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ARNE JENSEN

TOHRU OZAWA

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Classical and quantum scattering for Stark Hamiltonians with slowly decaying potentials

by

Arne JENSEN ([1])

Department of Mathematics and Computer Science, Institute for Electronic Systems, Aalborg University, Fredrik Bajers Vej 7, DK-9220 Aalborg Ø, Denmark

and

Tohru OZAWA

Research Institute for Mathematical Sciences, Kyoto University, Kyoto 606, Japan

ABSTRACT. — A discrepancy between classical and quantum scattering for Stark Hamiltonians is shown to exist for slowly decaying potentials. Let $H_0 = -(1/2)\Delta + x_1$ and $H = H_0 + V(x)$ on $L^2(\mathbb{R}^n)$. For $V(x) \sim c|x|^{-\gamma}$, $0 < \gamma \leqq 1/2$, as $|x| \to \infty$, the usual quantum wave operators between H_0 and H do not exist. In classical one-dimensional scattering the classical wave operators exist and are asymptotically complete for the corresponding classical problem for $V(x) = O(\log(1+|x|))^{-\alpha})$, $\alpha > 1$, as $x \to -\infty$.

RÉSUMÉ. — Nous montrons un désaccord entre la diffusion classique et quantique pour les hamiltoniens de Stark qui possèdent des potentiels à décroissance lente. Soient $H_0 = -1/2\Delta + x_1$ et $H = H_0 + V(x)$ sur $L^2(\mathbb{R}^n)$. Pour $V(x) \sim c|x|^{-\gamma}$, $0 < \gamma \leqq 1/2$ si $|x| \to \infty$, l'opérateur d'onde habituel n'existe pas entre H_0 et H. Dans le cas de la diffusion classique à une dimension, les opérateurs d'onde classique existent et sont asymptotiquement complets pour $V(x) = O((\log(1+|x|))^{-\alpha})$, $\alpha > 1$, si $x \to -\infty$.

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1. INTRODUCTION

In recent developments in scattering theory analogies between classical and quantum mechanics have played an important role. This is in particular the case with the geometric method where one decomposes phase space (or configuration space) into certain regions where the motion of the quantum particle is closely approximated by the corresponding motion of free classical particles. It is therefore of interest to investigate a case where this analogy breaks down.

In this paper we study in detail the quantum scattering for the free Stark Hamiltonian $H_0 = -(1/2)\Delta + x_1$ on $L^2(\mathbb{R}^n)$ and its perturbation $H = H_0 + V$ by a slowly decaying potential, and the corresponding classical problem in one dimension. Our main result can be summarized briefly as follows: There is a discrepancy between quantum and classical scattering theory. In quantum scattering theory the usual wave operators $W_\pm = s\text{-}\lim_{t \to \pm\infty} e^{itH} e^{-itH_0}$ do not exist for potentials with a behavior $\lambda |x|^{-\gamma}$, $0 < \gamma \leqq 1/2$, as $x_1 \to -\infty$. In the corresponding one-dimensional scattering problem the classical wave operators exist and are asymptotically complete for potentials with a decrease as slow as $O((\log(1+|x|))^{-\alpha})$, $\alpha > 1$, as $x \to -\infty$. Thus the concept of a long range potential is different in the quantum and classical case. This is in contrast to the case of usual Schrödinger operators (zero electric field), where the borderline between long range and short range behavior is the Coulomb potential $c|x|^{-1}$ both in the classical and the quantum case.

This paper is organized as follows: In section 2 we give our result on non-existence of quantum wave operators and discuss the concept of a Stark short range potential in some detail. In section 3 we prove existence and completeness of classical wave operators for the one-dimensional problem with a slowly decaying potential. Section 4 contains various remarks.

2. ON LONG RANGE POTENTIALS IN QUANTUM SCATTERING FOR STARK HAMILTONIANS

In this section we give results that characterize a class of long range potentials for the free Stark Hamiltonian. To explain the concept of a long range potential we first give a review of the concept of a short range potential.

We start by recalling the usual framework for quantum scattering. Let $H_0 = -(1/2)\Delta + x_1$ denote the free Stark Hamiltonian on $\mathcal{H} = L^2(\mathbb{R}^n)$. It is essentially self-adjoint on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$. The domain is denoted by $\mathscr{D}(H_0)$. The spectrum is the real line, *i.e.* $\sigma(H_0)=\mathbb{R}$, and it is purely absolutely continuous.

