$$\int_{\mathrm{S}^2_\omega}\int_{r^1_* \leqq r_* \leqq r^2_*} |\mathrm{U}(\mathrm{T})|^2 r^2\, dr\, d\omega \leqq \int_{\mathrm{S}^2_\omega}\int_{r^1_* - \mathrm{T} \leqq r_* \leqq r^2_* + \mathrm{T}} |\mathrm{U}(0)|^2 r^2\, dr\, d\omega.$$

Proof. – By a classical argument of density we consider only the case where U(0) is in $\mathrm{D}(\mathrm{H}_{|\tilde{\mathscr{H}}})$. Then we have in strong sense

$$\frac{d}{dt}|\mathrm{U}|^2 = 2\, \mathscr{R}e(-i\overset{*}{\mathrm{U}}\mathrm{HU}) \tag{59}$$

where $\overset{*}{\mathrm{U}}$ is the transposate complex conjugate of U. By using (32) we obtain

$$2\,\mathcal{R}e(-i\overset{*}{\mathrm{U}}\,\mathrm{HU})=2\,\mathcal{R}e\left\{\frac{1}{\alpha^2 r^2}\frac{\partial}{\partial r_*}[\alpha^2 r^2(\mathrm{E}^{\hat\theta}\,\bar{\mathrm{B}}^{\hat\varphi}-\mathrm{E}^{\hat\varphi}\,\bar{\mathrm{B}}^{\hat\theta})]\right. \tag{60}$$

$$\left.+\frac{\alpha}{r\sin\theta}\left[\frac{\partial}{\partial\theta}[\sin\theta(\mathrm{E}^{\hat\varphi}\,\bar{\mathrm{B}}^{\hat r}-\mathrm{E}^{\hat r}\,\bar{\mathrm{B}}^{\hat\varphi})]+\frac{\partial}{\partial\varphi}(\mathrm{E}^{\hat r}\,\bar{\mathrm{B}}^{\hat\theta}-\mathrm{E}^{\hat\theta}\,\bar{\mathrm{B}}^{\hat r})\right]\right\}$$

where $\bar{\mathrm{X}}^{\hat\mu}$ is the complex conjugate of $\mathrm{X}^{\hat\mu}$. We integrate equation (59) on the domain

$$\{(t,r,\omega);\ 0\leqq t\leqq \mathrm{T},\ r_*^1-\mathrm{T}+t\leqq r_*\leqq r_*^2+\mathrm{T}-t,\ \omega\in\mathrm{S}^2\}$$

with the measure $r^2\,dr\,d\omega\,dt$; we obtain by direct integrations

$$\int_{\mathrm{S}^2_\omega}\int_{r_*^1-\mathrm{T}\leqq r_*\leqq r_*^2+\mathrm{T}}|\mathrm{U}(0)|^2 r^2\,dr\,d\omega=\int_{\mathrm{S}^2_\omega}\int_{r_*^1\leqq r_*\leqq r_*^2}|\mathrm{U}(\mathrm{T})|^2 r^2\,dr\,d\omega$$
$$+\int_0^{\mathrm{T}}\int_{\mathrm{S}^2_\omega}\alpha^2 r^2[|\mathrm{E}^{\hat r}|^2+|\mathrm{B}^{\hat r}|^2+(|\mathrm{E}^{\hat\theta}|^2$$
$$+|\mathrm{B}^{\hat\varphi}|^2-2\,\mathcal{R}e\,\mathrm{E}^{\hat\theta}\,\bar{\mathrm{B}}^{\hat\varphi})](t,r_*=r_*^2+\mathrm{T}-t,\omega)\,dt\,d\omega$$
$$+\int_0^{\mathrm{T}}\int_{\mathrm{S}^2_\omega}\alpha^2 r^2[|\mathrm{E}^{\hat r}|^2+|\mathrm{B}^{\hat r}|^2+(|\mathrm{E}^{\hat\theta}|^2$$
$$+|\mathrm{B}^{\hat\varphi}|^2+2\,\mathcal{R}e\,\mathrm{E}^{\hat\theta}\,\bar{\mathrm{B}}^{\hat\varphi})](t,r_*=r_*^1-\mathrm{T}-t,\omega)\,dt\,d\omega. \tag{61}$$

We conclude by noting these last both integrals are non negative.

Q.E.D

Proof of Theorem 2.3. – Let $\mathrm{U}={}^t(\mathrm{E},\mathrm{B})$ be in $\mathcal{H}$ such that

$$\mathrm{HU}=\lambda\,\mathrm{U},\qquad \lambda\in\mathbb{R} \tag{62}$$
$$\nabla_{\mathrm{V}}.\mathrm{E}=\nabla_{\mathrm{V}}.\mathrm{B}=0. \tag{63}$$

For X = E or B we put

$$\left.\begin{aligned}\mathrm{A}^0=r^2\mathrm{X}^{\hat r},\qquad \mathrm{A}^+&=-2^{-1/2}\,ra(\mathrm{X}^{\hat\varphi}+i\mathrm{X}^{\hat\theta}),\\ \mathrm{A}^-&=2^{-1/2}\,r\alpha(\mathrm{X}^{\hat\varphi}-i\mathrm{X}^{\hat\theta}).\end{aligned}\right\} \tag{64}$$

(62) and (63) imply

$$(\nabla_{\mathrm{V}}\times\alpha)(\nabla_{\mathrm{V}}\times\alpha)\mathrm{X}=\lambda^2\mathrm{X} \tag{65}$$
$$\nabla_{\mathrm{V}}.\mathrm{X}=0. \tag{66}$$

By a direct calculation, we express equation (65) in terms of A^0, A^+, A^- by using constraint of free divergence (66):

$$-\lambda^2\mathrm{A}^0-\partial_{r_*}^2\mathrm{A}^0=\frac{\alpha^2}{r^2}\Delta_{\mathrm{S}^2}\mathrm{A}^0 \tag{67}$$

$$-\lambda^2 A^+ - \partial_{r_*}^2 A^+ = \frac{\alpha^2}{r^2}\left\{\Delta_{S^2} A^+ - 2i\frac{\cos\theta}{\sin^2\theta}\partial_\varphi A^+ - \frac{1}{\sin^2\theta}A^+ - \frac{i}{r\sqrt{2}}\left(2-\frac{3r_0}{r}\right)\left[\partial_\theta A^0 - \frac{i}{\sin\theta}\partial_\varphi A^0\right]\right\} \quad (68)$$

$$-\lambda^2 A^- - \partial_{r_*}^2 A^- = \frac{\alpha^2}{r^2}\left\{\Delta_{S^2} A^- + 2i\frac{\cos\theta}{\sin^2\theta}\partial_\varphi A^- - \frac{1}{\sin^2\theta}A^- - \frac{i}{r\sqrt{2}}\left(2-\frac{3r_0}{r}\right)\left[\partial_\theta A^0 + \frac{i}{\sin\theta}\partial_\varphi A^0\right]\right\} \quad (69)$$

where Δ_{S^2} is the Laplace-Beltrami operator on S^2:

$$\Delta_{S^2} = \frac{1}{\sin\theta}\partial_\theta(\sin\theta\,\partial_\theta) + \frac{1}{\sin^2\theta}\partial_\varphi^2 \quad (70)$$

and equation (66) becomes

$$\partial_{r_*} A^0 + \frac{i}{\sqrt{2}}\left\{\frac{1}{\sin\theta}(\partial_\theta \sin\theta + i\partial_\varphi)A^+ + \frac{1}{\sin\theta}(\partial_\theta\sin\theta - i\partial_\varphi)A^-\right\} = 0. \quad (71)$$

Because U is in $\tilde{\mathscr{H}}$, we have

$$A^0 \in L^2\left(\mathbb{R}_{r_*} \times S^2_\omega, \frac{\alpha^2}{r^2}\,dr_*\,d\omega\right) \quad (72)$$

$$A^+,\ A^- \in L^2(\mathbb{R}_{r_*} \times S^2_\omega,\ dr_*\,d\omega). \quad (73)$$

Following J. M. Gel'fand and Z. Ya. Šapiro [11] we expand A^0, A^+, A^- in series of generalised spherical functions $T^l_{m,n}$:

$$A^0(r_*, \theta, \varphi) = \sum_{l=0}^{\infty}\sum_{n=-l}^{l} a^0_{l,n}(r_*)\, T^l_{0,n}\left(\frac{\pi}{2}-\varphi, \theta, 0\right) \quad (74)$$

$$A^\pm(r_*, \theta, \varphi) = \sum_{l=0}^{\infty}\sum_{n=-l}^{l} a^\pm_{l,n}(r_*)\, T^l_{\pm 1,n}\left(\frac{\pi}{2}-\varphi, \theta, 0\right) \quad (75)$$

where $(T^l_{m,n})_{l,n}$ is an orthogonal basis of $L^2(S^2)$ and

$$a^\nu_{l,n}(r_*) = \int_{S_2} A^\nu(r_*, \theta, \varphi)\,\bar{T}^l_{\nu n}\left(\frac{\pi}{2}-\varphi, \theta, 0\right) d\omega, \qquad \nu = 0, +1, -1. \quad (76)$$

We recall that

$$T^l_{m,n}\left(\frac{\pi}{2}-\varphi, \theta, 0\right) = e^{-in((\pi/2)-\varphi)}\, u^l_{m,n}(\theta) \quad (77)$$

and $u^l_{m,\,n}$ satisfies the differential equations:

$$\frac{1}{\sin\theta}\,\partial_\theta(\sin\theta\,\partial_\theta u^l_{m,\,n})+\left[l(l+1)-\frac{n^2-2mn\cos\theta+m^2}{\sin^2\theta}\right]u^l_{m,\,n}=0 \tag{78}$$

$$\partial_\theta u^l_{m,\,n}-\frac{n-m\cos\theta}{\sin\theta}\,u^l_{m,\,n}=-i((l+m)(l-m+1))^{1/2}\,u^l_{m-1,\,n} \tag{79}$$

$$\partial_\theta u^l_{m,\,n}+\frac{n-m\cos\theta}{\sin\theta}\,u^l_{m,\,n}=-i((l+m+1)(l-m))^{1/2}\,u^l_{m+1,\,n}. \tag{80}$$

By using these relations we find, after elementary and tedious calculations, that $a^\nu_{l,\,n}$ are solution of

$$\partial^2_{r_*}a^0+\lambda^2 a^0=l(l+1)\frac{\alpha^2}{r^2}\,a^0 \tag{81}$$

$$\partial^2_{r_*}a^+ +\lambda^2 a^+ =l(l+1)\frac{\alpha^2}{r^2}\,a^+ +\frac{\alpha^2}{r^2}\,a^+ +\frac{\alpha^2}{r^3}\left(2-\frac{3r_0}{r}\right)a^0 \tag{82}$$

$$\partial^2_{r_*}a^- +\lambda^2 a^- =l(l+1)\frac{\alpha^2}{r^2}\,a^- +\frac{\alpha^2}{r^2}\,a^+ +\frac{\alpha^2}{r^3}\left(2-\frac{3r_0}{r}\right)a^0 \tag{83}$$

$$\partial_{r_*}a^0+[l(l+1)]^{1/2}\,2^{-1/2}(a^+ +a^-)=0 \tag{84}$$

$$a^0\in L^2\left(\mathbb{R},\frac{\alpha^2}{r^2}\,dr_*\right) \tag{85}$$

$$a^\pm\in L^2(\mathbb{R},dr_*) \tag{86}$$

and for simplicity we have put

$$a^\nu=a^\nu_{l,\,n},\qquad \nu=0,\,+,\,-$$

First we consider the case $\lambda\neq 0$. Because $\alpha^2 r^{-2}$ is in $L^1(\mathbb{R}, dr_*)$, it is well known that any solution of

$$u''+\lambda^2 u=l(l+1)\frac{\alpha^2}{r^2}\,u \tag{87}$$

is a linear combination of particular solutions u_+, u_- satisfying

$$u_\pm(r_*)-e^{\pm i\lambda r_*}\to 0,\qquad r_*\to+\infty \tag{88}$$

$$u'_\pm(r_*)\mp i\lambda e^{i\lambda r_*}\to 0,\qquad r_*\to+\infty. \tag{89}$$

Therefore (87) admits no nonnull solution such that

$$u \text{ or } u'\in L^2(\mathbb{R},dr_*). \tag{90}$$

Following (84) and (86), $\partial_{r_*}a^0$ in in $L^2(\mathbb{R}, dr_*)$ then we deduct from (81) a^0 is null. Now (82) and (83) imply that a^+, a^- are solutions of (87) in $L^2(\mathbb{R}, dr_*)$ and therefore a^+, a^- are so null. We conclude that $\lambda\neq 0$ is not eigenvalue of H on $\mathscr{H}^{(0)}$.

Now we consider the case $\lambda=0$. If $l\neq 0$ we investigate the differential equation

$$u''=l(l+1)\frac{\alpha^2}{r^2}u. \tag{91}$$

We write

$$u(r_*)=f(r_*)+g(r_*)r_*, \qquad u'(r_*)=g(r_*). \tag{92}$$

By putting $\psi(r_*)={}^t(f(r_*), g(r_*))$ (91) becomes

$$\psi'=V\psi \tag{93}$$

where

$$V(r_*)=l(l+1)\frac{\alpha^2}{r^2}\begin{pmatrix} -r_* & -r_*^2 \\ 1 & r_* \end{pmatrix}. \tag{94}$$

We recall that α^2 is exponentially decreasing as $r_*\to-\infty$, then $V\in L^1(]-\infty, r_*^1]_{r_*}, dr_*)$ for any r_*^1 and for any complex c, c' the integral equation

$$\psi(r_*)=\begin{pmatrix} c \\ c' \end{pmatrix}+\int_{-\infty}^{r_*} V(s)\psi(s)\,ds \tag{95}$$

admits a unique solution $\psi={}^t(f, g)$ in $L^\infty(]-\infty, r_*^1])$; moreover $|f(r_*)-c|$ and $|g(r_*)-c'|$ tends exponentially to 0 as $r_*\to-\infty$. Taking $c=1$, $c'=0$ we obtain a solution u_1 of (91) with

$$|u_1(r_*)-1|+|u_1'(r_*)|\leqq Ce^{-\gamma|r_*|}, \qquad 0<\gamma, \quad r_*\to-\infty, \tag{96}$$

and by choosing $c=0$, $c'=1$ we find a solution u_2 of (91) with

$$|u_2(r_*)-r_*|+|u_2'(r_*)-1|\leqq Ce^{-\gamma|r_*|}, \qquad 0<\gamma, \quad r_*\to-\infty. \tag{97}$$

Obviously the Wronskian $W(r_*)=u_1' u_2-u_2' u_1$ is constant and thanks to (96) (97) we have

$$W(r_*)=1.$$

Therefore, any solution u of (91) can be written

$$u=\lambda_1 u_1+\lambda_2 u_2, \qquad \lambda_i\in\mathbb{C} \tag{98}$$

and

$$\lambda_2=\lim_{r_*\to-\infty} u'(r_*). \tag{99}$$

We see that if u' is in $L^2(\mathbb{R}, dr_*)$, then $\lambda_2=0$. So, we deduct from (81) (84) (86) that

$$a^0=\lambda_1 u_1, \qquad \lambda_1\in\mathbb{C}. \tag{100}$$

Now, we have

$$u_1(r_*)\to 1, \qquad r_*\to-\infty \tag{101}$$

and

$$u_1'(r_*)=\int_{-\infty}^{r_*} l(l+1)\frac{\alpha^2}{r^2}u_1(s_*)\,ds_*. \tag{102}$$

Let R_* be defined by

$$R_*=\operatorname{Sup}\{r_*,\ s_*\leqq r_*\Rightarrow u_1(s_*)>0\}. \tag{103}$$

If R_* is finite, (102) gives

$$u_1'(R_*)>0$$

that is in contradiction with (103). Therefore u_1 is strictly positive and u_1' is strictly creasing and positive and so

$$u_1'\notin L^2(\mathbb{R},dr_*). \tag{104}$$

We conclude that a^0 is null.

Now a^+ and a^- are solutions of (91) again:

$$a^{\pm}=\lambda_1^{\pm}u_1+\lambda_2^{\pm}u_2 \tag{105}$$

and

$$a^{\pm}(r_*)\cong\lambda_1^{\pm}+\lambda_2^{\pm}r_*,\qquad r_*\to-\infty. \tag{106}$$

Then (86) implies a^+, a^- are null.

At last we consider the case $\lambda=0$, $l=0$. By (82) and (83) we have

$$a^{\pm}=\lambda_1^{\pm}+\lambda_2^{\pm}r_*,\qquad \lambda_i^{\pm}\in\mathbb{C} \tag{107}$$

and by (86) a^+, a^- are null again. On the other hand, (84) shows that

$$a^0=\text{Const}. \tag{108}$$

We conclude that any $U\in\mathscr{H}^{(0)}$, such that $HU=0$, has the form

$$U={}^t(ar^{-2},0,0,br^{-2},0,0),\qquad a,b\in\mathbb{C}; \tag{109}$$

that ends the proof.

Q.E.D.

Proof of Corollary 2.4. – We remark $H^k f$ is in $\mathscr{H}$ for any $k\geqq1$ and f in $(C_0^\infty(V))^6$ and

$$\mathscr{E}_k=\{H^k f,\ f\in(C_0^\infty(V))^6\}\subset D(H^k_{|\mathscr{H}}).$$

Let U be in $D(H^k_{|\mathscr{H}})$ orthogonal to $\mathscr{E}_k$ for the graph norm of H^k:

$$\forall f\in(C_0^\infty(V))^6,\quad \langle U,H^k f\rangle_{\mathscr{H}}+\langle H^k U,H^{2k}f\rangle_{\mathscr{H}}=0. \tag{110}$$

Then we have

$$\forall f\in(C_0^\infty(V))^6,\quad \langle H^k U+H^{3k}U,f\rangle_{\mathscr{D}',\mathscr{D}(V)}=0 \tag{111}$$

where the last bracket is taken in distributions sense on V. We deduct that

$$H^k U+H^{3k}U=0\quad in\ (\mathscr{D}'(V))^6.$$

Therefore

$$H^k U \in D(H^{2k}), \quad (H^{2k}+1)(H^k U)=0. \tag{112}$$

Hence $H^k U$ is null and we conclude thanks to Theorem 2.3 that U is null and $\mathscr{E}_k$ is dense in $D(H^k_{|\mathscr{H}})$ which is dense in $\mathscr{H}$.

Q.E.D.

3. WAVE OPERATORS AT INFINITY

The Schwarzschild universe is asymptotically flat and far from the Black-Hole we compare hamiltonian H with classical electromagnetic hamiltonian H_0:

$$H_0 = i\begin{pmatrix} 0 & \text{curl} \\ -\text{curl} & 0 \end{pmatrix},$$

$$\text{curl} = \begin{pmatrix} 0 & -\frac{1}{\rho\sin\theta}\,\partial_\varphi & \frac{1}{\rho\sin\theta}\,\partial_\theta \sin\theta \\ -\frac{1}{\rho\sin\theta}\,\partial_\varphi & 0 & -\frac{1}{\rho}\,\partial_\rho \rho \\ -\frac{1}{\rho}\,\partial_\theta & -\frac{1}{\rho}\,\partial_\rho \rho & 0 \end{pmatrix} \tag{113}$$

in the Minkowski space-time $\mathscr{M}$ with the metric

$$ds^2_{\mathscr{M}} = dt^2 - d\rho^2 - \rho^2(d\theta^2 + \sin^2\theta\, d\varphi^2), \qquad 0 \leqq \rho. \tag{114}$$

For any choice of $\rho=\rho(r)$ we can verify the difference $H-H_0$ is a long range type perturbation, but because the radial null geodesics (15) are straight like their flat analogs, we avoid the long range interaction between the gravitational and electromagnetic fields by choosing

$$\rho = r_* \geqq 0. \tag{115}$$

We introduce the usual finite energy Hilbert spaces

$$\tilde{\mathscr{H}}_0 = \{ U_0 = {}^t(E_0, B_0) \in [L^2(\mathbb{R}^+_{r_*} \times S^2_\omega, r_*^2\, dr_*\, d\omega)]^6 \} \tag{116}$$

$$\mathscr{H}_0 = \{ U_0 \in \tilde{\mathscr{H}}_0, \text{ div } E_0 = \text{div } B_0 = 0 \} \tag{117}$$

where

$$\text{div } X = \frac{1}{r_*^2}\,\partial_{r_*}(r_*^2 X^{\hat{r}}) + \frac{1}{r_*\sin\theta}\,[\partial_\theta(\sin\theta X^{\hat{\theta}}) + \partial_\varphi X^{\hat{\varphi}}]. \tag{118}$$

In order to compare the dynamics far from the Black-Hole in the Schwarzschild and Minkowski space time, we choose a cut-off function χ_0 satisfying

$$\chi_0 \in C^\infty(\mathbb{R}^+_{r_*}), \qquad \exists\, a, b, \qquad 0<a<b, \tag{119}$$
$$0 \leqq r_* \leqq a \;\Rightarrow\; \chi_0(r_*)=0, \qquad b<r_* \;\Rightarrow\; \chi_0(r_*)=1.$$

We construct identification operators between $\mathscr{H}$ and $\mathscr{H}_0$ by putting

$$\mathscr{I}_0:\ \mathscr{H}_0 \to \mathscr{H}, \qquad \mathscr{I}_0 U_0 = \chi_0 U_0 \quad \textit{for } r_* \geqq 0,$$
$$\mathscr{I}_0 U_0 = 0 \quad \textit{for } r_* \leqq 0, \tag{120}$$
$$\mathscr{I}_0^*:\ \mathscr{H} \to \mathscr{H}_0, \qquad \mathscr{I}_0^* U = \chi_0 U\,|_{r_* \geqq 0}. \tag{121}$$

We define the classical wave operators without Dollard's modification

$$W_0^\pm U_0 = s-\lim_{t \to +\infty} e^{itH} \mathscr{I}_0 e^{-itH_0} U_0 \quad \textit{in } \mathscr{H}. \tag{122}$$

In this section we prove $W_0^\pm$ are well defined on $\mathscr{H}_0$; the key of the proof is

1. the spherical invariance of Maxwell equations that implies a t^{-2} decay of radial components of fields,

2. our choice (115) which cancels long range effects, then we can use Cook's method to obtain the following

THEOREM 3.1. – $W_0^\pm$ *are defined from* $\mathscr{H}_0$ *to* $\mathscr{H}$, *are independent on* χ_0 *satisfying* (119) *and*

$$\| W_0^\pm \|_{\mathscr{L}(\mathscr{H}_0,\, \mathscr{H})} \leqq 1.$$

We deduct from this result, the existence of outgoing fields.

