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G. VODEV

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Polynomial bounds on the number of scattering poles for symmetric systems

by

G. VODEV

Section of Mathematical Physics, Institute of Mathematics,
Bulgarian Academy of Sciences, 1090 Sofia, Bulgaria

ABSTRACT. – In the case of symmetric first order systems in $\mathbb{R}^n$, $n \geqq 3$, odd, it is shown that the number $N(r)$ of the scattering poles in the disk of radius r satisfies the estimate

$$N(r) \leqq C r^{n+1} + C$$

RÉSUMÉ. – Dans le cas de systèmes symétriques de premier ordre dans $\mathbb{R}^n$, $n \geqq 3$, impair, on montre que le nombre $N(r)$ des pôles de la diffusion dans le disque de rayon r satisfaise l'estimation

$$N(r) \leqq C r^{n+1} + C$$

1. INTRODUCTION

The purpose of this paper is to obtain a polynomial bound on the number of the scattering poles associated to the problem

$$E(x)\,\partial_t u = \left(\sum_{j=1}^{n} A_j(x)\,\partial_{x_j} + B(x)\right) u \qquad \text{in } \mathbb{R}_t \times \mathbb{R}_x^n. \tag{1.1}$$

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where $n \geqq 3$, odd, $E(x) \in C^0(\mathbb{R}^n; \operatorname{Hom} \mathbb{C}^d)$, $A_j(x) \in C^1(\mathbb{R}^n; \operatorname{Hom} \mathbb{C}^d)$ are Hermitian $(d \times d)$ matrices, and $B(x) \in C^0(\mathbb{R}^n; \operatorname{Hom} \mathbb{C}^d)$. Furthermore, we make the following assumptions:

(*a*) There exists a constant $c>0$ so that $E(x) \geqq c I$ for all $x \in \mathbb{R}^n$, I being the identity $(d \times d)$ matrix;

(*b*) There exist constant Hermitian $(d \times d)$ matrices A_j^0, $j=1, \ldots, n$, and $\rho_0>0$ so that $E(x)=I$, $A_j(x)=A_j^0$ and $B(x)=0$ for $|x| \geqq \rho_0$;

(*c*) $$B(x)+B(x)^* = \sum_{j=1}^{n} \partial_{x_j} A_j(x), \qquad \forall x \in \mathbb{R}^n;$$

(*d*) $$\operatorname{Rank} \sum_{j=1}^{n} A_j(x) \xi_j = d, \qquad \forall (x, \xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus 0.$$

Under the above assumptions, it is well known that the operator $E(x)^{-1}\left(\sum_{j=1}^{n} A_j(x) \partial_{x_j} + B(x) \right)$ has a skew selfadjoint realization, which will be denoted by G, in the Hilbert space H which is by definition the space $L^2(\mathbb{R}^n; \mathbb{C}^d)$ equipped with the scalar product

$$(f, g)_H = \int_{\mathbb{R}^n} \langle E f, g \rangle \, dx,$$

where $\langle \ , \ \rangle$ denotes the scalar product in $\mathbb{C}^d$. Then, the solutions to (1.1) are expressed by the unitary group $U(t) = \exp(tG)$. Denote by G_0 the skew selfadjoint realization of the operator $\sum_{j=1}^{n} A_j^0(x) \partial_{x_j}$ in the Hilbert space $H_0 = L^2(\mathbb{R}^n; \mathbb{C}^d)$ and set $U_0(t) = \exp(t G_0)$. Note that the assumption (*d*) means that the operator G and G_0 are elliptic. Then, it is well known (*see* [5]) that the scattering matrix relating the unitary groups $U_0(t)$ and $U(t)$ has a meromorphic continuation to the entire complex plane $\mathbb{C}$. Moreover, the poles of this continuation, called scattering poles, coincide, with multiplicity, with the poles of the meromorphic continuation of the cutoff resolvent $R_\chi(z) = \chi R(z) \chi$ from $\{ z \in \mathbb{C} : \operatorname{Re} z > 0 \}$ to the entire $\mathbb{C}$, where $R(z) = (G - z)^{-1}$ for $\operatorname{Re} z > 0$, $\chi \in C_0^\infty(\mathbb{R}^n)$ is such that $\chi = 1$ for $|x| \leqq \rho_0 + 1$, $\chi = 0$ for $|x| \geqq \rho_0 + 2$. Let $\{ z_j \}$ be these poles repeated according to multiplicity, and set