Let V denote a symmetric operator with $\mathscr{D}(V)\supset\mathscr{D}(H_0)$ and assume that V is H_0-bounded with relative bound less than one. Then $H=H_0+V$ is self-adjoint with $\mathscr{D}(H)=\mathscr{D}(H_0)$. V is called the potential. The Møller wave operators are defined as

$$W_{\pm}\varphi=\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}\varphi \tag{2.1}$$

for $\varphi\in\mathscr{D}(W_{\pm})=\{\varphi\in\mathscr{H};\ \lim_{t\to\pm\infty} e^{itH}e^{-itH_0}\varphi\,\text{exist}\}$. *See* e.g. [7], [11] for general results on wave operators. The wave operator are said to be asymptotically complete, in Range $(W_{\pm})=\mathscr{H}_p(H)^{\perp}$, where $\mathscr{H}_p(H)$ denotes the closed subspace spanned by the L^2-eigenvectors of H.

We introduce the following definition :

DEFINITION 2.1. – The potential V is said to be Stark short range (or Stark-SR), if the wave operators $W_{\pm}$ exist and are asymptotically complete.

We state briefly some sufficient conditions for a potential to be Stark-SR. Let $A=i\partial/\partial x_1=-p_1$. The following result is proved in [5]:

THEOREM 2.2. – *Let* V *be a potential as above. Assume*

(i) $(H+i)^{-1}-(H_0+i)^{-1}$ *is compact.*

(ii) *There exist integers* $k,\ l\geqq 0$ *and a real number* $\mu>1$ *such that the operator*

$$(H+i)^{-k}V(H_0+i)^{-l}(1+A^2)^{\mu/2} \tag{2.2}$$

extends to a bounded operator on $\mathscr{H}$.

Then V *is Stark-SR.*

The condition (ii) requires momentum decay. One can give other explicit conditions for a multiplication operator. The first characterizations of $V(x)$ as a Stark short range potential were obtained in [1], [13] (existence), and [2] (completeness). More general results were obtained by Yajima [15] and Simon [12]. We state the following result from [15] : Let $\chi_{\pm}\in C^{\infty}(\mathbb{R})$ satisfy $0\leqq\chi_{\pm}\leqq 1$, $\chi_{+}+\chi_{-}=1$, and $\chi_{+}(x_1)=1$ for $x_1\geqq 1$, $\chi_{+}(x_1)=0$ for $x_1\leqq 0$.

PROPOSITION 2.3. – *Let* V *be a real-valued function on* $\mathbb{R}^n$. *Assume for some* $\delta>1/2$

$$V(x)=\{(1+x_1^2)^{-\delta/2}\chi_{-}(x_1)+(1+x_1^2)^{1/2}\chi_{+}(x_1)\}(V_1(x)+V_2(x))$$

with $V_1 \in L^\infty(\mathbb{R}^n)$, $\lim_{|x| \to \infty} V_1(x) = 0$, and $V_2 \in L^2_{loc}(\mathbb{R}^n)$ such that for some ν *with* $0 < \nu < 4$,

$$\lim_{|x| \to \infty} \left((1 + x_1^2) \int_{|x-y| \leq 1} |V_2(y)|^2 |x-y|^{\nu - n} dy \right) = 0.$$

Then V *is Stark-SR.*

Stated in rough asymptotic form, the conditions of Proposition 2.3 are essentially

$$V(x) = \begin{cases} O(|x_1|^{-\delta}), & \text{as } x_1 \to -\infty, \\ o(x_1), & \text{as } x_1 \to +\infty, \\ o(1), & \text{as } |x'| \to \infty, \end{cases}$$

where we wrote $x = (x_1, ') \in \mathbb{R} \times \mathbb{R}^{n-1}$.

There are other classes of potentials that are Stark-SR without having this asymptotic behavior. The decay in the $x_1 \to -\infty$ direction can be replaced by an "oscillation" condition. We recall the following result from [3], [4] :

PROPOSITION 2.4. – *Assume* $n = 1$. *Let* V *be a real-valued function on* $\mathbb{R}$, *decomposable as* $V = V_1 + U$, *where* V_1 *satisfies the condition in Proposition* 2.3, *and where* $U = W''$ *with* $W \in C^4(\mathbb{R})$, *bounded with four bounded derivatives. Then* V *is Stark-SR.*

The results above characterize some classes of short range potentials for the free Stark Hamiltonian H_0. We should note that potentials satisfying the conditions in either Proposition 2.3 or Proposition 2.4 also satisfy the two conditions (i) and (ii) in Theorem 2.2.