PROPOSITION 3.2. – *if* $U_0^\pm \in \mathscr{H}_0$ *satisfy*

$$e^{-itH_0} U_0^\pm = 0 \qquad \textit{for} \quad 0 \leqq r_* \leqq \pm t + C$$

then we have

$$e^{-itH} W_0^\pm U_0^\pm = 0 \qquad \textit{for} \quad r_* \leqq \pm t + C.$$

In section 5 we will prove the asymptotic completeness of $W_0^\pm$, *i.e.* the electromagnetic field in the Schwarzschild space-time is asymptotic far from the Black-Hole, to a free field in the Minkowski space-time. Here we state only this fundamental result. We introduce wave operator W_0:

$$W_0 U = s-\lim_{t \to +\infty} e^{itH_0} \mathscr{I}_0^* e^{-itH} U \quad \textit{in } \mathscr{H}_0. \tag{123}$$

THEOREM 3.3. – W_0 *is defined from* $\mathscr{H}$ *to* $\mathscr{H}_0$ *is independent on* χ_0 *satisfying* (119) *and*

$$\| W_0 \|_{\mathscr{L}(\mathscr{H},\, \mathscr{H}_0)} \leqq 1.$$

Remark 3.4. – Theorems 3.1 and 3.3 are true again for the non smooth cut off function χ_0 such that

$$\exists a,\quad 0<a,\qquad 0\leqq r_*\leqq a \;\Rightarrow\; \chi_0(r_*)=0,\qquad a<r_* \;\Rightarrow\; \chi_0(r_*)=1.$$

To prove that it is sufficient to approximate χ_0 with smooth functions satisfying (119) and we use the previous results and the decay of the local energy.

Proof of Theorem 3.1. – Given U_0 in $\mathscr{H}_0$ we put

$$W_0(t)U_0=e^{itH}\mathscr{I}_0 e^{-itH_0}U_0. \tag{124}$$

We have

$$\|W_0(t)\|_{\mathscr{L}(\mathscr{H}_0,\,\tilde{\mathscr{H}})}\leqq\|\chi_0\|_{L^\infty(\mathbb{R}^+_{r_*})} \tag{125}$$

therefore it is sufficient to prove the existence of $W_0^{\pm}U_0$ on a dense subspace X of $\mathscr{H}_0$. We recall a tempered distribution on euclidian space $\mathbb{R}^3$ is a regular wave packet if its Fourier transform is C^∞ with compact support and 0 is not in this support. For U_0 in $\mathscr{H}_0$ and $\varepsilon>0$ we choose φ, regular wave packet such that

$$\|U_0-\varphi\|_{\tilde{\mathscr{H}}_0}\leqq\varepsilon.$$

Then $U_{0,\,\varepsilon}=-\Delta^{-1}$ curl curl φ is a regular wave packet in $\mathscr{H}_0$ which satisfies

$$\|U_0-U_{0,\,\varepsilon}\|_{\mathscr{H}_0}\leqq c\,\varepsilon.$$

Therefore

$$X=\{U_0\in\mathscr{H}_0,\ U_0 \textit{ regular wave packet}\} \tag{126}$$

is dense in $\mathscr{H}_0$. Now given U_0 in X, we define

$$\tilde{W}_0(t)U_0=e^{itH}M\,\mathscr{I}_0 e^{-itH_0}U_0 \tag{127}$$

where

$$M=\operatorname{diag}\left(\alpha^{-2},\frac{r_*}{\alpha r},\frac{r_*}{\alpha r},\alpha^{-2},\frac{r_*}{\alpha r},\frac{r_*}{\alpha r}\right) \tag{128}$$

and

$$D(t)=W_0(t)U_0-\tilde{W}_0(t)U_0. \tag{129}$$

Given $\varepsilon>0$, we introduce function $\chi(t, r_*, \omega)$ by putting

$$\begin{aligned}\omega\in S^2,\quad r_*\in[(1-\varepsilon)(t),(1+\varepsilon)(t)] \;&\Rightarrow\; \chi(t,r_*,\omega)=1\\ \omega\in S^2,\quad r_*\notin[(1-\varepsilon)(t),(1+\varepsilon)(t)] \;&\Rightarrow\; \chi(t,r_*,\omega)=0.\end{aligned} \tag{130}$$

We have

$$\|D(t)\|_{\tilde{\mathscr{H}}}\leqq\|(1-M)\mathscr{I}_0\chi(t,.)e^{-itH_0}U_0\|_{\tilde{\mathscr{H}}} +\|(1-M)\mathscr{I}_0(1-\chi(t,.))e^{-itH_0}U_0\|_{\tilde{\mathscr{H}}}. \tag{131}$$

It is well known by a stationary phase argument (to *see* e.g. [22]) that for any regular wave packet we have

$$\operatorname{Sup}_{r_*,\,\omega} |\chi(t,.,.)e^{-itH_0}U_0| \leqq C(1+|t|)^{-1} \tag{132}$$

$$\forall N \in \mathbb{N},\quad \exists C_N>0,$$
$$\operatorname{Sup}_{\omega} |(1-\chi(t,r_*,.))e^{-itH_0}U_0| \leqq C_N(1+r_*+|t|)^{-N}. \tag{133}$$

Since $(1-M)\mathscr{I}_0$ is bounded, (133) implies

$$\|(1-M)\mathscr{I}_0(1-\chi(t,.))e^{-itH_0}U_0\|_{\tilde{\mathscr{H}}} = O(|t|^{-N}). \tag{134}$$

We verify also that for any $\delta>0$

$$\operatorname{Sup}_{r_*,\,\omega} |(1-M)\mathscr{I}_0\chi(t,.)| \leqq C_\delta(1+|t|)^{-1+\delta} \tag{135}$$

and therefore

$$\|(1-M)\mathscr{I}_0\chi(t,.)e^{-itH_0}U_0\|_{\tilde{\mathscr{H}}} = O(|t|^{-1+\delta}). \tag{136}$$

We conclude from (131) (134) (136) that $D(t)$ tends to 0 as $|t|\to\infty$ and to prove the existence of $W_0^\pm U_0$, it is sufficient to show that $\tilde{W}_0 U_0$ has a limit in $\tilde{\mathscr{H}}$ as $|t|\to\infty$. For that, we apply Cook's method and we establish that

$$\left\| \frac{d}{dt}\tilde{W}_0(t)U_0(t)U_0 \right\|_{\tilde{\mathscr{H}}} \in L^1(\mathbb{R}_t, dt). \tag{137}$$

We have

$$\left\| \frac{d}{dt}\tilde{W}_0(t)U_0 \right\|_{\tilde{\mathscr{H}}} = \|(HM\mathscr{I}_0 - M\mathscr{I}_0H_0)e^{-itH_0}U_0\|_{\tilde{\mathscr{H}}} \tag{138}$$

and

$$HM\mathscr{I}_0 - M\mathscr{I}_0H_0 = i\begin{pmatrix} O & A \\ -A & 0 \end{pmatrix} \tag{139}$$

$$A = \begin{pmatrix} 0 & -\chi_0\cdot\left(\dfrac{r_*^2}{r^2}-\alpha^{-2}\right)\dfrac{1}{r_*\sin\theta}\partial_\varphi & \chi_0\cdot\left(\dfrac{r_*^2}{r^2}-\alpha^{-2}\right)\dfrac{\partial_\theta \sin\theta}{r_*\sin\theta} \\ 0 & 0 & -\dfrac{r_*}{r\alpha}(\partial_{r_*}\chi_0) \\ 0 & \dfrac{r_*}{r\alpha}(\partial_{r_*}\chi_0) & 0 \end{pmatrix}.$$

We note

$$e^{-itH_0}U_0 = {}^t(E_0, B_0) = {}^t(E_0^{\hat{r}_*}, E_0^{\hat{\theta}}, E_0^{\hat{\varphi}}, B_0^{\hat{r}_*}, B_0^{\hat{\theta}}, B_0^{\hat{\theta}}). \tag{140}$$

So we have

$$\left\|\frac{d}{dt}\tilde{W}_0(t)U_0\right\|_{\tilde{\mathscr{H}}}\leqq\sum_{X=E_0,\,B_0}\left\|\left(\frac{r_*^2}{r^2}-\alpha^{-2}\right)\chi_0(r_*)(\operatorname{curl}X(t))^{\hat{r}_*}\right\|_{L^2(V,\,r^2\,dr\,d\omega)}$$
$$+\sum_{X=E_0,\,B_0}\sum_{\hat{\mu}=\hat{\theta},\,\hat{\varphi}}\left\|\frac{r_*}{r\alpha}\chi_0'(r_*)X^{\hat{\mu}}(t)\right\|_{L^2(V,\,r^2\,dr\,d\omega)}\tag{141}$$

On the one hand, χ_0' has a compact support in $]0,+\infty[_{r_*}$, thus (133) implies

$$\left\|\frac{r_*}{r\alpha}\chi_0'(r_*)X^{\hat{\mu}}(t)\right\|_{L^2(V,\,r^2\,dr\,d\omega)}\leqq C_N(1+|t|)^{-N}\tag{142}$$

on the other hand, Curl X is so a regular wave packet, hence (133) gives

$$\left\|(1-\chi(t))\left(\frac{r_*^2}{r^2}-\alpha^{-2}\right)\chi_0(r_*)(\operatorname{curl}X(t))^{\hat{r}_*}\right\|_{L^2(V,\,r^2\,dr\,d\omega)}=O(|t|)^{-N})\tag{143}$$

where χ is defined by (130).

Now the spherical invariance of Maxwell's equations in the Minkowski space time has an important consequence: the radial components decay as $|t|^{-2}$, indeed $X(t)$ satisfies

$$\partial_t^2X-\Delta_xX=0,\qquad \operatorname{div}X=0,\qquad X=E_0,\ B_0,\ \operatorname{curl}E_0,\ \operatorname{curl}B_0$$

where Δ_x is the laplacian in the euclidian space. Hence

$$\partial_t^2(r_*\,\omega.X)-\Delta_x(r_*\,\omega.X)=0$$

$r_*\,\omega.X$ is a regular wave packet and thanks to (132), (133) we have:

$$|X^{\hat{r}_*}(t,r_*,\omega)|=|\omega.X(t,r_*,\omega)|=O(|t|^{-2}).\tag{144}$$

We remark that

$$\sup_{r_*,\,\omega}\left|\chi(t)\left(\frac{r_*^2}{r^2}-\alpha^{-2}\right)\right|=O(|t|^{-1+\delta}),\qquad 0<\delta<\frac{1}{2}.\tag{145}$$

We deduct from (144) and (145) that

$$\left\|\chi(t)\left(\frac{r_*^2}{r^2}-\alpha^{-2}\right)\chi_0(r_*)(\operatorname{Curl}X(t))^{\hat{r}_*}\right\|_{L^2(V,\,r^2\,dr\,d\omega)}=O(|t|^{-3/2+\delta}).\tag{146}$$

Now (137) follows from (142), (143), (146).

It remains to prove

$$W_0^{\pm}U_0\in\mathscr{H}.\tag{147}$$

We note

$$e^{-itH}W_0^{\pm}U_0={}^t(E,B)(t).$$

From (58) we have

$$\frac{d}{dt}\nabla_V . E(t) = \frac{d}{dt}\nabla_V . B(t) = 0. \tag{148}$$

Let (Φ_1, Φ_2) be in $[C_0^\infty(V)]^2$. We put

$$f(t) = \langle \nabla_V . E(t), \Phi_1 \rangle + \langle \nabla_V . B(t), \Phi_2 \rangle = f(0) \tag{149}$$

where $\langle\ ,\ \rangle$ is the bracket of distribution on V defined for $T \in L^1_{loc}(V, r^2\, dr\, d\omega)$ by

$$\langle T, \Phi \rangle = \int T \Phi r^2\, dr\, d\omega.$$

Then we have

$$f(t) = \langle e^{-itH} W_0^\pm U_0, {}^t(\nabla_V^* \bar{\Phi}_1, \nabla_V^* \bar{\Phi}_2) \rangle_{\tilde{\mathscr{H}}} \tag{150}$$

where

$$\nabla_V^* \Phi = {}^t\left(-\frac{\alpha}{r^2}\partial_r(r^2\Phi),\ -\frac{1}{r}\partial_\theta \Phi,\ -\frac{1}{r\sin\theta}\partial_\varphi \Phi \right). \tag{151}$$

The definition of $W_0^\pm$ assures that

$$f(t) = \langle \mathscr{I}_0 e^{-itH_0} U_0, {}^t(\nabla_V^* \bar{\Phi}_1, \nabla_V^* \bar{\Phi}_2)) \rangle_{\tilde{\mathscr{H}}} + \varepsilon_\pm(t)$$

with

$$\varepsilon_\pm(t) \to 0, \qquad t \to \pm\infty. \tag{152}$$

Because U_0 is in $\mathscr{H}_0$, we have

$$e^{-itH_0} U_0 \to 0 \quad \textit{in } \mathscr{H}_0\textit{-weak}$$

hence

$$\mathscr{I}_0 e^{-itH_0} U_0 \to 0 \quad \textit{in } \tilde{\mathscr{H}}\textit{-weak} \tag{153}$$

We conclude from (148) (149) (152) (153) that

$$W_0^\pm U_0 \in \tilde{\mathscr{H}}^{(0)}. \tag{154}$$

At last we consider

$$I = \langle W_0^\pm U_0, {}^t(Ar^{-2}, 0, 0, Br^{-2}, 0,) \rangle_{\tilde{\mathscr{H}}}, \qquad A, B \in \mathbb{C}.$$

Because ${}^t(Ar^{-2}, 0, 0, Br^{-2}, 0, 0)$ is in $\mathbb{H}^2$ we have

$$I = \langle e^{-itH} W_0^\pm U_0, {}^t(Ar^{-2}, 0, 0, Br^{-2}, 0, 0) \rangle_{\tilde{\mathscr{H}}}.$$

We write again

$$I = \varepsilon_\pm(t) + \langle \mathscr{I}_0 e^{-itH_0} U_0, {}^t(Ar^{-2}, 0, 0, Br^{-2}, 0, 0) \rangle_{\tilde{\mathscr{H}}} \tag{155}$$

where $\varepsilon_\pm$ satisfies (152). Now (144) implies

$$\langle \mathscr{I}_0 e^{-itH_0} U_0, {}^t(Ar^{-2}, 0, 0, Br^{-2}, 0, 0) \rangle_{\tilde{\mathscr{H}}} = O(|t|^{-1}). \tag{156}$$

We conclude from (155) (156) that $W_0^\pm U_0$ is in $\mathscr{H}$. Now we have

$$\| e^{it H} \mathscr{I}_0 e^{-it H_0} U_0 \|_{\tilde{\mathscr{H}}}^2 \leqq C \int_0^b \int_{S^2} | e^{-it H_0} U_0 |^2 r^2 \alpha^2 \, dr_* \, d\omega + \int_b^\infty \int_{S^2} | e^{-it H_0} U_0 |^2 (r^2 \alpha^2 - r_*^2) \, dr_* \, d\omega + \int_b^\infty \int_{S^2} | e^{-it H_0} U_0 |^2 r_*^2 \, dr_* \, d\omega.$$

Thanks to (132) (133) the both first integrals tend to 0 as $|t| \to \infty$, hence

$$\| W_0^\pm U_0 \|_{\tilde{\mathscr{H}}} \leqq \| U_0 \|_{\mathscr{H}_0}.$$

At last we show that $W_0^\pm U_0$ does not depend on χ_0, if χ_0' satisfies (119) and $\mathscr{I}_0'$ is the associated operator, we evaluate

$$\| e^{it H} \mathscr{I}_0 e^{-it H_0} U_0 - e^{it H} \mathscr{I}_0' e^{-it H_0} U_0 \|_{\tilde{\mathscr{H}}} = \| (\mathscr{I}_0 - \mathscr{I}_0') e^{-it H_0} U_0 \|_{\tilde{\mathscr{H}}}.$$

This quantity decaies to zero as $|t|$ tends to infinity because $\chi_0 - \chi_0'$ has a compact support and the local energy of $e^{-it H_0} U_0$ tends to zero. Therefore $e^{it H} \mathscr{I}_0 e^{-it H_0} U_0$ and $e^{it H} \mathscr{I}_0' e^{-it H_0} U_0$ admit the same limit and $W_0^\pm$ is independent on χ_0.

Q.E.D.

Proof of Proposition 3.2. – Let $U_0^\pm$ be in $\mathscr{H}_0$ such that

$$e^{-it H_0} U_0^\pm = 0 \qquad \textit{for} \quad 0 \leqq r_* \leqq \pm t + C. \tag{157}$$

Given $\varepsilon > 0$, we choose τ satisfying

$$\| W_0^\pm U_0^\pm - e^{i\tau H} \mathscr{I}_0 e^{-it H_0} U_0^\pm \|_{\tilde{\mathscr{H}}} \leqq \varepsilon, \quad \pm\tau > |t|. \tag{158}$$

We write

$$\left[\int_{r_* \leqq \pm t + C} \int_{S^2} | e^{-it H} W_0^\pm U_0^\pm |^2 r^2 \, dr \, d\omega \right]^{1/2} \leqq \left[\int_{r_* \leqq \pm t + C} \int_{S^2} | e^{-it H} (W_0^\pm U_0^\pm - e^{i\tau H} \mathscr{I}_0 e^{-i\tau H_0} U_0^\pm) |^2 r^2 \, dr \, d\omega \right]^{1/2} + \left[\int_{r_* \leqq \pm t + C} \int_{S^2} | e^{-i(t-\tau) H} \mathscr{I}_0 e^{-i\tau H_0} U_0^\pm |^2 r^2 \, dr \, d\omega \right]^{1/2}.$$

By using (158) and proposition II.5, we obtain

$$\left[\int_{r_* \leqq \pm t + C} \int_{S^2} | e^{-it H} W_0^\pm U_0^\pm |^2 r^2 \, dr \, d\omega \right]^{1/2} \leqq \varepsilon + \left[\int_{r_* \leqq \pm \tau + C} \int_{S^2} | \mathscr{I}_0 e^{-i\tau H_0} U_0^\pm |^2 r^2 \, dr \, d\omega \right]^{1/2} \tag{159}$$

but this last integral is null because

$$\mathscr{I}_0 e^{-i\tau H_0} U_0^\pm = 0 \qquad \textit{for} \quad r_* \leqq 0 \quad \textit{and} \quad r_* \leqq \pm\tau + C.$$

We conclude $e^{-itH} W_0^{\pm} U_0^{\pm}$ is null for $r_* \leqq \pm t + C$.

Q.E.D.

4. WAVE OPERATORS AT THE HORIZON

Hamiltonian H degenerates as r tends to r_0, but $r\alpha H (r\alpha)^{-1}$ admits a formal limit H_1:

$$H_1 = i\begin{pmatrix} 0 & h_1 \\ -h_1 & 0 \end{pmatrix}, \qquad h_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\partial_{r_*} \\ 0 & \partial_{r_*} & O \end{pmatrix}. \tag{160}$$

We shall see that H_1 is essentially the dynamic in the Rindler space-time that approximates the Schwarzschild metric near the horizon. We introduce the Hilbert space of finite energy data:

$$\tilde{\mathscr{H}}_1 = \{ U_1 = {}^t(E_1^{\hat{r}_*}, E_1^{\hat{\theta}}, E_1^{\hat{\varphi}}, B_1^{\hat{r}_*}, B_1^{\hat{\theta}}, B_1^{\hat{\varphi}}) \in [L^2(\mathbb{R}_{r_*} \times S^2_\omega, dr_* \, d\omega)]^6 \}. \tag{161}$$

We define the subspaces $\mathscr{H}_1^{+(-)}$ of data with a left (right) polarization

$$\mathscr{H}_1^{\pm} = \{ U_1 \in \tilde{\mathscr{H}}_1, E_1^{\hat{r}_*} = B_1^{\hat{r}_*} = \pm E_1^{\hat{\theta}} + B_1^{\hat{\varphi}} = \pm E_1^{\hat{\varphi}} - B_1^{\hat{\theta}} = 0 \}. \tag{162}$$

The fields in $\mathscr{H}_1^{+(-)}$ behave like a plane wave, falling into the future (coming out of the past) horizon: obviously H_1 is a densely defined selfadjoined operator on $\mathscr{H}_1^{+(-)}$ and

$$U_1 \in \mathscr{H}_1^{+(-)} \quad \Rightarrow \quad [e^{-itH_1} U_1](r_*, \omega) = U_1(\pm t + r_*, \omega). \tag{163}$$

In order to compare the H and H_1-dynamics near the Black-Hole we choose a cut-off function χ_1 satisfying

$$\left.\begin{array}{c} \chi_1 \in C^\infty(\mathbb{R}_{r_*}), \quad \exists\, c, d, c<d, \\ r_* < c \;\Rightarrow\; \chi_1(r_*) = 1, \qquad r_* > d \;\Rightarrow\; \chi_1(r_*) = 0 \end{array}\right\} \tag{164}$$

and we construct identification operators between $\tilde{\mathscr{H}}$ and $\tilde{\mathscr{H}}_1$ by putting:

$$\mathscr{I}_1 : \; \tilde{\mathscr{H}}_1 \to \tilde{\mathscr{H}}, \qquad \mathscr{I}_1 U_1 = (r\alpha)^{-1} \chi_1 U_1 \tag{165}$$

$$\mathscr{I}_1^* : \; \tilde{\mathscr{H}} \to \tilde{\mathscr{H}}_1, \qquad \mathscr{I}_1^* U = r\alpha\chi_1 U. \tag{166}$$

We define classical wave operators

$$W_1^{\pm} U_1 = s-\lim_{t \to \pm\infty} e^{itH} \mathscr{I}_1 e^{-itH_1} U_1 \quad in\ \tilde{\mathscr{H}}. \tag{167}$$

In this section we prove that $W_1^{\pm}$ are well-defined on $\mathscr{H}_1^{\pm}$ because the Schwarzschild potential is exponentially decreasing as $r_* \to -\infty$ and we can use Cook's method.