$$N(r) = \# \{ z_j : |z_j| \leqq r \}.$$

Our main result is the following

THEOREM 1. — *Under the above assumptions, there exists a constant* $C>0$ *so that*

$$N(r) \leqq C r^{n+1} + C. \tag{1.2}$$

Note that in some cases the following sharper bound is known to hold:

$$N(r) \leqq C r^n + C. \tag{1.3}$$

In [7] Melrose proved (1.3) in the case of the Laplacian in exterior domains with Dirichlet or Robin boundary conditions, while in [14] Zworski proved (1.3) for the Schrödinger operator $-\Delta + V(x)$ with a potential $V \in L_0^\infty(\mathbb{R}^n)$. Recently, in [11], we have proved (1.3) for the number of the scattering poles associated to the operator

$$L = c(x)^{-1} \sum_{i,j=1}^{n} \partial_{x_i} (g_{ij}(x) \partial_{x_j}) \quad \text{in } \mathbb{R}^n,$$

where $n \geqq 3$, odd, $c(x) \in C^\infty(\mathbb{R}^n)$, $c(x) > 0$, $\forall x \in \mathbb{R}^n$; $g_{ij}(x) \in C^\infty(\mathbb{R}^n)$ are such that the matrix $\{g_{ij}(x)\}$ is a strongly positive Hermitian one for all $x \in \mathbb{R}^n$; finally, $c(x) = 1$, $g_{ij}(x) = \delta_{ij}$ for $|x| \geqq \rho_0$ with some $\rho_0 > 0$, δ_{ij} being Kronecker's symbol. In these three papers, however, the fact that the unperturbed generator is the Laplacian Δ in $\mathbb{R}^n$ has been essentially exploited. This suggests that the sharper bound (1.3) could hold in the case of the operator G if the characteristics $\lambda_j(\xi)$ of G_0, which are by definition the eigenvalues of the matrix $\sum_{j=1}^{n} A_j^0 \xi_j$, $\xi \in \mathbb{R}^n \setminus 0$, are of constant multiplicity and hence of class $C^\infty(\mathbb{R}^n \setminus 0)$. In this work we make no restrictions on the $\lambda_j(\xi)$. In particular, they may be of nonconstant multiplicity and hence nonsmooth. Note that in this generality it is hardly possible to improve (1.2) to the bound (1.3). In the present paper we propose an approach different from the ones in [3], [4], [6], [7], [10], [11] and [14], based on an application of Huygens' principle for $U_0(t)$, only. Note that in our case Huygen's principle holds, as n is odd and by the assumption (*d*) the $\lambda_j(\xi)$ do not vanish in $\mathbb{R}^n \setminus 0$.

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2. REPRESENTATION OF THE CUTOFF RESOLVENT

By Theorem 1.3.5. of [2], given any integer $m \geqq 1$ there exists a function $\Phi_m(t) \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} \Phi_m \subset [1; 2]$, $\int \Phi_m \, dt = 1$ and

$$|\partial_t^k \Phi_m(t)| \leqq C^{k+1} m^k \quad \text{for} \quad k \leqq m, \tag{2.1}$$

with a constant $C>0$ independent of k and m. Setting

$$\varphi_m(t)=\int_t^\infty \Phi_m(s)\,ds,$$

we have $\varphi_m\in C^\infty(\mathbb{R})$, $d\varphi_m(t)/dt=-\Phi_m(t)$, $\varphi_m=0$ for $t\geqq 2$, $\varphi_m=1$ for $t\leqq 1$. Set $F_m(t)=-(\partial_t-G)\varphi_m(t)U(t)$. It is easy to see that $F_m(t)=-(d\varphi_m/dt)U(t)=\Phi_m(t)U(t)$. Let $V_m(t)$ denote the solution to the equation