The definition of a long range potential can be given in the form of the negation of Definition 2.1. This is not entirely satisfactory, since usually other spectral properties should be preserved. For example, we would like to have the property that $\sigma_{ac}(H) = \sigma_{ac}(H_0)$ where $\sigma_{ac}(.)$ denotes the absolutely continuous spectrum, and also the invariance of the essential spectra: $\sigma_{ess}(H) = \sigma_{ess}(H_0)$. It was conjectured in [13] that the borderline for the short range potential should be the behavior $O(|x_1|^{-1/2})$ as $x_1 \to -\infty$. For the purely homogeneous potential of the form $\lambda |x|^{-\gamma}$, $\lambda \neq 0$, $0 < \gamma \leq 1/2$ ($\gamma < 1/2$ if $n = 1$), this conjecture was proved in [10]. Here we present a generalization of this result:

THEOREM 2.5. – *Let* $V = V_0(x) + V_1$, *where* $V_0(x) = \lambda |x|^{-\gamma}$, $\lambda \in \mathbb{R} \setminus \{0\}$, $0 < \gamma \leq 1/2$ ($\gamma < 1/2$ *if* $n = 1$), *and* V_1 *is a symmetric operator satisfying the conditions in Theorem* 2.2. *Let* $\varphi \in \mathcal{H}$ *with Fourier transform* $\hat{\varphi} \in C_0^\infty(\mathbb{R}^n)$. *Assume that the limit*

$$\lim_{t \to \pm\infty} e^{itH} e^{-itH_0} \varphi$$

exists in $\mathcal{H}$. *Then* $\varphi = 0$. *The same result holds for* $t \to -\infty$.

Proof. — The proof is obtained by modifying the first step of the proof in [10]. Let $\varphi \in \mathscr{H}$ with $\hat{\varphi} \in C_0^\infty(\mathbb{R}^n)$ be given. We note that $\varphi \in \mathscr{D}((H_0 - i)^k)$ for any integers $k \geqq 0$. Let $u_+ = \lim_{t \to \pm\infty} e^{itH} e^{-itH_0} \varphi = W_+ \varphi$. Using the general arguments in [7], [11], we get that $u_+ \in \mathscr{D}((H-i)^k)$ and furthermore, $(H-i)^k u_+ = W_+((H_0 - i)^k \varphi)$.

Let $W(t) = e^{itH} e^{-itH_0}$. Redoing the first step in the proof of the theorem in [10], we get

$$((W(t) - W(s))\varphi, u_+) = i\int_s^t (V e^{-i\tau H_0}\varphi, e^{-i\tau H} u_+)\, d\tau$$
$$= i\int_s^t (V_0 e^{-i\tau H_0}\varphi, e^{-i\tau H} u_+)\, d\tau + i\int_s^t (V_1 e^{-i\tau H_0}\varphi, e^{-i\tau H} u_+)\, d\tau. \quad (2.3)$$

The first term in (2.3) is treated as in [10]. Namely, by a homogeneity argument and the propagation estimate used in [1], we have for some constant $C > 0$

$$(\lambda/|\lambda|)\,\mathrm{Im}\, i\int_s^t (V_0 e^{-i\tau H_0}\varphi, e^{-i\tau H} u_+)\, d\tau \geqq (|\lambda|/4)\|\varphi\|^2 \int_s^t \tau^{-2\gamma}\, d\tau - C\|\varphi\|.\|(1+|x|)^{n+1}\varphi\| s^{-2\gamma}$$

for all $t > s > T$ with $T > 0$ sufficiently large.

To treat the second term of (2.3) we recall the propagation estimate used in [3], [5]: For $s \geqq 0$ we have

$$\|(1+A^2)^{-s/2} e^{-itH_0}(1+A^2)^{-s/2}\| \leqq C(1+t^2)^{-s/2}, \qquad t \in \mathbb{R},$$

where $\|.\|$ denotes the operator norm on $\mathscr{H}$. We rewrite the integrand in the second term in (2.3) to get

$$(V_1 e^{-i\tau H_0}\varphi, e^{-i\tau H} u_+) = ((H+i)^{-k} V_1 (H_0+i)^{-l}(1+A^2)^{\mu/2} \times (1+A^2)^{-\mu/2} e^{-i\tau H_0}(1+A^2)^{-\mu/2}.(1+A)^{\mu/2}(H_0+i)^l \varphi, e^{-i\tau H}(H-i)^k u_+).$$

Hence we have

$$|(V_1 e^{-i\tau H_0}\varphi, e^{-i\tau H} u_+)| \leqq C\|(1+A^2)^{\mu/2}(H_0+i)^l\varphi\|.\|(H_0-i)^k\varphi\|(1+t^2)^{-\mu/2}.$$

Collecting these estimates, we have

$$(\lambda/|\lambda|)\,\mathrm{Im}((W(t) - W(s))\varphi, u_+) \geqq (|\lambda/4)\|\varphi\|^2 \int_s^t \tau^{-2\gamma}\, d\tau - C(s^{-2\gamma} + s^{1-\mu}).$$

The left hand side of the inequality above is bounded by $2\|\varphi\|^2$ and the integral in the first term of the right hand side diverges as $t \to +\infty$. Therefore we have $\varphi = 0$.