THEOREM 4.1. — W_1^+ *(resp.* W_1^-*) is defined from* $\mathscr{H}_1^+$ *(resp.* $\mathscr{H}_1^-$*) to* $\mathscr{H}$*, is independent on* χ_1 *satisfying* (164) *and*

$$\| W_1^{\pm} \|_{\mathscr{L}(\mathscr{H}_1^{\pm}, \mathscr{H})} \leqq 1.$$

We deduct from this result, the existence of infalling fields, similar to the disappearing solutions in the classical scattering theory for the dissipative obstacles [20]:

PROPOSITION 4.2. – *If* $U_1 \in \mathscr{H}_1^{\pm}$ *v satisfies*

$$U_1(r_*, \omega)=0 \qquad \textit{for} \quad r_* \geqq C$$

then we have

$$e^{-itH} W_1^{\pm} U_1=0 \qquad \textit{for} \quad r_* \pm t \geqq C.$$

In section 5 we shall prove the asymptotic completeness of $W_1^{\pm}$: let W_1 be the wave operator

$$W_1 U = s - \lim_{t \to \pm\infty} e^{itH_1} \mathscr{I}_1^* e^{-itH} U \quad \textit{in} \ \tilde{\mathscr{H}}_1. \tag{168}$$

THEOREM 4.3. – W_1 *is defined from* $\mathscr{H}$ *to* $\mathscr{H}_1^+$, *is independent on* χ_1 *satisfying* (164) *and*

$$\| W_1 \|_{\mathscr{L}(\mathscr{H}, \mathscr{H}_1^+)} \leqq 1.$$

The physical meaning of this fundamental result of completeness is the famous "impedence condition" of Damour ([5], [6], [7]) and Znajeck [24]. Moreover, the asymptotic profile of regular fields satisfies a dissipative condition of infalling left-polarization:

THEOREM 4.4. – *Let* U *be in* $\mathscr{H}$ *such that*

$$U = H f,\ f \in [C_0^\infty(]r_0, +\infty[\times S^2)]^6. \tag{169}$$

We note

$$e^{-itH} U = {}^t(E^{\hat{r}_*}, \ldots, B^{\hat{\varphi}}).$$

Then, for any $s \in \mathbb{R}$, *there exist* $e^{\hat{r}_*}, \ldots, b^{\hat{\varphi}}$ *in* $L^2(S^2)$ *such that, as*

$$r \to r_0, \qquad t + r_* = s \tag{170}$$

the following limits exist in $L^2(S^2)$:

$$r^2 E^{\hat{r}_*} \to e^{\hat{r}_*}, \qquad r^2 B^{\hat{r}_*} \to b^{\hat{r}_*} \tag{171}$$

$$r \alpha E^{\hat{\theta}} \to e^{\hat{\theta}}, \qquad r \alpha B^{\hat{\theta}} \to b^{\hat{\theta}} \tag{172}$$

$$r \alpha E^{\hat{\varphi}} \to e^{\hat{\varphi}}, \qquad r \alpha B^{\hat{\varphi}} \to b^{\hat{\varphi}}. \tag{173}$$

Moreover we have

$$e^{\hat{\theta}} = -b^{\hat{\varphi}}, e^{\hat{\varphi}} = b^{\hat{\theta}}, \tag{174}$$

$$\partial_s e^{\hat{r}_*} + \frac{1}{\sin\theta} [\partial_\theta (\sin\theta\, e^{\hat{\theta}} + \partial_\varphi e^{\hat{\varphi}}] = 0. \tag{175}$$

Remark that, following corollary 2.4, the subspace of data satisfying (169) is dense in $\mathscr{H}$.

So, the Black-Hole horizon is rather similar to a dissipative spherical membrane in the euclidian space with surface resistivity 377 Ω (impedence of vacuum): (174) is formally the impedence condition and (175) the charge conservation law, but we emphasize that unlike the euclidian case for which the dissipative condition is posed at each time and is necessary to solve the mixed problem, impedence property (174) is a consequence of Maxwell equations satisfied at infinity of infalling null geodesics.

Now, we discuss about the Rindler approximation.

Given $\omega_0=(\theta_0, \varphi_0)\in S^2$, we make the change of variables:

$$\begin{aligned} &T=2\,r_0\,\alpha\,\mathrm{sh}\,(t/2\,r_0)\\ &Z=2\,r_0\,\alpha\,\mathrm{ch}\,(t/2\,r_0)\\ &X=r_0\,(\theta-\theta_0)\\ &Y=r_0\sin\theta_0\,(\varphi-\varphi_0). \end{aligned} \tag{176}$$

Then Schwarschild metric (8) becomes

$$ds^2_{\mathscr{S}}=dT^2-dX^2-dY^2-dZ^2$$
$$-\left(1-\frac{r_0^4}{r^4}\right)\alpha^{-2}\,dr^2-\left(1-\frac{r_0^2}{r^2}\right)r^2\,d\theta^2-\left(\sin^2\theta-\frac{r_0^2}{r^2}\sin^2\theta_0\right)r^2\,d\varphi^2. \tag{177}$$

We recall that the Rindler space time is the flat manifold $\mathscr{R}_{\omega_0}$

$$\mathscr{R}_{\omega 0}=\{(T, X, Y, Z)\in\mathbb{R}^4, \qquad |T|<Z\} \tag{178}$$

with the induced Minkowski metric:

$$ds^2_{\mathscr{R}_{\omega_0}}=dT^2-dX^2-dY^2-dZ^2. \tag{179}$$

Hence (177) and (179) give

$$ds^2_{\mathscr{S}}=ds^2_{\mathscr{R}_{\omega_0}}+O\,(\alpha^2\,ds^2_{V}, (\theta-\theta_0)\,ds^2_{V}) \tag{180}$$

and we may approximate a neighborhood of (r_0, ω_0) in the Schwarschild Universe thanks to the Rindler space time $\mathscr{R}_{\omega_0}$.

Now, an electromagnetic field $(\mathbf{E}, \mathbf{B})$ in $\mathscr{R}_{\omega_0}$ satisfies the usual Maxwell's equations (22):

$$\begin{aligned} &\partial_T E^{\hat X}=\partial_Y B^{\hat Z}-\partial_Z B^{\hat Y}\\ &\partial_T E^{\hat Y}=\partial_Z B^{\hat X}-\partial_X B^{\hat Z}\\ &\partial_T E^{\hat Z}=\partial_X B^{\hat Y}-\partial_Y B^{\hat X}\\ &\partial_T B^{\hat X}=-\partial_Y E^{\hat Z}+\partial_Z E^{\hat Y}\\ &\partial_T B^{\hat Y}=-\partial_Z E^{\hat X}+\partial_X E^{\hat Z}\\ &\partial_T B^{\hat Z}=-\partial_X E^{\hat Y}+\partial_Y E^{\hat X}\\ &\partial_X E^{\hat X}+\partial_Y E^{\hat Y}+\partial_Z E^{\hat Z}=0\\ &\partial_X B^{\hat X}+\partial_Y B^{\hat Y}+\partial_Z B^{\hat Z}=0. \end{aligned} \tag{181}$$

Obviously the Cauchy problem for (181) in $\mathscr{R}_{\omega_0}$ is well-posed and no boundary condition is necessary at $Z=0$. Particular solutions are the restrictions at $\mathscr{R}_{\omega_0}$ of plane waves in the half Minkowski space time $\mathscr{M}^{1/2}$:

$$\mathscr{M}^{1/2}=\mathbb{R}_T\times\mathbb{R}_X\times\mathbb{R}_Y\times\mathbb{R}_Z^+$$

satisfying equations (181) in $\mathscr{M}^{1/2}$ and on the one hand

$$\left.\begin{array}{llll} E(T,X,Y,Z)=E(T,Z), & E^{\hat{Z}}=0, & T\in\mathbb{R}, & 0\leqq Z \\ B(T,X,Y,Z)=B(T,Z), & B^{\hat{Z}}=0, & T\in\mathbb{R}, & 0\leqq Z \end{array}\right\} \tag{182}$$

and on the other hand, the impedence condition on the boundary $\{Z=0\}$:

$$E^{\hat{X}}=-B^{\hat{Y}},\qquad E^{\hat{Y}}=B^{\hat{X}},\qquad Z=0,\quad T\in\mathbb{R}. \tag{183}$$

In fact (181) (182) (183) imply that the field satisfies everywhere the polarization condition

$$E^{\hat{X}}=-B^{\hat{Y}},\qquad E^{\hat{Y}}=B^{\hat{X}},\qquad Z\geqq 0,\quad T\in\mathbb{R}. \tag{184}$$

We call *incoming plane waves* in the Rindler space-time, any solution of (181) in $\mathscr{R}_{\omega_0}$, satisfying (182) and (184) for $|T|\leqq Z$. We see easily that for such an incoming plane wave, we have in (t, r_*) coordinates:

$$\partial_t u-\left(\frac{r}{r_0}\right)^2\partial_{r_*}u=0,\qquad t\in\mathbb{R},\quad r_*\in\mathbb{R},\quad u=E^{\hat{X}},E^{\hat{Y}},B^{\hat{X}},B^{\hat{Y}}. \tag{185}$$

On the horizon $\{r=r_0\}$, equation (185) becomes

$$\partial_t u-\partial_{r_*}u=0 \tag{186}$$

which is exactly the H_1-dynamic on $\mathscr{H}_1^+$: for $U_1\in\mathscr{H}_1^+$ and $e^{-itH_1}U_1={}^t(E_1,B_1)$,

$$u=E_1^{\hat{\theta}},E_1^{\hat{\varphi}},B_1^{\hat{\theta}},B_1^{\hat{\varphi}}$$

is solution of (186).

Therefore, to compare the electromagnetic field in the Schwarzschild space-time and in the Rindler approximation, it is sufficient to compare the solutions of (185) and (186).

We introduce Hilbert space $\tilde{\mathscr{H}}_{\mathscr{R}}$:

$$\tilde{\mathscr{H}}_{\mathscr{R}}=\left[L^2\left(\mathbb{R}_{r_*}\times S^2_\omega,\frac{r_0^2}{r^2}dr_*\,d\omega\right)\right]^6$$

and the subspace of incoming data:

$$\mathscr{H}_{\mathscr{R}}^+=\{U_{\mathscr{R}}={}^t(E^{\hat{X}},E^{\hat{Y}},E^{\hat{Z}},B^{\hat{X}},B^{\hat{Y}},B^{\hat{Z}})\in\tilde{\mathscr{H}}_{\mathscr{R}},\ E^{\hat{Z}}=B^{\hat{Z}}=E^{\hat{X}}+B^{\hat{Y}}=E^{\hat{Y}}-B^{\hat{X}}=0\}. \tag{187}$$

Obviously $i(r/r_0)^2\partial_{r_*}$ is a selfadjoint densely defined operator on $\tilde{\mathscr{H}}_{\mathscr{R}}$ and $\mathscr{H}_{\mathscr{R}}^+$. We define an identification operator between $\tilde{\mathscr{H}}$ and $\tilde{\mathscr{H}}_{\mathscr{R}}$ by putting

$$\mathscr{I}_{\mathscr{R}}:\quad \mathrm{U}={}^t(\mathrm{E}^{\hat{r}_*}, \mathrm{E}^{\hat{\theta}}, \mathrm{E}^{\hat{\varphi}}, \mathrm{B}^{\hat{r}_*}, \mathrm{B}^{\hat{\theta}}, \mathrm{B}^{\hat{\varphi}}) \in \tilde{\mathscr{H}} \to \mathscr{I}_{\mathscr{R}}\mathrm{U}=\mathrm{U}_{\mathscr{R}} = r\alpha\chi_1\, {}^t(\mathrm{E}^{\hat{\theta}}, \mathrm{E}^{\hat{\varphi}}, \mathrm{E}^{\hat{r}_*}, \mathrm{B}^{\hat{\theta}}, \mathrm{B}^{\hat{\varphi}}, \mathrm{B}^{\hat{r}_*}) \in \tilde{\mathscr{H}}_{\mathscr{R}}.$$

We define wave operator $\mathrm{W}_{\mathscr{R}}$:

$$\mathrm{U}\in\mathscr{H},\quad \mathrm{W}_{\mathscr{R}}\mathrm{U}=s-\lim_{t\to\pm\infty} e^{-t\,(r^2/r_0^2)\,\partial_{r_*}}\mathscr{I}_{\mathscr{R}}\, e^{-it\mathrm{H}}\mathrm{U} \quad in\ \tilde{\mathscr{H}}_{\mathscr{R}}. \tag{188}$$

THEOREM 4.5. — *$\mathrm{W}_{\mathscr{R}}$ is well-defined from $\mathscr{H}$ to $\mathscr{H}_{\mathscr{R}}^{+}$ is independent on χ_1 and*

$$\|\mathrm{W}_{\mathscr{R}}\|_{\mathscr{L}(\mathscr{H},\, \mathscr{H}_{\mathscr{R}}^{+})}\leq 1.$$

Remark 4.6. — Theorems 4.1, 4.3 and 4.5 are true again for the non smooth cut off function χ_1 such that

$$\exists c,\quad r_*\leq c \;\Rightarrow\; \chi_1(r_*)=1,\qquad c<r_* \;\Rightarrow\; \chi_1(r_*)=0.$$

To prove that it is sufficient to approximate χ_1 with smooth functions satisfying (164) and we use the previous results and the decay of the local energy.

Proof of Theorem 4.1. — Because

$$\|e^{it\mathrm{H}}\mathscr{I}_1 e^{-it\mathrm{H}_1}\mathrm{U}_1\|_{\tilde{\mathscr{H}}}\leq\|\chi_1\|_{\mathrm{L}^\infty}\|\mathrm{U}_1\|_{\tilde{\mathscr{H}}_1} \tag{189}$$

it is sufficient to prove the existence of $\mathrm{W}_1^{\pm}\mathrm{U}_1$ for

$$\mathrm{U}_1\in\mathscr{H}_1^{\pm}\cap[\mathrm{C}_0^\infty(\mathbb{R}_{r_*}\times \mathrm{S}_\omega^2)]^6 \tag{190}$$

such that $\mathrm{U}_1(r_*, .)$ is null for $|r_*|>\mathrm{R}$ and so, in a neighbourhood of $\theta=0$, $\theta=\pi$. We use Cook's method and evaluate

$$f_{\pm}(t)=\left\|\frac{d}{dt}(e^{it\mathrm{H}}\mathscr{I}_1 e^{-it\mathrm{H}_1}\mathrm{U}_1)\right\|_{\tilde{\mathscr{H}}}=\|(\mathrm{H}\mathscr{I}_1-\mathscr{I}_1\mathrm{H}_1)e^{-it\mathrm{H}_1}\mathrm{U}_1\|_{\tilde{\mathscr{H}}}. \tag{191}$$

We obtain easily that

$$f_{\pm}^2(t)\leq\int_{\mathbb{R}}\int_{\mathrm{S}^2}\mathrm{F}(\pm t+r_*,\omega)\,\alpha^2\,dr_*\,d\omega + \int_c^d\int_{\mathrm{S}^2}\mathrm{G}(\pm t+r_*,\omega)\,dr_*\,d\omega \tag{192}$$

where $\mathrm{F}, \mathrm{G}\in\mathrm{L}^1(\mathbb{R}_{r_*}\times \mathrm{S}_\omega^2, dr_*\,d\omega)$, $\operatorname{supp}\mathrm{F}\cup\operatorname{supp}\mathrm{G}\subset[-\mathrm{R},\mathrm{R}]_{r_*}\times\mathrm{S}_\omega^2$, .

We make the change of variables $s=\pm t+r_*$:

$$f_{\pm}^2(t)\leq\left(1-\frac{r_0}{r}\right)\Bigg|_{r_*=\mathrm{R}\mp t}\int_{-\mathrm{R}}^{\mathrm{R}}\int_{\mathrm{S}^2}\mathrm{F}(s,\omega)\,ds\,d\omega + \int_{[c,\,d]\cap[-\mathrm{R}\mp t,\,\mathrm{R}\mp t]}\int_{\mathrm{S}^2}\mathrm{G}(s,\omega)\,ds\,d\omega.$$

The last integral is null for $|t|$ large enough and $\left(1-\frac{r_0}{r}\right)_{|r_*=R\mp t}$ is exponentially decreasing by respect to t as $\pm t\to+\infty$. Therefore

$$f_\pm(t)\in L^1(\mathbb{R}^\pm, dt) \tag{193}$$

and $W_1^\pm U_1$ exists in $\tilde{\mathscr{H}}$. The proof that $W_1^\pm U_1$ is in $\mathscr{H}$ is the same that for $W_0^\pm U_0$; we use only that

$$U_1\in\mathscr{H}_1^\pm \quad\Rightarrow\quad \mathscr{I}_1 e^{-itH_1}U_1\to 0 \quad \textit{in weak } \tilde{\mathscr{H}} \textit{ as } \pm t\to+\infty. \tag{194}$$

By the same way we prove that $W_1^\pm$ is independent on χ_1 by noting that the local energy of $e^{-itH_1}U_1$ tends to zero. At last we choose χ_1 satisfying (164) and

$$\|\chi_1\|_{L^\infty(\mathbb{R})}=1$$

hence (189) gives

$$\|W_1^\pm\|_{\mathscr{L}(\mathscr{H}_1^\pm,\mathscr{H})}\leqq 1.$$

Q.E.D.

Proof of Proposition 4.2. – Let $U_1^\pm$ be in $\mathscr{H}_1^\pm$ satisfying

$$e^{-itH_1}U_1^\pm=0, \qquad r_*\geqq C\mp t. \tag{195}$$

Given $\varepsilon>0$, we choose τ such that

$$\|W_1^\pm U_1^\pm - e^{i\tau H}\mathscr{I}_1 e^{-i\tau H_1}U_1^\pm\|_{\tilde{\mathscr{H}}}\leqq\varepsilon,\ \pm\tau>|t|. \tag{196}$$

We write

$$\begin{aligned}\left[\int_{r_*\geqq C\mp t}\int_{S^2}|e^{-itH}W_1^\pm U_1^\pm|^2 r^2\,dr\,d\omega\right]^{1/2} \\ \leqq\left[\int_{r_*\geqq C\mp t}\int_{S^2}|e^{-itH}(W_1^\pm U_1^\pm - e^{i\tau H}\mathscr{I}_1 e^{-i\tau H_1}U_1^\pm)|^2 r^2\,dr\,d\omega\right]^{1/2} \\ +\left[\int_{r_*\geqq C\mp t}\int_{S^2}|e^{-i(t-\tau)H}\mathscr{I}_1 e^{-i\tau H_1}U_1^\pm|^2 r^2\,dr\,d\omega\right]^{1/2}.\end{aligned} \tag{197}$$

Following (196) and Proposition 2.5, we have

$$\begin{aligned}\left[\int_{r_*\geqq C\mp t}\int_{S^2}|e^{-itH}W_1^\pm U_1^\pm|^2 r^2\,dr\,d\omega\right]^{1/2} \\ \leqq\varepsilon+\left[\int_{r_*\geqq C\mp\tau}|\mathscr{I}_1 e^{-i\tau H_1}U_1^\pm|^2 r^2\,dr\,d\omega\right]^{1/2}\end{aligned} \tag{198}$$

and the last integral is null because (195). We conclude that

$$e^{-itH}W_1^\pm U_1^\pm=0, \qquad r_*\geqq C\mp t.$$

Q.E.D.

Proof of Theorem 4.5. – For U given in $\mathscr{H}$, Theorem 4.3 assures there exists $U_1 = U_1(r_*, \omega) \in \mathscr{H}_1^+$ such that

$$\| \mathscr{I}_1 e^{-it\mathrm{H}} \mathrm{U} - \chi_1 \mathrm{U}_1 (t + r_*, \omega) \|_{\tilde{\mathscr{H}}_1} \to 0, \qquad t \to +\infty. \tag{199}$$

Therefore, to prove $W_{\mathscr{R}} U$ is well defined, it is sufficient to verify that for any

$$f \in \mathrm{L}^2(\mathbb{R}_{r_*} \times \mathrm{S}^2_\omega, dr_*\, d\omega),$$

$$s-\lim_{t \to \pm\infty} e^{t(r^2/r_0^2)\,\partial_{r_*}} \chi_1 e^{-t\partial_{r_*}} f \text{ exists in } \mathrm{L}^2\left(\mathbb{R}_{r_*} \times \mathrm{S}^2_\omega, \frac{r_0^2}{r^2} dr_*\, d\omega\right). \tag{200}$$

We may consider only the case where

$$f \in \mathrm{C}_0^\infty(\mathbb{R}_{r_*} \times \mathrm{S}^2_\omega), \qquad |r_*| > \mathrm{R} \;\Rightarrow\; f(r_*, \omega) = 0. \tag{201}$$

We have

$$\left\| \frac{d}{dt}\left(e^{t(r^2/r_0^2)\,\partial_{r_*}} \chi_1 e^{-t\partial_{r_*}} f\right) \right\|_{\mathrm{L}^2(\mathbb{R}_{r_*} \times \mathrm{S}^2_\omega, (r_0^2/r^2)\, dr_* d\omega)}$$
$$\leqq \mathrm{C}\left[\int_{[cd] \cap [-\mathrm{R}-t, \mathrm{R}-t]} \int_{\mathrm{S}^2_\omega} |f(r_*, \omega)|^2 \, dr_*\, d\omega + \left(1 - \frac{r_0}{r}\right)_{|r_* = \mathrm{R}-t} \iint |\partial_{r_*} f(r_*, \omega)|^2 \, dr_*\, d\omega \right]^{1/2}. \tag{202}$$

The first integral is null for $t > \mathrm{R} - c$ and $\left(1 - \frac{r_0}{r}\right)_{|r_* = \mathrm{R}-t}$ is exponentially decreasing by respect to $t \to +\infty$. We conclude that

$$\left\| \frac{d}{dt}\left(e^{t(r^2/r_0^2)\,\partial_{r_*}} \chi_1 e^{-t\partial_{r_*}} f\right) \right\|_{\mathrm{L}^2(\mathbb{R}_{r_*} \times \mathrm{S}^2_\omega, (r_0^2/r^2)\, dr_* d\omega)} \in \mathrm{L}^1(\mathbb{R}_t^+)$$

and limit (200) exists. If χ_1 and χ_1' satisfy (164) and define $W_{\mathscr{R}}$ and $W'_{\mathscr{R}}$, we have

$$\| W_{\mathscr{R}} U - W'_{\mathscr{R}} U \|_{\mathscr{H}^+_{\mathscr{R}}} \leqq \lim_{t \to +\infty} \| (\chi_1 - \chi_1') e^{-it\mathrm{H}} \mathrm{U} \|_{\tilde{\mathscr{H}}}$$

and this last limit is null because (199). Hence $W_{\mathscr{R}}$ is independent on χ_1; by choosing χ_1 with

$$\| \chi_1 \|_{\mathrm{L}^\infty} \leqq 1$$

we obtain

$$\| W_{\mathscr{R}} \|_{\mathscr{L}(\mathscr{H}, \mathscr{H}_{\mathscr{R}}^{+})} \leq 1.$$

Q.E.D.

