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On the homomorphisms of sum logics

by

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ABSTRACT. — We show that every σ -homomorphism of logics $h: L_1 \rightarrow L_2$, where $(L_1, \mathcal{S}(L_1))$ and (L_2, \mathcal{S}_2) are sum logics [where $\mathcal{S}(L_1)$ is the set of all σ -additive states on L_1 and \mathcal{S}_2 is a quite full set of σ -additive states on L_2] is necessarily a lattice σ -homomorphism. Some applications of this statement to Hilbert space logics and projection logics of von Neumann algebras are given.

RÉSUMÉ. — Nous prouvons que tout σ -homomorphisme de logiques $h: L_1 \rightarrow L_2$ où $(L_1, \mathcal{S}(L_1))$ et (L_2, \mathcal{S}_2) sont des logiques sommes [$\mathcal{S}(L_1)$ est l'ensemble de tous les états σ -additifs sur L_1 et \mathcal{S}_2 est un ensemble fort d'états σ additifs sur L_2] est nécessairement un σ -homomorphisme de treillis. Nous donnons des applications de ce résultat aux logiques d'espaces de Hilbert et aux logiques d'algèbres de Von-Neumann.

1. INTRODUCTION

By a (quantum) logic we will mean an orthomodular σ -lattice $L(0, 1, ', \wedge, \vee)$ (see [13] for detailed definition). Two elements a, b of L

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are *orthogonal* (written $a \perp b$) if $a \leq b'$, and $a, b \in L$ are *compatible* if there are mutually orthogonal elements a_1, b_1, c in L such that $a = a_1 \vee c$ and $b = b_1 \vee c$.

Let L_1 and L_2 be two logics. A map $h: L_1 \rightarrow L_2$ is said to be a *homomorphism* (of logics) if (i) $h(1) = 1$, (ii) $h(a') = h(a)'$ for any $a \in L_1$, (iii) $h(a \vee b) = h(a) \vee h(b)$ for any $a, b \in L_1$, $a \perp b$. A homomorphism $h: L_1 \rightarrow L_2$ is said to be a σ -*homomorphism* if (iii') $h\left(\bigvee_{i=1}^{\infty} a_i\right) = \bigvee_{i=1}^{\infty} h(a_i)$ for any sequence $(a_i)_{i \in \mathbb{N}}$ of mutually orthogonal elements of L_1 . A homomorphism $h: L_1 \rightarrow L_2$ is a *complete* homomorphism if (iii'') $h\left(\bigvee_{i \in I} a_i\right) = \bigvee_{i \in I} h(a_i)$ for any set $(a_i)_{i \in I}$ of pairwise orthogonal elements of L_1 .

A logic homomorphism $h: L_1 \rightarrow L_2$ is a *lattice homomorphism* if $h(a \vee b) = h(a) \vee h(b)$ [dually, $h(a \wedge b) = h(a) \wedge h(b)$] for any $a, b \in L_1$. A σ -homomorphism (complete homomorphism) of logics which is a lattice homomorphism, is a lattice σ -homomorphism (complete lattice homomorphism).

It is easy to see that if $h: L_1 \rightarrow L_2$ is a logic homomorphism, then $h(0) = 0$, $a, b \in L_1$ and $a \leq b$ imply $h(a) \leq h(b)$ in L_2 and $a, b \in L_1$ and $a \perp b$ imply $h(a) \perp h(b)$ in L_2 . Moreover, if $a \leftrightarrow b$, then $h(a) \leftrightarrow h(b)$ and $h(a \vee b) = h(a) \vee h(b)$, $h(a \wedge b) = h(a) \wedge h(b)$. In particular, if L_1 is a Boolean (σ -) algebra, then every (σ -) homomorphism from L_1 into a logic L_2 is a lattice (σ -) homomorphism.

In general, a logic σ -homomorphism need not be a lattice σ -homomorphism. A simple reasoning shows that for a two-dimensional Hilbert space H , there is a two-valued homomorphism from $L(H)$ into itself, which is not a lattice homomorphism. It is easy to find a logic homomorphism from a horizontal sum of three four-element Boolean algebras into the Boolean algebra with three atoms that is even a bijection but it is not a lattice homomorphism.

The aim of this paper is to show that every σ -homomorphism of logics $h: L_1 \rightarrow L_2$ between two sum logics $(L_1, \mathcal{S}(L_1))$ and (L_2, \mathcal{S}) , is necessarily a lattice σ -homomorphism. We note that a special case of this statement is proved in [12]. Since a projection logic of a von Neumann algebra of operators acting on a separable complex Hilbert space which does not contain any I_2 -factor as a direct summand is a sum logic in the sense of our definition (see, e. g., the review paper [9]), we obtain as a consequence that every logic σ -homomorphism from such a projection logic into a projection logic of any von Neumann algebra is necessarily a lattice σ -homomorphism. In particular, if H and K are complex separable Hilbert spaces and $\dim H \neq 2$, then using Hamhalter's result (see [7]), we find that

every homomorphism of projection logics $h: L(H) \rightarrow L(K)$ is a lattice σ -homomorphism.