$$(\partial_t-G_0)V_m(t)=-F_m(t),\qquad V_m(0)=0. \tag{2.2}$$

By Duhamel's formula, we have

$$V_m(t)=-\int_0^t U_0(t-s)F_m(s)\,ds \tag{2.3}$$

Writing (2.2) in the form

$$(\partial_t-G)(V_m(t)-\varphi_m(t)U(t))=-QV_m(t),$$

where $Q=G-G_0$, we obtain using Duhamel's formula once more:

$$V_m(t)-\varphi_m(t)U(t)=-U(t)-\int_0^t U(t-s)QV_m(s)\,ds. \tag{2.4}$$

Now we are going to take Fourier-Laplace transform of this identity. Before doing so, however, let us recall the definition of this transform. Given two Hilbert spaces X and Y, $\mathscr{L}(X, Y)$ will denote the space of all linear bounded operators acting from X into Y. Let $P(t)\in\mathscr{L}(X, Y)$ be an operator-valued function such that $P(t)f\in L^1_{loc}(\mathbb{R}^+; Y)$, $\forall f\in X$, and $\|P(t)\|_{\mathscr{L}(X,Y)}\leqq C$, $\forall t$, with C independent of t. Then, the Fourier-Laplace transform $\hat{P}(z)\in\mathscr{L}(X, Y)$ of $P(t)$ is given by

$$\hat{P}(z)=\int_0^\infty e^{-tz}P(t)\,dt$$

for $z\in\mathbb{C}$, $\operatorname{Re} z>0$, and is holomorphic in this region.

Now, since $R(z)=-\hat{U}(z)$ for $\operatorname{Re} z>0$, by (2.4) we get

$$\hat{V}_m(z)-\widehat{\varphi_m U}(z)=R(z)+R(z)Q\hat{V}_m(z)\qquad \text{for}\quad \operatorname{Re} z>0.$$

Multiplying the both sides of this identity by χ, since $Q=\chi Q$, we obtain

$$R_\chi(z)(1+Q\hat{V}_m(z)\chi)=\chi\hat{V}_m(z)\chi-\chi\widehat{\varphi_m U}(z)\chi\qquad \text{for}\quad \operatorname{Re} z>0 \tag{2.5}$$

Since $\varphi_m=0$ for $t\geqq 2$, clearly $\widehat{\varphi_m U}(z)$ extends analytically to the entire $\mathbb{C}$ with values in $\mathscr{L}(H, H)$. In what follows given a compact operator $\mathscr{A}$, $\mu_j(\mathscr{A})$ will denote the characteristic values of $\mathscr{A}$, *i. e.* the eigenvalues of $(\mathscr{A}^*\mathscr{A})^{1/2}$, ordered, with multiplicity, to form a nonincreasing sequence. Also, we shall always suppose that $m\geqq n+1$. We need now the following

LEMMA 2. – *The operator-valued functions* $Q\hat{V}_m(z)\chi$ *and* $\chi\hat{V}_m(z)\chi$ *have analytic continuations to the entire complex plane* $\mathbb{C}$ *with values in the trace class operators in* $\mathscr{L}(H, H)$. *Moreover, there exists a constant* $C>0$ *independent of m and z so that*

$$\|Q\hat{V}_m(z)\chi\|_{\mathscr{L}(H,H)}\leqq C z^{-1}, \qquad \text{for} \quad z\in\mathbb{R}, \quad z\geqq 1; \tag{2.6}$$

$$\mu_j(Q\hat{V}_m(z)\chi)\leqq C\exp(C|z|), \quad \forall j, \quad \forall z\in\mathbb{C}; \tag{2.7}$$

$$\mu_j(Q\hat{V}_m(z)\chi)\leqq C^{m+1}(|z|^m+m^m)\,e^{C|z|}j^{-m/n}, \quad \forall j, \forall z\in\mathbb{C}. \tag{2.8}$$