5. ASYMPTOTIC COMPLETENESS - SCATTERING OPERATOR

This part is devoted to the proof of Theorems 3.3, 4.3 and 4.4.

We can resume our results by defining Scattering Operator S. We introduce Wave Operators W^-, W:

$$(U_1, U_0) \in \mathscr{H}_1^- \times \mathscr{H}_0, W^-(U_1, U_0) = W_1^- U_1 + W_0^- U_0 \tag{203}$$

$$U \in \mathscr{H}, W(U) = (W_1 U, W_0 U) \in \mathscr{H}_1^+ \times \mathscr{H}_0 \tag{204}$$

$$S = W W^-. \tag{205}$$

THEOREM 5.1. — W^- *is isometric from* $\mathscr{H}_1^- \times \mathscr{H}_0$ *onto* $\mathscr{H}$.
W *is isometric from* $\mathscr{H}$ *onto* $\mathscr{H}_1^+ \times \mathscr{H}_0$.
S *is isometric from* $\mathscr{H}_1^- \times \mathscr{H}_0$ *onto* $\mathscr{H}_1^+ \times \mathscr{H}_0$.

The ideas of the proof of the asymptotic completeness are following:

— because the fields are without stationnary part, we may use a vector potential and the problem is to investigate a vector wave equation in Schwarzschild metric,

— thanks to the spherical invariance, we can make a separation of variables and, roughly speaking, the problem is reduced to the study of the one dimensional scalar wave equation

$$\partial_t^2 u - \partial_{r_*}^2 u = -l(l+1)\frac{a^2}{r^2} u$$

— and we adopt the approach of Dimock [7]: because the potential α^2/r^2 is short range type, the classical scattering theory of Kato and Birman assures that

$$u(t, r_*) \cong u_1(t + r_*) + u_0(t - r_*), \qquad t \to \infty;$$

— u_0 and u_1 are respectively the asymptotic profiles of free fields which are asymptotic to the given electromagnetic field in Schwarzsxhild space-time, respectively at the flat infinity, and at the horizon.

Proof of Theorem 4.4. — We start by investigating the vector wave equation

$$\partial_t^2 X - (\nabla_V \times \alpha)(\nabla_V \times \alpha) X = 0 \tag{206}$$

with the constraint of free divergence

$$\nabla_V . X = 0. \tag{207}$$

Following (64), we split X into radial and transverse components A^0, A^+, A^- and we expand A^0, A^+, A^- in series of generalised spherical functions $T^l_{m,n}$ (74) (75):

$$A^0(t, r_*, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{n=-l}^{l} a^0_{l,n}(t, r_*) T^l_{0,n}\left(\frac{\pi}{2} - \varphi, \theta, 0\right) \tag{208}$$

$$A^{\pm}(t, r_*, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{n=-l}^{l} a^{\pm}_{l,n}(t, r_*) T^l_{\pm 1,n}\left(\frac{\pi}{2} - \varphi, \theta, 0\right). \tag{209}$$

For simplicity we omit subscript l, n and now a^{ν} are solutions of the scalar one dimensional wave equation

$$\partial_t^2 a^0 - \partial_{r_*}^2 a^0 = -l(l+1)\frac{\alpha^2}{r^2} a^0 \tag{210}$$

$$\partial_t^2 a^+ - \partial_{r_*}^2 a^+ = -l(l+1)\frac{\alpha^2}{r^2} a^+ - \left(\frac{l(l+1)}{2}\right)^{1/2} \frac{\alpha^2}{r^3}\left(2 - \frac{3r_0}{r}\right) a^0 \tag{211}$$

$$\partial_t^2 a^- - \partial_{r_*}^2 a^- = -l(l+1)\frac{\alpha^2}{r^2} a^- - \left(\frac{l(l+1)}{2}\right)^{1/2} \frac{\alpha^2}{r^3}\left(2 - \frac{3r_0}{r}\right) a^0 \tag{212}$$

$$\partial_{r_*} a^0 + \left(\frac{l(l+1)}{2}\right)^{1/2} (a^+ + a^-) = 0 \tag{213}$$

and if

$$X \in C^k(\mathbb{R}_t, \mathscr{H}) \tag{214}$$

then

$$a^0 \in C^k\left(\mathbb{R}_t, L^2\left(\mathbb{R}_{r_*}, \frac{\alpha^2}{r^2} dr_*\right)\right) \tag{215}$$

$$a^{\pm} \in C^k(\mathbb{R}_t, L^2(\mathbb{R}_{r_*}, dr_*)). \tag{216}$$

We note that for $l=0$, a^0, a^+, a^- are the free solutions of

$$\partial_t^2 u - \partial_{r_*}^2 u = 0. \tag{217}$$

Now we consider the case $l \neq 0$, then a^0 and $a^+ - a^-$ are solutions of the wave equation

$$\partial_t^2 u + \mu^2 u = 0 \tag{218}$$

where

$$\mu = \left[-\partial_{r_*}^2 + l(l+1)\frac{\alpha^2}{r^2}\right]^{1/2} \tag{219}$$

is self-adjoint on $L^2(\mathbb{R}, dr_*)$ with dense domain

$$H^1(\mathbb{R}) = \{u \in L^2(\mathbb{R}, dr_*),\ \partial_{r_*} u \in L^2(\mathbb{R}, dr_*)\}. \tag{220}$$

We compare the solutions of (217) and (218):

LEMMA 5.2. — *Let u be a solution of* (218) *with initial data*

$$u|_{t=0}=u_0\in \mathrm{H}^1(\mathbb{R})$$
$$\partial_t u|_{t=0}=\mu^2 v\in \mathrm{L}^2(\mathbb{R}),\qquad v\in \mathrm{L}^2(\mathbb{R}).$$

Then there exists u_+, u_- in $\mathrm{H}^1(\mathbb{R})$ such that

$$\|u_\pm\|_{\mathrm{H}^1(\mathbb{R})}\leqq c(\|u_0\|_{\mathrm{H}^1(\mathbb{R})}+\|\mu^2 v\|_{\mathrm{L}^2(\mathbb{R})}) \tag{221}$$

$$\lim_{t\to\pm\infty}[\|u(t,r_*)-u_+(r_*+t)-u_-(r_*-t)\|_{\mathrm{H}^1(\mathbb{R})} + \|\partial_t u(t,r_*)-\partial_{r_*}u_+(r_*+t)+\partial_{r_*}u_-(r_*-t)\|_{\mathrm{L}^2(\mathbb{R})}]=0. \tag{222}$$

Now let U be satisfying (169). Then E and B are the solutions of (206) (207) with X = E or B and

$$\mathrm{X}|_{t=0}\in[\mathrm{C}_0^\infty(\mathbb{R}_{r_*}\times \mathrm{S}^2_\omega)]^3 \tag{223}$$
$$\partial_t \mathrm{X}|_{t=0}=(\nabla_\mathrm{V}\times\alpha)(\nabla_\mathrm{V}\times\alpha)\mathrm{Y},\qquad \mathrm{Y}\in[\mathrm{C}_0^\infty(\mathbb{R}_{r_*}\times \mathrm{S}^2_\omega)]^6. \tag{224}$$

We may apply lemma 5.2 to a^0 and a^+-a^-: there exists $f^0_{l,n}$, $g_{l,n}\in \mathrm{H}^1(\mathbb{R}_{r_*})$ such that

$$\sum_{l=0}^{\infty}\sum_{n=-l}^{l}\|f^0_{l,n}\|^2_{\mathrm{H}^1(\mathbb{R})}<+\infty \tag{225}$$

$$\sum_{l=0}^{\infty}\sum_{n=-l}^{l}\|g_{l,n}\|^2_{\mathrm{H}^1(\mathbb{R})}<+\infty \tag{226}$$

$$\forall l,n,\quad \lim_{t\to\pm\infty}[\|\chi_1(r_*)(a^0_{l,n}(t,r_*)-f^0_{l,n}(t+r_*))\|_{\mathrm{H}^1(\mathbb{R})} + \|\chi_1(r_*)(\partial_t a^0_{l,n}(t,r_*)-\partial_{r_*}f^0_{l,n}(t+r_*))\|_{\mathrm{L}^2(\mathbb{R})})]=0 \tag{227}$$

$$\forall l,n,\quad \lim_{t\to\pm\infty}[\|\chi_1(r_*)(a^+_{l,n}(t,r_*)-a^-_{l,n}(t,r_*)-g_{l,n}(t+r_*))\|_{\mathrm{H}^1(\mathbb{R})} + \|\chi_1(r_*)(\partial_t a^+_{l,n}(t,r_*)-\partial_t a^-_{l,n}(t,r_*)-\partial_{r_*}g_{l,n}(t+r_*))\|_{\mathrm{L}^2(\mathbb{R})}]=0. \tag{228}$$

Now thanks to equations (213) (225) (227) there exists so $h_{l,n}\in \mathrm{L}^2(\mathbb{R})$ such that

$$\sum_{l=0}^{\infty}\sum_{n=-l}^{l}\|h_{l,n}\|^2_{\mathrm{L}^2(\mathbb{R})}<\infty \tag{229}$$

$$\forall l,n,\ \lim_{t\to\pm\infty}\|\chi_1(r_*)(a^+_{l,n}(t,r_*)+a^-_{l,n}(t,r_*)-h_{l,n}(t+r_*))\|_{\mathrm{L}^2(\mathbb{R})}=0. \tag{230}$$

We deduct from (298) and (230) there exists $f^+_{l,n}$, $f^-_{l,n}$ in $\mathrm{L}^2(\mathbb{R})$ such that

$$\sum_{l=0}^{\infty}\sum_{n=-l}^{l}\|f^\pm_{l,n}\|^2_{\mathrm{L}^2(\mathbb{R})}<+\infty \tag{231}$$

$$\forall l,n,\ \lim_{t\to\pm\infty}\|\chi_1(r_*)(a^\pm_{l,n}(t,r_*)-f^\pm_{l,n}(t+r_*))\|_{\mathrm{L}^2(\mathbb{R})}=0. \tag{232}$$

Because $\partial_t X$ satisfies so (206) (207) (223) (224) the same proof assures the existence of $f^{0'}_{l,n}$ in $H^1(\mathbb{R})$ such that

$$\sum_{l=0}^{\infty}\sum_{n=-l}^{l}\| f^{0'}_{l,n}\|^2_{H^1(\mathbb{R})} < +\infty \tag{233}$$

$$\forall l,n,\ \lim_{t\to\pm\infty}\ [\|\chi_1(r_*)(\partial_t a^0_{l,n}(t,r_*)-f^{0'}_{l,n}(r_*+t))\|_{H^1(\mathbb{R})} + \|\chi_1(r_*)(\partial_t^2 a^0_{l,n}(t,r_*)-\partial_{r_*} f^{0'}_{l,n}(r_*+t))\|_{L^2(\mathbb{R})}]=0. \tag{234}$$

In fact (227) and (234) imply

$$f^{0'}_{l,n}=\partial_{r_*} f^0_{l,n} \tag{235}$$

and

$$\forall l,n,\ \lim_{t\to\pm\infty}\ \|\chi_1(r_*)(\partial^2_{r_*} a^0_{l,n}(t,r_*)-l(l+1)\frac{\alpha^2}{r^2}a^0_{l,n}(t,r_*) - \partial^2_{r_*} f^0_{l,n}(r_*+t))\|_{L^2(\mathbb{R})}=0. \tag{236}$$

Because $\frac{\alpha^2}{r^2}$ tends to 0 as $r_*\to-\infty$ (227) gives

$$\lim_{t\to\pm\infty}\ \|\chi_1(r_*)\frac{\alpha^2}{r^2}a^0_{l,n}(t,r_*)\|_{L^2(\mathbb{R})}=0$$

and thus

$$\forall l,n,\quad \lim_{t\to\pm\infty}\ \|\chi_1(r_*)(\partial^2_{r_*} a^0_{l,n}(t,r_*)-\partial^2_{r_*} f^0_{l,n}(r_*+t))\|_{L^2(\mathbb{R})}=0. \tag{237}$$

We obtain by (213) and (232), (237):

$$\forall l,n,\quad \lim_{t\to\pm\infty}\ \|\chi_1(r_*)(a^{\pm}_{l,n}(t,r_*)-f^{\pm}_{l,n}(r_*+t))\|_{H^1(\mathbb{R})}=0. \tag{238}$$

Because the spherical invariance of equations (206) (207) we have the same results for $\Delta_{S^2}X$; the initial data are multiplied by $-l(l+1)$ and in particular we have

$$\sum_{l=0}^{\infty}\sum_{n=-l}^{l}\| l(l+1) f^{0'}_{l,n}\|^2_{H^1(\mathbb{R})} < +\infty.$$

We conclude by (210) (213) the following improvement of (229):

$$\sum_{l=0}^{\infty}\sum_{n=-l}^{l}\| h_{l,n}\|^2_{H^1(\mathbb{R})} < +\infty$$

and finally

$$\sum_{l=0}^{\infty}\sum_{n=-l}^{l}\| f^{\pm}_{l,n}\|^2_{H^1(\mathbb{R})} < +\infty. \tag{239}$$

Now we define for $\nu = 0, +, -$

$$A_1^\nu(t, r_*, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{n=-l}^{l} f_{l,n}^\nu(r_* + t) T_{\nu,n}^l\left(\frac{\pi}{2} - \varphi, \theta, 0\right). \tag{240}$$

We introduce Sobolev Hilbert space $H^1(\mathbb{R}_{r_*}; L^2(S_\omega^2))$

$$U \in H^1(\mathbb{R}_{r_*}; L^2(S_\omega^2)) \quad \Leftrightarrow \quad u, \partial_{r_*} u \in L^2(\mathbb{R}_{r_*} \times S_\omega^2). \tag{241}$$

(225) and (239) prove that

$$\nu = 0, +, -, A_1^\nu \in C^0(\mathbb{R}_t; H^1(\mathbb{R}_{r_*}; L^1(S_\omega^2))) \cap C^1(\mathbb{R}_{r_*} \times S_\omega^2)) \tag{242}$$

and (227) (238) and the Lebesgue theorem imply that for $\nu = 0, +, -$,

$$\lim_{t \to \pm\infty} [\| \chi_1(r_*)(A^\nu(t, r_*, \omega) - A_1^\nu(t, r_*, \omega)) \|_{H^1(\mathbb{R}_{r_*}; L^2(S_\omega^2))} + \| \chi_1(r_*)(\partial_t A^\nu(t, r_*, \omega) - \partial_t A_1^\nu(t, r_*, \omega)) \|]_{L^2(\mathbb{R}_{r_*} \times S_\omega^2)} = 0. \tag{243}$$

We recall the classical Sobolev imbedding:

$$H^1(\mathbb{R}_{r_*}; L^2(S_\omega^2)) \subset (C^0 \cap L^\infty)(\mathbb{R}_{r_*}; L^2(S_\omega^2))$$

thus we deduct from (243)

$$\lim_{t \to +\infty} \| \chi_1(r_*)(A^\nu(t, r_*, \omega) - A_1^\nu(t, r_*, \omega)) \|_{(C^0 \cap L^\infty)(\mathbb{R}_{r_*}; L^2(S_\omega^2))} = 0 \tag{244}$$

and because

$$A_1^\nu(t, r_*, \omega) = A_1^\nu(t + r_*, \omega)$$

we have for any $s \in \mathbb{R}$

$$\lim_{r_* \to -\infty} \| A^\nu(t = s - r_*, r_*) - A_1^\nu(s) \|_{L^2(S_\omega^2)} = 0. \tag{245}$$

For $X = E$ we put

$$e^{\hat{r}_*} = A_1^0, \qquad e^{\hat{\theta}} = i 2^{-1/2}(A_1^+ + A_1^-), \qquad e^{\hat{\varphi}} = -2^{1/2}(A_1^+ - A_1^-). \tag{246}$$

$b^{\hat{r}_*}$, $b^{\hat{\theta}}$, $b^{\hat{\varphi}}$ are defined by the same way for $X = B$. Limits (171) (172) (173) are so a consequence of (245) and we have for $\nu = \theta, \varphi$

$$\| \chi_1(r_*)(\partial_t(r\alpha E^{\hat{\nu}})(t, r_*, \omega) - \partial_{r_*} e^{\hat{\nu}}(t + r_*, \omega)) \|_{L^2(\mathbb{R}_{r_*} \times S_\omega^2)} \to 0, \quad t \to +\infty \tag{247}$$

$$\| \chi_1(r_*)(\partial_{r_*}(r\alpha B^{\hat{\nu}})(t, r_*, \omega) - \partial_{r_*} b^{\hat{\nu}}(t + r_*, \omega)) \|_{L^2(\mathbb{R}_{r_*} \times S_\omega^2)} \to 0, \quad t \to +\infty. \tag{248}$$

$\partial_t(r\alpha E^{\hat{\nu}})$ and $\partial_{r_*}(r\alpha B^{\hat{\nu}})$ are related by Maxwell's equations:

$$\partial_t(r\alpha E^{\hat{\theta}}) = \frac{\alpha^2}{r^2 \sin\theta} \partial_\varphi(r^2 E^{\hat{r}_*}) - \partial_{r_*}(r\alpha B^{\hat{\varphi}}) \tag{249}$$

$$\partial_t(r\alpha E^{\hat{\varphi}}) = -\frac{\alpha^2}{r^2} \partial_\theta(r^2 E^{\hat{r}_*}) + \partial_{r_*}(r\alpha B^{\hat{\theta}}) \tag{250}$$

and we have

$$\sup_{t>0} \| r^2 \, E^{\hat{r}_*}(t) \|_{L^2(\mathbb{R}_{r_*} \times S^2_\omega)} < +\infty. \tag{251}$$

Let ψ be in $C_0^\infty(\mathbb{R}_{r_*} \times S^2_\omega)$. We have for t large enough:

$$|\langle \partial_{r_*}(e^{\hat{\theta}} + b^{\hat{\varphi}}), \psi \rangle| + |\langle \partial_{r_*}(e^{\hat{\varphi}} + b^{\hat{\theta}}), \psi \rangle|$$
$$\leqq \varepsilon(t) + \left| \langle \frac{\alpha^2}{r^2} \,.\, r^2 \, E^{\hat{r}_*} \Big|_{r_*+t=s}, \frac{1}{\sin\theta} \partial_\varphi \psi \rangle \right|$$
$$+ \left| \langle \frac{\alpha^2}{r^2} \,.\, r^2 \, E^{\hat{r}_*} \Big|_{r_*+t=s}, \frac{1}{\sin\theta} \partial_\theta (\sin\theta \psi) \rangle \right|$$

where $\langle \;,\; \rangle$ is the distributions bracket on $\mathbb{R}_{r_*} \times S^2_\omega$ and thanks to (247) (248)

$$\varepsilon(t) \to 0, \qquad t \to +\infty.$$

(251) implies

$$\frac{\alpha^2}{r^2} \,.\, r^2 \, E^{\hat{r}_*} \big|_{r_*+t=s} \to 0 \quad \textit{in } \mathscr{D}'(\mathbb{R}_{r_*} \times S^2_\omega), \qquad t \to +\infty.$$

We obtain

$$\partial_{r_*}(e^{\hat{\theta}} + b^{\hat{\varphi}}) = \partial_{r_*}(e^{\hat{\varphi}} - b^{\hat{\theta}}) = 0.$$

$e^{\hat{v}}$, $b^{\hat{v}}$ being in L^2, polarization property (174) is proved.

The constraint of free divergence assures that

$$\partial_{r_*}(r^2 \, E^{\hat{r}_*}) + \frac{1}{\sin\theta} [\partial_\theta \sin\theta \, r \alpha \, E^{\hat{\theta}} + \partial_\varphi (r \alpha \, E^{\hat{\varphi}})] \equiv 0.$$

We have shown that

$$\partial_{r_*}(r^2 \, E^{\hat{r}_*})(t, r_* - t, \omega) \to \partial_{r_*} e^{\hat{r}_*}(r_*, \omega)$$
$$r \alpha \, E^{\hat{v}}(t, r_* - t, \omega) \to e^{\hat{v}}(r_*, \omega), \qquad v = \theta, \varphi.$$

in distributions sense on $\mathbb{R}_{r_*} \times S^2_\omega$ as $t \to +\infty$. Hence we deduct charge conservation law (175):

$$\partial_{r_*} e^{\hat{r}_*} + \frac{1}{\sin\theta} [\partial_\theta \sin\theta \, e^{\hat{\theta}} + \partial_\varphi e^{\hat{\varphi}}] = 0.$$

Q.E.D.