Q.E.D.

Remark 2.6. — (i) The result in Theorem 2.5 can also be proved by using the chain rule for wave operators and propagation estimates for $H_0 + \tilde{V}_0$, where $\tilde{V}_0 \in C^\infty(\mathbb{R}^n)$ and $\tilde{V}_0(x) = V_0(x)$ for $|x| \geqq 1$.

(ii) The result in [10] can be strengthened slightly: Assume that $\lim_{t \to \pm\infty} e^{it(H_0+V_0)} e^{-itH_0} \varphi$ exists for some $\varphi \in \mathcal{H}$. Then $\varphi = 0$.

We can then also strengthen the result in Theorem 2.5 similarly under the additional assumption that $k=0$ in (2.2).

3. ONE-DIMENSIONAL CLASSICAL SCATTERING THEORY FOR STARK HAMILTONIANS

In this section we prove a result on one-dimensional classical Stark Hamiltonians. The main result is asymptotic completeness in the classical scattering theory for a class of potentials with slow decay as $x \to -\infty$.

Throughout this section we consider only the one-dimensional case. Some remarks on higher dimensions can be found in section 4.

The free classical Stark Hamiltonian is the function

$$H_0(x, p) = (1/2) p^2 + x$$

on phase space $\mathbb{R}^2$. Newton's equation is

$$\begin{aligned} \ddot{x}(t) &= -1, \\ x(0) &= \xi, \\ \dot{x}(0) &= \nu, \end{aligned} \tag{3.1}$$

with the solution

$$y(t) = -(1/2) t^2 + t\nu + \xi, \tag{3.2}$$

where we used the dot notation for the time-derivative. We introduce the following assumptions for slowly decaying potentials.

ASSUMPTION 3.1. — V is a real-valued differentiable function on $\mathbb{R}$ such that $V, V' \in L^\infty(\mathbb{R})$ and V' is Lipschitz continuous.

Furthermore, there exist $C>0$, $k \geqq 1$, $\alpha > 1$, $x_0 < 0$ such that

$$|V(x)| \leqq C \varphi_\alpha(x) \quad \text{for all } x \leqq x_0 \tag{3.3}$$

with

$$\varphi_\alpha(x) = \prod_{j=1}^{k-1} (1+\underbrace{\log(1+\log(\ldots(1+\log}_{j-\text{times}}(1+|x|))\ldots)))^{-1}$$
$$\times (1+\underbrace{\log(1+\log(\ldots(1+\log}_{k-\text{times}}(1+|x|))\ldots)))^{-\alpha} \quad (3.4)$$

with the convention $\prod_{j=1}^{0} (\ldots) = 1$.

ASSUMPTION 3.2. — V is a real-valued continuous function on $\mathbb{R}$. There exist $\beta > 1$, $\rho_0 > 1$, and $C > 0$, such that for all $\rho \geqq \rho_0$, $|x_1| \geqq \rho$, $|x_2| \geqq \rho$

$$|V(x_1) - V(x_2)| \leqq C\,\varphi_\beta(\rho)\,|x_1 - x_2|$$

where φ_β is a function of the form (3.4) for some $k \geqq 1$.

We note that a function φ_α from (3.4) satisfies

$$\int_t^\infty s^{-1}\,\varphi_\alpha(s)\,ds < \infty$$

and

$$\int_t^\infty \int_\tau^\infty s^{-2}\,\varphi_\alpha(s)\,ds\,d\tau < \infty$$

provided t is sufficiently large.

Let V satisfy Assumption 3.1. The full Hamiltonian considered here is $H(x, p) = H_0(x, p) + V(x)$. We look at solutions to Newton's equation

$$\begin{aligned} \ddot{x}(t) &= -1 - V'(x(t)), \\ x(0) &= x_0. \\ \dot{x}(0 &= v_0. \end{aligned} \quad (3.5)$$

It is well-known that there exists a unique solution to (3.5), defined for all $t \in \mathbb{R}$. The initial conditions (x_0, v_0) in (3.5) are classified according to the large time behavior of the solution. We write $(x_0, v_0) \in \mathcal{M}_{\text{bound}}$ and call the solution to (3.5) a bound state, if

$$\sup_{t \in \mathbb{R}} (|x(t)| + |\dot{x}(t)|) < \infty,$$

i.e. the orbit lies in a bounded subset of phase space. We write $(x_0, v_0) \in \mathcal{M}_{\text{scat}}$ and call the solution to (3.5) a scattering state, if there exist two free solutions $y^\pm(t)$ [*see* (3.2)] such that

$$\lim_{t \to \pm\infty} (|x(t) - y^\pm(t)| + |\dot{x}(t) - \dot{y}^\pm(t)|) = 0.$$