Assume for a moment that the conclusions of Lemma 2 are fulfilled. By (2.6), $1+Q\hat{V}_m(z)\chi$ is invertable in $\mathscr{L}(H, H)$ for $z\in\mathbb{R}$, $z\gg 1$. Hence, by the analytic Fredholm theorem, since $Q\hat{V}_m(z)\chi$ is an entire family of compact operators, we conclude that $(1+Q\hat{V}_m(z)\chi)^{-1}$ is a meromorphic function on $\mathbb{C}$ with values in $\mathscr{L}(H, H)$. Now, by (2.5) we deduce that $R_\chi(z)$ has a meromorphic continuation to $\mathbb{C}$ with values in $\mathscr{L}(H, H)$ and the poles of this continuation, with multiplicity, are among the poles of $(1+Q\hat{V}_m(z)\chi)^{-1}$. Hence, introducing the entire function

$$h_m(z)=\det(1+Q\hat{V}_m(z)\chi),$$

we conclude that the poles of $R_\chi(z)$, with multiplicity, are among the zeros of $h_m(z)$. Now, to obtain (1.2) we need the following

LEMMA 3. – *There exists a constant* $C>0$ *independent of m so that*

$$|h_m(z)|\leqq C\exp(C m^{n+1}) \qquad \textit{for} \quad |z|=m. \tag{2.9}$$

Proof. – We shall derive (2.9) from (2.7) and (2.8). First, it is easy to see by (2.8) that there exists a constant $C>0$ independent of m so that

$$\mu_j(Q\hat{V}_m(z)\chi)\leqq C j^{-(n+1)/n} \qquad \text{for} \quad |z|=m, \quad \text{if } j\geqq C m^n. \tag{2.10}$$

Indeed, by (2.8), for $|z|=m$, $j\geqq q^n m^n$, with $q>0$ to be chosen below, we have

$$\mu_j(Q\hat{V}_m(z)\chi)\leqq 2C q^{n+1}(C'/q)^m j^{-(n+1)/n},$$

where $C'>0$ depends on C and n only. Now, taking $q=C'$ yields (2.10). By Weyl's convexity estimate, in view of (2.7) and (2.10), for $|z|=m$, we have

$$\begin{aligned}|h_m(z)|&\leqq\prod_{j=1}^{\infty}(1+\mu_j(Q\hat{V}_m(z)\chi))\\ &\leqq\Big(\prod_{1\leqq j\leqq C m^n} C\exp(Cm)\Big)\exp\Big(\sum_{j\geqq C m^n}\mu_j(Q\hat{V}_m(z)\chi)\Big)\\ &\leqq\exp(C' m^{n+1})\exp\Big(C\sum_{j=1}^{\infty}j^{-(n+1)/n}\Big),\end{aligned}$$

which is the desired estimate. The proof of Lemma 3 is complete.

Proof of Theorem 1. — By Jensen's inequality (*see* [9]) and the analysis before Lemma 3 we can conclude that

$$N(m/2) \leqq C_1 \sup_{|z|=m} \log|h_m(z)|, \tag{2.11}$$

with a constant $C_1 > 0$ independent of m. This combined with (2.9) yield

$$N(m/2) \leqq C_2 m^{n+1} + C_2, \tag{2.12}$$

with $C_2 > 0$ independent of m. Now, since (2.12) holds for any integer $m \geqq n+1$, this implies (1.2) at once.

3. PROOF OF LEMMA 2

Taking Fourier-Laplace transform of (2.3) we get

$$\hat{V}_m(z) = R_0(z)\hat{F}_m(z) \quad \text{for} \quad \operatorname{Re} z > 0, \tag{3.1}$$

where $R_0(z) = (G_0 - z)^{-1}$. Clearly, $\hat{F}_m(z)$ has an analytic continuation to the entire $\mathbb{C}$ with values in $\mathcal{L}(H, H)$. We need now the following.

LEMMA 4. — *The operator-valued function* $\hat{F}_m(z)\chi$ *takes values in the trace class operators in* $\mathcal{L}(H, H)$. *Moreover, there exists a constant* $C > 0$ *so that*

$$\mu_j(\hat{F}_m(z)\chi) \leqq C^{m+1}(|z|^m + m^m) e^{C|z|} j^{-m/n}, \quad \forall j, \forall z \in \mathbb{C}. \tag{3.2}$$