Proof of Lemma 5.2. — We verify easily that

$$u(t, r_*) = e^{it\mu} f + e^{-it\mu} g \tag{252}$$

with

$$f = \frac{1}{2}(u_0 - i \mu v), \qquad g = \frac{1}{2}(u_0 + i \mu v). \tag{253}$$

We note that f and g are in $H^1(\mathbb{R})$.
We introduce self-adjoint operator μ_1 on $L^2(\mathbb{R}_{r_*})$:

$$\mu_1 = (-\partial_{r_*}^2)^{1/2}. \tag{254}$$

J. Dimock [7] proved that for any $h \in L^2(\mathbb{R}_{r_*})$ the following limits exist in this space:

$$h_1^{\pm} = \lim_{t \to \pm\infty} e^{-it\mu_1} e^{it\mu} h, \qquad h^{\pm} = \lim_{t \to \pm\infty} e^{-it\mu} e^{it\mu_1} h. \tag{255}$$

It is a direct consequence of the invariance principle for the wave operators and of fact that $\mu^2 - \mu_1^2$ is positive and a short range perturbation [22]. Now let f_1 be

$$f_1 = s - \lim_{t \to \pm\infty} e^{-it\mu_1} e^{it\mu} f \quad in \ L^2(\mathbb{R}). \tag{256}$$

We have

$$\| \mu_1 e^{-it\mu_1} e^{it\mu} f \|_{L^2} = \| \mu_1 e^{it\mu} f \|_{L^2} \leqq \| \mu e^{it\mu} f \|_{L^2} = \| \mu f \|_{L^2}.$$

Hence f_1 is in $H^1(\mathbb{R})$ and

$$\mu_1 e^{-it\mu_1} e^{it\mu} f \to \mu_1 f_1 \quad in \ L^2(\mathbb{R})\ weak, \qquad t \to +\infty \tag{257}$$

$$\| \mu_1 f_1 \|_{L^2} \leqq \| \mu f \|_{L^2}. \tag{258}$$

We have so

$$f = s - \lim_{t \to \pm\infty} e^{-it\mu} e^{it\mu} f_1 \quad in \ L^2(\mathbb{R})$$

and

$$\| \mu e^{-it\mu} e^{it\mu_1} f_1 \|_{L^2} = \| \mu e^{it\mu_1} f_1 \|_{L^2} \leqq \| \mu_1 f_1 \|_{L^2} + \| (\mu - \mu_1) e^{it\mu_1} f_1 \|_{L^2}.$$

Thus

$$\| \mu f \|_{L^2} \leqq \| \mu_1 f_1 \|_{L^2} + \liminf_{t \to \pm\infty} \| (\mu - \mu_1) e^{it\mu_1} f_1 \|_{L^2} \tag{259}$$

and

$$\mu e^{-it\mu} e^{it\mu_1} f_1 \to \mu f \quad in \ L^2(\mathbb{R})\ weak, \qquad t \to +\infty. \tag{260}$$

Now we estimate $(\mu - \mu_1) e^{it\mu_1} f_1$; we use the integral representation

$$(\mu - \mu_1) e^{it\mu_1} f_1 = \frac{1}{\pi} \int_0^{\infty} \lambda^{-1/2} [(\mu^2 + \lambda)^{-1} \mu^2 - (\mu_1^2 + \lambda)^{-1} \mu_1^2] e^{it\mu_1} f_1 \, d\lambda \tag{261}$$

On the one hand we write.

$$\begin{aligned} \lambda^{-1/2} \| [(\mu^2 + \lambda)^{-1} \mu^2 - (\mu_1^2 + \lambda)^{-1} \mu_1^2] e^{it\mu_1} f_1 \|_{L^2} \\ = \lambda^{-1/2} \| (\mu^2 + \lambda)^{-1} (\mu^2 - \mu_1^2)(1 - (\mu_1^2 + \lambda)^{-1} \mu_1^2) e^{it\mu_1} f_1 \|_{L^2} \\ \leqq \lambda^{-3/2} \| (\mu^2 - \mu_1^2) e^{it\mu_1} (1 - (\mu_1^2 + \lambda)^{-1} \mu_1^2) f_1 \|_{L^2} \\ \leqq c \left\| \frac{\alpha^2}{r^2} [f_1'(r_* + t) + f''_1(r_* - t)] \right\|_{L^2(\mathbb{R}_{r_*})} \end{aligned}$$

for some f'_1, f''_1 in $L^2(\mathbb{R}_{r_*})$; because α^2/r^2 tends to zero as $|r_*| \to \infty$, we obtain

$$\forall \lambda > 0, \quad \lim_{t \to \pm\infty} \lambda^{-1/2} \| [(\mu^2+\lambda)^{-1}\mu^2 - (\mu_1^2+\lambda)^{-1}\mu_1^2] e^{it\mu_1} f_1 \|_{L^2} = 0. \tag{262}$$

On the other hand for $0 < \lambda \leqq 1$ we have

$$\lambda^{-1/2} \| [(\mu^2+\lambda)^{-1}\mu^2 - (\mu_1^2+\lambda)^{-1}\mu_1^2] e^{it\mu_1} f_1 \|_{L^2} \leqq 2 \| f_1 \|_{L^2} \lambda^{-1/2} \tag{263}$$

and for $\lambda \geqq 1$

$$\lambda^{-1/2} \| [(\mu^2+\lambda)^{-1}\mu^2 - (\mu_1^2+\lambda)^{-1}\mu_1^2] e^{it\mu_1} f_1 \|_{L^2} \leqq c\, \lambda^{-3/2}. \tag{264}$$

We apply the dominated convergence theorem to conclude that

$$\lim_{t \to \pm\infty} \| (\mu - \mu_1) e^{it\mu_1} f_1 \|_{L^2} = 0. \tag{265}$$

(258) (259) and (265) give

$$\| \mu_1 f_1 \|_{L^2} = \| \mu f \|_{L^2} \tag{266}$$

and

$$\lim_{t \to \pm\infty} \| \mu e^{-it\mu} e^{it\mu_1} f_1 \|_{L^2} = \| \mu_1 f_1 \|_{L^2}. \tag{267}$$

It follows from (260) (266) (267) that

$$\mu e^{-it\mu} \mu e^{it\mu_1} f_1 \to \mu f \quad in\ L^2(\mathbb{R}), \qquad t \to +\infty$$

hence

$$\| \mu e^{it\mu_1} f_1 - \mu e^{it\mu} f \|_{L^2} \to 0, \qquad t \to +\infty. \tag{268}$$

Because $\mu \geqq \mu_1$ we deduct that

$$\| \mu_1 (e^{it\mu_1} f_1 - e^{it\mu} f) \|_{L^2} \to 0, \qquad t \to +\infty$$

thus

$$\| e^{it\mu_1} f_1 - e^{it\mu} f \|_{H^1(\mathbb{R})} \to 0 \qquad t \to +\infty \tag{269}$$

and by applying (265) (268) again

$$\| \partial_t (e^{it\mu_1} f_1) - \partial_t (e^{it\mu} f) \|_{L^2(\mathbb{R})} \to 0, \qquad t \to +\infty. \tag{270}$$

Obviously we have the same results for $e^{-it\mu} g$ and by recalling that

$$e^{it\mu_1} f_1(r_*) = f'_1(r_* + t) + f''_1(r_* - t) \tag{271}$$

(222) is a consequence of (269) (270) (271). At last (221) is proved by (253) (258).

Q.E.D.

Proof of Theorem 4.3. — Because

$$\| e^{itH_1} \mathcal{J}_1^* e^{-itH} U \|_{\tilde{\mathscr{H}}_1} \leqq \| \chi_1 \|_{L^\infty} \| U \|_{\mathscr{H}} \tag{272}$$

it is sufficient to take U in the dense domain $H[C_0^\infty(V)]^6$. Following (243) (246) we have

$$\|\chi_1(r_*)[r^2 E^{\hat{r}_*}(t, r_*, \omega) - e^{\hat{r}_*}(t+r_*, \omega)]\|_{L^2(\mathbb{R}_{r_*}\times S^2_\omega)} \to 0, \qquad t\to+\infty, \tag{273}$$

$$\nu=\hat{\theta}, \hat{\varphi}, \|\chi_1(r_*)[r\alpha E^{\hat{\nu}}(t, r_*, \omega) - e^{\hat{\nu}}(t+r_*, \omega)]\|_{L^2(\mathbb{R}_{r_*}\times S^2_\omega)} \to 0, \quad t\to+\infty, \tag{274}$$

$$\|\chi_1(r_*)[r^2 B^{\hat{r}_*}(t, r_*, \omega) - b^{\hat{r}_*}(t+r_*, \omega)]\|_{L^2(\mathbb{R}_{r_*}\times S^2_\omega)} \to 0, \qquad t\to+\infty, \tag{275}$$

$$\nu=\hat{\theta}, \hat{\varphi}, \|\chi_1(r_*)[r\alpha B^{\hat{\nu}}(t, r_*, \omega) - b^{\hat{\nu}}(t+r_*, \omega)]\|_{L^2(\mathbb{R}_{r_*}\times S^2_\omega)} \to 0, \qquad t\to+\infty, \tag{276}$$

$$e^{\hat{\theta}} = -b^{\hat{\varphi}}, \qquad e^{\hat{\varphi}} = b^{\hat{\theta}}. \tag{277}$$

(273) and (275) imply

$$\|\chi_1(r_*)r\alpha X^{\hat{r}_*}(t, r_*, \omega)\|_{L^2(\mathbb{R}_{r_*}\times S^2_\omega)} \to 0, \qquad t\to+\infty, \quad X=E, B. \tag{278}$$

By putting

$$U_1 = {}^t(0, e^{\hat{\theta}}, e^{\hat{\varphi}}, 0, b^{\hat{\theta}}, b^{\hat{\varphi}})$$

we obtain

$$U_1 \in \mathscr{H}_1, \quad \|\chi_1 e^{-itH_1}U_1 - \mathscr{I}_1^* e^{-itH}U\|_{\tilde{\mathscr{H}}_1} \to 0, \qquad t\to+\infty, \tag{279}$$

therefore $W_1 U$ is well defined in $\mathscr{H}_1^+$.

Now if χ_1' satisfies (164), we deduct from (273) (274) (275) (276) that

$$\|(\chi_1 - \chi_1')r\alpha e^{-itH}U\|_{\tilde{\mathscr{H}}_0} \to 0, \qquad t\to+\infty.$$

We conclude that W_1 is independent on χ_1 an by choosing $0 \leqq \chi_1 \leqq 1$, (272) shows that

$$\|W_1\|_{\mathscr{L}(\mathscr{H}, \mathscr{H}_1^+)} \leqq 1$$

Q.E.D.

Proof of Theorem 3.3. — Because

$$\|e^{itH_0}\mathscr{I}_0^* e^{-itH}U\|_{\tilde{\mathscr{H}}_0} \leqq \|\chi_0\|_{L^\infty}\|U\|_{\mathscr{H}} \tag{280}$$

it is sufficient to consider

$$U \in H[C_0^\infty(V)]^6.$$

Now thanks to lemma 5.2 we may replace in the proof of Theorem 4.4 χ_1 by χ_0 and $t+r_*$ by r_*-t, then we obtain a result similar to Theorem 4.4 for the flat infinity: the electromagnetic field admits an asymptotic profile as $r_*-t=s=\text{Const.}$, $t\to+\infty$; more precisely there exists $e^{\hat{\nu}}$, $b^{\hat{\nu}}$ satisfying (273) to (278) by replacing χ_1 by χ_0 and $t+r_*$ by r_*-t. That implies there exists $U_1 \in \tilde{\mathscr{H}}_1$ such that

$$U_1 = {}^t(e^{\hat{r}_*}, e^{\hat{\theta}}, e^{\hat{\varphi}}, b^{\hat{r}_*}, b^{\hat{\theta}}, b^{\hat{\varphi}}) \in \tilde{\mathscr{H}}_1$$

$$e^{\hat{\theta}} = b^{\hat{\varphi}}, \qquad e^{\hat{\varphi}} = -b^{\hat{\theta}}, \tag{281}$$

$$\partial_{r_*}e^{\hat{r}_*} + \frac{1}{\sin\theta}[\partial_\theta \sin\theta\, e^{\hat{\theta}} + \partial_\varphi e^{\hat{\varphi}}] = 0 \tag{282}$$

$$\partial_{r_*} b^{\hat{r}_*} + \frac{1}{\sin\theta}[\partial_\theta \sin\theta\, b^{\hat{\theta}} + \partial_\varphi b^{\hat{\varphi}}] = 0 \tag{283}$$

$$\| \chi_0(r_*)(r^2 \mathrm{E}^{\hat{r}_*}(t, r_*, \omega) - e^{\hat{r}_*}(r_* - t, \omega)) \|_{L^2(\mathbb{R}_{r_*} \times S^2_\omega)} \to 0, \qquad t \to +\infty \tag{284}$$

$$\left.\begin{aligned} &\| \chi_0(r_*)(r\alpha \mathrm{E}^{\hat{\nu}}(t, r_*, \omega) - e^{\hat{\nu}}(r_* - t, \omega)) \|_{L^2(\mathbb{R}_{r_*} \times S^2_\omega)} \to 0, \\ &t \to +\infty, \qquad \nu = \hat{\theta}, \hat{\varphi} \end{aligned}\right\} \tag{285}$$

$$\| \chi_0(r_*)(r^2 \mathrm{B}^{\hat{r}_*}(t, r_*, \omega) - b^{\hat{r}_*}(r_* - t, \omega)) \|_{L^2(\mathbb{R}_{r_*} \times S^2_\omega)} \to 0, \qquad t \to +\infty \tag{286}$$

$$\left.\begin{aligned} &\| \chi_0(r_*)(r\alpha \mathrm{B}^{\hat{\nu}}(t, r_*, \omega) - b^{\hat{\nu}}(r_* - t, \omega)) \|_{L^2(\mathbb{R}_{r_*} \times S^2_\omega)} \to 0, \\ &t \to +\infty, \qquad \nu = \hat{\theta}, \hat{\varphi}. \end{aligned}\right\} \tag{287}$$

Now we introduce the Hilbert space

$$\mathscr{H}_0^0 = \left\{ \mathrm{U}_0^0 = {}^t(e_0^{\hat{r}_*}, e_0^{\hat{\theta}}, e_0^{\hat{\varphi}}, b_0^{\hat{r}_*}, b_0^{\hat{\theta}}, b_0^{\hat{\varphi}});\right.$$

$$\left.\begin{aligned} &e_0^{\hat{r}_*}, b_0^{\hat{r}_*} \in L^2(\mathbb{R}^+_{r_*} \times S^2_\omega, r_*^{-2}\, dr_*\, d\omega), \\ &e_0^{\hat{\theta}}, e_0^{\hat{\varphi}}, b_0^{\hat{\theta}}, b_0^{\hat{\varphi}} \in L^2(\mathbb{R}^+_{r_*} \times S^2_\omega, dr_*\, d\omega), \\ &\partial_{r_*} \mathrm{X}^{\hat{r}_*} + \frac{1}{\sin\theta}[\partial_\theta \sin\theta\, \mathrm{X}^{\hat{\theta}} + \partial_\varphi \mathrm{X}^{\hat{\varphi}}] = 0, \mathrm{X} = e_0, b_0 \end{aligned}\right\} \tag{288}$$

and operator the H_0^0

$$\mathrm{H}_0^0 = i\begin{pmatrix} 0 & h_0^0 \\ -h_0^0 & 0 \end{pmatrix}, \quad h_0^0 = \begin{pmatrix} 0 & -\dfrac{1}{\sin\theta}\partial_\varphi & \dfrac{1}{\sin\theta}\partial_\theta \sin\theta \\ \dfrac{1}{r_*^2 \sin\theta}\partial_\varphi & 0 & -\partial_{r_*} \\ -\dfrac{1}{r_*^2}\partial_\theta & \partial_{r_*} & 0 \end{pmatrix}. \tag{289}$$

Obviously H_0^0 is self-adjoint on $\mathscr{H}_0^0$ and unitarily equivalent to H_0 on $\mathscr{H}_0$. We define so

$$\mathrm{H}_1^1 = i\begin{pmatrix} 0 & h_1^1 \\ -h_1^1 & 0 \end{pmatrix}, \qquad h_1^1 = \begin{pmatrix} 0 & -\dfrac{1}{\sin\theta}\partial_\varphi & \dfrac{1}{\sin\theta}\partial_\theta \sin\theta \\ 0 & 0 & -\partial_{r_*} \\ 0 & \partial_{r_*} & 0 \end{pmatrix} \tag{290}$$

H_1^1 is self-adjoint on Hilbert space $\mathscr{H}_1^1$:

$$\mathscr{H}_1^1 = \{ \mathrm{U}_1 \in \tilde{\mathscr{H}}_1;\ (281)(282)(283)\ \textit{are verified} \}. \tag{291}$$

We see easily that

$$\mathrm{H}_1^1 = -i\partial_{r_*}, \qquad \mathrm{U}_1 \in \mathscr{H}_1^1 \Rightarrow (e^{-it\mathrm{H}_1^1}\mathrm{U}_1)(r_*, \omega) = \mathrm{U}_1(r_* - t, \omega). \tag{292}$$

We introduce the cut-off operator

$$\mathscr{I}_{0,1}:\ \mathscr{H}_1^1 \to \mathscr{H}_0^0 : (\mathscr{I}_{0,1}\mathrm{U}_1)(r_*, \omega) = \chi_0(r_*, \omega)\, \mathrm{U}_1(r_*, \omega), \quad 0 \leqq r_*. \tag{293}$$

LEMMA 5.3. — *The wave operator*

$$W_{0,1}U_1 = s - \lim_{t\to\pm\infty} e^{itH_0^0}\mathscr{I}_{0,1}e^{-itH_1^1}U_1 \quad in\ \mathscr{H}_0^0$$

is well defined from $\mathscr{H}_1^1$ *to* $\mathscr{H}_0^0$.

Proof. — Because

$$\| e^{itH_0^0}\mathscr{I}_{0,1}e_1^{-itH^1}U_1\|_{\mathscr{H}_0^0} \leqq c(\chi_0)\|U_1\|_{\mathscr{H}_1^0}$$

we consider only the case

$$U_1 \in \mathscr{H}_1^1 \cap [C_0^\infty(\mathbb{R}_{r_*}\times S_\omega^2)]^6.$$

Now thanks to (292)

$$\left\|\frac{d}{dt}(e^{itH_0^0}\mathscr{I}_{0,1}e^{-itH_1^1}U_1)\right\|_{\mathscr{H}_0^0} \leqq \|r_*^{-2}\chi_0(r_*)f(r_*-t))\|_{L^2(\mathbb{R}_{r_*}\times S_\omega^2; dr_*\,dw)} + \left\|\left(\frac{d}{dr_*}\chi_0\right)(r_*)g(r_*-t)\right\|_{L^2(\mathbb{R}_{r_*}\times S_\omega^2; dr_*\,d\omega)}$$

where

$$g, f \in C_0^\infty(\mathbb{R}_{r_*}\times S_\omega^2).$$

Because g, f and $\frac{d}{dr_*}\chi_0$ have compact support we obtain

$$\left\|\frac{d}{dt}(e^{itH_0^0}\mathscr{I}_{0,1}e^{-itH_1^1}U_1)\right\|_{\mathscr{H}_0^0} = O(t^{-2})$$

Q.E.D.

Now we can end the proof of theorem.

Lemma 5.3 assures there exists $U_0 \in \mathscr{H}_0$ such that by noting

$$e^{-itH_0}U_0 = {}^t(E_0, B_0)$$

$$\|\chi_0(r_*)(r_*^2 E_0^{\hat{r}_*}(t, r_*) - e^{\hat{r}_*}(r_*-t, \omega))\|_{L^2(\mathbb{R}_{r_*}^+\times S_\omega^2)} \to 0, \quad t\to+\infty, \tag{294}$$

$$\left.\begin{array}{l}\|\chi_0(r_*)(r_* E_0^{\hat{v}}(t, r_*, \omega) - e^{\hat{v}}(r_*-t, \omega))\|_{L^2(\mathbb{R}_{r_*}^+\times S_\omega^2)} \to 0,\\ t\to+\infty, \quad \hat{v} = \hat{\theta}, \hat{\varphi},\end{array}\right\} \tag{295}$$

$$\begin{array}{l}\|\chi_0(r_*)(r_*^2)(r_*^2 B_0^{\hat{r}_*}(t, r_*, \omega) - b^{\hat{r}_*}(r_*-t, \omega))\|_{L^2(\mathbb{R}_{r_*}^+\times S_\omega^2)} \to 0,\\ t\to+\infty,\end{array} \tag{296}$$

$$\left.\begin{array}{l}\|\chi_0(r_*)(r_* B_0^{\hat{v}}(t, r_*, \omega) - b^{\hat{v}}(r_*-t, \omega))\|_{L^2(\mathbb{R}_{r_*}^+\times S_\omega^2)} \to 0,\\ t\to+\infty, \quad \hat{v} = \hat{\theta}, \hat{\varphi}.\end{array}\right\} \tag{297}$$

We remark that for $f \in L^2(\mathbb{R}_{r_*}\times S_\omega^2)$

$$\left\|\chi_0(r_*)\left(1-\frac{r\alpha}{r_*}\right)f(r_*-t)\right\|_{L^2(\mathbb{R}_{r_*}\times S_\omega^2)} \to 0, \quad t\to+\infty. \tag{298}$$

The limits (284) to (287) and (294) to (298) allow to conclude that

$$\| \mathcal{J}_0^* e^{-itH} U - e^{-itH_0} U_0 \|_{\tilde{\mathcal{H}}_0} \to 0, \qquad t \to +\infty$$

that is, $W_0 U$ is well defined in $\mathcal{H}_0$. Now if χ_0' is another cut-off function satisfying (119), estimates (284) to (287) show that

$$\| (\chi_0(r_*) - \chi_0'(r_*)) e^{-itH} U \|_{\tilde{\mathcal{H}}} \to 0, \qquad t \to +\infty.$$

That proves $W_0 U$ does not depend on χ_0. Finally, by choosing $|\chi_0| \leqq 1$ we obtain

$$\| W_0 \|_{\mathcal{L}(\mathcal{H}, \mathcal{H}_0)} \leqq 1.$$

Q.E.D.