Asymptotic completeness in classical scattering theory means that there are no other types of asymptotic behavior, up to a set of measure zero, *i.e.* that we have

$$\mathbb{R}^2 = \mathcal{M}_{\text{bound}} \cup \mathcal{M}_{\text{scat}} \cup \mathcal{M}_0 \,(\text{disjoint})$$

with $\mathcal{M}_0$ a set of Lebesgue measure zero. *See* [11] for further discussion and references.

We have the following result:

THEOREM 3.3. — *Let* V *satisfy Assumption* 3.1. *Then the classical scattering problem for* (3.5) *is asymptotically complete.*

Proof. — We shall only give the essential step in the proof. We consider the $t \to +\infty$ case. The other case is treated analogously. Let $x(t)$ be a solution which is not bounded in time, *i.e.* we assume

$$\limsup_{t \to +\infty} (|x(t)| + |\dot{x}(t)|) = +\infty.$$

Arguing as in [3], we conclude from conservation of energy and boundedness of V that we actually have

$$\lim_{t \to +\infty} x(t) = -\infty,$$
$$\lim_{t \to +\infty} \dot{x}(t) = -\infty.$$

Using Assumption 3.1, we can find $C_1 > 0$ such that

$$|V(x)(1 + V'(x))| \leqq C_1$$

for all $x \in \mathbb{R}$, and then we can fix $t_0 > 0$ such that for all $t \geqq t_0$ we have $\dot{x}(t) < 0$ and

$$|V(x(t))(1 + V'(x(t)))|(\dot{x}(t))^{-2} \leqq 1/2. \tag{3.6}$$

We integrate Newton's equation from t_0 to t to get

$$\begin{aligned}
\dot{x}(t) - \dot{x}(t_0) &= -(t - t_0) - \int_{t_0}^{t} V'(x(s))\,ds \\
&= -(t - t_0) - \int_{t_0}^{t} V'(x(s))\,\dot{x}(s)(\dot{x}(s))^{-1}\,ds \\
&= -(t - t_0) - V(x(t))(\dot{x}(t))^{-1} + V(x(t_0))(\dot{x}(t_0))^{-1} \\
&\quad - \int_{t_0}^{t} V'(x(s))\,\ddot{x}(s)(\dot{x}(s))^{-2}\,ds \\
&= -(t - t_0) - V(x(t))(\dot{x}(t))^{-1} + V(x(t_0))(\dot{x}(t_0))^{-1} \\
&\quad + \int_{t_0}^{t} V(x(s))(1 + V'(x(s)))(\dot{x}(s))^{-2}\,ds,
\end{aligned} \tag{3.7}$$

where we first used integration by parts, and then Newton's equation. Using (3.6) in (3.7), we get

$$\dot{x}(t) \leqq -(1/4)\,t \tag{3.8}$$

for all $t \geqq t_1$ for some $t_1 \geqq t_0$. Going back to (3.7), we conclude the existence of the limit

$$\lim_{t \to +\infty} (\dot{x}(t) + t) = v^+. \tag{3.9}$$

The estimate (3.8) allows us to rewrite (3.7) as

$$\dot{x}(t) + t = v^+ - V(x(t))(\dot{x}(t))^{-1} + \int_t^\infty V(x(s))(1 + V'(x(s)))(\dot{x}(s))^{-2} ds. \tag{3.10}$$

We note the following expression for v^+:

$$v^+ = x(t_0) + t_0 + V(x(t_0))(\dot{x}(t_0))^{-1} + \int_{t_0}^\infty V(x(s))(1 + V'(x(s))(\dot{x}(s))^{-2} ds.$$

We use (3.9) to find $t_2 \geqq t_1$ such that for all $t \geqq t_2$

$$\dot{x}(t) + t \leqq v^+ + 1$$

Integrating both sides of the inequality above, we have

$$x(t) - x(t_2) + (1/2)t^2 - (1/2)t_2^2 \leqq (v^+ + 1)(t - t_2).$$

Thus there exists $t_3 \geqq t_2$ such that for all $t \geqq t_3$

$$x(t) \leqq -(1/4)t^2. \tag{3.11}$$

We can now integrate both sides of (3.10) once more to get

$$x(t) + (1/2)t^2 = x(t_3) + (1/2)t_3^2 + (t - t_3)v^+ - \int_{t_3}^t V(x(s))(\dot{x}(s))^{-1} ds + \int_{t_3}^t \int_\tau^\infty V(x(s))(1 + V'(x(s)))(\dot{x}(s))^{-2} ds\, d\tau. \tag{3.12}$$