Assuming that the conclussions of Lemma 4 are fulfilled we shall complete the proof of Lemma 2. By the finite speed of propagation of the solutions to the problem (1.1) it is easy to see that there exists a constant $\rho > 0$, independent of m, so that $\Phi_m(t) U(t) \chi f = 0$ for $|x| \geqq \rho$, $\forall t \in \mathbb{R}$, $\forall f \in H$. Hence, choosing a function $\chi_1 \in C_0^\infty(\mathbb{R}^n)$ such that $\chi_1 = 1$ for $|x| \leqq \rho + 1$, $\chi_1 = 0$ for $|x| \geqq \rho + 2$, we deduce

$$\hat{F}_m(z)\chi = \chi_1 \hat{F}_m(z)\chi, \quad \forall z \in \mathbb{C}. \tag{3.3}$$

On the other hand, by Huygens' principle, there exists a constant $T > 0$ so that $\chi U_0(t) \chi_1 = 0$ for $t \geqq T$. Hence,

$$\chi R_0(z) \chi_1 = -\int_0^T e^{-tz} \chi U_0(t) \chi_1 \, dt \quad \text{for} \quad \operatorname{Re} z > 0.$$

Hence, $\chi R_0(z) \chi_1$ can be continued analytically to the entire $\mathbb{C}$ with values in $\mathcal{L}(H_0, H_0)$ and for this continuation we have

$$\| \chi R_0(z) \chi_1 \|_{\mathcal{L}(H_0, H_0)} \leqq C \exp(C|z|), \quad \forall z \in \mathbb{C}, \tag{3.4}$$

with some constant $C>0$. Now we shall show that so is true for the operator $QR_0(z)\chi_1$. Since

$$G_0 R_0(z)=1+z R_0(z) \quad \text{for} \quad \operatorname{Re} z>0,$$

we have

$$\begin{aligned} QR_0(z)\chi_1 &= Q(G_0-1)^{-1}(G_0-1)\chi R_0(z)\chi_1 \\ &= Q(G_0-1)^{-1}(\chi\chi_1+z\chi R_0(z)\chi_1+\chi_2 R_0(z)\chi_1) \\ &\qquad \text{for } \operatorname{Re} z>0, \end{aligned}$$

where $\chi_2=[G_0, \chi]-\chi$ is a matrix-valued function with entries of class $C_0^\infty(\mathbb{R}^n)$. Here [,] denotes the commutator. Since G_0 is elliptic, we have $Q(G_0-1)^{-1}\in\mathscr{L}(H_0, H_0)$. Now, the above representation gives the desired analytic continuation of $QR_0(z)\chi_1$ as well as the estimate

$$\| QR_0(z)\chi_1 \|_{\mathscr{L}(H_0, H_0)} \leqq C(1+(1+|z|)\|\chi R_0(z)\chi_1\|_{\mathscr{L}(H_0, H_0)}) \tag{3.5}$$

for all $z\in\mathbb{C}$. By (3.4) and (3.5) we get

$$\| QR_0(z)\chi_1 \|_{\mathscr{L}(H_0, H_0)} \leqq C\exp(C|z|), \qquad \forall z\in\mathbb{C}. \tag{3.6}$$

Now, in view of (3.1), (3.3) and Lemma 4 we can conclude that $Q\hat{V}_m(z)\chi$ and $\chi\hat{V}_m(z)\chi$ can be continued analytically to the entire $\mathbb{C}$ with values in the trace class operators in $\mathscr{L}(H, H)$. For a later use observe that it follows from (3.1) and (3.3) that

$$\| Q\hat{V}_m(z)\chi \|_{\mathscr{L}(H, H)} \leqq C\| QR_0(z)\chi_1 \|_{\mathscr{L}(H_0, H_0)} \| \hat{F}_m(z) \|_{\mathscr{L}(H, H)}, \tag{3.7}$$

with a constant $C>0$ independent of z and m.

To prove (2.6) observe that for $z\in\mathbb{R}$, $z\geqq 1$, by (3.5) we have

$$\begin{aligned} \| QR_0(z)\chi_1 \|_{\mathscr{L}(H_0, H_0)} &\leqq C(1+(1+z)\| R_0(z) \|_{\mathscr{L}(H_0, H_0)}) \\ &\leqq C(1+(1+z)z^{-1})\leqq 3C. \end{aligned}$$

Now (2.6) follows from this estimate, (3.7) and the estimate

$$\| \hat{F}_m(z) \|_{\mathscr{L}(H, H)} \leqq \int_0^\infty e^{-tz}|\Phi_m(t)|\,dt \leqq C'\int_0^\infty e^{-tz}\,dt = C' z^{-1},$$

for $z\in\mathbb{R}$, $z\geqq 1$, with a constant C' independent of z and m.