Proof of Theorem 5.1. – It remains to verify W^-, W are isometric. We have

$$\| W^-(U_0, U_1) \|_{\mathcal{H}}^2 = \lim_{t \to \pm\infty} \| \mathcal{J}_0 e^{-itH_1} U_1 + \mathcal{J}_0 e^{-itH_0} U_0 \|_{\tilde{\mathcal{H}}}^2.$$

Thanks to the properties of χ_0, χ_1 and the decay of the local energy of free fields we have

$$\lim_{t \to \pm\infty} \| \mathcal{J}_1 e^{-itH_1} U_1 + \mathcal{J}_0 e^{-itH_0} U_0 \|_{\tilde{\mathcal{H}}}^2$$
$$= \lim_{t \to \pm\infty} (\| \mathcal{J}_1 e^{-itH_1} U_1 \|_{\tilde{\mathcal{H}}}^2 + \| \mathcal{J}_0 e^{-itH_0} U_0 \|_{\tilde{\mathcal{H}}}^2) = \| U_1 \|_{\mathcal{H}_1^-}^2 + \| U_0 \|_{\mathcal{H}_0}^2.$$

Q.E.D.

we obtain so

$$\| W(U) \|_{\mathcal{H}_1^+ \times \mathcal{H}_0}^2 = \lim_{t \to \pm\infty} (\| \mathcal{J}_1^* e^{-itH} U \|_{\tilde{\mathcal{H}}_1}^2 + \| \mathcal{J}_0^* e^{-itH} U \|_{\tilde{\mathcal{H}}_0}^2)$$
$$= \lim_{t \to \pm\infty} (\| \chi_1 e^{-itH} U \|_{\tilde{\mathcal{H}}}^2 + \| \tilde{\chi}_0 e^{-itH} U \|_{\tilde{\mathcal{H}}}^2)$$

where

$$\tilde{\chi}_0(r_*) = \chi_0(r_*) \qquad \text{for} \quad r_* > 0$$
$$\tilde{\chi}_0(r_*) = 0 \qquad \text{for} \quad r_* \leqq 0.$$

Because the local energy of $e^{-itH} U$ decaies we have again

$$\| W(U) \|_{\mathcal{H}_1^+ \times \mathcal{H}_0}^2 = \lim_{t \to \pm\infty} \| (\chi_1 + \tilde{\chi}_0) e^{-itH} U \|_{\tilde{\mathcal{H}}}^2 = \| U \|_{\mathcal{H}}^2.$$

Q.E.D.

6. MEMBRANE PARADIGM

The Membrane Paradigm [14] [15] states that, in order to describe long range effects of Black-Hole, we may replace the horizon by a stretched horizon that is a true time like boundary with the impedence condition.

For $\varepsilon > 0$, we define the absolute space $(V_\varepsilon, ds^2_{V_\varepsilon})$

$$V_\varepsilon =]r_0+\varepsilon, +\infty[_r \times S^2_\omega, \qquad ds^2_{V_\varepsilon} = ds^2_V|_{V_\varepsilon} \tag{299}$$

The boundary of V_ε is called the streched horizon

$$\partial V_\varepsilon = \{r_0+\varepsilon\} \times S^2_\omega. \tag{300}$$

We keep the notations of part 2 and by noting G_ε the restriction of $-iH$ at the distribution space on V_ε, Maxwell's equations in $\mathbb{R}_t \times V_\varepsilon$ are

$$\frac{dU}{dt} = G_\varepsilon U \quad in\ V_\varepsilon \tag{301}$$

$$\frac{\alpha}{r^2}\partial_r(r^2 X^{\hat{r}}) + \frac{1}{r\sin\theta}[\partial_\theta \sin\theta X^{\hat{\theta}} + \partial_\varphi X^{\hat{\varphi}}] = 0 \quad in\ V_\varepsilon, \qquad X=E, B. \tag{302}$$

We impose the impedence condition on the stretched horizon

$$E^{\hat{\theta}} = -B^{\hat{\varphi}}, \quad E^{\hat{\varphi}} = B^{\hat{\theta}} \quad for\ r=r_0+\varepsilon. \tag{303}$$

We introduce the Hilbert spaces

$$\tilde{\mathscr{H}}_\varepsilon = \{U_{|V_\varepsilon}, U \in \tilde{\mathscr{H}}\} \tag{304}$$

$$\mathscr{H}_\varepsilon = \{U_{|V_\varepsilon}, U \in \mathscr{H}\}. \tag{305}$$

We consider operator G with domain $D(G_\varepsilon) \subset \tilde{\mathscr{H}}_\varepsilon$, determined as the closure of the set of functions U in $C^1_0(\bar{V}_\varepsilon, \mathbb{C}^6)$ having a compact support in $\bar{V}_\varepsilon = [r_0+\varepsilon, +\infty[_r \times S^2_\omega$ and satisfying condition (303), with respect the graph norm

$$\|U\|^2_{D(G_\varepsilon)} = \|U\|^2_{\tilde{\mathscr{H}}_\varepsilon} + \|G_\varepsilon U\|^2_{\tilde{\mathscr{H}}_\varepsilon}.$$

THEOREM 6.1. – *Operator* G_ε *with domain* $D(G_\varepsilon)$ *generates a contraction semigroup* $\mathscr{V}_\varepsilon(t)$ *in* $\tilde{\mathscr{H}}_\varepsilon$; $\mathscr{H}_\varepsilon$ *is invariant with respect to* $\mathscr{V}_\varepsilon(t)$.

Therefore, given $U \in \mathscr{H}_\varepsilon$, $\mathscr{V}_\varepsilon(t)U$, $t>0$, is a weak solution of (301) (302); moreover, if U is in dense subspace $D(G_\varepsilon) \cap \mathscr{H}_\varepsilon$, we have

$$\mathscr{V}_\varepsilon(t)U \in C^0([0, +\infty[_t, D(G_\varepsilon) \cap \mathscr{H}_\varepsilon)$$

and boundary condition (303) is satisfied in strong sense thanks to the following result:

LEMMA 6.2. – *The map*

$$U \in [C^1_0(\bar{V}_\varepsilon, \mathbb{C}^6)] \to (E^{\hat{\theta}}, E^{\hat{\varphi}}, B^{\hat{\theta}}, B^{\hat{\varphi}})_{|r=r_0+\varepsilon}$$

can be extended to a continuous map from $\{U \in \tilde{\mathscr{H}}_\varepsilon;\ G_\varepsilon U \in \tilde{\mathscr{H}}_\varepsilon\}$ *to Sobolev space* $[H^{-1/2}(S^2_\omega)]^4$.

Moreover the classical charge conservation law is satisfied on the streched horizon:

PROPOSITION 6.3. – *Let* U *be in* $D(G_\varepsilon)$; *we note*

$$t \geq 0, \qquad \mathscr{V}_\varepsilon(t)U = {}^t(E^{\hat{r}}, E^{\hat{\theta}}, E^{\hat{\varphi}}, B^{\hat{r}}, B^{\hat{\theta}}, B^{\hat{\varphi}})$$

then we have

$$\nu = r, \theta, \varphi, \qquad E^{\hat{\nu}}|_{r=r_0+\varepsilon} \in C^1([0, +\infty[_t, H^{1/2}(S^2_\omega))$$

and

$$\left.\frac{dE^{\hat{r}}}{dt}\right|_{r=r_0+\varepsilon} + \varepsilon^{1/2}(r_0+\varepsilon)^{3/2}\frac{1}{\sin\theta}(\partial_\theta \sin\theta\, E^{\hat{\theta}} + \partial_\varphi E^{\hat{\varphi}})\Big|_{r=r_0+\varepsilon} = 0.$$

The existence of a limit of $E^{\hat{r}}$ on $\{r=r_0+\varepsilon\}\times S^2_\omega$ is a consequence of the constraint of free divergence.

Now we assume $\varepsilon>0$ small enough for that

$$r^\varepsilon_* = r_0+\varepsilon+r_0 \operatorname{Log}\varepsilon < 0. \tag{306}$$

In order to compare the electromagnetic fields in V_ε and in the Minkowski space-time, we introduce cut-off function χ_0 satisfying (119) and identifications operators:

$$\left.\begin{array}{c}\mathcal{I}_0 : \tilde{\mathcal{H}}_0 \to \tilde{\mathcal{H}}_\varepsilon; \\ \mathcal{I}_0 U_0 = \chi_0 U_0 \quad \text{for } r_* \geqq 0, \qquad \mathcal{I}_0 U_0 = 0 \quad \text{for } r^\varepsilon_* \leqq r_* \leqq 0,\end{array}\right\} \tag{307}$$

$$\mathcal{I}^*_0 : \tilde{\mathcal{H}}_\varepsilon \to \tilde{\mathcal{H}}_0; \qquad \mathcal{I}^*_0 U = \chi_0 U|_{r_*\geqq 0}. \tag{308}$$

We define wave operators

$$W^\varepsilon_- U_0 = s-\lim_{t\to\pm\infty} \mathcal{V}_\varepsilon(t)\mathcal{I}_0 e^{itH_0} U_0 \quad \text{in } \tilde{\mathcal{H}}_\varepsilon \tag{309}$$

$$W^\varepsilon U = s-\lim_{t\to\pm\infty} e^{itH_0}\mathcal{I}^*_0 \mathcal{V}_\varepsilon(t) U \quad \text{in } \tilde{\mathcal{H}}_0. \tag{310}$$

THEOREM 6.4. — W^ε_- *is well defined from* $\mathcal{H}_0$ *to* $\mathcal{H}_\varepsilon$*;*
W^ε *is well defined from* $\mathcal{H}_\varepsilon$ *to* $\mathcal{H}_0$*;*
W^ε_-, W^ε *do not depend on* χ_0 *and*

$$\| W^\varepsilon_- \|_{\mathcal{L}(\mathcal{H}_0, \mathcal{H}_\varepsilon)} \leqq 1, \qquad \| W^\varepsilon \|_{\mathcal{L}(\mathcal{H}_\varepsilon, \mathcal{H}_0)} \leqq 1.$$

Therefore, despite the long range gravitational interaction, the classical wave operators exist and the scattering by the stretched horizon is described by the operator

$$S_\varepsilon = W^\varepsilon W^\varepsilon_- : \mathcal{H}_0 \to \mathcal{H}_0. \tag{311}$$

The following result is the mathematical foundation of the Membrane Paradigm:

THEOREM 6.5. — *For* $U^-_0 \in \mathcal{H}_0$

$$S_\varepsilon U^-_0 \to W_0 W^-_0 U^-_0 \quad \text{in } \mathcal{H}_0, \qquad \varepsilon \to 0. \tag{312}$$

Remark 6.6. — Theorems 6.5 and 6.6 are true again for the non smooth cut off function χ_0 such that

$$\exists a, \quad 0<a, \quad 0 \leqq r_* \leqq a \ \Rightarrow\ \chi_0(r_*)=0, \quad a<r_* \ \Rightarrow\ \chi_0(r_*)=1.$$

To prove that it sufficient to approximate χ_0 with smooth functions satisfying (119) and we use the previous results and the decay of the local energy.

(312) is meaningfull of viewpoint of numerical analysis; indeed, impedence condition (303) is the first order absorbing condition introduced by B. Hanouzet, M. Sesques [12] by generalising for the Maxwell system in euclidian space the absorbing boundary conditions for the wave equation of A. Bayliss, E. Turkel [2]. Hence we may make numerical calculations of fields by using boundary condition (303) on the stretched horizon. On physical view point it is very natural that condition (303) is absorbing: it is well known in geometrical optics that the reflecting coefficient between two media is minimal if these media have the same impedence; (303) means the stretched horizon has the vacuum impedence; hence it is a transparent boundary.

Proof of Theorem 6.1. — It is a direct consequence of the classical results about the symmetric systems with dissipative boundary conditions (e. g. to *see* [20]); we recall only the key point is operator G_ε is accretive *i. e*:

$$\forall U \in D(G_\varepsilon), \quad \operatorname{Re}\langle G_\varepsilon U, U\rangle_{\tilde{\mathscr{H}}_\varepsilon} \leqq 0; \tag{313}$$

indeed we obtain easily by Green's formula:

$$\operatorname{Re}\langle G_\varepsilon U, U\rangle_{\tilde{\mathscr{H}}_0} \leqq 2\varepsilon(r_0+\varepsilon)\int_{r=r_0+\varepsilon} |E^{\hat{\theta}}|^2 + |E^{\hat{\varphi}}|^2\, d\omega. \tag{314}$$

So we show that the adjoint of G_ε is accretive, hence G_ε generates a semigroup of contraction. Now by (58) the restriction of $V_\varepsilon(t)$ at $\mathscr{H}_\varepsilon$ is a semigroup of contraction on $\mathscr{H}_\varepsilon$ again.

Q.E.D.

Proof of Lemma 6.2. — Given $e^{\hat{\theta}}$, $e^{\hat{\varphi}}$, $b^{\hat{\theta}}$, $b^{\hat{\varphi}}$ in $H^{1/2}(S^2_\omega)$, take $\tilde{e}^{\hat{\theta}}$, $\tilde{e}^{\hat{\varphi}}$, $\tilde{b}^{\hat{\theta}}$, $\tilde{b}^{\hat{\varphi}}$ in $H^1_{\text{comp}}(V_\varepsilon)$ so that for $X=e, b$, $\nu=\theta, \varphi$,

$$\tilde{X}^{\hat{\nu}}|_{\partial V_\varepsilon} = X^{\hat{\nu}}, \qquad \|\tilde{X}^{\hat{\nu}}\|_{H^1(V_\varepsilon)} \leqq c\,\|X^{\hat{\nu}}\|_{H^{1/2}(\partial V_\varepsilon)}. \tag{315}$$

We put

$$\tilde{U} = (0, \tilde{e}^{\hat{\theta}}, -\tilde{e}^{\hat{\varphi}}, 0, -\tilde{b}^{\hat{\theta}}, \tilde{b}^{\hat{\varphi}}).$$

Green's formula implies

$$\langle G_\varepsilon U, \tilde{U}\rangle_{\tilde{\mathscr{H}}_0} + \langle U, G_\varepsilon \tilde{U}\rangle_{\tilde{\mathscr{H}}_0}$$

$$= \varepsilon(r_0+\varepsilon)\int_{\partial V_\varepsilon} [B^{\hat{\varphi}} e^{\hat{\theta}} + B^{\hat{\theta}} e^{\hat{\varphi}} + E^{\hat{\theta}} b^{\hat{\varphi}} + E^{\hat{\varphi}} b^{\hat{\theta}}]\, d\omega. \tag{316}$$

Thus we get

$$\left|\left\langle {}^t(\mathrm{E}^{\hat\theta},\,\mathrm{E}^{\hat\varphi},\,\mathrm{B}^{\hat\theta},\,\mathrm{B}^{\hat\varphi})_{|r=r_0+\varepsilon},\ {}^t(e^{\hat\theta},\,e^{\hat\varphi},\,b^{\hat\theta},\,b^{\hat\varphi})_{|r=r_0+\varepsilon}\right\rangle_{\mathrm{H}^{-1/2},\,\mathrm{H}^{1/2}(\mathrm{S}^2_\omega)}\right|$$
$$\leqq c_\varepsilon \left\| {}^t(e^{\hat\theta},\,e^{\hat\varphi},\,b^{\hat\theta},\,b^{\hat\varphi})\right\|_{\mathrm{H}^{1/2}(\mathrm{S}^2_\omega)}\left(\|\mathrm{U}\|_{\tilde{\mathscr{H}}_0}+\|\mathrm{G}_\varepsilon\mathrm{U}\|_{\tilde{\mathscr{H}}_0}\right)$$

therefore

$$\left\| {}^t(\mathrm{E}^{\hat\theta},\,\mathrm{E}^{\hat\varphi},\,\mathrm{B}^{\hat\theta},\,\mathrm{B}^{\hat\varphi})_{|r=r_0+\varepsilon}\right\|_{\mathrm{H}^{-1/2}(\mathrm{S}^2_\omega)}\leqq c_\varepsilon\left(\|\mathrm{U}\|_{\tilde{\mathscr{H}}_0}+\|\mathrm{G}_\varepsilon\mathrm{U}\|_{\tilde{\mathscr{H}}_0}\right).$$

Q.E.D.

Proof of Proposition 6.3. – For $\Phi\in\mathrm{H}^{1/2}(\mathrm{S}^2_\omega)$ we choose $\tilde\Phi\in\mathrm{H}^1_{\mathrm{comp}}(\mathrm{V}_\varepsilon)$ so that

$$\|\tilde\Phi\|_{\mathrm{H}^1(\mathrm{V}_\varepsilon)}\leqq c\,\|\Phi\|_{\mathrm{H}^{1/2}(\mathrm{S}^2_\omega)}. \tag{317}$$

We have

$$0=\int_{\mathrm{V}_\varepsilon}\left\{\frac{\alpha}{r^2}\partial_r(r^2\,\mathrm{E}^{\hat r})+\frac{1}{r\sin\theta}\left[\partial_\theta\sin\theta\,\mathrm{E}^{\hat\theta}+\partial_\varphi\,\mathrm{E}^{\hat\varphi}\right]\right\}\tilde\Phi\, r^2\sin\theta\,dr\,d\varphi\,d\theta.$$

By integration by part we get

$$\varepsilon^{1/2}(r_0+\varepsilon)^{3/2}\int_{\mathrm{S}^2_\omega}\mathrm{E}^{\hat r}|_{r=r_0+\varepsilon}\,\Phi\,d\omega=-\int_{\mathrm{V}_\varepsilon}\left\{\mathrm{E}^{\hat r}\,\partial_r(\alpha\tilde\Phi)+\mathrm{E}^{\hat\theta}\frac{1}{r}\partial_\theta\tilde\Phi\right.$$
$$\left.+\,\mathrm{E}^{\hat\varphi}\frac{1}{r\sin\theta}\partial_\varphi\tilde\Phi\right\}r^2\sin\theta\,dr\,d\theta\,d\varphi$$

thus

$$\left|\left\langle \mathrm{E}^{\hat r}|_{r=r_0+\varepsilon},\ \Phi\right\rangle_{\mathrm{H}^{-1/2},\,\mathrm{H}^{1/2}(\mathrm{S}^2_\omega)}\right|\leqq c_\varepsilon\,\|\mathrm{U}\|_{\mathscr{H}_\varepsilon}\,\|\tilde\Phi\|_{\mathrm{H}^1(\mathrm{V}_\varepsilon)}$$

then we deduct from (317) that

$$\mathrm{E}^{\hat r}|_{r=r_0+\varepsilon}\in\mathrm{C}^0(\mathbb{R}^+_t,\,\mathrm{H}^{-1/2}(\mathrm{S}^2_\omega)).$$

Now for U in $\mathrm{D}(\mathrm{G}_\varepsilon)$ we have

$$0\leqq t,\quad \mathrm{G}_\varepsilon\mathscr{V}_\varepsilon(t)\,\mathrm{U}\in\mathscr{H}_\varepsilon$$

and as previous

$$\partial_t\mathrm{E}^{\hat r}|_{r=r_0+\varepsilon}=(\mathrm{G}_\varepsilon\mathscr{V}_\varepsilon(t)\,\mathrm{U})^{\hat r}|_{r_0+\varepsilon}\in\mathrm{C}^0(\mathbb{R}^+_t,\,\mathrm{H}^{-1/2}(\mathrm{S}^2_\omega)),$$

that is to say

$$\frac{d}{dt}\mathrm{E}^{\hat r}+\varepsilon^{1/2}(r_0+\varepsilon)^{-3/2}\frac{1}{\sin\theta}(\partial_\theta\sin\theta\,\hat{\mathrm{E}}+\partial_\varphi\,\mathrm{E}^{\hat\varphi})\,|_{r=r_0+\varepsilon}=0.$$

Q.E.D.

Proof of Theorem 6.4. – To prove the existence of W^ε_- we can repeat exactly the arguments of the proof of Theorem 3.1.

To establish $W^\varepsilon U$ exists its is sufficient to consider

$$U \in G^2_\varepsilon [C^\infty_0 (V_\varepsilon)]^6. \tag{318}$$

We split the electric and magnetic fields into radial and transverse components that we expand in series of generalized spherical functions (208), (209); we note $a^\nu_X = a^\nu_X(t, r_*)$ the coefficients of $T^l_{\nu, n}$ in (208) (209), associated to $X = E, B$, $\nu = 0, +, -$.

a^ν_X sarisfying equations (210) (211) (212) (213) in $\mathbb{R}_t \times]r^\varepsilon_*, +\infty[$. By using Maxwell's equations, impedence condition (303) and relations (79) (90) we obtain the boundary conditions at r^ε_*:

$$(\partial_t - \partial_{r_*}) a^0_X = 0, \qquad t > 0, \quad r_* = r^\varepsilon_*, \quad X = E, B, \tag{319}$$

$$(\partial_t - \partial_{r_*}) a^\pm_E = \mp \frac{i}{\sqrt{2}} \sqrt{l(l+1)} \frac{a^2}{r^2} a^0_B, \; t > 0, \; r_* = r^\varepsilon_*. \tag{320}$$

$$(\partial_t - \partial_{r_*}) a^\pm_B = \pm \frac{i}{\sqrt{2}} \sqrt{l(l+1)} \frac{a^2}{r^2} a^0_E, \qquad t > 0, \quad r_* = r^\varepsilon_*. \tag{321}$$

Now equations (210) (320) give

$$(\partial_t - \partial_{r_*}) \left(a^+_E - a^-_E - i \frac{\sqrt{2}}{\sqrt{l(l+1)}} (\partial_t + \partial_{r_*}) a^0_B\right) \Big|_{r_* = r^\varepsilon_*} = 0$$

and with (213) (321)

$$(\partial_t - \partial_{r_*}) \left(a^+_E - a^-_E - i \frac{\sqrt{2}}{\sqrt{l(l+1)}} \partial_t a^0_B\right) = i (\partial_t - \partial_{r_*}) (a^+_B + a^-_B) = 0.$$

The same result holds for $a^+_B - a^-_B + i \frac{\sqrt{2}}{\sqrt{l(l+1)}} \partial_t a^0_E$, $a^+_E + a^-_E$.