2. HOMOMORPHISMS OF SUM LOGICS

Let L be a logic. A *state* on L is a map $s: L \rightarrow [0, 1]$ such that $s(1) = 1$, and $s(\bigvee_{i \in \mathbb{N}} a_i) = \sum_{i \in \mathbb{N}} s(a_i)$ for any sequence $(a_i)_{i \in \mathbb{N}}$ of mutually orthogonal elements in L . Let $\mathcal{S}(L)$ denote the set of all states on L . A logic L is called *quite full* if for any $a, b \in L$ such that $a \not\leq b$ there is $s \in \mathcal{S}(L)$ such that $s(a) = 1$ and $s(b) \neq 1$.

An *observable* on a logic L is a σ -homomorphism from the σ -algebra $\mathcal{B}(\mathbb{R})$ of Borel subsets of the real line \mathbb{R} into L . If x is an observable and $s \in \mathcal{S}(L)$, then the map $s_x: \mathcal{B}(\mathbb{R}) \rightarrow [0, 1]$ defined by $s_x(E) = s(x(E))$ is a probability measure on $\mathcal{B}(\mathbb{R})$. The expectation of an observable x in a state s is then defined by $s(x) = \int ts_x(dt)$, if the integral exists. An observable x is *bounded* if there is a compact subset $C \subset \mathbb{R}$ such that $s(C) = 1$. An observable x is a *proposition observable* if $x(\{0, 1\}) = 1$. To every $a \in L$, there is a unique proposition observable q_a such that $q_a(\{1\}) = a$.

If x is a bounded observable on L , then $s(x)$ exists and is finite for every $s \in \mathcal{S}(L)$. In [5], there is proved that also the opposite statement is true: If $\mathcal{S} \subset \mathcal{S}(L)$ is σ -convex and quite full, then an observable x is bounded if and only if $s(x)$ exists and is finite for all $s \in \mathcal{S}$.

Let x, y be bounded observables on a logic L with a quite full set of states \mathcal{S} . We say that a bounded observable z is a *sum* of x and y if $s(x) + s(y) = s(z)$ for all $s \in \mathcal{S}(L)$. A couple (L, \mathcal{S}) , where L is a logic and \mathcal{S} is a σ -convex quite full set of states is said to be a *sum logic* if for every pair x, y of bounded observables on L there is a unique sum z . We shall write $z = x + y$ if z is the sum of x and y . For more details about sum logics we refer to [6].

Now we are ready to state and prove our main result.

THEOREM 1. — *Let $(L_1, \mathcal{S}(L_1))$ and (L_2, \mathcal{S}) , $\mathcal{S} \subset \mathcal{S}(L_2)$ be sum logics. Then every logic σ -homomorphism $h: L_1 \rightarrow L_2$ is a lattice σ -homomorphism.*

Proof. — Let $h: L_1 \rightarrow L_2$ be a σ -homomorphism. Let x be an observable on L_1 . Define $h(x): \mathcal{B}(\mathbb{R}) \rightarrow L_2$ by $h(x)(E) = h(x(E))$. Then $h(x)$ is an observable on L_2 . For every $s \in \mathcal{S}(L_2)$, the map $s \cdot h: L_1 \rightarrow [0, 1]$ defined by $s \cdot h(a) = s(h(a))$ is a state on L_1 . If x is a bounded observable on L_1 , then for any $s \in \mathcal{S}(L_2)$, $s(h(x)) = s \cdot h(x)$, where $s \cdot h \in \mathcal{S}(L_1)$, and therefore $s \cdot h(x)$ exists and is finite. This implies that $h(x)$ is bounded. For $a \in L_1$, let q_a denote the proposition observable such that $q_a(\{1\}) = a$. Then

$h(q_a) (\{1\}) = h(q_a(\{1\})) = h(a)$, similarly $h(q_a(\{0\})) = h(a)'$. This proves that $h(q_a) = q_{h(a)}$.

Now let x, y be bounded observables on L_1 and let $x + y = z$. Since for every $s \in \mathcal{S}$, $s.h \in \mathcal{S}(L_1)$, we find $s.h(z) = s.h(x) + s.h(y)$, i.e., $s(h(z)) = s(h(x)) + s(h(y))$ for all $s \in \mathcal{S}$, which entails $h(z) = h(x) + h(y)$.

By Gudder [6], for any pair a, b of elements of a sum logic L we have $(q_a + q_b) (\{2\}) = a \wedge b$. Let $a, b \in L_1$. Then

$$\begin{aligned} h(a \wedge b) &= h((q_a + q_b) (\{2\})) = h(q_a + q_b) (\{2\}) \\ &= (h(q_a) + h(q_b)) (\{2\}) \\ &= (q_{h(a)} + q_{h(b)}) (\{2\}) = h(a) \wedge h(b). \end{aligned}$$

Hence $h(a \wedge b) = h(a) \wedge h(b)$ holds for all $a, b \in L_1$. This proves that h is a lattice σ -homomorphism.