To prove (2.7) observe that

$$\| \hat{F}_m(z) \|_{\mathscr{L}(H, H)} \leqq \int_0^\infty e^{t|z|}|\Phi_m(t)|\,dt \leqq C''\int_1^2 e^{t|z|}\,dt \leqq C'' e^{2|z|}, \qquad \forall z\in\mathbb{C},$$

with a constant $C''>0$ independent of m and z. Now (2.7) follows from this estimate, (3.6), (3.7) and the well known inequality $\mu_j(\mathscr{A})\leqq\|\mathscr{A}\|$, $\forall j$. Finally, note that (2.8) follows from (3.2), (3.6) and the well known inequality $\mu_j(\mathscr{A}\mathscr{B})\leqq\|\mathscr{A}\|\mu_j(\mathscr{B})$, $\forall j$.

4. PROOF OF LEMMA 3

Set $S_m(z)=(G-1)^m \hat{F}_m(z)$. We have

$$S_m(z)=\int_0^\infty e^{-tz}\Phi_m(t)(G-1)^m U(t)\,dt$$
$$=\int_0^\infty e^{-tz}\Phi_m(t)(\partial_t-1)^m U(t)\,dt$$
$$=\int_0^\infty U(t)(-\partial_t-1)^m(e^{-tz}\Phi_m(t))\,dt. \quad (4.1)$$

Setting $p_m(t,z)=(-\partial_t-1)^m(e^{-tz}\Phi_m(t))$, we have

$$p_m(t,z)=(-1)^m\sum_{k=0}^m\binom{m}{k}\partial_t^k(e^{-tz}\Phi_m(t))$$
$$=(-1)^m\sum_{k=0}^m\binom{m}{k}\sum_{j=0}^k\binom{k}{j}(\partial_t^{k-j}e^{-tz})\,\partial_t^j\Phi_m(t)$$
$$=(-1)^m e^{-tz}\sum_{k=0}^m\binom{m}{k}\sum_{j=0}^k\binom{k}{j}(-z)^{k-j}\partial_t^j\Phi_m(t).$$

Clearly, $p_m(t,z)$ is an entire function in z, and hence, by (4.1), $S_m(z)$ is an entire $\mathscr{L}(H, H)$-valued function. Furthermore, by (2.1), we get

$$|p_m(t,z)|\leqq e^{|tz|}\sum_{k=0}^m\binom{m}{k}\sum_{j=0}^k\binom{k}{j}|z|^{k-j}C^{j+1}m^j$$
$$=C e^{|tz|}\sum_{k=0}^m\binom{m}{k}(|z|+Cm)^k$$
$$=C e^{|tz|}(1+|z|+Cm)^m\leqq e^{|tz|}C_1^{m+1}(|z|^m+m^m)$$

with some constant $C_1>0$ independent of t, z and m. This together with (4.1) lead to the estimate

$$\|S_m(z)\|_{\mathscr{L}(H,H)}\leqq\int_1^2|p_m(t,z)|\,dt\leqq C_1^{m+1}(|z|^m+m^m)e^{2|z|}. \quad (4.2)$$

Set $\Omega=\{x\in\mathbb{R}^n:|x|\leqq\rho+3\}$ and denote by H_Ω the Hilbert space obtained as a closure of $C_0^\infty(\Omega;\mathbb{C}^d)$ with respect to the norm of H. Then the operator G restricted on $C_0^\infty(\Omega;\mathbb{C}^d)$ has a skew selfadjoint realization, which will be denoted by G_Ω, in H_Ω. It follows from the definition of G_Ω that if $f\in H$ and $\operatorname{supp} f\subset\Omega$, then $G_\Omega f=G f$. To complete the proof of Lemma 3 we need the following

LEMMA 4. — *The operator* $(G_\Omega-1)^{-1}\in\mathscr{L}(H_\Omega, H_\Omega)$ *is compact. Moreover, there exists a constant* $C>0$ *so that*

$$\mu_j((G_\Omega-1)^{-m})\leqq C^m j^{-m/n},\qquad \forall j,\ \forall m. \quad (4.3)$$

Remark. – It follows from (4.3) that the operator $(G_\Omega-1)^{-m}$ is trace class for $m \geqq n+1$.