Using (3.11) and Assumption 3.1, we see that the following limit exists:

$$\lim_{t \to +\infty} (x(t) + (1/2)t^2 - t v^+) = \xi^+.$$

We also get the expression

$$\xi^+ = (1/2)t_3^2 - t_3 v^+ + x(t_3) - \int_{t_3}^\infty V(x(s))(\dot{x}(s))^{-1} ds + \int_{t_3}^\infty \int_\tau^\infty V(x(s))(1 + V'(x(s)))(\dot{x}(s))^{-2} ds\, d\tau. \tag{3.13}$$

This completes the essential step of the proof. The remaining arguments are well-known. *See*, e.g. [3], [11].

Q.E.D.

THEOREM 3.4. — *Let* V *satisfy Assumptions* 3.1 *and* 3.2. *Let* $(\xi, \nu) \in \mathbb{R}^2$ *and let* $y(t) = -(1/2)t^2 + t\nu + \xi$. *Then there exist solutions* $x^+(t)$ *and* $x^-(t)$ *to Newton's equation* (3.5) *such that*

$$\lim_{t \to \pm\infty} (|x^\pm(t) - y(t)| + |\dot{x}^\pm(t) - \dot{y}(t)|) = 0. \tag{3.14}$$

Proof. — We prove the result in the $t \to +\infty$ case. The other case is treated analogously. We omit the superscript + in the sequel. We look for the solution in the form

$$x(t) = y(t) + u(t),$$

where $u \in C^2(\mathbb{R})$ and $|u(t)| + |\dot{u}(t)| \to 0$ as $t \to +\infty$. If $x(t)$ satisfies Newton's equation, then $u(t)$ must satisfy

$$\ddot{u}(t) = -V'(y(t) + u(t)). \tag{3.15}$$

We transform this differential equation into a pair of integral equations by the arguments in the proof of Theorem 3.3. A solution is then found by using a fixed point argument. To be more specific, we proceed as follows:

Let $t_0 \geqq 1$ be a parameter to be fixed later. Let

$$\begin{aligned}\mathcal{B}(t_0) = \{(u, w);\ & u, w \in C((t_0, \infty), \mathbb{R}), \\ & |u(t)| + |w(t)| \to 0 \text{ as } t \to \infty, \\ & |u(t)| + |w(t)| \leqq 1/2 \text{ for all } t \geqq t_0\}\end{aligned}$$

and define on $\mathcal{B}(t_0)$ the metric

$$d((u_1, w_1), (u_2, w_2)) = \|u_1 - u_2\|_\infty + \|w_1 - w_2\|_\infty.$$

With this metric $\mathcal{B}(t_0)$ becomes a complete metric space.

Define a map by $J(u, w) = (\tilde{u}, \tilde{w})$, where

$$\begin{aligned}\tilde{u}(t) = \int_t^\infty & V(y(s) + u(s))(\dot{y}(s) + w(s))^{-1}\,ds \\ & + \int_t^\infty \int_\tau^\infty V(y(s) + u(s))(1 + V'(y(s) + u(s))) \\ & \qquad \times (\dot{y}(s) + w(s))^{-2}\,ds\,d\tau. \qquad (3.16)\\ \tilde{w}(t) = -V&(y(t) + u(t))(\dot{y}(t) + w(t))^{-1} \\ & - \int_t^\infty V(y(s) + u(s))(1 + V'(y(s) + u(s))) \\ & \qquad \times (\dot{y}(s) + w(s))^{-2}\,ds. \qquad (3.17)\end{aligned}$$

To continue the proof we need the following lemma:

LEMMA 3.5. — *There exists* $T_0 \geqq 1$ *such that for all* $t_0 \geqq T_0$ *the map* J *is well-defined on* $\mathcal{B}(t_0)$ *and takes* $\mathcal{B}(t_0)$ *into itself. There exists a constant* c_0, $0 < c_0 < 1$, *such that for all* $t_0 \geqq T_0$ *and all* $(u_1, w_1), (u_2, w_2) \in \mathcal{B}(t_0)$ *we* *have*

$$d(\mathrm{J}(u_1, w_1), \mathrm{J}(u_2, w_2)) \leqq c_0\, d((u_1, w_1), (u_2, w_2)), \tag{3.18}$$

i.e. J *is a contraction on* $\mathscr{B}(t_0)$.