3. APPLICATIONS

PROPOSITION 1. — *Let H, K be separable, complex Hilbert spaces ($\dim H \geq 3$) and let $h: L(H) \rightarrow L(K)$ be a homomorphism of logics. Then h is a lattice σ -homomorphism.*

Proof. — See [7].

PROPOSITION 2. — *Let H, K be separable, complex Hilbert spaces ($\dim H \geq 3$) and let $h: L(H) \rightarrow L(K)$ be a homomorphism. Then $\dim h([x]) = \dim h([y])$ for every $x, y \in H$ (where $[x]$ denotes the one-dimensional subspace generated by x). In particular, there is a finite or countably infinite cardinal r such that $\dim K = r \dim H$.*

COROLLARY. — *A homomorphism between two separable complex Hilbert space logics $L(H), L(K)$ ($\dim H \geq 3$) is necessarily injective.*

The proof follows by Theorem 1 and Matolcsi [11], Propositions 3 and 4.

Recall that a set S is of *non measurable cardinality* if there does not exist any probability measure defined on all subsets of S vanishing at all points. In the opposite case we say that S is of measurable cardinality.

PROPOSITION 3. — *Let H, K be Hilbert spaces ($\dim H \geq 3$). Then every σ -homomorphism $h: L(H) \rightarrow L(K)$ is a complete lattice homomorphism if and only if $\dim H$ is a non measurable cardinal.*

Proof. — Assume that $\dim H$ is non measurable. Let x be a unit vector of K . Then $m_x^h(M) = \langle h(M)x, x \rangle$, $M \in L(H)$, defines a completely additive state on $L(H)$ (see, e.g., [9]). Since x is arbitrary, we conclude that h

is a complete homomorphism of logics. Applying Theorem 1, we obtain that h is a complete lattice homomorphism.

Now suppose that $\dim H$ is a measurable cardinal. Let $(x_t)_{t \in T}$ be an orthonormal basis in H . Then there exists a probability measure μ on 2^T vanishing on all one-point sets.

Define

$$m(M) = \int_T \|Me_t\|^2 d\mu(t), \quad M \in L(H).$$

Then m is a state on $L(H)$ with the properties $m(H) = 1$ and $m([e_t]) = 0$ for all $t \in T$. Indeed,

$$\begin{aligned} m(H) &= \mu(T) = 1, \\ m([e_t]) &= \int \|e_t\|^2 d\mu(t) = \mu(\{t\}) = 0. \end{aligned}$$

(See also [3]). This implies that m is a σ -additive, but not completely additive measure on $L(H)$. Define

$$K_m(M, N) = \int_T \langle Me_t, Ne_t \rangle d\mu(t).$$

For $\alpha_i \in \mathbb{C}$, $i \leq n$ ($n \in \mathbb{N}$) we have

$$\begin{aligned} \sum \alpha_i \bar{\alpha}_j K_m(M_i, M_j) &= \int \sum \alpha_i \bar{\alpha}_j \langle M_i e_t, M_j e_t \rangle d\mu(t) \\ &= \int \left\| \sum \alpha_i M_i e_t \right\|^2 d\mu(t) \geq 0. \end{aligned}$$

This shows that K_m is a reproducing kernel. By [12], there is a vector-valued measure ξ defined on $L(H)$ with values in a Hilbert space K , such that $m(M) = \|\xi(M)\|^2$ ($M \in L(H)$). By [10], there is a σ -homomorphism $h: L(H) \rightarrow L(K)$ of logics and a vector $v \in K$ such that $\xi(M) = h(M)v$ ($M \in L(H)$). This entails that h is not a complete homomorphism.

In a similar manner as Proposition 2, we can prove the following statement.

PROPOSITION 4. — *Every σ -homomorphism $h: L(H) \rightarrow L(K)$, where $2 \neq \dim H$ is non measurable, is injective. Moreover, $\dim h([x]) = \dim h([y])$ for all $x, y \in H$, and there is a cardinal α ($\alpha = \dim[x]$) such that $\dim K = \alpha \dim H$.*

The following proposition is a generalization of the theorem obtained by Jajte, Paszkiewicz [8]. Using Proposition 1 and results of Aerts, Daubechies ([1], [2]) and Wright [14], we obtain an alternative proof.

PROPOSITION 5. — Let H, K be two separable Hilbert spaces with dimensions greater or equal to three and let $h: L(H) \rightarrow L(K)$ be a homomorphism. Then there exists a family of maps $(\varphi_j)_{j \in J}$ from H to K such that

- each φ_j is an isometry or anti-isometry;
- K is the orthogonal sum $K = \bigoplus_{j \in J} \varphi_j(H)$;
- for all $M \in L(H)$, $h(M) = \bigvee_{j \in J} \varphi_j^{-1}(M)$.

PROPOSITION 6 (a generalization of Dye's theorem). — Let \mathcal{M} be a von Neumann algebra with no direct summand of type I_2 . Then any σ -homomorphism of logics between the projection logic of \mathcal{M} and that of a von Neumann algebra \mathcal{N} is implemented by the direct sum of a \star -isomorphism and a \star -antiisomorphism between \mathcal{M} and \mathcal{N} .

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