Suppose that the conclusions of Lemma 4 are fulfilled. Then, in view of (3.3) and since supp $\chi_1 \subset \Omega$, we have

$$\begin{aligned}\hat{F}_m(z)\chi &= (G_\Omega-1)^{-m}(G_\Omega-1)^m \chi_1 \hat{F}_m(z)\chi \\ &= (G_\Omega-1)^{-m}(G-1)^m \chi_1 \hat{F}_m(z)\chi = (G_\Omega-1)^{-m}(G-1)^m \hat{F}_m(z)\chi \\ &(G_\Omega-1)^{-m} S_m(z)\chi = (G_\Omega-1)^{-m}\chi_1 S_m(z)\chi.\end{aligned}$$

Now, by this identity and the above remark we conclude that $\hat{F}_m(z)\chi$ forms an entire family of trace class operators in $\mathscr{L}(H, H)$. Moreover,

$$\begin{aligned}\mu_j(\hat{F}_m(z)\chi) &\leqq \mu_j((G_\Omega-1)^{-m}) \| \chi_1 S_m(z)\chi \|_{\mathscr{L}(H_\Omega, H_\Omega)} \\ &\leqq C\mu_j((G_\Omega-1)^{-m}) \| S_m(z) \|_{\mathscr{L}(H, H)},\end{aligned}$$

which together with (4.2) and (4.3) imply (3.2) at once.

5. PROOF OF LEMMA 4

Set $\mathscr{A}=(G_\Omega-1)^{-1}$. Since the operator G_Ω is elliptic, $\mathscr{A}$ sends H_Ω into the Sobolev space $H^1(\Omega; \mathbb{C}^d)$. Hence, by Rellich's compactness theorem, $\mathscr{A}$ is compact. Moreover, if Δ_Ω is the selfadjoint realization of the Laplacian Δ with domain $D(\Delta)=C_0^\infty(\Omega)$ in the Hilbert space $L^2(\Omega)$, we have $(1-\Delta_\Omega)^{1/2}\mathscr{A} \in \mathscr{L}(H_\Omega, H_\Omega)$. Hence,

$$\mu_j(\mathscr{A}) \leqq \mu_j((1-\Delta_\Omega)^{-1/2}) \| (1-\Delta_\Omega)^{1/2}\mathscr{A} \|_{\mathscr{L}(H_\Omega, H_\Omega)}.$$

On the other hand, it is well known that

$$\mu_j((1-\Delta_\Omega)^{-1/2}) \leqq C_\Omega j^{-1/n}.$$

Hence,

$$\mu_j(\mathscr{A}) \leqq C j^{-1/n}. \tag{5.1}$$

Now we shall show that (5.1) implies (4.3). Clearly, for the adjoint operator $\mathscr{A}^*$ of $\mathscr{A}$ we have $\mathscr{A}^*=(-G_\Omega-1)^{-1}$, and hence $\mathscr{A}^*\mathscr{A}=\mathscr{A}\mathscr{A}^*$. This immediately yields

$$(\mathscr{A}^{m*}\mathscr{A}^m)^{1/2}=((\mathscr{A}^*\mathscr{A})^{1/2})^m \text{ for any integer } m \geqq 1. \tag{5.2}$$

Setting $\mathscr{B}=(\mathscr{A}^*\mathscr{A})^{1/2}$, we clearly have that $\mathscr{B}$ is a selfadjoint positively definite compact operator. Hence $\mu_j(\mathscr{B}^m)=\mu_j(\mathscr{B})^m$, which together with (5.2) imply

$$\mu_j(\mathscr{A}^m)=\mu_j(\mathscr{A})^m. \tag{5.3}$$

Now (4.3) is an immediate consequence of (5.1) and (5.3). This completes the proof of Lemma 4, and hence the proof of (1.2).