Proof of Lemma 3.5. – For a fixed $(\xi, \nu) \in \mathbb{R}^2$ we can find $s_0 \geqq 1$ and $C > 0$ such that

$$|y(s)+u(s)| \geqq C s^2,$$
$$|\dot{y}(s)+w(s)| \geqq C s,$$

for all $s \geqq s_0$ and all $(u, w) \in \mathscr{B}(s_0)$. Using Assumption 3.1, we get from (3.16) for all $t \geqq t_0 \geqq s_0$

$$|\tilde{u}(t)| \leqq C \int_{t_0}^{\infty} s^{-1} \varphi_\alpha(s)\, ds + C \int_{t_0}^{\infty} \int_{\tau}^{\infty} s^{-2} \varphi_\alpha(s)\, ds\, d\tau. \tag{3.19}$$

Thus $\tilde{u}$ is a well-defined continuous function, and $\tilde{u}(t) \to 0$ as $t \to \infty$. From (3.17) we get for all $t \geqq t_0 \geqq s_0$

$$|\tilde{w}(t)| \leqq C t_0^{-1} + C \int_{t_0}^{\infty} s^{-2} \varphi_\alpha(s)\, ds. \tag{3.20}$$

Thus $\tilde{w}$ is a well-defined continuous function, and $\tilde{w}(t) \to 0$ as $t \to \infty$. The estimates (3.19) and (3.20) then show that $(\tilde{u}, \tilde{w}) \in \mathscr{B}(t_0)$, if $t_0 \geqq T_0$, and T_0 is sufficiently large.

Using Assumption 3.2, we get after some straightforward computations that (3.18) holds, provided T_0 is chosen sufficiently large.

Q.E.D.

Proof of Theorem 3.4 *continued.* – For $t_0 \geqq T_0$, Lemma 3.5 and the contraction map theorem imply the existence and uniqueness of $(u, w) \in \mathscr{B}(t_0)$ with

$$(u, w) = \mathrm{J}(u, w). \tag{3.21}$$

It remains to verify that $u \in C^2((t_0, \infty))$ and satisfies (3.15). We write out (3.12) explicitly using the definition of J to get for $t \geqq t_0$

$$u(t) = \int_t^{\infty} \mathrm{V}(y(s)+u(s))\,(\dot{y}(s)+w(s))^{-1}\, ds + \int_t^{\infty} \int_{\tau}^{\infty} \mathrm{V}(y(s)+u(s))\,(1+\mathrm{V}'(y(s)+u(s))) \times (\dot{y}(s)+w(s))^{-2}\, ds\, d\tau. \tag{3.22}$$

and

$$w(t) = -V(y(t)+u(t))(\dot{y}(t)+w(t))^{-1} - \int_t^\infty V(y(s)+u(s))(1+V'(y(s)+u(s))) \times (\dot{y}(s)+w(s))^{-2}\,ds. \quad (3.23)$$

It follows immediately from (3.22) and (3.23) that $u \in C^1((t_0, \infty))$ with

$$\dot{u}(t) = w(t). \quad (3.24)$$

We introduce the abbreviations

$$f(t) = -V(y(t)+u(t)),$$

$$g(t) = -\int_t^\infty V(y(s)+u(s))(1+V'(y(s)+u(s)))(\dot{y}(s)+w(s))^{-2}\,ds.$$

Then f, $g \in C^1((t_0, \infty))$, and we can rewrite (3.23) as

$$w(t) = f(t)(-t+v+w(t))^{-1} + g(t)$$

or

$$w(t)^2 + (-t+v-g(t))\,w(t) + (t-v)\,g(t) - f(t) = 0. \quad (3.25)$$

The discriminant is given by

$$D(t) = (-t+v-g(t))^2 - 4((t-v)\,g(t) - f(t)).$$

Since both f and g are bounded functions, there exists $t_1 \geqq t_0$ such that for all $t \geqq t_1$ we have $D(t) \geqq (1/2)\,t^2$. It follows that either solution to (3.25) is in $C^1((t_1, \infty))$. Hence $u \in C^2((t_1, \infty))$ and $\ddot{u} = \dot{w}$. We differentiate both sides of (3.23) and make a substitution $w = \dot{u}$ to get

$$\ddot{u}(t) = \dot{w}(t) = -V'(y(t)+u(t)) + V(y(t)+u(t))(-1+\ddot{u}(t))(\dot{y}(t)+\dot{u}(t))^{-2} + V(y(t)+u(t))(1+V'(y(t)+u(t)))(\dot{y}(t)+\dot{u}(t))^{-2}.$$

This expression can be rewritten as

$$(\ddot{u}(t) + V'(y(t)+u(t)))(1 - V(y(t)+u(t))(\dot{y}(t)+\dot{u}(t))^{-2}) = 0.$$

Taking $t \geqq t_2 \geqq t_1$, we get

$$|V(y(t)+u(t))(\dot{y}(t)+\dot{u}(t))^{-2}| \leqq 1/2,$$

and hence, for $t \geqq t_2$

$$\ddot{u}(t) = -V'(y(t)+u(t)).$$

By uniqueness and global existence of solutions to (3.5), we extend this result to all $t \in \mathbb{R}$. This concludes the proof of Theorem 3.4.