We conclude that

$$u \in \left\{ a^0_E, \; a^0_B, \; a^+_E + a^-_E, \; a^+_B + a^-_B, \; \partial_t a^0_E, \; \partial_t a^0_B, \right.$$
$$\left. a^+_E - a^-_E - i \frac{\sqrt{2}}{\sqrt{l(l+1)}} \partial_t a^0_B, \; a^+_B - a^-_B + i \frac{\sqrt{2}}{\sqrt{l(l+1)}} \partial_t a^0_E \right\}$$

is solution of

$$\partial^2_t u - \partial^2_{r_*} u = - l(l+1)) \frac{\alpha^2}{r^2} u, \qquad t > 0, \quad r_* > r^\varepsilon_*, \tag{323}$$

$$\partial_t u - \partial_{r_*} u = 0, \qquad t > 0, \quad r_* = r^\varepsilon_*. \tag{324}$$

But (324) is a perfecty transparent condition, so

$$u = \tilde{u} \big|_{r_* > r^\varepsilon_*}$$

where $\tilde{u}$ is solution of

$$\partial_t^2 \tilde{u} - \partial_{r_*}^2 \tilde{u} = -l(l+1)\frac{\tilde{\alpha}^2}{r^2}\tilde{u}, \qquad t>0, \quad r_* \in \mathbb{R}, \tag{325}$$

$$\tilde{u}(0, r_*) = u(0, r_*),\ r_* > r_*^\varepsilon \qquad \textit{and} \qquad \tilde{u}(0, r_*) = 0, \quad r_* \leqq r_*^\varepsilon, \tag{326}$$

$$\partial_t \tilde{u}(0, r_*) = \partial_t u(0, r_*),\ r_* > r_*^\varepsilon \qquad \textit{and} \qquad \partial_t \tilde{u}(0, r_*) = 0, \quad r_* \leqq r_*^\varepsilon \tag{327}$$

with

$$\tilde{\alpha}\big|_{r_* \geqq r_*^\varepsilon} = \alpha\big|_{r_* \geqq r_*^\varepsilon}, \qquad \tilde{\alpha}\big|_{r_* < r_*^\varepsilon} = 0. \tag{328}$$

Now we may apply lemma 5.2 for

$$\tilde{\mu}^2 = -\partial_{r_*}^2 + l(l+1)\frac{\tilde{\alpha}^2}{r^2}. \tag{329}$$

Hence there exists an asymptotic profile $f \in H^1(]r_*^\varepsilon, \infty[)$ so that

$$\lim_{t \to +\infty} \left(\| u(t, r_*) - f(r_* - t) \|_{H^1(]r_*^\varepsilon, \infty[)} + \| \partial_t u(t, r_*) + \partial_{r_*} f(r_* - t) \|_{L^2(]r_*^\varepsilon, \infty[)} \right) = 0$$

and we achieve the proof of existence of $W^\varepsilon u$ like for $W_0 U$.

Q.E.D.

Proof of Theorem 6.5. – We can consider only the case

$$U_0^- \in \mathscr{H}_0 \cap (C_0^\infty(\mathbb{R}_{r_*}^+ \times S_\omega^2))^6, \qquad \operatorname{supp} U_0^- \subset [0, R]_{r_*} \times S_\omega^2. \tag{330}$$

Then by Huygens principle

$$\operatorname{supp} e^{-itH_0} U_0^- \subset [-R + |t|, \infty[_{r_*} \times S_\omega^2.$$

If

$$U = W_0^- U_0^-$$

Proposition 3.2 states

$$\operatorname{supp} e^{-itH} U \subset [-R - t, \infty[_{r_*} \times S_\omega^2. \tag{331}$$

It follows that for ε small enough so that

$$r_*^\varepsilon < -R \tag{332}$$

we have

$$W_0^- U_0^- \big|_{r_* > r_*^\varepsilon} = W_-^\varepsilon U_0^-, \ W_0^- U_0^- \big|_{r_* < r_*^\varepsilon} = 0. \tag{333}$$

By the asymptotic completeness we write

$$e^{-itH} U = \mathscr{I}_1 e^{-itH_1} U_1 + M \mathscr{I}_0 e^{-itH_0} U_0^+ + \theta(t) \tag{334}$$

where M is defined by (128) and

$$U_1 \in \mathscr{H}_1^+, \quad U_0^+ \in \mathscr{H}_0, \qquad \|\theta(t)\|_{\tilde{\mathscr{H}}} \to 0, \qquad t \to +\infty. \tag{335}$$

Given $\eta>0$ we choose $U'_1\in\mathscr{H}_1^+$, $U'_0\in\mathscr{H}_0$, $T\in\mathbb{R}$, such that

$$\|U_i-U'_i\|_{\tilde{\mathscr{H}}_i}\leqq\eta,\qquad U'_i\in C_0^\infty,\qquad U'_i(r_*,\omega)=0\quad for\ |r_*|>R'\tag{336}$$

$$T\leqq t\quad\Rightarrow\quad\|\theta(t)\|_{\tilde{\mathscr{H}}}\leqq\eta.\tag{337}$$

Thus

$$\|W_0\,W_0^-\,U_0^- - U'_0\|_{\mathscr{H}_0}\leqq\eta.\tag{338}$$

We introduce

$$\mathscr{I}_\varepsilon:\tilde{\mathscr{H}}\to\tilde{\mathscr{H}}_\varepsilon;\qquad\mathscr{I}_\varepsilon U=U|_{r_*\geqq r_*^\varepsilon}.\tag{339}$$

Our main result is to prove that

$$\|W^\varepsilon\mathscr{I}_\varepsilon U-U'_0\|_{\mathscr{H}_0}\leqq 7\,\eta,\qquad U=W_0^-\,U_0.\tag{340}$$

We assume $\varepsilon>0$ so that

$$r_*^\varepsilon<-T-\mathrm{Max}(R,R').\tag{341}$$

Then for $0\leqq t\leqq T$

$$\mathscr{V}_\varepsilon(t)\,\mathscr{I}_\varepsilon U=\mathscr{I}_\varepsilon e^{-itH}U.\tag{342}$$

Notice that $\mathscr{I}_\varepsilon M\mathscr{I}_0\,e^{-itH_0}U'_0$ satisfies boundary condition (303) for $t\geqq T$ and

$$\frac{d}{dt}\mathscr{I}_\varepsilon M\mathscr{I}_0\,e^{-itH_0}U'_0=G_\varepsilon\mathscr{I}_\varepsilon M\mathscr{I}_0\,e^{-itH_0}U'_0+i\,\mathscr{I}_\varepsilon(HM\mathscr{I}_0-M\mathscr{I}_0H_0)\,e^{-itH_0}U'_0.\tag{343}$$

Since

$$\mathscr{I}_\varepsilon(HM\mathscr{I}_0-M\mathscr{I}_0H_0)\,e^{-itH_0}U'_0\in D(G_\varepsilon),\qquad t\geqq T$$

we obtain for $t\geqq T$

$$\mathscr{I}_\varepsilon M\mathscr{I}_0\,e^{-itH_0}U'_0=\mathscr{V}_\varepsilon(t-T)\,\mathscr{I}_\varepsilon M\mathscr{I}_0\,e^{-itH_0}U'_0+\int_T^t\mathscr{V}_\varepsilon(t-s)\,[i\,\mathscr{I}_\varepsilon(HM\mathscr{I}_0-M\mathscr{I}_0H_0)\,e^{-isH_0}U'_0]\,ds.\tag{344}$$

On the other hand $\mathscr{I}_\varepsilon\mathscr{I}_1\,e^{-itH_1}U'_1$ satisfies so boundary condition (303) and

$$\frac{d}{dt}\mathscr{I}_\varepsilon\mathscr{I}_1\,e^{-itH_1}U'_1=G_\varepsilon\mathscr{I}_\varepsilon\mathscr{I}_1\,e^{-itH_1}U'_1+i\,\mathscr{I}_\varepsilon(H\mathscr{I}_1)\,e^{-itH_1}\tag{345}$$

with

$$\mathscr{I}_\varepsilon(H\mathscr{I}_1-\mathscr{I}_1H_1)\,e^{-itH_1}U'_1\in D(G_\varepsilon)$$

thus for $t \geqq T$

$$\mathscr{I}_\varepsilon \mathscr{I}_1 e^{-itH_1} U'_1 = \mathscr{V}_\varepsilon (t-T) \mathscr{I}_\varepsilon \mathscr{I}_1 e^{-iTH_1} U'_1 + \int_T^t \mathscr{V}_\varepsilon (t-s) [i \mathscr{I}_\varepsilon (H \mathscr{I}_1 - \mathscr{I}_1 H_1) e^{-isH_1} U'_1] \, ds. \quad (346)$$

From (342) (344) (346) we deduct

$$\begin{aligned} \mathscr{V}_\varepsilon (t) \mathscr{I}_\varepsilon U - \mathscr{I}_\varepsilon (M \mathscr{I}_0 e^{-itH_0} U'_0 + \mathscr{I}_1 e^{-itH_1} U'_1) &= \mathscr{V}_\varepsilon (t-T) (e^{-iTH} U - M \mathscr{I}_0 e^{-iTH_0} U'_0 - \mathscr{I}_1 e^{-iTH_1} U'_1) \\ &\quad + i \int_T^t \mathscr{V}_\varepsilon (t-s) \mathscr{I}_\varepsilon [(HM \mathscr{I}_0 - M \mathscr{I}_0 H_0) \\ &\quad \times e^{-isH_0} U'_0 + (H \mathscr{I}_1 - \mathscr{I}_1 H_1) e^{-isH_1} U'_1] \, ds. \quad (347) \end{aligned}$$

We evaluate

$$\begin{aligned} \| \mathscr{V}_\varepsilon (t) \mathscr{I}_\varepsilon U - \mathscr{I}_\varepsilon \mathscr{I}_0 e^{-itH_0} U'_0 \|_{\tilde{\mathscr{H}}_\varepsilon} &\leqq \| (1-M) \mathscr{I}_0 e^{-itH_0} U'_0 \|_{\tilde{\mathscr{H}}} + \| \mathscr{I}_\varepsilon \mathscr{I}_1 e^{-itH_1} U'_1 \|_{\tilde{\mathscr{H}}_\varepsilon} \\ &\quad + \| U_0 - U'_0 \|_{\tilde{\mathscr{H}}_0} + \| U_1 - U'_1 \|_{\tilde{\mathscr{H}}_1} + \| \theta (t) \|_{\tilde{\mathscr{H}}} \\ &\quad \int_T^t \| (HM \mathscr{I}_0 - M \mathscr{I}_0 H_0) e^{-isH_0} U'_0 \|_{\tilde{\mathscr{H}}} \, ds \\ &\quad + \int_T^t \| (H \mathscr{I}_1 - \mathscr{I}_1 H_1) e^{-isH_1} U'_1 \|_{\tilde{\mathscr{H}}} \, ds. \quad (348) \end{aligned}$$

By using respectively (134) (136) (163) (336) (337) (137) (138) (191) (193), we see that each norm of right side of (348) is smaller than η for t large enough

$$\exists T_1 / t \geqq T_1 \quad \Rightarrow \quad \| \mathscr{V}_\varepsilon (t) \mathscr{I}_\varepsilon U - \mathscr{I}_\varepsilon \mathscr{I}_0 e^{-itH_0} U'_0 \|_{\tilde{\mathscr{H}}} \leqq 7 \eta;$$

that proves (340) and we obtain finally with (338):

$$\| W_0 W_0^- U_0^- - W^\varepsilon W_0^\varepsilon U_0^- \|_{\mathscr{H}_0} \leqq 8 \eta.$$

Q.E.D.

7. CHARACTERISTIC CAUCHY PROBLEM

We can interpret the whole scattering theory in terms of characteristic Cauchy problem with data specified on the horizons.

We introduce the Kruskal coordinates:

$$u = -2 r_0 e^{-(t-r_*)/2r_0}, \qquad v = 2 r_0 e^{(t+r_*)/2r_0}, \quad (349)$$

and we use the Penrose compactification:

$$\xi_- = 2 \operatorname{Arctg} u, \ \xi_+ = 2 \operatorname{Arctg} v. \quad (350)$$

So we put:

$$(\xi_+, \xi_-, \theta, \varphi) = \mathrm{T}(t, r_*, \theta, \varphi), \tag{351}$$

$$\mathrm{T}(\mathscr{S}) = \mathrm{K} =]0, \pi[_{\xi_+} \times]-\pi, 0[_{\xi_-} \times S^2_\omega \tag{352}$$

Then the Schwarzschild metric becomes

$$ds^2_{\mathscr{S}} = \frac{r_0}{r} e^{-r/r_0}\, du\, dv - r^2 (d\theta^2 + \sin^2\theta\, d\varphi^2)$$

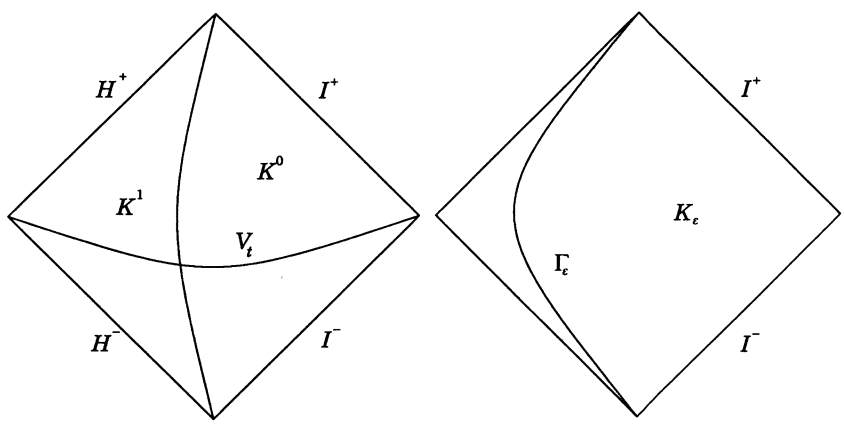
$$= \frac{1}{4} \frac{1}{\cos^2 \xi_+/2 \cos^2 \xi_-/2} \frac{r_0}{r} e^{-r/r_0}\, ds^2_{\mathrm{K}}. \tag{353}$$

where ds^2_{K} is the metric

$$ds^2_{\mathrm{K}} = d\xi_+\, d\xi_- - 4 \frac{r^3}{r_0} e^{r/r_0} \cos^2 \xi_+/2 \cos^2 \xi_-/2\, (d\theta^2 + \sin^2\theta\, d\varphi^2) \tag{354}$$

on manifold K.

Therefore we can represent the Schwarzschild space-time by the famous Penrose diagram.



Penrose conformal diagrams for Schwarzschild Black-Holes.

We have introduced the classical notations for which

$$on\ \mathrm{H}^\pm,\ t = \pm\infty,\ r = r_0\ or\ \xi_\mp = 0 \tag{355}$$

$$on\ \mathrm{I}^\pm,\ t = \pm\infty,\ r = +\infty\ or\ \xi_\pm = \pm\pi. \tag{356}$$

$\mathrm{H}^{+(-)}$ is the future (past) black-hole horizon and $\mathrm{I}^{+(-)}$ is the future (past) flat horizon. Given a real c we define neighbourhoods K^0 of flat infinity $\mathrm{I}^- \cup \mathrm{I}^+$ and K^1 of black hole horizon $\mathrm{H}^- \cup \mathrm{H}^+$ by putting:

$$\mathrm{K}^0 = \mathrm{T}(\mathbb{R}_t \times \{r_* \geqq c\} \times S^2),\ \mathrm{K}^1 = \mathrm{T}(\mathbb{R}_t \times \{r_* \leqq c\} \times S^2), \\ \mathrm{K} = \mathrm{K}^0 \cup \mathrm{K}^1. \tag{357}$$

We split manifold K with Cauchy hypersurfaces V_t:

$$t \in \mathbb{R}, \quad V_t = T(\{t\} \times \mathbb{R}_t \times S^2), \qquad K = \bigcup_{-\infty < t < \infty} V_t, \tag{358}$$

and we put

$$\begin{aligned} V_t^0 &= V_t \cap K^0 = T(\{t\} \times \{r_* \geqq c\} \times S^2), \\ V_t^1 &= V_t \cap K^1 = T(\{t\} \times \{r_* \leqq c\} \times S^2). \end{aligned} \tag{359}$$

If (E, B) is an electromagnetic field in the Schwarzschild space-time, we define

$$f(\xi_+, \xi_-, \theta, \varphi) = \alpha\, r\, {}^t(E^{\hat{r}}, E^{\hat{\theta}}, E^{\hat{\varphi}}, B^{\hat{r}}, B^{\hat{\theta}}, B^{\hat{\varphi}}) \circ T^{-1}(\xi_+, \xi_-, \theta, \varphi) \tag{360}$$

Because Maxwell's equations are conformal invariant and $ds^2_{\mathscr{S}}$ and ds^2_K are conformally related metrics, f satisfies an hyperbolic system $\mathscr{L}$ and an constraint equation $\mathscr{D}$ (free divergence) equivalent to Maxwell's equations on (K, ds^2_K)

$$\mathscr{L}(\partial_{\xi_+}, \partial_{\xi_-}, \partial_\theta, \partial_\varphi) f = 0, \tag{361}$$

$$\mathscr{D}(\partial_{\xi_+}, \partial_{\xi_-}, \partial_\theta, \partial_\varphi) f = 0. \tag{362}$$

In fact we may consider $\mathscr{D}$ as a differential operator on V_t for each t.

We say f is a *finite energy solution* if (E, B) given by (360) is a finite redshifted energy field, *i.e.* (E, B) $\in \tilde{\mathscr{H}}$ and satisfies (30), (31). Therefore we introduce

$$\| f \|_{\mathscr{H}(V_t)} = \| f|_{V_t} \|_{\mathscr{H}(V_t)} = \| (E, B)(t) \|_{\tilde{\mathscr{H}}}, \tag{363}$$

$$\| f \|_{\mathscr{H}(V_t^\nu)} = \| f|_{V_t^\nu} \| = \| \chi_\nu(r_*) (E, B)(t) \|_{\tilde{\mathscr{H}}}, \qquad \nu = 0, 1, \tag{364}$$

where cut off function χ_ν is given by:

$$\chi_1(r_*) = 1 \quad \textit{for } r_* < c, \qquad \chi_1(r_*) = 0 \quad \textit{for } r_* > c, \qquad \chi_0 = 1 - \chi_1. \tag{365}$$

The classical results of Leray on the hyperbolic systems assure the Cauchy problem for (361) in K with Cauchy data specified on $V_{t=t_0}$ is well posed; moreover, Theorem 2.1 implies that if for some real t_0

$$\| f \|_{\mathscr{H}(V_{t_0})} < \infty \qquad \textit{and} \qquad \mathscr{D} f|_{V_{t_0}} = 0, \tag{366}$$

then for each t in $\mathbb{R}$

$$\| f \|_{\mathscr{H}(V_t)} = \| f \|_{\mathscr{H}(V_{t_0})} < \infty \qquad \textit{and} \qquad \mathscr{D} f|_{V_t} = 0. \tag{367}$$

The study of the characteristic Cauchy problem does not follow from the standard results of existence of solutions of Maxwell's equations on a globally hyperbolic curved spacetime, since timelike and spacelike infinity are singular in the Penrose conformal spacetime. Actually, this study is equivalent to the Scattering Theory.

We start by defining Hilbert spaces of data with right polarization on horizons H^-, I^+, and left polarization on horizons H^+, I^-:

$$\mathscr{H}^R(H^-)=\{\Phi^1_-(\xi_+=0,\xi_-,\omega)={}^t(0,e^\theta,e^\varphi,0,b^\theta,b^\varphi)(\xi_-,\omega);\\ e^\theta,e^\varphi,b^\theta,b^\varphi\in L^2(]-\pi,0[_{\xi_-}\times S^2_\omega,d\mu(\xi_-,\omega)),\ e^\theta=b^\varphi,\ e^\varphi=-b^\theta\},\quad (368)$$

$$\mathscr{H}^R(I^+)=\{\Phi^0_+(\xi_+=\pi,\xi_-,\omega)={}^t(0,e^\theta,e^\varphi,0,b^\theta,b^\varphi)(\xi_-,\omega);\\ e^\theta,e^\varphi,b^\theta,b^\varphi\in L^2(]-\pi,0[_{\xi_-}\times S^2_\omega,d\mu(\xi_-,\omega)),\ e^\theta=b^\varphi,\ e^\varphi=-b^\theta\},\quad (369)$$

$$\mathscr{H}^L(H^+)=\{\Phi^1_+(\xi_+,\xi_-=0,\omega)={}^t(0,e^\theta,e^\varphi,0,b^\theta,b^\varphi)(\xi_+,\omega);\\ e^\theta,e^\varphi,b^\theta,b^\varphi\in L^2(]0,\pi[_{\xi_+}\times S^2_\omega,d\mu(\xi_+,\omega)),\ e^\theta=-b^\varphi,\ e^\varphi=b^\theta\},\quad (370)$$

$$\mathscr{H}^L(I^-)=\{\Phi^0_-(\xi_+,\xi_-=-\pi,\omega)={}^t(0,e^\theta,e^\varphi,0,b^\theta,b^\varphi)(\xi_+,\omega);\\ e^\theta,e^\varphi,b^\theta,b^\varphi\in L^2(]0,\pi[_{\xi_+}\times S^2_\omega,d\mu(\xi_+,\omega)),\ e^\theta=-b^\varphi,\ e^\varphi=b^\theta\},\quad (371)$$

where

$$d\mu(\xi_\pm,\omega)=r_0\left|\frac{1}{tg\,\xi_\pm/2}+\mathrm{tg}\,\xi_\pm/2\right|d\xi_\pm\,d\omega.\quad (372)$$

Given Φ^0_-, Φ^1_-, respectively defined on I^- and H we call solution of the characteristic Cauchy problem with past data (Φ^0_-,Φ^1_-), any finite energy solution f of (361) (362) such that

$$\|f(\xi_+,\xi_-,\omega)-\hat{\Phi}^1_-(\xi_+,\xi_-,\omega)\|_{\mathscr{H}(V^1_t)}\to 0,\qquad t\to-\infty,\quad (373)$$

$$\|f(\xi_+,\xi_-,\omega)-\hat{\Phi}^0_-(\xi_+,\xi_-,\omega)\|_{\mathscr{H}(V^0_t)}\to 0,\qquad t\to-\infty,\quad (374)$$

where $\hat{\Phi}^\nu_-$ is the plane wave defined on K and related to Φ^ν_-:

$$(\xi_+,\xi_-,\omega)\in K,\quad \hat{\Phi}^0_-(\xi_+,\xi_-,\omega)=\Phi^0_-(\xi_+,-\pi,\omega),\\ \hat{\Phi}^1_-(\xi_+,\xi_-,\omega)=\Phi^1_-(0,\xi_-,\omega);\quad (375)$$

We say that f has a limit on future horizons if there exists Φ^ν_+ such that

$$\|f(\xi_+,\xi_-,\omega)-\hat{\Phi}^1_+(\xi_+,\xi_-,\omega)\|_{\mathscr{H}(V^1_t)}\to 0,\qquad t\to+\infty,\quad (376)$$

$$\|f(\xi_+,\xi_-,\omega)-\hat{\Phi}^0_+(\xi_+,\xi_-,\omega)\|_{\mathscr{H}(V^0_t)}\to 0,\qquad t\to+\infty,\quad (377)$$

where

$$\left.\begin{aligned}(\xi_+,\xi_-,\omega)\in K,\quad \hat{\Phi}^0_+(\xi_+,\xi_-,\omega)=\Phi^0_+(\pi,\xi_-,\omega),\\ \hat{\Phi}^1_+(\xi_+,\xi_-,\omega)=\Phi^1_+(\xi_+,0,\omega).\end{aligned}\right\}\quad (378)$$

Now the existence of W^- implies that the characteristic Cauchy problem is well posed and the asymptotic completeness assures the solution has a limit on the future horizons:

THEOREM 7.1. – *For* $\Phi^1_-\in\mathscr{H}^R(H^-)$ *and* $\Phi^0_-\in\mathscr{H}^L(I^-)$ *there exists a unique solution* f *of* (361), (362) *satisfying characteristic boundaries conditions* (373) (374) *and*

$$\forall t\in\mathbb{R},\quad \|f\|^2_{\mathscr{H}(V_t)}=\|\Phi^1_-\|^2_{\mathscr{H}^R(H^-)}+\|\Phi^0_-\|^2_{\mathscr{H}^L(I^-)}.\quad (379)$$

Moreover, there exists $\Phi_+^1 \in \mathscr{H}^L(H^+)$, $\Phi_+^0 \in \mathscr{H}^R(I^+)$ *satisfying* (376) (377). *The linear map*

$$S_K: \quad (\Phi_-^1, \Phi_-^0) \to (\Phi_+^1, \Phi_+^0)$$

is an isometry from $\mathscr{H}^R(H^-) \times \mathscr{H}^L(I^-)$ *onto* $\mathscr{H}^L(H^+) \times \mathscr{H}^R(I^+)$.