Note added in proof. – In fact, in the rigth-hand side of (2.11) it should be added a term of the form $-C\log|a_m|$, where $C>0$ is independent of m and $a_m \neq 0$ is such that

$h_m(z) = z^M(a_m + o(z))$, as $z \to 0$, with some integer $M \geqq 0$. Unfortunately, it is not very clear how estimate $|a_m|$ from below for large m. This difficulty can be avoided in the following way. By (2.6), there exists $z_0 > 0$, independent of m, so that $\|Q\hat{V}_m(z_0)\chi\| \leqq 1/2$. Hence $\|(1 + Q\hat{V}_m(z_0)\chi)^{-1}\| \leqq 2$. It is easy to see that

$$h_m(z_0)^{-1} = \det(1 - (1 + Q\hat{V}_m(z_0)\chi)^{-1} Q\hat{V}_m(z_0)\chi).$$

Using all this, in the same way as in the proof of lemma 3, one can obtain $|h_m(z_0)|^{-1} \leqq C\exp(Cm^{n+1})$. Denoting by $N(z_0, r)$ the number of the scattering poles in a disk of radius r centered at z_0, by Jensen's inequality we have

$$N(z_0, 3m/5) \leqq C \sup_{|z - z_0| = 4m/5} \log|h_m(z)| - C\log|h_m(z_0)|, \qquad C > 0.$$

It is easy to see that for $m \geqq 10z_0$ we have

$$N(m/2) \leqq N(z_0, 3m/5) \quad \text{and} \quad \sup_{|z - z_0| = 4m/5} |h_m(z)| \leqq \sup_{|z| = m} |h_m(z)|.$$

Now (2.12) follows from the above estimates and (2.9).

REFERENCES

[1] C. Bardos, J. Guillot and J. Ralston, La relation de Poisson pour l'équation des ondes dans un ouvert non borné application à la théorie de la diffusion, *Comm. Partial Diff. Eq.*, T. **7**, 1982, pp. 905-958.

[2] L. Hörmander, *The Analysis of Linear Partial Differential Operators I*, Springer Verlag, Berlin, 1983.

[3] A. Intissar, A Polynomial Bound on the Number of Scattering Poles for a Potential in Even Dimensional Space $\mathbb{R}^n$, *Comm. Partial Diff. Eq.*, T. **11**, 1986, pp. 367-396.

[4] A. Intissar, *On the Value Distribution of the Scattering Poles Associated to the Schrödinger Operator* $H = (-i\vec{\nabla} + \vec{b}(x))^2 + \alpha(x)$ *in* $\mathbb{R}^n$, $n \geqq 3$, preprint.

[5] P. D. Lax and R. S. Phillips, *Scattering Theory*, Academic Press, 1967.

[6] R. B. Melrose, Polynomial Bounds on the Number of Scattering Poles, *J. Funct. Anal.*, T. **53**, 1983, pp. 287-303.

[7] R. B. Melrose, Polynomial Bounds on the Distribution of Poles in Scattering by an Obstacle, *Journées « Equations aux Dérivées Partielles »*, Saint-Jean-de-Monts, 1984.

[8] R. B. Melrose, Weyl Asymptotics for the Phase in Obstacle Scattering, *Comm. Partial Diff. Eq.*, T. **13**, 1988, pp. 1431-1439.

[9] E. C. Titchmarsch, *The Theory of Functions*, Oxford University Press, 1968.

[10] G. Vodev, Polynomial Bounds on the Number of Scattering Poles for Metric Perturbations of the Laplacian in $\mathbb{R}^n$, $n \geqq 3$, Odd, *Osaka J. Math.* (to appear).

[11] G. Vodev, Sharp Polynomial Bounds on the Number of Scattering Poles for Metric Perturbations of the Laplacian in $\mathbb{R}^n$, preprint.

[12] M. Zworski, Distribution of Poles for Scattering on the Real Line, *J. Funct. Anal.*, T. **73**, 1987, pp. 277-296.

[13] M. Zworski, Sharp Polynomial Bounds on the Number of Scattering Poles of Radial Potentials, *J. Funct. Anal.*, T. **82**, 1989, pp. 370-403.

[14] M. Zworski, Sharp Polynomial Bounds on the Number of Scattering Poles, *Duke Math. J.*, T. **59**, 1989, pp. 311-323.

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