Q.E.D.

Let $(\xi, \nu) \in \mathbb{R}^2$ and let $x^{\pm}$ be the solutions from Theorem 3.4 satisfying (3.14). We define the classical wave operators by

$$\Omega_{\pm}(\xi, \nu) = (x^{\pm}(0), \dot{x}^{\pm}(0)). \tag{3.26}$$

We can reformulate the results of Theorems 3.3 and 3.4 as

COROLLARY 3.6. – *The classical wave operators* $\Omega_{\pm}$ *defined by* (3.26) *are bijections from* $\mathbb{R}^2$ *onto* $\mathcal{M}_{\text{scat}}$.

Remark 3.7. – The decay condition in Assumption 3.1 is almost optimal for asymptotic completeness. This can be seen by examining the expression (3.13) for a potential of the type $V \in C^2(\mathbb{R})$, $V(x) = \varphi_1(x)$, $x < x_0$, $V(x) = 0$, $x > x_0 + 1$, where φ_1 is the function from (3.4) with $\alpha = 1$. In this case we find that the limit

$$\lim_{t \to +\infty} (x(t) + (1/2)t^2 - t\nu^+)$$

does not exist.

It is possible to extend the result of Theorem 3.3 to include the class of "oscillating" potentials from [3].

THEOREM 3.8. – *Let* $V = V_1 + U$, *where* V_1 *satisfies Assumption* 3.1 *and* $U = W'$, $W \in C^2(\mathbb{R})$ *with* $W, W', W'' \in L^{\infty}(\mathbb{R})$ *and* W'' *is Lipschitz continuous. Then the classical scattering problem for* (3.5) *is asymptotically complete.*

Proof. – The result follows by combining the proofs of Theorem 3.3 above and Theorem 4.1 in [3].

Q.E.D.

The conclusion of Theorem 3.4 has not been obtained for the class of potentials considered in Theorem 3.8.

4. REMARKS

If we compare the quantum scattering results in section 2 with the classical scattering results in section 3, then we observe a remarkable discrepancy between the two cases. The classical scattering is asymptotically complete for potentials with slow logarithmic decay, e.g. $(\log(1+|x|))^{-\alpha}$, $\alpha > 1$, as $x \to -\infty$, whereas in the quantum case no state is asymptotic to a free state (in the sense of $\lim_{t \to \pm\infty} \| e^{-itH}\varphi - e^{-itH_0}\psi \| = 0$) for potentials with decay $|x|^{-\gamma}$, $0 < \gamma \leqq 1/2$, as $x \to -\infty$. This seems to be the first discrepancy of this type, which has been observed. For the case of Schrödinger operators with magnetic fields a discrepancy in the opposite direction has been observed. For magnetic fields with decay rate

$B(x)=O(|x|^{-\delta})$, $3/2<\delta\leqq 2$, as $|x|\to\infty$, the quantum system is asymptotically complete, whereas the corresponding classical system need not be asymptotically complete, *see* [9].

The quantum scattering for Stark Hamiltonians with long range potentials satisfying $V\in C^{\infty}(\mathbb{R}^n)$, $|\partial_x^\alpha V(x)|\leqq C(1+|x|)^{-\varepsilon-|\alpha|/2}$ for some $0<\varepsilon<1/2$ and all multi-indices α has been studied in [14] by using the stationary modifier analogous to the one used for ordinary Schrödinger operators in [8]. A time-dependent modified free evolution of Dollard-type has been constructed in [6], and asymptotic completeness proved in the one-dimensional case. The results in [6] show that a very mild modification of the free evolution is needed to get the existence and completeness of the modified wave operators.

It would be of considerable interest to study the classical system in the higher dimensional cases, to see whether the completeness holds for potentials with slow decay (in the sense of Assumption 3.1) as $x_1\to-\infty$. Preliminary computations indicate that this may be the case. Details of these computations will be given elsewhere.

In [1] the classical scattering problem for Stark Hamiltonians is discussed in $\mathbb{R}^n$ for a force with decay $O(|x_1|^{-1-\varepsilon})$. Some growth in the x'-variable is permitted. As the authors remark, this class is too general to hope for completeness to hold.

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