At last we interpret the Membrane Paradigm. We note Γ_ε the stretched horizon:

$$\Gamma_\varepsilon = T(\mathbb{R}_t \times \{r = r_0 + \varepsilon\} \times S^2), \qquad K_\varepsilon = T(\mathbb{R}_t \times \{r > r_0 + \varepsilon\} \times S^2), \quad (380)$$

and we note for $t \in \mathbb{R}$

$$\| f^\varepsilon \|_{\mathscr{H}(V_t \cap K_\varepsilon)} = \| f^\varepsilon |_{V_t \cap K_\varepsilon} \|_{\mathscr{H}(V_t \cap K_\varepsilon)} = \| \chi_\varepsilon(r)(E, B)(t) \|_{\tilde{\mathscr{H}}} \quad (381)$$

where

$$\chi_\varepsilon(r) = 0 \quad \textit{for } r < r_0 + \varepsilon, \quad \chi_\varepsilon(r) = 1 \quad \textit{for } r \geqq r_0 + \varepsilon \quad (382)$$

We specify on Γ_ε the impedence condition for f given by (360):

$$E^{\hat{\theta}} = -B^{\hat{\varphi}}, \; E^{\hat{\varphi}} = B^{\hat{\theta}} \quad (383)$$

Theorem 6.2 means the mixed characteristic Cauchy problem is well posed in K_ε and the solution has a limit on I^+:

THEOREM 7.2. – *For* $\Phi_-^0 \in \mathscr{H}^L(I^-)$ *there exists a unique solution* f^ε *of* (361) (362) *in* K_ε *satisfying* (383) *and*

$$\| f^\varepsilon(\xi_+, \xi_-, \omega) - \hat{\Phi}_-^0(\xi_+, \xi_-, \omega) \|_{\mathscr{H}(V_t \cap K_\varepsilon)} \to 0, \qquad t \to -\infty, \quad (384)$$

$$\forall t \in \mathbb{R}, \qquad \| f^\varepsilon \|_{\mathscr{H}(V_t \cap K_\varepsilon)} \leqq \| \Phi_-^0 \|_{\mathscr{H}^L(I^-)}. \quad (385)$$

Moreover there exists $\Phi_+^{0,\varepsilon} \in \mathscr{H}^R(I^+)$ *satisfying:*

$$\| f^\varepsilon(\xi_+, \xi_-, \omega) - \hat{\Phi}_+^0(\xi_+, \xi_-, \omega) \|_{\mathscr{H}(V_t \cap K_\varepsilon)} \to 0, \qquad t \to +\infty. \quad (386)$$

The Membrane Paradigm states that $\Phi_+^{0,\varepsilon}$ approximates the limit on I^+ of the solution in K with past data, null on H^-, equal to Φ_-^0 on I^-:

THEOREM 7.3. – *With the notations of Theorem* 7.2 *we have*

$$\Phi_+^{0,\varepsilon} \to S_K(0, \Phi_-^0) \quad \textit{in } \mathscr{H}^R(I^+), \qquad \varepsilon \to 0. \quad (387)$$

Proof of Theorem 7.1. – We prove immediately the uniqueness of the solution: assume

$$\Phi_-^1 = 0 \qquad \textit{and} \qquad \Phi_-^0 = 0, \quad (388)$$

then (373) and (374) give

$$\| f \|_{\mathscr{H}(V_t)} \to 0, \qquad t \to -\infty,$$

and we conclude by (367) f is null.

We shall use the fact that a free electromagnetic field in the Minkowski spacetime has an asymptotic profile and is caracterised by it:

LEMMA 7.4. — *Let* χ_0 *be satisfying* (365) *with* $c>0$. *We define*

$$\mathscr{I}_{1,0}: \tilde{\mathscr{H}}_0 \to \tilde{\mathscr{H}}_1, \tag{389}$$

$$\left.\begin{aligned} \mathscr{I}_{1,0} \mathrm{U}_0(r_*, \omega) &= r_* \chi_0(r_*) \mathrm{U}_0(r_*, \omega) \quad \text{if} \quad r_* \geqq 0, \\ \mathscr{I}_{1,0} \mathrm{U}_0(r_*, \omega) &= 0 \quad \text{if} \quad r_* < 0. \end{aligned}\right\} \tag{390}$$

We introduce wave operators

$$\mathrm{W}_{1,0} \mathrm{U}_0 = s - \lim_{t \to +\infty} e^{it\mathrm{H}_1} \mathscr{I}_{1,0} e^{-it\mathrm{H}_0} \mathrm{U}_0 \quad \text{in } \tilde{\mathscr{H}}_1, \tag{391}$$

$$\mathrm{W}^{\pm}_{1,0} \mathrm{U}_1 = s - \lim_{t \to \pm\infty} e^{it\mathrm{H}_0} \mathscr{I}^*_{1,0} e^{-it\mathrm{H}_1} \mathrm{U}_1 \quad \text{in } \tilde{\mathscr{H}}_0; \tag{392}$$

Then $\mathrm{W}_{1,0}$ *is defined from* $\mathscr{H}_0$ *to* $\mathscr{H}_1^-$ *and* $\mathrm{W}^{\pm}_{1,0}$ *is defined from* $\mathscr{H}_1^{\mp}$ *to* $\mathscr{H}_0$.

These results are classical; we can prove them easily by using Cook's method (to see so in e. g. [20] a proof by the Lax-Phillips method involving the Radon transform).

Given $\Phi^1_- \in \mathscr{H}^{\mathrm{R}}(\mathrm{H}^-)$ and $\Phi^0_- \in \mathscr{H}^{\mathrm{L}}(I^-)$ we define

$$\mathrm{U}_1(r_*, \omega) = \Phi^1_-(\xi_+ = 0, \xi_- = 2 \operatorname{Arctg}(-2 r_0 e^{r_*/2r_0}), \omega), \tag{393}$$

$$\mathrm{U}_{1,0}(r_*, \omega) = \Phi^0_-(\xi_+ = 2 \operatorname{Arctg}(2 r_0 e^{r_*/2r_0}), \xi_- = -\pi, \omega), \tag{394}$$

Then we have:

$$\mathrm{U}_1 \in \mathscr{H}_1^-, \quad \| \mathrm{U}_1 \|_{\mathscr{H}_1^-} = \| \Phi^1_- \|_{\mathscr{H}^{\mathrm{R}}(\mathrm{H}^-)}, \tag{395}$$

$$\mathrm{U}_{1,0} \in \mathscr{H}_1^+, \quad \| \mathrm{U}_{1,0} \|_{\mathscr{H}_1^+} = \| \Phi^0_- \|_{\mathscr{H}^{\mathrm{L}}(\mathrm{I}^-)}, \tag{396}$$

and we put

$$\mathrm{U}_0 = \mathrm{W}^-_{1,0}(\mathrm{U}_{1,0}). \tag{397}$$

We verify that $\hat{\Phi}^1_-$, $\hat{\Phi}^0_-$ defined by (375) satisfies:

$$(\xi_+, \xi_-, \omega) \in \mathrm{V}_t \Rightarrow \hat{\Phi}^1_-(\xi_+, \xi_-, \omega) = [e^{-it\mathrm{H}_1} \mathrm{U}_1] \circ \mathrm{T}^{-1}(\xi_+, \xi_-, \omega). \tag{398}$$

$$(\xi_+, \xi_-, \omega) \in \mathrm{V}_t \Rightarrow \hat{\Phi}^0_-(\xi_+, \xi_-, \omega) = [e^{-it\mathrm{H}_1} \mathrm{U}_{1,0}] \circ \mathrm{T}^{-1}(\xi_+, \xi_-, \omega). \tag{399}$$

Now we establish that the finite energy solution of the characteristic Cauchy problem is given by:

$$f = [r \alpha e^{-it\mathrm{H}} \mathrm{W}^-(\mathrm{U}_1, \mathrm{U}_0)] \circ \mathrm{T}^{-1} \tag{400}$$

where operator W^- is defined by (203).

We have

$$\| f(\xi_+, \xi_-, \omega) - \hat{\Phi}^1_-(\xi_+, \xi_-, \omega) \|_{\mathscr{H}(\mathrm{V}^1_t)} = \| \chi_1(r_*) [r \alpha e^{-it\mathrm{H}} \mathrm{W}^-(\mathrm{U}_1, \mathrm{U}_0) - e^{-it\mathrm{H}_1} \mathrm{U}_1] \|_{\tilde{\mathscr{H}}_1}. \tag{401}$$

Thanks to remark 4.6 this last quantity tends to 0 as $t \to -\infty$, that proves (373). On the other hand

$$\begin{aligned}
&\| f(\xi_+, \xi_-, \omega) - \hat{\Phi}^0_- (\xi_+, \xi_-, \omega) \|_{\mathscr{H}(V_t^0)} \\
&\quad = \| \chi_0 (r_*) [r \alpha e^{-itH} W^- (U_1, U_0) - e^{-itH_1} U_{1,0}] \|_{\tilde{\mathscr{H}}_1} \\
&\quad \leqq \| \chi_0 (r_*) [r \alpha e^{-itH} W^- (U_1, U_0) - r_* e^{-itH_0} U_0] \|_{\tilde{\mathscr{H}}_1} \\
&\qquad + \| \chi_0 (r_*) [r_* e^{-itH_0} U_0 - e^{-itH_1} U_{1,0}] \|_{\tilde{\mathscr{H}}_1} \\
&\quad \leqq C \{ \chi_0 (r_*) [e^{-itH} W^- (U_1, U_0) - e^{-itH_0} U_0] \|_{\tilde{\mathscr{H}}_0} \\
&\qquad + \| \chi_0 (r_*) [1 - r_*/r \alpha] e^{-itH_0} U_0 \|_{\tilde{\mathscr{H}}_0} \\
&\qquad + \| \chi_0 (r_*) [r_* e^{-itH_0} U_0 - e^{-itH_1} U_{1,0}] \|_{\tilde{\mathscr{H}}_1} \}. \qquad (402)
\end{aligned}$$

By using remark 3.4, the decay of the local energy of $e^{-itH_0} U_0$ and (391), we conclude that (374) is satisfied. Now by (373), (374) we have

$$\| f \|^2_{\mathscr{H}(V_t)} = \| \Phi^1_- \|^2_{\mathscr{H}^R(H^-)} + \| \Phi^0_- \|^2_{\mathscr{H}^L(I^-)} + \varepsilon_- (t) \qquad (403)$$

with:

$$\varepsilon_- (t) \to 0 \quad as \ t \to -\infty. \qquad (404)$$

Hence (367) gives (379).

Now we put

$$(U_1^+, U_0^+) = S (U_1, U_0) \in \mathscr{H}_1^+ \times \mathscr{H}_0, \qquad U_{1,0}^+ = W_{1,0} U_0^+ \in \mathscr{H}_1^-, \ (405)$$

$$U_1^+ (r_*, \omega) = \Phi_+^1 (\xi_+ = 2 \, \text{Arctg} (2 r_0 e^{r_*/2r_0}), \xi_- = 0, \omega), \qquad (406)$$

$$U_{1,0}^+ (r_*, \omega) = \Phi_+^0 (\xi_+ = \pi, \xi_- = 2 \, \text{Arctg} (-2 r_0 e^{r_*/2r_0}), \omega). \qquad (407)$$

Then we have:

$$\Phi_+^1 \in \mathscr{H}^L (H^+), \qquad \Phi_+^0 \in \mathscr{H}^R (I^+), \qquad (408)$$

$$(\xi_+, \xi_-, \omega) \in V_t \ \Rightarrow \ \hat{\Phi}_+^1 (\xi_+, \xi_-, \omega) = [e^{-itH_1} U_1^+] \circ T^{-1} (\xi_+, \xi_-, \omega). \quad (409)$$

$$(\xi_+, \xi_-, \omega) \in V_t \ \Rightarrow \ \hat{\Phi}_+^0 (\xi_+, \xi_-, \omega) = [e^{-itH_1} U_{1,0}^+] \circ T^{-1} (\xi_+, \xi_-, \omega). \quad (410)$$

$$\| f(\xi_+, \xi_-, \omega) - \hat{\Phi}_+^1 (\xi_+, \xi_-, \omega) \|_{\mathscr{H}(V_t^1)} = \| \chi_1 (r_*) [r \alpha e^{-itH} W^+ (U_1^+, U_0^+) - e^{-itH_1} U_1^+] \|_{\tilde{\mathscr{H}}_1}, \quad (411)$$

$$\| f(\xi_+, \xi_-, \omega) - \hat{\Phi}_+^0 (\xi_+, \xi_-, \omega) \|_{\mathscr{H}(V_t^0)} = \| \chi_0 (r_*) [r \alpha e^{-itH} W^+ (U_1^+, U_0^+) - e^{-itH_1} U_{1,0}^+] \|_{\tilde{\mathscr{H}}_1}, \quad (412)$$

where

$$(U_1^+, U_0^+) \in \mathscr{H}_1^+ \times \mathscr{H}_0, \ W^+ (U_1^+, U_0^+) = W_1^+ U_1^+ + W_0^+ U_0^+. \quad (413)$$

By changing t into-t in estimates (401) (402) we deduct (376) (377) from (411) (412); moreover

$$\| f\|^2_{\mathscr{H}(V_t)} = \| \Phi^1_+ \|^2_{\mathscr{H}^L(H^+)} + \| \Phi^0_+ \|^2_{\mathscr{H}^R(I^+)} + \varepsilon_+(t),$$
$$\varepsilon_+(t) \to 0 \quad as \ t \to \infty, \tag{414}$$

then (403) and (414) prove that S_k is isometric. Q.E.D.

Proof of Theorem 7.2. – If $\Phi^0_- = 0$, (384) implies

$$\| f\|_{\mathscr{H}(V_t \cap K_\varepsilon)} \to 0, \qquad t \to -\infty; \tag{415}$$

on the other hand, (313) implies that any solution of (361) (362) (383) satisfies

$$t_0 < t_1, \quad \| f\|_{\mathscr{H}(V_{t_1} \cap K_\varepsilon)} \leqq \| f\|_{\mathscr{H}(V_{t_0} \cap K_\varepsilon)}; \tag{416}$$

therefore $f=0$ and the uniqueness of the solution is proved.

We introduce U_0 defined by (394) and (397) and we define f^ε on K_ε by putting:

$$f^\varepsilon = [r\alpha \mathscr{V}_\varepsilon(t) W^\varepsilon_- U_0] \circ T^{-1} \tag{417}$$

where wave operator W^ε_- is defined by (309). As previous we have

$$\begin{aligned}
&\| f^\varepsilon(\xi_+, \xi_-, \omega) - \hat{\Phi}^0_-(\xi_+, \xi_-, \omega)\|_{\mathscr{H}(V_t \cap K_\varepsilon)} \\
&\quad = \| \chi_0(r_*)[r\alpha \mathscr{V}_\varepsilon(t) W^\varepsilon_- U_0 - e^{-itH_1} U_{1,0}]\|_{\tilde{\mathscr{H}}_1} \\
&\quad \leqq \| \chi_0(r_*)[r\alpha \mathscr{V}_\varepsilon(t) W^\varepsilon_- U_0 - r_* e^{-itH_0} U_0]\|_{\tilde{\mathscr{H}}_1} \\
&\qquad + \| \chi_0(r_*)[r_* e^{-itH_0} U_0 - e^{-itH_1} U_{1,0}]\|_{\tilde{\mathscr{H}}_1} \\
&\quad \leqq C\{ \| \chi_0(r_*)[\mathscr{V}_\varepsilon(t) W^\varepsilon_- U_0 - e^{-itH_0} U_0]\|_{\tilde{\mathscr{H}}_0} \\
&\qquad + \| \chi_0(r_*)[1 - r_*/r\alpha] e^{-itH_0} U_0\|_{\tilde{\mathscr{H}}_0} \\
&\qquad + \| \chi_0(r_*)[r_* e^{-itH_0} U_0 - e^{-itH_1} U_{1,0}]\|_{\tilde{\mathscr{H}}_1}\}.
\end{aligned} \tag{418}$$

By using remark 6.6, the decay of the local energy of $e^{-itH_0} U_0$ and (391),we conclude that (384) is satisfied. Now by (416), (384) we have

$$\| f\|^2_{\mathscr{H}(V_t)} \leqq \| \Phi^0_- \|^2_{\mathscr{H}^L(I^-)}. \tag{419}$$

Now we put

$$U_0^{+,\varepsilon} = S_\varepsilon(U_0) \tag{420}$$

$$U_{1,0}^{+,\varepsilon} = W_{1,0} U_0^{+,\varepsilon} \in \mathscr{H}_1^-, \tag{421}$$

$$U_{1,0}^{+,\varepsilon}(r_*, \omega) = \Phi_+^{0,\varepsilon}(\xi_+, \xi_- = 2\,\mathrm{Arctg}(-2r_0 e^{r_*/2r_0}), \omega). \tag{422}$$

Then we have:

$$\Phi_{+}^{0,\varepsilon} \in \mathcal{H}^{R}(I^{+}), \tag{423}$$

$$\begin{aligned}
&\| f^{\varepsilon}(\xi_{+}, \xi_{-}, \omega) - \hat{\Phi}_{+}^{0,\varepsilon}(\xi_{+}, \xi_{-}, \omega) \|_{\mathcal{H}(V_t \cap K_\varepsilon)} \\
&\quad = \| \chi_0(r_*) [r \alpha \mathcal{V}_{\varepsilon}(t) W_{-}^{\varepsilon} U_0 - e^{-itH_1} U_{1,0}^{+,\varepsilon}] \|_{\tilde{\mathcal{H}}_1} \\
&\quad \leqq \| \chi_0(r_*) [r \alpha \mathcal{V}_{\varepsilon}(t) W_{-}^{\varepsilon} U_0 - r_* e^{-itH_0} U_0^{+,\varepsilon}] \|_{\tilde{\mathcal{H}}_1} \\
&\qquad + \| \chi_0(r_*) [r_* e^{-itH_0} U_0^{+,\varepsilon} - e^{-itH_1} U_{1,0}^{+,\varepsilon}] \|_{\tilde{\mathcal{H}}_1} \\
&\quad \leqq C \{ \| \chi_0(r_*) [\mathcal{V}_{\varepsilon}(t) W_{-}^{\varepsilon} U_0 - e^{-itH_0} U_0^{+,\varepsilon}] \|_{\tilde{\mathcal{H}}_0} \\
&\qquad + \| \chi_0(r_*) [1 - r_*/r \alpha] e^{-itH_0} U_0^{+,\varepsilon} \|_{\tilde{\mathcal{H}}_0} \\
&\qquad + \| \chi_0(r_*) [r_* e^{-itH_0} U_0^{+,\varepsilon} - e^{-itH_1} U_{1,0}^{+,\varepsilon}] \|_{\tilde{\mathcal{H}}_1} \}.
\end{aligned} \tag{424}$$

Hence we conclude from (420) and (391) that $\Phi_{+}^{0,\varepsilon}$ satisfies (386).

Q.E.D.

Proof of Theorem 7.3. — We evaluate

$$\Delta_{\varepsilon} = \| \Phi_{+}^{0,\varepsilon}(\xi_{+} = \pi, \xi_{-}, \omega) - \Phi_{+}^{0}(\xi_{+} = \pi, \xi_{-}, \omega) \|_{\mathcal{H}^{R}(I^{+})} \tag{425}$$

where Φ_{+}^{0}, $\Phi_{+}^{0,\varepsilon}$ are defined by (407) and (422).
We have

$$\Delta_{\varepsilon} = \| U_{1,0}^{+,\varepsilon} - U_{1,0}^{+} \|_{\tilde{\mathcal{H}}_1} = \| S_{\varepsilon} U_0 - S U_0 \|_{\mathcal{H}_0}. \tag{426}$$

Therefore Theorem 6.5 implies $\Delta_{\varepsilon} \to 0$, $\varepsilon \to 0$.

Q.E.D.

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